UNBIASED ESTIMATES FOR CERTAIN BINOMIAL SAMPLING PROBLEMS WITH APPLICATIONS[1]

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1. Introduction. The purpose of this paper is to present some theorems with applications concerning unbiased estimation of the parameter p (fraction defective) for samples drawn from a binomial distribution. The estimate constructed is applicable to samples whose items are drawn and classified one at a time until the number of defectives i, and the number of nondefectives j, simultaneously agree with one of a set of preassigned number pairs. When this agreement takes place, the sampling operation ceases and an unbiased estimate of the proportion p of defectives in the population may be made. Some examples of this kind of sampling are ordinary single sampling in which n items are observed and classified as defective or nondefective; curtailed single sampling where it is desired to cease sampling as soon as the decision regarding the lot being inspected can be made, that is as soon as the number of defectives or nondefectives attain one of a fixed pair of preassigned values; double, multiple, and sequential sampling. In the cases of double and multiple sampling the subsamples may be curtailed when a decision is reached, while for sequential sampling the process may be truncated, i.e. an upper bound may be set on the amount of sampling to be done. In section 3 expressions are given for the unique unbiased estimates of p for single, curtailed single, curtailed double, and sequential sampling.

One or two of the illustrative examples of section 3 may be of interest because their rather bizarre results suggest that some estimate other than an unbiased estimate may be preferable; but the discussion of estimates other than unbiased ones is outside the scope of this paper.

2. The estimate $\hat{p}$. For the purposes of the present paper the word point will refer only to points in the xy-plane with nonnegative integral coordinates.

We shall need the following nomenclature. A region R is a set of points containing (0, 0). The point (x_2, y_2) is *immediately beyond* (x_1, y_1) if either $x_2 = x_1 + 1, y_2 = y_1$ or $x_2 = x_1, y_2 = y_1 + 1$. A *path in R* from the point α_0 to the point α_n is a finite sequence of points $\alpha_0, \alpha_1, \cdots, \alpha_n$ such that α_i $(i > 0)$ is immediately beyond α_{i-1}, and $\alpha_j \in R$ with the possible exception of α_n. A *boundary* point, that is, an element of the boundary B of R, is a point not in R which is the last point α_n of a path from the origin. *Accessible* points are the points in R which can be reached by paths from the origin, while *inaccessible* points are the points which cannot be reached by any path from the origin.

[1] This paper was originally written by Mosteller and Savage. A communication from M. A. Girshick revealed that he had independently discovered for the sequential probability ratio test the estimate $\hat{p}(\alpha)$ given here and demonstrated its uniqueness. For purposes of publication it seemed appropriate to present the results in a single paper.

All points are thus divided into three mutually exclusive categories: accessible, inaccessible, and boundary points. The *index of a point* is the sum of its coordinates, and the *index of a region* is the least upper bound of the indices of its accessible points. A *finite* region is a region for which the indices of the accessible points are less than some number n. In particular a region containing only a finite number of points is finite.

Paths may be thought of as arising by a random process such that a path reaching $\alpha_i = (x, y)$, $\alpha_i \in R$, will be extended to $\alpha_{i+1} = (x, y + 1)$ with probability p or to $\alpha'_{i+1} = (x + 1, y)$ with probability $q = 1 - p$. We exclude $p = 0, 1$ unless these values are specifically mentioned. When a path is extended to a boundary point of R the process ceases. It is clear from the definitions that for a finite region R, paths from the origin cannot include more points than $n + 2$ where n is the index of the region. This means that a path from the origin cannot escape from a finite region and that the probability that it strikes some boundary point is unity. It is clear that each path from the origin to a boundary point or an accessible point has probability $p^y q^x$, if the point has coordinates (x, y). We will need the following statements which are immediate consequences of the discussion above:

A. *The probability of a boundary point or an accessible point being included in a path from the origin is* $P(\alpha) = k(\alpha)p^y q^x$, *where* $k(\alpha)$ *is the number of paths from the origin to the point. We shall call* $P(\alpha)$ *the probability of the point.*

B. *For a finite region* $\sum_{\alpha \in B} P(\alpha) = 1$, i.e. *the sum of the probabilities of the boundary points is unity.*

Any region for which $\sum_{\alpha \in B} P(\alpha) = 1$ will be called a *closed* region.

Of course, all finite regions are closed; but it is convenient to have a condition such as that supplied by the following theorem guaranteeing the closure of some infinite regions as well.

THEOREM 1. *A sufficient condition*[2] *that a region R be closed is that* $\liminf_{n\to\infty} A(n)/\sqrt{n} = 0$, *where* $A(n)$ *is the number of accessible points of index* n.

PROOF. We consider the ascending sequence of finite regions R_n, each consisting of the points of R whose indices are less than n. The boundary B_n of R_n can be written as the set theoretic union $K_n \cup A_n$, where K_n is $B_n \cap B$, and A_n are the accessible points of R of index n. If $\alpha \in B_n$ and $P_n(\alpha)$ is the probability of α with respect to R_n, it is easily seen that for $\alpha \in K_n$, $P_n(\alpha) = P(\alpha)$. Since every point of B is ultimately contained in the ascending sequence K_n,

$$\sum_{\alpha \in B} P(\alpha) = \lim_{n\to\infty} \sum_{\alpha \in K_n} P(\alpha) = \lim_{n\to\infty} \sum_{\alpha \in K_n} P_n(\alpha) \leq 1,$$

the inequality being a consequence of statement B. But $\sum_{\alpha \in A_n} P_n(\alpha)$ is monotonically decreasing because $\sum_{\alpha \in K_n} P_n(\alpha)$ is monotonically increasing with n while $\sum_{\alpha \in B_n} P_n(\alpha) = 1$, from statement B.

[2] If it is desired to admit $p = 0, 1$, the existence of boundary points $(x,0)$ or $(0,y)$ respectively must be postulated.

If we can show $\lim_{n\to\infty} \sum_{\alpha \epsilon A_n} P_n(\alpha) = 0$ under the condition of the theorem, the proof is complete. For any point $\alpha \epsilon A_n$, $P_n(\alpha) = k_n(\alpha)p^y q^{n-y}$ which for fixed p is $O(1/\sqrt{n})$. The sum over A_n is $O(A(n)/\sqrt{n})$ and therefore since the hypothesis of the theorem implies that $A(n)/\sqrt{n}$ attains arbitrarily small values for arbitrarily large values of n, the sum in question decreases monotonically to zero.

COROLLARY. *If the number of accessible points of R of index n is bounded, the region is closed.*

That the condition given in Theorem 1 is not a necessary condition may be seen by examining the region R consisting of all points except points of the form $(2x + 1, 2y + 1)$ and $(3, 0)$ and $(0, 3)$.

THEOREM 2. *If R is closed and R contains S, S is closed.*

PROOF. The proof is essentially similar to that of Theorem 1.

Any reasonable estimate of p will be a function defined on the boundary points, because the boundary points constitute, so to speak, a sufficient statistic for p. That is, the probability of any path from (0, 0) given the boundary point α at which it terminates is independent of p, and is in fact $1/k(\alpha)$.

We shall construct an unbiased estimate of p for closed regions R, that is a function $\hat{p}(\alpha)$, $\alpha \epsilon B$, such that $\sum_{\alpha \epsilon B} \hat{p}(\alpha)P(\alpha) = p$ (absolutely convergent).[3]

CONSTRUCTION. *Let $k^*(\alpha)$ be the number of paths in R from the point* (0, 1) *to the boundary point α, and let $\hat{p}(\alpha) = k^*(\alpha)/k(\alpha)$.* We remark that the definitions imply $k^*((0, 1)) = 1$, when (0, 1) is a boundary point.

THEOREM 3. *For any closed region R $\hat{p}(\alpha)$ is an unbiased estimate of p.*

PROOF:

$$\sum_{\alpha \epsilon B} \hat{p}(\alpha) P(\alpha) = \sum_{\alpha \epsilon B} \frac{k^*(\alpha)}{k(\alpha)} k(\alpha) p^y q^x = \sum_{\alpha \epsilon B} k^*(\alpha) p^y q^x.$$

If (0, 1) is a boundary point, then $k^*((0, 1)) = 1$ and $k^*(\alpha) = 0$, $\alpha \neq (0, 1)$, in which case the sum in question consists of the single term p. If (0, 1) is not a boundary point, consider the region R' obtained by deleting (0,1) from R, and $k'(\alpha)$, the number of paths in R' from the origin to the boundary point α of R.

$$k^*(\alpha) = k(\alpha) - k'(\alpha)$$

$$\sum_{\alpha \epsilon B} k^*(\alpha) p^y q^x = \sum_{\alpha \epsilon B} k(\alpha) p^y q^x - \sum_{\alpha \epsilon B} k'(\alpha) p^y q^x = 1 - \sum_{\alpha \epsilon B} k'(\alpha) p^y q^x.$$

Now R' is closed (Theorem 2); except for (0, 1) every boundary point of R' is

[3] Even if such a sum were p for a region which was not closed, we would not call the estimate an unbiased estimate.

easily seen to be a boundary point of R; and $k'(\alpha)$ vanishes except for the boundary points of R'. Therefore

$$p + \sum_{\alpha \epsilon B} k'(\alpha) p^y q^x = 1,$$

and the proof is complete.

It is clear from the construction that $0 \leq \hat{p}(\alpha) \leq 1$; this is rather satisfying, since an estimate of p outside of these bounds would be received with some misgivings.

Theorem 3 may be generalized to yield unbiased estimates of linear combinations of functions of the form $p^t q^u$ provided the points (u, t) are not inaccessible points. We need only let the point (u, t) play the role of $(0, 1)$. Even though the point (u, t) is inaccessible it may be possible to represent $p^t q^u$ as a polynomial, none of whose terms correspond to inaccessible points.

It is clear from Theorem 1 that $\hat{p}(\alpha)$ is an unbiased estimate of p for the usual sequential binomial tests, but the computation may be quite heavy. It should be noted that the coordinate system used here differs slightly from the coordinate system customarily used in sequential analysis. The custom is to let the x coordinate represent the number of items inspected, whereas we use it to represent the number of nondefectives; this is the only difference between the coordinates. We understand that in applications the customary procedure seems preferable, but we find the present coordinates more convenient for the purposes of ths article.

In general $\hat{p}$ is not the only unbiased estimate of p. A necessary condition for uniqueness is that the region be *simple*, that is that all the points between any two accessible points on the line $x + y = n$ be accessible points. In other words no accessible points of index n shall be separated on the line $x + y = n$ by inaccessible points or boundary points.

THEOREM 4. *A necessary condition that the estimate $\hat{p}$ be the unique unbiased estimate for the closed region R is that R be simple.*

PROOF. For a region that is not simple we shall construct a function $m(\alpha)$ not identically zero, such that

$$\sum_{\alpha \epsilon B} m(\alpha) P(\alpha) = 0. \tag{1}$$

But $\hat{p}(\alpha) + m(\alpha)$ will be an unbiased estimate of p different from $\hat{p}$.

Suppose we have a closed region R which is not simple. We consider the lowest index n where the accessible points are separated. There will be at least one uninterrupted sequence of points between some pair of accessible points that are not accessible points. It is easy to see that all the points of this uninterrupted sequence are boundary points of R. Let this sequence be the points $\alpha_i = (x_0 - i, y_0 + i)$, $i = 0, 1, \cdots, t$, $x_0 + y_0 = n$. To begin the construction of $m(\alpha)$ let $m(\alpha_j) = (-1)^j/k(\alpha)$, $0 \leqq j \leqq t$. The coordinates of the point α'' above the top point of the sequence are $(x_0 - t, y_0 + t + 1)$, and the number of paths from α'' to any point on the boundary is $l''(\alpha)$, where if α'' is a boundary point the number of paths $l''(\alpha'') = 1$; similarly $\alpha' = (x_0 + 1, y_0)$ and $l'(\alpha)$ is the number of paths from α' to the boundary point α with the same convention if α' is a boundary point. To complete the construction of $m(\alpha)$, let $m(\alpha) = -[l'(\alpha) + (-1)^t l''(\alpha)]/k(\alpha)$ for boundary points not members of the sequence under consideration. Before proceeding to check equation (1), we show that

$$\sum_{\alpha \epsilon B} l'(\alpha) p^y q^x = p^{y_0} q^{x_0+1} \; ; \sum_{\alpha \epsilon B} l''(\alpha) p^y q^x = p^{y_0+t+1} q^{x_0-t}. \tag{2}$$

Because of symmetry we need only carry out the demonstration for the first sum. If α' is a boundary point $l'(\alpha') = 1$, and for all other points α $l'(\alpha) = 0$, and the sum is the single term $p^{y_0}q^{x_0+1}$. If α' is not a boundary point consider the region obtained by deleting α' from R and the corresponding $k'(\alpha)$, the number of paths from (0, 0) to the boundary points of the new closed region R'. Every boundary of R' except α' is a boundary point of R. Let us extend the definition of $k'(\alpha)$ to the whole boundary of R by defining $k'(\alpha) = 0$ for α not in the boundary B' of R'. Then it is easy to see that

$$k(\alpha) = k'(\alpha')l'(\alpha) + k'(\alpha).$$

Now

$$\begin{aligned} 1 &= \sum_{\alpha \epsilon B} k(\alpha)p^y q^x \\ &= k'(\alpha') \sum_{\alpha \epsilon B} l'(\alpha)p^y q^x + \sum_{\alpha \epsilon B} k'(\alpha)p^y q^x \\ &= k'(\alpha') \sum_{\alpha \epsilon B} l'(\alpha)p^y q^x + 1 - k'(\alpha')p^{y_0} q^{x_0+1} \end{aligned}$$

establishing equation (2).

We now check that $m(\alpha)$ satisfies equation (1):

$$\begin{aligned} \sum_{\alpha \epsilon B} m(\alpha)k(\alpha)p^y q^x &= \sum_{j=0}^{t} (-1)^j p^{y_0+j} q^{x_0-j} - \sum_{\alpha \epsilon B} l'(\alpha)p^y q^x - \sum_{\alpha \epsilon B} (-1)^t l''(\alpha)p^y q^x \\ &= \sum_{j=0}^{t} (-1)^j p^{y_0+j} q^{x_0-j} - p^{y_0} q^{x_0+1} - (-1)^t p^{y_0+t+1} q^{x_0-t} \\ &= p^{y_0} q^{x_0-t} \left(\sum_{j=0}^{t} (-1)^j p^j q^{t-j} - q^{t+1} - (-1)^t p^{t+1} \right) \\ &= 0. \end{aligned}$$

THEOREM 5. *A necessary condition that $\hat{p}(\alpha)$ be a unique unbiased estimate of p for the closed region R is that there be no closed region R' whose boundary is a proper subset of the boundary of R.*

PROOF. Again supposing that the condition is not satisfied we shall construct a function $m(\alpha)$ not identically zero such that equation (1) is satisfied. Let $k'(\alpha)$ be the number of paths in R' to α in B of R, understanding, of course, that $k'(\alpha) = 0$ if α is not in B' of R'. Consider $m(\alpha) = 1 - k'(\alpha)/k(\alpha)$, $m(\alpha)$ is not identically zero because $k'(\alpha)$ vanishes for at least one α, but $k(\alpha)$ does not. From the closure of R and R' it is obvious that $m(\alpha)$ satisfies equation (1).

Two simple examples will suffice to show that neither simplicity nor the condition of Theorem 5 is alone sufficient to insure the uniqueness of $\hat{p}$. The region consisting of the points whose coordinates are given in the following configuration and whose boundary points are

x				
(0, 3)	x			
(0, 2)	x			
(0, 1)	(1, 1)	x	x	
(0, 0)	(1, 0)	(2, 0)	(3, 0)	x

indicated by the x's satisfies the condition of Theorem 5 but is not simple. On the other hand the region consisting of all points for which $y < 3$, except for the two points (1, 0), (1, 1) is simple but does not satisfy the conditions of Theorem 5, because the region consisting of all points except (1, 0) with $y < 3$ can play the role of R'.

The authors are unable to decide whether the two conditions together guarantee the uniqueness of $\hat{p}$ as an unbiased estimate of p, and supply the following sufficient condition which is adequate for many practical purposes.

THEOREM 6. *A sufficient condition that a closed region have $\hat{p}(\alpha)$ a unique unbiased estimate of p is that the region be simple and that there exist g, h $(0 < g, h \leqq 1)$ such that for all boundary points $| gx - hy | < M$.*

PROOF. If there were an unbiased estimate of p different from $\hat{p}$, subtracting it from p would yield an equation of the form (sum absolutely convergent):

$$\sum_{\alpha \in B} m(\alpha) p^{y} q^{x} = 0, \tag{3}$$

where $m(\alpha)$ is not identically zero. But this will be shown to be impossible. If $m(\alpha)$ were not identically zero, there would be an α_0 such that $m(\alpha_0) \neq 0$ and 1) $m(\alpha) = 0$ for all boundary points of index less than that of α_0, and 2) one of the coordinates of α_0 is less than the corresponding coordinate of any other boundary point for which $m(\alpha) \neq 0$. This follows easily from the simplicity requirement which implies that the boundary points of index n are broken into two sets *a*) those whose y coordinates are less than the y coordinates of the accessible points of index n, and *b*) those whose x coordinates are less than the x coordinates of the accessible points of index n.[4] Since the situations *a*) and *b*) are symmetrical we suppose without loss of generality that α_0 is a boundary point whose y coordinate is less than that of any other boundary point with $m(\alpha) \neq 0$. Equation (3) may be written

$$m(\alpha_0) p^{y_0} q^{x_0} + p^{y_0+1} \sum_{\substack{\alpha \in B \\ \alpha \neq \alpha_0}} m(\alpha) p^{y-y_0-1} q^{x} = 0, \tag{4}$$

[4] It will be seen as the proof proceeds that if there are no boundary points to which alternative a) applies, the restriction $g > 0$ may be removed and replaced by $g \geqq 0$, similarly if there are no boundary points to which b) applies the condition $h > 0$ may be replaced by $h \geqq 0$.

where the exponents appearing in the sum are nonnegative. But it will be shown that for sufficiently small p

$$q^{x_0} \mid m(\alpha_0) \mid > p \mid \sum_{\substack{\alpha \in B \\ \alpha \neq \alpha_0}} m(\alpha) p^{y-y_0-1} q^x \mid, \tag{5}$$

which contradicts equation (4). Now

$$\begin{aligned} \mid \Sigma m(\alpha) p^{y-y_0-1} q^x \mid &\leq \Sigma \mid m(\alpha) \mid p^{y-y_0-1} q^x \\ &\leq \Sigma \mid m(\alpha) \mid p^{y-y_0-1} q^{x-(hy_0+h+M+gx-hy)/g} \\ &= q^{-M/g} \Sigma \mid m(\alpha) \mid (pq^{h/g})^{y-y_0-1} \\ &\leq q^{-[h(y_0+1)+2M]/g} \Sigma \mid m(\alpha) \mid p^{y-y_0-1} q^x, \end{aligned} \tag{6}$$

where all the summations range over the values indicated in (5). The summation indicated in (5) is thus seen to be dominated by a convergent power series in $pq^{h/g}$.

Thus Theorem 6 shows that $\hat{p}$ is a unique estimate for the sequential binomial tests.

THEOREM 7. *A necessary and sufficient condition that $\hat{p}$ be the unique unbiased estimate of p for a closed finite region R is that R be simple.*

PROOF. The proof follows immediately from Theorems 4 and 6.

3. Applications and illustrative examples.

A. Single sampling. In single sampling a random sample of n items is drawn from a lot containing items each of which is either defective or nondefective. It is customary to estimate p, the proportion defective by the unbiased estimate i/n, where i is the number of defectives observed. The boundary of the region defined by a single sampling plan consists of all points of index n. Now $k((n-i, i)) = \binom{n}{i}$ and $k^*((n-i, i-1)) = \binom{n-1}{i-1}$. *Consequently the unique unbiased estimate of p is*

$$\hat{p}((n-i, i)) = \binom{n-1}{i-1} \Big/ \binom{n}{i} = i/n,$$

the result above.

It may be of interest to note that an unbiased estimate of the variance pq/n of the proportion $\hat{p}$, is $\binom{n-2}{i-1} \Big/ \left[\binom{n}{i} n\right] = \dfrac{i(n-i)}{n^2(n-1)}$, $(n > 1)$; this estimate is obtained by the method suggested immediately following Theorem 3.

B. Curtailed single sampling. In single sampling schemes, there is usually given a rejection number c as well as the sample size n. If c or more defectives are found in the sample the lot is rejected, but if less than c defectives are found in the sample the lot is accepted. It is customary to inspect all the items in the sample even if the final decision to accept or reject the lot is known before the completion of the inspection of the sample. One reason sometimes mentioned for this procedure is that an unbiased estimate for p is not known when the inspection is halted as soon as a decision is reached. We provide the unbiased estimate in the following paragraph.

In curtailed single sampling the boundary points when rejecting are (x, c), $c + x \leqq n$, when accepting $(n - c + 1, y)$, $y \leqq c - 1$. The region is a rectangular array and obviously simple. *The unique unbiased estimate along the horizontal line corresponding to rejection with* $c > 1$ *therefore is*

$$\hat{p}((x, c)) = \binom{c - 2 + x}{c - 2} \Big/ \binom{c + x - 1}{c - 1} = \frac{c - 1}{c + x - 1},$$

or in words, *one less than the number of defectives observed divided by one less than the number of observations. The unique unbiased estimate along the vertical line corresponding to acceptance for* $c > 1$ *is*

$$\hat{p}((n - c + 1, y)) = \binom{n - c + i - 1}{n - c} \Big/ \binom{n - c + i}{n - c} = \frac{i}{n - c + i}$$

that is, the number of defectives observed divided by one less than the number of observations. We reserved the case $c = 1$ because it is rather illuminating. The construction of Theorem 3 works as usual, and we note that $\hat{p}((0, 1)) = 1$, $\hat{p}((n, 0)) = 0$ as we might expect, but $\hat{p}((x, 1)) = 0$, $0 < x < n$.

It is somewhat startling to find that the only unbiased estimate of p for curtailed single sampling with $c = 1$ provides zero estimates unless a defective is observed on the first item. We remark that the variance of this estimate is pq. In other words, curtailed single sampling with $c = 1$ is no better for estimation purposes than a sample of size one when the unbiased estimate $\hat{p}$ is used.

A limiting case of curtailed sampling when n is unbounded has been considered by Haldane[5] as a useful technique in connection with estimates of the frequency of occurrence of rare events. The region would not be closed unless $p = 0$ were excluded. In our nomenclature there is a "rejection number" c $(c > 1)$, and we continue sampling and inspecting until c defectives have been observed. The unbiased estimate[6] is $(c - 1)/(j - 1)$, where j is the total number of observations, and of course this is the estimate given by Haldane.

C. A general curtailed double sampling plan. The following example will illustrate the sort of calculations involved in computing p for multiple and sequential plans. A sample of size n_1 is drawn and items are inspected until 1) $r_1 (1 < r_1 \leqq n_1)$ defectives are found, or 2) $n_1 - a + 1$ $(a \geqq 0)$ nondefectives are found, or 3) the sample is exhausted with neither of these events occurring. If case 3) arises, a second sample of size n_2 is drawn and inspection proceeds until a grand total of $r_2 (r_1 \leqq r_2 \leqq n_1 + n_2)$ defectives is found or $n_1 + n_2 - r_2 + 1$

[5] J. B. S. Haldane, *Nature*, Vol. 155 (1945), No. 3924.

[6] For the uniqueness, see footnote [4].

nondefectives are found. In this scheme we call r_1 and r_2 rejection numbers and a an acceptance number. *The unique unbiased estimate $\hat{p}$ is as follows:*

$$\text{(a)} \qquad \hat{p}((j, r_1)) = \frac{r_1 - 1}{r_1 + j - 1}, \qquad j = 0, 1, \cdots, n_1 - r_1;$$

$$\text{(b)} \qquad \hat{p}((n_1 - a + 1, i)) = \frac{i}{n_1 - a + i}, \qquad i = 0, 1, \cdots, a;$$

$$\text{(c)} \quad \hat{p}((x, r_2)) = \frac{\Sigma\binom{x_0 + y_0 - 1}{x_0}\binom{x - x_0 + r_2 - y_0 - 1}{r_2 - y_0 - 1}}{\Sigma\binom{x_0 + y_0}{x_0}\binom{x - x_0 + r_2 - y_0 - 1}{r_2 - y_0 - 1}},$$

$$n_1 - r_1 < x \leqq n_1 + n_2;$$

$$\text{(d)} \; \hat{p}((n_1 + n_2 - r_2 + 1, y)) = \frac{\Sigma\binom{x_0 + y_0 - 1}{x_0}\binom{n_1 + n_2 - r_2 + y - y_0 - x_0}{y - y_0}}{\Sigma\binom{x_0 + y_0}{x_0}\binom{n_1 + n_2 - r_2 + y - y_0 - x_0}{y - y_0}},$$

$$a < y \leqq n_1 + n_2;$$

where the summations extend from $y_0 = a + 1$ to $y_0 = r_1 - 1$, and $x_0 + y_0 = n_1$. In the above equations (a) and (b) are the estimates corresponding to rejection and acceptance on the basis of the first sample, while (c) and (d) correspond to rejection and acceptance when a second sample has been drawn. Rather than use the sums indicated in (c) and (d), some may find it preferable to make the estimation entirely on the basis of the first sample. If there is no curtailing, the procedure of estimation is equivalent to single sampling, and the estimate is again i/n_1 as mentioned in paragraph A above. *If the first sample is curtailed and the estimate is made on the basis of the results of the first sample only, the unique unbiased estimate is given by formula (a) when rejecting, by formula (b) when accepting, and by i/n_1 when a second sample is to be drawn.* It will be noted that (a) and (b) are identical with the expressions derived in paragraph B over the range of values for which they are valid.

D. The sequential probability ratio test. Using the nomenclature of sequential analysis,[7] the criterion for a decision is given by two parallel straight lines in the dn-plane

$$\text{(7)} \qquad \begin{aligned} d_1 &= h_1 + sn \text{ (lower line)} \\ d_2 &= h_2 + sn \text{ (upper line)}, \end{aligned}$$

where d is the number of defectives and n is the number of observations. The acceptance and rejection numbers for any n are given by a_n and r_n, respectively,

[7] See, for example, *Sequential Analysis of Statistical Data: Applications*, Section 2, Columbia University Press, 1945.

where a_n is the largest positive integer less than or equal to d_1, and r_n is the smallest integer greater than or equal to d_2. We let $k_a(n)$ be the number of paths from the origin which end in a decision to accept on the nth observation; $k_r(n)$ is similarly defined when rejection occurs on the nth observation. We also require an auxiliary sequential test with acceptance and rejection numbers $a'_{n-1} = a_n - 1$, $r'_{n-1} = r_n - 1$ (which is equivalent to replacing h_1 and h_2 by $h_1 + 1 - s$ and $h_2 - 1 + s$ in the equations (7)), and with $k'_a(n)$ and $k'_r(n)$ the number of paths from the origin which lead to acceptance or rejection on the nth observation for the new test. A graphical comparison of the two plans shows that: *The unique unbiased estimate of p is*

$$\hat{p}(n) = k'_a(n-1)/k_a(n)$$

when the original test leads to a decision to accept, and

$$\hat{p}(n) = k'_r(n-1)/k_a(n)$$

when the original test leads to a decision to reject on the nth observation.

E. Regions with narrow throats. Let us consider the case of a closed region which has only one accessible point of index n, $n > 0$ (n being the lowest index not zero at which this phenomenon occurs). The number of paths from the origin to this accessible point α' we will denote m, while the number of paths from α' to α, boundary points of index greater than n, will be denoted $l(\alpha)$. Then the total number of paths to α from the origin is $ml(\alpha)$. We use the construction preceding Theorem 3 to get $\hat{p}(\alpha)$. The number of paths from (0,1) to α is similarly $m^*l(\alpha)$, so for such points $\hat{p}(\alpha) = m^*/m$. In other words, if a closed region has a narrow throat such as that described, $\hat{p}(\alpha)$ for α of index higher than that of the accessible point α' are independent of the shape of the region beyond the line $x + y = n$, and in fact they are all identical. The curtailed single sample with $c = 1$ is a particular case of a region with a narrow throat.

4. Estimation based on data from several experiments. In the previous discussion we have been concerned with estimation based on the result of a single experiment. Various kinds of acceptance sampling plans have been suggested as examples of the possible experiments. Acceptance sampling is one of many activities where data toward the estimation of p are often accumulated in a series of experiments. It has been pointed out by John Tukey that when information is available from several experiments the estimate $\hat{p}$ will no longer be the unique unbiased estimate of p. Little has been done on this problem of combining information from several experiments, but to illustrate the point, we will discuss a very simple example in terms of acceptance sampling.

Let us suppose that two large lots of the same size are inspected according to the following curtailed single sampling plan: if a defective occurs at the first or second observation, sampling is stopped and the lot is rejected; if the first two items inspected are nondefective, we accept the lot.

The total number of defective and of nondefective items in the two samples form a sufficient statistic for p. In a single application of the sampling plan the boundary points with their probabilities are (0, 1), p; (1, 1), pq; (2, 0), q^2. From this information we can generate the possible totals of defectives and of nondefectives which may arise when samples are drawn from two lots, with their probabilities by expanding

$$(8) \qquad (p + pq + q^2)^2 = p^2 + p^2q^2 + q^4 + 2\,p^2q + 2\,pq^2 + 2pq^3,$$

where a term on the right of the form mp^yq^x is the probability that in two samples there will be x nondefectives and y defectives altogether. On the basis of the observed number pair (x, y), which may be regarded as a possible terminal point α for the two experiments performed successively, we wish to form an unbiased estimate $e((x, y)) = e(\alpha)$. For the estimate e to be unbiased the condition $\Sigma\, e(\alpha)P(\alpha) = p$ must be satisfied, where in the present example the $P(\alpha)$ are the six terms on the right of equation (8), and the $e(\alpha)$ are the estimates with which the six probabilities are associated.

In the example under consideration the condition for unbiasedness will be satisfied if and only if $e((0, 2)) = 1$, $e((4, 0)) = 0$, $e((1, 2)) = \frac{1}{2}$, $e((2, 1)) = [1 - e((2, 2))]/2$, $e((3, 1)) = e((2, 2))/2$. Consequently a one parameter family of unbiased estimates is available. Unfortunately the popular condition that the variance be a minimum depends on the true value of p; in fact the variance is minimized just when $e((2, 2)) = 1/(2 + p)$. So an unbiased estimate of uniformly minimum variance does not exist. In practical applications to acceptance sampling one might meet this difficulty by choosing a value of p near zero for such a minimization scheme.

However it is clear that the last word has yet to be said about how best to estimate p when one is faced with the results of several experiments.

5. Conclusion. We would like to call attention to a few problems raised by but not solved in this paper: 1) find a necessary and sufficient condition that $\hat{p}$ be the unique unbiased estimate for p; 2) suggest criteria for selecting one unbiased estimate when more than one is possible; 3) evaluate the variance of $\hat{p}$.

In this connection, in a forthcoming paper by M. A. Girshick, it will be shown for certain regions, for example for those of the sequential probability ratio test, that the variance of $\hat{p}(\alpha)$,

$$\sigma_{\hat{p}}^2 \geq pq/E(x + y),$$

where $E(x + y)$ is the expected number of observations required to reach a boundary point.