UNBIASED ESTIMATION: FUNCTIONS OF LOCATION AND SCALE PARAMETERS

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1. Summary. Unbiased estimators for functions of a location parameter θ and a scale parameter ρ are expressed as unknown functions in integral equations of convolution type, and are then obtained by integral transform methods. An outline of the paper is contained in Section 3. The main results consist in the application of various derived expressions to the exponential distribution with parameters θ and ρ, the gamma and Weibull distributions with parameter ρ, and to general distributions with truncation parameter θ. In the latter case, a simple formula is given for a minimum variance unbiased estimator of any absolutely continuous function of θ; this extends slightly a result of Davis [3] concerning distributions of exponential type. Throughout the paper particular attention is paid to the estimation of the probability that a single observation will lie in a certain Borel set, when this probability is regarded as a function of the parameters θ and/or ρ. Extensions to sample points of m observations and Borel sets in m-space are in most cases immediate.

2. Introduction. Estimation of location and scale parameters was first studied systematically by Pitman [15] through the use of fiducial functions. He showed that for a random sample $X_1, X_2, \cdots, X_n$ from a density $f(x - \theta)$ the estimator

$$(2.1) \qquad \phi(X_1, X_2, \cdots, X_n) = \frac{\int \theta f(X_1 - \theta) f(X_2 - \theta) \cdots f(X_n - \theta)\, d\theta}{\int f(X_1 - \theta) f(X_2 - \theta) \cdots f(X_n - \theta)\, d\theta}$$

has minimum variance among the class of all estimators $U(X_1, X_2, \cdots, X_n)$ with the translation property, that is to say estimators satisfying the condition

$$(2.2) \qquad U(X_1 + c, X_2 + c, \cdots, X_n + c) = U(X_1, X_2, \cdots, X_n) + c$$

for all real c. This was his main result concerning unbiased estimation. The estimator for ρ, when the X_i's have density $\rho f(\rho x)$,

$$(2.3) \qquad \phi(X_1, X_2, \cdots, X_n) = \frac{\int \frac{1}{\rho^{n+2}} f\left(\frac{X_1}{\rho}\right) f\left(\frac{X_2}{\rho}\right) \cdots f\left(\frac{X_n}{\rho}\right) d\rho}{\int \frac{1}{\rho^{n+3}} f\left(\frac{X_1}{\rho}\right) f\left(\frac{X_2}{\rho}\right) \cdots f\left(\frac{X_n}{\rho}\right) d\rho},$$

was shown to have a negative bias, although it possesses optimal properties

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among the class of estimators with the multiplicative property, that is among those estimators satisfying the condition

$$U(cX_1, cX_2, \cdots, cX_n) = cU(X_1, X_2, \cdots, X_n) \tag{2.4}$$

for all positive c. Sufficient statistics[2] need not exist in order to apply his results, although their existence simplifies the labor; this is also a feature of the present paper. Pitman's work has been extended by Girshick and Savage [6] and others in the direction of minimax estimation.

The aim of the present paper is to consider only unbiased estimators, but to allow the functions to be estimated to be of a more general nature than those which Pitman considered. The methods which will be used are related to those of Washio, Morimoto, and Ikeda [16], in that they also use integral transform theory to obtain their results. Washio et al deal with the Koopman-Pitman family of densities (see Koopman [12], and Pitman [14]) which possess sufficient statistics, and for which the range does not depend on the parameter. The joint density for n independent random variables from a density of this type can be expressed as

$$p(x_1, x_2, \cdots, x_n \mid \tau) = K(\tau)e^{B(\tau)T(x_1,x_2,\cdots,x_n)+U(x_1,x_2,\cdots,x_n)},$$

where τ is a real parameter, T is a sufficient statistic for τ, and the positive sample space is independent of τ (except possibly for a set in n-space with Lebesgue measure zero). Now, the density of T can be expressed as

$$f(t \mid \tau) = k(\tau)e^{-\tau t+v(t)},$$

where the positive sample space of T is also independent of τ. Theorem 1 of Washio, Morimoto, and Ikeda [16] gives an estimator for a function $m(\tau)$. This estimator may be denoted by $\phi(T)$, where

$$\int (\phi(t)e^{v(t)})e^{-\tau t}\, dt = \frac{m(\tau)}{k(\tau)}, \tag{2.5}$$

with integration taking place over the positive sample space of T. Two alternative sets of conditions restricting the functions $m(\tau)$ and $k(\tau)$ are contained in the hypothesis of their Theorem 1.

At this point it is possible to discuss the differences between their methods and those of the present paper, with respect to distributions which possess a single parameter. First, the family of densities which they consider is broader than the one we consider in that τ can be any parameter, instead of a parameter of location or scale, but narrower by virtue of the fact that we shall treat cases in which the range of the density depends on the parameter. Secondly, their hypotheses are designed to insure that $m(\tau)/k(\tau)$ is itself a bilateral Laplace transform, since in that event $\phi(t)e^{v(t)}$ is determined (up to a set with probability measure

[2] By the existence of sufficient statistics is meant the existence of a small number of "simple" continuous real-valued functions of the sample that are jointly sufficient in the Halmos-Savage sense.

zero for all τ) by inversion, and then the unbiased estimator obtained from it after multiplication by $e^{-v(t)}$. In contrast to this we shall in each case express our estimator $\phi(T)$ as a simple transformation of the unknown function of some integral equation of convolution type. The function $m(\tau)/k(\tau)$, or some simple transformation thereof, will then be required to possess a Laplace, bilateral Laplace, or Mellin transform, depending on what type of parameter is studied.

Side results in reference [16] concerning the actual process of inversion are offered, parallel to the statistical development. These results depend on the concept of a bounded linear translatable operation. This notion was dealt with by Kitagawa in a series of papers, including one (see Kitagawa [10]) which refers directly to the work in reference [16], and is also of independent interest in operational calculus.

3. Notation and outline of results. Throughout, X will denote the basic random variable, and $X_1, X_2, \cdots, X_n$ will be a random sample from the X distribution. X_S and X_L will denote the smallest and largest observations, respectively, in such a sample. The following notation will be used for densities:[3]

A density of X which is completely specified will be denoted by $f(x)$.

For those situations in which the density of X has a location parameter θ we denote the density by $f_\theta(x)$. Whenever the location parameter is actually a truncation parameter we will let

$$f_\theta(x) = k_1(\theta)h_1(x) \qquad \theta < x < \infty, -\infty < \theta < \infty;$$

$$f_\theta(x) = k_2(\theta)h_2(x) \qquad -\infty < x < \theta, -\infty < \theta < \infty.$$

For the case of a translation parameter the notation will be

$$f_\theta(x) = f(x - \theta).$$

A density of X with a positive scale parameter ρ will be written as

$$f_\rho(x) = \rho f(\rho x).$$

For the two parameter problems we have the notation

$$f_{\theta,\rho}(x) = \rho f(\rho(x - \theta)).$$

Throughout, $g_\theta(x)$, $g_\rho(x)$, and $g_{\theta,\rho}(x)$ will denote densities of statistics.

In the scale parameter case one more often sees the notation $\frac{1}{\rho} f\left(\frac{x}{\rho}\right)$, as for example in Pitman's paper previously referred to. It is, however, slightly easier to cast the estimation problem into the framework within which Mellin transforms apply when the above form is used.

At various stages the Laplace, bilateral Laplace, and Mellin transforms will be used. The symbol x will generally denote the argument of a function to be

[3] All densities will be assumed positive in some interval of the real line, and moreover piecewise continuous in the interior of that interval. Also, a density specified analytically over part of its domain will be assumed to vanish over the rest of its domain.

transformed, while s will be used for the argument in the transform of a function. Thus, for a function $\zeta(x)$ we shall write

$$\mathfrak{L}[\zeta(x); s] = \int_0^{\infty} e^{-sx}\zeta(x)\,dx, \qquad \text{Re}(s) > s_0;$$

$$\mathfrak{B}[\zeta(x); s] = \int_{-\infty}^{+\infty} e^{-sx}\zeta(x)\,dx, \qquad s_0 < \text{Re}(s) < s_1;$$

$$\mathfrak{M}[\zeta(x); s] = \int_0^{\infty} x^{s-1}\zeta(x)\,dx, \qquad s_0 < \text{Re}(s) < s_1.$$

For the inverse function associated with a transform $\delta(s)$ of one of the above types we shall write, respectively,

$$\mathfrak{L}^{-1}[\delta(s); x], \qquad \mathfrak{B}^{-1}[\delta(s); x], \qquad \mathfrak{M}^{-1}[\delta(s); x],$$

all functions being unique up to a set of Lebesgue measure zero. Analytic expressions for the real and complex inversion of the above transforms may be found in Widder [17]; however, they will not be used here since all problems considered can be solved by the tables of the Bateman Manuscript Project [1].

Finally, $\phi(X_1, X_2, \cdots, X_n)$ will denote the unbiased estimator in a given situation; if ϕ depends explicitly on some statistic $T(X_1, X_2, \cdots, X_n)$, the expression $\phi(T)$ will be used. The function of θ and/or ρ which is to be estimated will be written as $\xi(\theta, \rho)$. As one example consider the formulation used in Section 2 to introduce some of the results in reference [16]. There we have

$$(3.1) \qquad \xi(\tau) = \frac{m(\tau)}{k(\tau)}, \qquad \phi(x) = e^{-v(x)}\mathfrak{B}^{-1}\left[\frac{m(s)}{k(s)}; x\right];$$

the estimator is then expressed as $\phi(T)$ since in their case it depends explicitly on the sufficient statistic $T(X_1, X_2, \cdots, X_n)$.

Section 4, on the estimation of functions of a scale parameter ρ, begins with some simple observations concerning densities with scale parameters and statistics which are homogeneous functions of the sample. A formula is then derived which provides unbiased estimators for many functions $\xi(\rho)$ with Mellin transforms. The result is specialized to the case $\xi(\rho) = P(X \varepsilon A \mid \rho)$. All estimators depend explicitly on some homogeneous statistic $H(X_1, X_2, \cdots, X_n)$.

Densities which may be factored into a product of a function of θ and a function of x, with the range having θ as one endpoint, are very common in applications. These are the densities which are referred to as having a truncation parameter θ. Either X_S or X_L will be a sufficient statistic. Section 5 contains the derivation, and some applications, of results which provide minimum variance unbiased estimators for a wide class of functions $\xi(\theta)$. The formulas used in this section are extremely simple, due to a fortunate relationship involving conditional expectations, requiring only a single differentiation for their application. The question of the estimation of θ for the cases of truncation at either or both endpoints of the range of a density of the Koopman-Pitman family was investigated by Davis[4] [3] in 1951. Our work constitutes a simplification and generalization of part of his study. Connections are discussed at the end of Section 5.

Section 6 is devoted to the case of a translation parameter θ. For those situations in which the density of X is positive over $(-\infty, \infty)$ or (θ, ∞) the derived estimators have the translation property. In the former instance the problem can be handled by the methods of reference [16] whenever a sufficient statistic exists; this is not so for the latter case. It is shown that when the positive sample space is (θ, b), for some fixed $b > \theta$, there exists no unbiased estimator with the translation property. For this case a formula is provided for an unbiased estimator of θ based on a single observation. Results are in general less satisfactory for the translation parameter case because of the lack of sufficient statistics; Pitman's estimator for $\xi(\theta) = \theta$ is usually difficult to compute. Kolmogorov [11] derived the minimum variance unbiased estimator for $P(X \varepsilon A \mid \theta)$ in the case $X = \mathfrak{N}(\theta, \sigma_0^2)$. The estimator was obtained again by Washio et al [16] as an application of their theorems; they also derived the minimum variance unbiased estimator of $[P(X \varepsilon A \mid \theta)]^m$. We shall use this example as one of the illustrations of our methods, and point out that it is a special case of a slightly more general result concerning stable laws. Still another way of arriving at Kolmogorov's result has appeared in the literature as one of the cases considered by Hirshman-Widder [9]. Their approach is briefly described.

In Section 7 almost all results are applied to the very important cases of the exponential, gamma, and Weibull distributions. Some of the applications are presented in the framework of a recent paper by Birnbaum and Saunders [2] which clarifies the need for the latter two distributions in Life-Testing. The exponential case has been studied by Epstein and Sobel [4]; they consider estimators based on the first r of n ordered observations. In the present paper, minimum variance unbiased estimators are found in particular for $[P(X \varepsilon A) \mid \rho]^m$, $m = 1, 2, \cdots, n-1$, in the gamma and Weibull cases and $[P(X \varepsilon A \mid \theta, \rho)]^m$, $m = 1, 2, \cdots$, in the exponential case. It is shown that some of the calculations can be carried out with the Table of the Incomplete Beta Function [13].

4. Unbiased estimation for $\xi(\rho)$. Let X be a random variable with density $f_\rho(x) = \rho f(\rho x)$. We wish to find an unbiased estimator for a function $\xi(\rho)$. First we shall make a few observations which justify the use of homogeneous statistics in problems of this kind.

If $H(X_1, X_2, \cdots, X_n)$ is homogeneous of degree $\alpha \neq 0$, with density $g(x)$ when $\rho = 1$, then for all values of ρ, $H(X_1, X_2, \cdots, X_n)$ has the density $\rho^\alpha g(\rho^\alpha x)$. This follows immediately from these considerations: the conditional density of $H(X_1, X_2, \cdots, X_n)$, given $\rho = 1$, is the same as the unconditional density of $H(\rho X_1, \rho X_2, \cdots, \rho X_n)$. Thus, $\rho^{-\alpha} H(\rho X_1, \rho X_2, \cdots, \rho X_n)$ must have density $\rho^\alpha g(\rho^\alpha x)$.

[4] The author would like to thank W. Hoeffding for bringing the paper of R. C. Davis to his attention.

If we restrict our attention to estimators depending explicitly on some homogeneous function H of degree α, for example[5] X_S, $\sum X_j$, $\prod X_j$, then the problem of estimating $\xi(\rho)$ becomes that of solving the integral equation

$$\text{(I)} \qquad \int_{-\infty}^{\infty} \phi(x)\rho^{\alpha}g(\rho^{\alpha}x)\,dx = \xi(\rho)$$

for the unknown function ϕ, and then expressing the estimator as

$$\phi(H(X_1, X_2, \cdots, X_n)).$$

Basic integral equations will be designated by roman numerals.

Since the case $\xi(\rho) = \rho^r$ can be handled in an elementary way, it will be treated first. A family of estimators is provided by the following theorem, the proof of which is immediate.

THEOREM 1: *If X has density $\rho f(\rho x)$, and $H(X_1, X_2, \cdots, X_n)$ is a homogeneous function of degree $\alpha \neq 0$, then an unbiased estimator is provided for*

$$\xi(\rho) = \rho^r$$

by[6]

$$\phi(H) = \frac{H^{-r/\alpha}}{E_1(H^{-r/\alpha})},$$

for all values of r and α for which the indicated expectation exists.

PROOF: Let H have the density $\rho^{\alpha}g(\rho^{\alpha}x)$ guaranteed by the preliminary statements in this section. Then, consider the integral

$$(4.1) \qquad \int_{-\infty}^{\infty} x^{-r/\alpha}\rho^{\alpha}g(\rho^{\alpha}x)\,dx.$$

The stated result follows after the substitution $y = \rho^{\alpha}x$.

We may wish to estimate the 100 pth percentile of the X distribution. This is equivalent to setting

$$(4.2) \qquad \xi(\rho) = \frac{b_p}{\rho}, \qquad \int_{-\infty}^{b_p} f(x)\,dx = p.$$

The quantity b_p is usually available in a table. The value of the theorem stems mainly from the fact that sufficient statistics frequently have the homogeneity property; for example, $H(X_1, X_2, \cdots, X_n) = \sum X_j^{\alpha}$ is sufficient for ρ in the Weibull density with (known) exponent α, namely

$$f_\rho(x) = \alpha\rho^{\alpha}x^{\alpha-1}e^{-(\rho x)^{\alpha}}, \qquad 0 < x < \infty, 0 < \rho < \infty, \alpha \geqq 1.$$

Subject to certain, not too stringent, conditions the integral equation (I) can be solved by use of the Mellin transform. The exact result is given by

THEOREM 2: *Let X have density $\rho f(\rho x)$, and H be a nonnegative,*[7] *homogeneous*

[5] Summations and products without indices will be assumed to have index running from 1 to n.

[6] The symbol E_ρ stands for expectation under the condition that the parameter value is ρ.

[7] In order to use the classical Mellin transform it is necessary to consider either a non-

statistic with density $\rho^\alpha g(\rho^\alpha x)$. Assume that both $g(x)$ and $\xi(\rho)$ have Mellin transforms. If there exists an unbiased estimator $\phi(H)$ for which the Mellin transform exists, then it is determined uniquely by

$$\phi(H) = \frac{1}{H}\mathfrak{M}^{-1}\left[\frac{\mathfrak{M}\left[\frac{1}{x}\xi(x^{1/\alpha}); s\right]}{\mathfrak{M}[g(x); s]}; \frac{1}{H}\right].$$

PROOF: Consider equation (I). Replace ρ by $\rho^{1/\alpha}$, and then make the variable change $y = 1/x$, which produces the equation

$$\text{(I}') \qquad \int_0^\infty \frac{1}{y}\phi\left(\frac{1}{y}\right) g\left(\frac{\rho}{y}\right)\frac{1}{y}\,dy = \frac{1}{\rho}\xi(\rho^{1/\alpha}).$$

This is a well-known expression of convolution type. Application of the Mellin transform to both sides yields

$$\text{(4.3)} \qquad \mathfrak{M}\left[\frac{1}{x}\phi\left(\frac{1}{x}\right); s\right]\cdot\mathfrak{M}[g(x); s] = \mathfrak{M}\left[\frac{1}{x}\xi(x^{1/\alpha}); s\right],$$

from which the conclusion follows.

In view of the importance in applications of estimating $\xi(\rho) = P(X \varepsilon A \mid \rho)$ the result concerning this function will be stated as a corollary. The function $\xi(\rho) = \rho f(\rho z)$, for fixed z, is easier to work with in most cases; from an unbiased estimator of it one can derive an unbiased estimator for $P(X \varepsilon A \mid \rho)$ (Kolmogorov [11], page 22).

COROLLARY: *Let the conditions of Theorem 2 be satisfied for the functions $H(X_1, X_2, \cdots, X_n)$ and $\xi(\rho) = \rho f(\rho z)$, where $f(x)$ is a density which vanishes for negative x, and z is a fixed positive number.*[8] *If there exists an unbiased estimator of $\xi(\rho)$ with a Mellin transform, it will be given by*

$$\phi(H) = \frac{\alpha}{H}\mathfrak{M}^{-1}\left[\frac{\mathfrak{M}[f(x); \alpha(s-1)+1]}{z^{\alpha(s-1)+1}\mathfrak{M}[g(x); s]}; \frac{1}{H}\right].$$

PROOF. The result follows from Theorem 2 and the fact that

$$\text{(4.4)} \qquad \begin{aligned}\mathfrak{M}\left[\frac{1}{x}\xi(x^{1/\alpha}); s\right] &= \mathfrak{M}[x^{1/\alpha - 1}f(x^{1/\alpha}z); s] \\ &= \frac{\alpha\mathfrak{M}[f(x); \alpha(s-1)+1]}{z^{\alpha(s-1)+1}}.\end{aligned}$$

negative or a nonpositive statistic H, so that its density will vanish on the left or right half of the reals respectively. The proof is conducted for nonnegative H; the obvious modification consisting of replacing H by $-H$ and $g(x)$ by $g(-x)$ will provide the result for nonpositive H.

[8] If the corollary is stated instead for a density $f(x)$ which vanishes for positive x and for a fixed negative number z, then the conclusion as stated can be modified by replacing $f(x)$ by $f(-x)$ and z by $-z$ in order to give the desired answer. The reasoning is the same as in footnote 7. Note also that the assumption of nonnegativity for H was carried over from Theorem 2, but can be removed in favor or nonpositivity by the alteration described in footnote 7.

Once we have an estimator for $\rho f(\rho z)$ for fixed z, estimation of $P(X \varepsilon A \mid \rho)$ can be performed if an interchange of limits of integration is valid. More precisely, denote the estimator above by $\phi(z \mid H)$, to emphasize the dependence on z. We always have

$$(4.5) \qquad P(X \varepsilon A \mid \rho) = \int_A \rho f(\rho z)\, dz = \int_A \int_0^\infty \phi(z \mid x) \rho^\alpha g(\rho^\alpha x)\, dx\, dz.$$

If now, for example, $\phi(z \mid x) \geqq 0$ for $x \geqq 0$ and $z \varepsilon A$, which is usually the case,

$$(4.6) \qquad \phi_A(H) = \int_A \phi(z \mid H)\, dz$$

will be an unbiased estimator for $P(X \varepsilon A \mid \rho)$. This is a consequence of Fubini's theorem. The method can also be replaced by the following scheme, if so desired. We can consider the unbiased estimator for $P(X \leqq z \mid \rho)$, for fixed z, which can usually be found in a manner similar to our derivation of $\phi(z \mid H)$. Call this estimator $\phi^*(z \mid H)$. Then, $P(X \varepsilon A \mid \rho)$ will be estimated by

$$(4.7) \qquad \phi_A(H) = \int_A d\phi^*(z \mid H).$$

This will be required later on when we speak of the location parameter situations. Thus, it will be seen in Section 7 that the gamma and Weibull distributions with scale parameter ρ can be treated by the Corollary, while in Section 5 all truncation parameter problems will be handled by the Stieltjes integral method. Finally, it should be noted that the probability that $(X_1, X_2, \cdots, X_m)$ lies in a set A_m in m-space can be estimated by the ordinary integral of the estimator $\phi(z_1, z_2, \cdots, z_m \mid H)$ of the function $\rho^m f(\rho z_1) f(\rho z_2) \cdots f(\rho z_m)$, or the Stieltjes integral of the estimator $\phi^*(z_1, z_2, \cdots, z_m \mid H)$ of

$$P(X_1 \leqq z_1, X_2 \leqq z_2, \cdots, X_m \leqq z_m \mid \rho),$$

over the set A_m. In connection with this paragraph see Kolmogorov ([11], Section 8).

5. Unbiased estimation for functions of a truncation parameter. Following the notation of Section 3, we define two types of densities with truncation parameters. We include densities over any range (a, b), finite or infinite.

Type I: $f_\theta(x) = k_1(\theta) h_1(x), \qquad a < \theta < x < b;$

Type II: $f_\theta(x) = k_2(\theta) h_2(x), \qquad a < x < \theta < b.$

In connection with these densities we assume the following: $h_1(x)$ and $h_2(x)$ are nonnegative, continuous, and integrable over (θ, b) and (a, θ), respectively, for θ in (a, b). $k_1(\theta)$ and $h_1(x)$ on the one hand, and $k_2(\theta)$, $h_2(x)$ on the other, have the obvious relations

$$(5.1) \qquad k_1(\theta) = \frac{1}{\int_\theta^b h_1(x)\, dx}, \qquad k_2(\theta) = \frac{1}{\int_a^\theta h_2(x)\, dx}$$

for $a < \theta < b$.

Any completely specified density $f(x)$ defined over (a, b), which satisfies the continuity and integrability conditions imposed on $h_1(x)$ and $h_2(x)$, will be of Type I or Type II as soon as a truncation parameter θ is introduced. Since (a, b) may possibly be $(-\infty, +\infty)$, the situation is one of some generality. If a random sample $X_1, X_2, \cdots, X_n$ is considered, then X_S is sufficient in the Type I case, and X_L is sufficient in the Type II case. The purpose of this section is to derive simple expressions which lead to minimum variance unbiased estimators for a wide class of functions $\xi(\theta)$. Results will be stated for a finite interval (a, b) and then amended to take care of the infinite case.

THEOREM 3: *Let X have a density $f_\theta(x)$ which is of Type I over some finite interval (a, b), and $\xi(\theta)$ be a function which is absolutely continuous over (a, b). A minimum variance unbiased estimator for $\xi(\theta)$ is given uniquely, by*

$$\phi(X_S) = \xi(X_S) - \frac{\xi'(X_S)}{nk_1(X_S)h_1(X_S)}.$$

PROOF: The density of X_S is easily shown to be

$$p_\theta(x) = n[k_1(\theta)]^n h_1(x)\left(\int_x^b h_1(t)\,dt\right)^{n-1}, \qquad \theta < x < b.$$

Completeness for the family $\{p_\theta(x)\}$ is equivalent to the proposition

$$\int_\theta^b \psi(x)h_1(x)\left(\int_x^b h_1(t)\,dt\right)^{n-1} dx \equiv 0$$

implies $\psi(x) = 0$ a.e. This follows from a well-known result in measure theory due to Lebesgue. We now obtain a simple unbiased estimator in order to apply the Rao-Blackwell-Lehmann-Scheffé method on the sufficient statistic X_S. The hypothesis allows the equation for a single observation,

$$\int_\theta^b \psi(x_1)k_1(\theta)h_1(x_1)\,dx_1 = \xi(\theta),$$

to be differentiated with respect to θ. This results in the relation

$$\psi(x_1) = \frac{\xi(x_1)k_1'(x_1) - k_1(x_1)\xi'(x_1)}{[k_1(x_1)]^2h_1(x_1)}. \tag{5.2}$$

One can easily show that

$$\phi(X_S) = E[\psi(X_1) \mid X_S] = \frac{1}{n}\psi(X_S) + \frac{n-1}{n}\frac{\int_{X_S}^b \psi(x_1)k_1(\theta)h_1(x_1)\,dx_1}{\int_{X_S}^b k_1(\theta)h_1(x_1)\,dx_1}. \tag{5.3}$$

Recalling the relation between $k_1(\theta)$ and $h_1(x)$, we see that the ratio of integrals in the right member is $\xi(X_S)$. From this fact and the simplifying relationship $k_1'(X_S) / [k_1(X_S)]^2h_1(X_S) = 1$, the stated conclusion now follows.

REMARK 1: For the case $b = \infty$ the same result is obtained if integrability is postulated for $[\xi(x_1) - k_1(x_1)\, \xi'(x_1)] \,/\, h_1(x_1)$ over every subinterval (θ, ∞); this holds also for $a = -\infty$.

REMARK 2: If the conditions above are satisfied for finite a and b, and the corresponding estimator, which we shall now denote by $\phi_b(X_S)$, is bounded by an integrable function $G(X_S)$ which has a finite second moment, the minimum variance unbiased estimator for the (a, ∞) case can be found by computing

$$\phi_\infty(X_S) = \operatorname*{Lim}_{b\to\infty} \phi_b(X_S),$$

whenever this limit exists. This follows from the fact that

$$\int_\theta^b \phi_b(x) n h_1(x) \left(\int_x^b h_1(t)\, dt\right)^{n-1} dx = \xi(\theta) \left(\int_\theta^b h_1(t)\, dt\right)^n,$$

since when $b \to \infty$, the right member approaches[9] $\xi(\theta) \,/\, [k_{1\infty}(\theta)]^n$ for each θ in (a, ∞), and Lebesgue's dominated convergence theorem applied to the left member yields

$$\int_\theta^\infty \phi_\infty(x) n [k_{1\infty}(\theta)]^n h_1(x) \left(\int_x^\infty h_1(t)\, dt\right)^{n-1} dx = \xi(\theta).$$

EXAMPLE 1: Consider a population of incomes, all of which are at least equal to a certain (unknown) minimum θ, but at most equal to a certain (known) maximum b. Let $\xi(\theta) = \theta$. An individual income might reasonably be assumed to follow the truncated Pareto law with density

$$f_\theta(x) = \frac{\frac{1}{\theta}\left(\frac{\theta}{x}\right)^2}{1 - \frac{\theta}{b}}, \qquad 0 < \theta < x < b.$$

In this example

$$k_1(\theta) = \frac{\theta}{1 - \frac{\theta}{b}}, \qquad h_1(x) = \frac{1}{x^2}.$$

An application of Theorem 3 shows immediately that

$$\phi(X_S) = X_S - \frac{1}{n} X_S \left(1 - \frac{X_S}{b}\right), \qquad 0 < X_S < b, \tag{5.4}$$

is the desired estimator. Notice that $h_1(x)$ is not integrable over $(0, b)$, and that this is not required. Remark 2 applies here, and

$$\phi_\infty(X_S) = \operatorname*{Lim}_{b\to\infty} \phi_b(X_S) = X_S \left(1 - \frac{1}{n}\right)$$

is minimum variance unbiased for the density $f_\theta(x) = \theta/x^2$, $0 < \theta < x < \infty$.

For a Type II density there exists an entirely similar theorem; statements analogous to remarks 1 and 2 also apply. We state the result for completeness.

THEOREM 4: *Let X have a density $f_\theta(x)$ which is of Type II over some finite*

[9] The symbol $k_{1_\infty}(\theta)$ denotes the limit of $k_1(\theta)$ as $b \to \infty$.

interval (a, b), *and* $\xi(\theta)$ *be a function which is absolutely continuous over* (a, b). *A minimum variance unbiased estimator for* $\xi(\theta)$ *is given uniquely by*

$$\phi(X_L) = \xi(X_L) + \frac{\xi'(X_L)}{nk_2(X_L)h_2(X_L)}.$$

EXAMPLE 2: For the Pareto law in Example 1 we can change the assumptions to a (known) minimum income a, and an unknown maximum θ. Then, the density becomes

$$f_\theta(x) = \frac{\frac{1}{\theta}\left(\frac{\theta}{x}\right)^2}{\frac{\theta}{a} - 1} \qquad a < x < \theta,$$

and the corresponding estimator for θ is

$$\phi(X_L) = X_L + \frac{1}{n} X_L\left(\frac{X_L}{a} - 1\right), \qquad a < X_L < \infty. \tag{5.5}$$

Note that if $a = 0$, $f_\theta(x)$ is no longer a density. The estimators for $P(X \varepsilon A \mid \theta)$ are given for both types of densities by the

COROLLARY: *Let* $X_1, X_2, \cdots, X_n$ *be a random sample from a density of Type I or Type II, and let*

$$\xi(\theta) = P(X \leqq z \mid \theta),$$

where $a < z < b$. *Minimum variance unbiased estimators obtained from Theorems 3 and 4 are*

Type I:

$$\phi^*(z \mid X_S) = \begin{cases} 0, & a < z < X_S < b, \\ \frac{1}{n} + \left(1 - \frac{1}{n}\right)\frac{\int_{X_S}^{z} h_1(x)\,dx}{\int_{X_S}^{b} h_1(x)\,dx}, & a < X_S < z < b. \end{cases}$$

Type II:

$$\phi^*(z \mid X_L) = \begin{cases} \left(1 - \frac{1}{n}\right)\frac{\int_{a}^{z} h_2(x)\,dx}{\int_{a}^{X_L} h_2(x)\,dx}, & a < z < X_L < b, \\ 1, & a < X_L < z < b. \end{cases}$$

We have the case mentioned, following the corollary to Theorem 2 in Section 4, in which a Stieltjes integral is required to estimate $P(X \varepsilon A \mid \theta)$. Both of the above estimators are mixed distribution functions with z as the variable; a mass of $1/n$ exists at $z = X_S$ in the first case and at $z = X_L$ in the second case. The following expressions can be written for the estimators, and cover all cases. Details will be omitted.[10]

[10] The symbols $A(X_S, b)$ and $A(a, X_L)$ denote set intersections, and I_A is the characteristic function of the set A.

Type I:

$$\phi_A(X_S) = \frac{1}{n} I_A(X_S) + \left(1 - \frac{1}{n}\right) \frac{\int_{A(X_S,b)} h_1(x)\,dx}{\int_{X_S}^{b} h_1(x)\,dx}$$

Type II:

$$\phi_A(X_L) = \frac{1}{n} I_A(X_L) + \left(1 - \frac{1}{n}\right) \frac{\int_{A(a,X_L)} h_2(x)\,dx}{\int_{a}^{X_L} h_2(x)\,dx}$$

Estimators can also be obtained for $P((X_1, X_2, \cdots, X_m) \varepsilon A_m \mid \theta)$; see the last paragraph of Section 4.

It is noteworthy that in these problems the distribution function of X is continuous, but the estimator of $P(X \leqq z \mid \theta)$ regarded as a function of z is the distribution function of a mixed distribution.

Davis [3] has looked into the question of estimation of a parameter θ in a truncated distribution. He assumes a distribution of the Koopman-Pitman family, with truncation points $a(\theta)$ and $b(\theta)$ which are continuous when regarded as functions of θ. Restriction to a subfamily of distributions for which X_S and/or X_L are sufficient takes his density into the factored form considered in the present section. The main part of the paper concerns results regarding a single sufficient statistic. He shows among other things that in those cases for which a "single" sufficient statistic (one 'simple' function—see footnote 2) exists, one of the end-points is a monotone decreasing function of the other; this extends the work of Pitman [14].

The point of contact with our work occurs in his section 4 when he estimates θ under the condition that either $a(\theta)$ or $b(\theta)$ is identically constant. In that event X_L or X_S, respectively, is a sufficient statistic for θ. As an example consider the case which he discusses of a density which is positive over the range $(a, b(\theta))$, with $b(\theta)$ a monotone function. Suppose one wishes to estimate $\xi^*(\theta)$ instead of just θ. Then, let $\xi(\theta) = \xi^*(b^{-1}(\theta))$. Theorem 4 yields the estimator

$$\phi(X_L) = \xi^*(b^{-1}(X_L)) + \frac{\frac{d}{dX_L} \xi^*(b^{-1}(X_L))}{nk_2(X_L)h_2(X_L)}$$

which, for the case $\xi^*(\theta) = \theta$, reduces to formula (8) of [3].

Hypotheses for our Theorems 3 and 4 are expressed somewhat differently than Davis' conditions, and in our derivation the Rao-Blackwell theorem occupies a more central role (see expression (5.3) and the remarks following).

6. Unbiased estimation for functions of a translation parameter. This section is divided into two parts; the first deals with densities having the form

$$f_\theta(x) = f(x - \theta), \qquad -\infty < x < \infty, -\infty < \theta < \infty;$$

the second deals with densities of the form

$$f_\theta(x) = \frac{f(x-\theta)}{\int_\theta^b f(t-\theta)\,dt}, \qquad -\infty < \theta < x < b,$$

where b is any fixed number.

Let $U(X_1, X_2, \cdots, X_n)$ denote any statistic with the translation property (see (2.2)). When X has density $f(x-\theta)$, U is known to have a density with form $g(x-\theta)$, $-\infty < x < \infty$. Thus, for an arbitrary function $\xi(\theta)$ we consider the estimation equation

$$\text{(II)} \qquad \int_{-\infty}^{+\infty} \phi(x) g(x-\theta)\,dx = \xi(\theta).$$

As in the case of equation (I) Section 4, this equation is essentially of convolution type. The following simple theorem may be stated.

THEOREM 5: *Let X have density $f(x-\theta)$, $U(X_1, X_2, \cdots, X_n)$ be a statistic with density $g(x-\theta)$, and $\xi(\theta)$ be a function with a bilateral Laplace transform. If there exists an unbiased estimator $\phi(U)$ with a bilateral Laplace transform, it will be determined uniquely by*

$$\phi(U) = \mathfrak{B}^{-1}\left[\frac{\mathfrak{B}[\xi(-x); s]}{\mathfrak{B}[g(x); s]}; -U\right].$$

PROOF: Replace x by $-x$ and θ by $-\theta$ in equation (II); then take the bilateral Laplace transform[11] of each side of the equation.

COROLLARY: *If X has density $f(x-\theta)$, and $U(X_1, X_2, \cdots, X_n)$ is a statistic with density $g(x-\theta)$, then the function*

$$\xi(\theta) = P(X \leqq z \mid \theta),$$

z fixed, has the unbiased estimator

$$\phi^*(z \mid U) = \mathfrak{B}^{-1}\left[\frac{\mathfrak{B}[f(x); s]}{s\mathfrak{B}[g(x); s]}; z - U\right],$$

provided the right member exists, and hence $P(X \varepsilon A \mid \theta)$ has the unbiased estimator $\phi_A(U) = \int_A d\phi^*(z \mid U)$.

PROOF:

$$\mathfrak{B}[\xi(-x); s] = \mathfrak{B}\left[\int_0^{z+x} f(y)\,dy; s\right] = \frac{1}{s} e^{sz}\mathfrak{B}[f(x); s].$$

Thus, from Theorem 5 we have

$$\text{(6.1)} \qquad \begin{aligned} \phi(-x) &= \mathfrak{B}^{-1}\left[\frac{e^{sz}\mathfrak{B}[f(x); s]}{s\mathfrak{B}[g(x); s]}; x\right] \\ &= \mathfrak{B}^{-1}\left[\frac{\mathfrak{B}[f(x); s]}{s\mathfrak{B}[g(x); s]}; x\right] * I_{(-z,\infty)}(x). \end{aligned}$$

[11] Recall that for the case of a density $g(x)$ which vanishes for $x < 0$, $\mathfrak{B}[g(x); s]$ can be replaced by $\mathfrak{L}[g(x); s]$

The latter expression can be rewritten in the form

$$\phi(-x) = \mathfrak{B}^{-1}\left[\frac{\mathfrak{B}[f(x)\ ;s]}{s\mathfrak{B}[g(x)\ ;s]}\ ;x+z\right],$$

which is equivalent to the conclusion of the corollary.

If $\phi^*(z \mid U)$ is itself an integral of a function $\phi(z \mid U)$, then the latter will be an unbiased estimator for $\xi(\theta) = f(z - \theta)$, and can be found by omitting the s in the denominator of the right member before inverting.

An extension to the vector case, analogous to those of the corollaries of sections 4 and 5, also holds for the present corollary.

EXAMPLE 3: The corollary allows another derivation for a result of Kolmogorov [11] (see also [16]). Let X be normal with mean θ and known variance σ_0^2; let $U(X_1, X_2, \cdots, X_n) = \bar{X}$. Then,

$$\mathfrak{B}[f(x); s] = e^{\sigma_0^2 s^2/2}, \qquad \mathfrak{B}[g(x); s] = e^{\sigma_0^2 s^2/2n}.$$

The estimator

$$\begin{aligned}\phi(z \mid \bar{X}) &= \mathfrak{B}^{-1}[e^{\sigma_0^2 s^2(1-1/n)/2}; z - \bar{X}] \\ &= \frac{1}{\sigma_0\sqrt{1-1/n}\,\sqrt{2\pi}}\, e^{-(z-\bar{X})^2/2\sigma_0^2(1-1/n)}\end{aligned} \tag{6.2}$$

is minimum variance unbiased for

$$\xi(\theta) = f(z-\theta) = \frac{1}{\sigma_0\sqrt{2\pi}}\, e^{-(z-\theta)^2/2\sigma_0^2}. \tag{6.3}$$

It can easily be shown that for $1 \leqq m < n$

$$f(z_1-\theta)f(z_2-\theta)\cdots f(z_m-\theta) = \left(\frac{1}{\sigma_0\sqrt{2\pi}}\right)^m \exp\left[-\sum_{j=1}^{m}(z_j-\theta)^2/2\sigma_0^2\right] \tag{6.4}$$

is estimated by

$$\begin{aligned}\phi(z_1, z_2, \cdots, z_m \mid \bar{X}) &= \frac{1}{(\sigma_0\sqrt{2\pi})^m\sqrt{\dfrac{1}{m}-\dfrac{1}{n}}}\cdot \\ &\exp\left[-\sum_{j=1}^{m}(z_j-\bar{z})^2/2\sigma_0^2\right]\exp\left[-(\bar{z}-\bar{X})^2/2\sigma_0^2\left(\frac{1}{m}-\frac{1}{n}\right)\right].\end{aligned} \tag{6.5}$$

In his derivation of the above result Kolmogorov [11] utilized the close connection between the "source solution" of the heat equation

$$\psi(z,t) = \frac{1}{\sqrt{2\pi t}}\, e^{-z^2/4t}, \qquad 0 < t < \infty, -\infty < z < +\infty, \tag{6.6}$$

and both the kernel of the equation and the function to be estimated. Hirshman and Widder [9] considered the heat equation in a similar, but not identical manner in order to solve convolution equations. They defined the Weierstrass transform of a function $\phi(x)$ as

$$\int_{-\infty}^{+\infty} \phi(x)\psi(\theta - x, 1)\, dx = \xi(\theta)\,. \tag{6.7}$$

Inversion is accomplished by the operator exp $(-tD^2)$, defined by

$$e^{-tD^2}\xi(x) = \frac{1}{2\pi i}\int_{a-i\infty}^{a+i\infty} \sqrt{\frac{\pi}{t}}\, e^{y^2/4t}\cdot e^{yD}\xi(x)\, dy\,.$$

Then, $e^{-tD^2}\xi(x)$ is shown to satisfy the heat equation, and

$$\phi(x) = \operatorname*{Lim}_{t\uparrow -1} e^{-tD^2}\xi(x). \tag{6.8}$$

Hirshman and Widder, in a series of papers summarized in [9], consider in general the convolution equation

$$\int_{-\infty}^{+\infty} \phi(x)\,\zeta(\theta - x)\, dx = \xi(\theta)\,, \tag{6.9}$$

where $\zeta(x)$ is a function with a bilateral Laplace transform $1/E(s)$, $E(s)$ having the form

$$E(s) = e^{-cs^2+bs}\prod_{k=1}^{\infty}\left(1 - \frac{s}{a_k}\right)e^{s/a_k}, \tag{6.10}$$

with $0 < c < \infty$, $-\infty < a_k < \infty$, $-\infty < b < \infty$, $\sum (a_k)^{-2} < \infty$.
Motivated by the methods of operational calculus, they assign a meaning to the operator $E(D)$, and then write the solution of their basic equation as $\phi(x) = E(D)\xi(x)$. Their work constitutes a rather elegant unification of many separate integral transform theories under the general heading of the convolution transform, but it is essentially non-statistical. This may perhaps best be seen from the fact that one of the cornerstones of their theory is the notion of a variation diminishing kernel; that is, a kernel ζ such that the number of sign changes for $\xi(x)$ in $(-\infty, +\infty)$ does not exceed the number of sign changes for $\phi(x)$ in $(-\infty, +\infty)$ when ϕ, ξ, and ζ are related as above. This concept has no special operational meaning in statistical problems, which accounts for the fact that Kolmogorov, as well as Washio, Morimoto, and Ikeda, who were interested in problems of statistical inference, were apparently unaware of some of their work. The function of the Weierstrass transform in the Hirshman-Widder theory is to take care of the factor exp $(-cs^2)$ in $E(s)$.

EXAMPLE 4: The normal distribution is a special member of the class of stable distributions. Let us recall three facts concerning this class (see Gnedenko and Kolmogorov [7], chapter 7). First, *the characteristic function must have the form* (Khinchin and Lévy)

$$C(t) = \exp\left[i\theta t - \beta\,|t|^{\alpha}\left(1 + i\gamma\,\mathrm{sgn}\,(t)\,\tan\frac{\pi\alpha}{2}\right)\right], \qquad -\infty < \theta < \infty, \beta \geqq 0\,,$$

$$0 < \alpha \leqq 2,\ \alpha \neq 1,\ |\gamma| \leqq 1,$$

and another form (which will not be used) for $\alpha = 1$. *Also, all nondegenerate stable distributions are continuous* (Khinchin), *and for* $1 < \alpha \leqq 2$ *have entire distribution functions* (Lapin). Thus, a nondegenerate stable distribution has a density f which satisfies our conditions. Let α, β, γ be known, with $1 < \alpha \leqq 2$, $U(X_1, X_2, \cdots, X_n) = \bar{X}$, and $\xi(\theta) = f(z - \theta)$. Theorem 5 and its corollary can also be stated in terms of characteristic functions (which is easier in this case than to convert $C(t)$ to a bilateral Laplace transform). Using this fact we obtain

$$\phi(z \mid \bar{X}) = C^{-1}\left[\exp\left\{-\beta(1 - n^{1-\alpha})\,|t|^{\alpha}\left(1 + i\gamma \operatorname{sgn}(t) \tan\frac{\pi\alpha}{2}\right)\right\}; z - \bar{X}\right] \tag{6.11}$$

Therefore,

$$\phi_A(\bar{X}) = \int_A \left(\frac{1}{1 - n^{1-\alpha}}\right)^{1/\alpha} f\left(\frac{z - \bar{X}}{(1 - n^{1-\alpha})^{1/\alpha}}\right) dz \tag{6.12}$$

is an unbiased estimator for $P(X \varepsilon A \mid \theta)$.

For a density of type $f(x - \theta)$, and any statistic U with the translation property, it is well known (and also follows from our discussion earlier), that if $E_\theta(U)$ exists for any value of θ, it will exist for all θ, and that $\phi(U) = U - E_0(U)$ is unbiased for $\xi(\theta) = \theta$. It appears reasonable that if the first moment of a density $f(x - \theta)$ fails to exist, there will be no unbiased estimator for θ based on a single observation X_1. The strongest result known to the author in this direction is: for a density $f(x - \theta) = 0$ when $x < \theta$ there is no estimator ϕ which is bounded in every interval $(0, c)$ and such that the order condition $\phi(x + y) = O(\phi(x) + \phi(y))$ holds for large x and y. The proof of this somewhat unnatural assertion will be omitted.

The existence of an unbiased estimator may well depend on the sample size. In fact, the following observation, stated as a Theorem, will be of some interest in this connection. The proof is simple and will be omitted.

THEOREM 6: If X has the density $f(x)$ satisfying the conditions $f(x) = 0$ for $x < 0$, and

$$f(x) = O\left(\frac{1}{x^{\alpha+1+k/n}}\right)$$

for large x, where k and n are positive integers, and $\alpha > 0$, then EX_S^k will exist when X_S is based on n observations.

From the unparametrized density of X_S,

$$p(x) = nf(x)[1 - F(x)]^{n-1}, \qquad 0 < x < \infty,$$

where $F(x)$ is the distribution function of X, an integration by parts shows that

$$E_0(X_S) = \int_0^\infty xp(x)\,dx = \int_0^\infty [1 - F(x)]^n\,dx \tag{6.13}$$

whenever the right member exists.

EXAMPLE 5: Let X have the half-Cauchy density $f(x) = (2/\pi)(1 + x^2)^{-1}$, $0 < x < \infty$, with distribution function $F(x) = (2/\pi) \tan^{-1} x$, $x > 0$. If we are interested in $E(X_S)$, then $k = 1$, and by Theorem 6 the smallest sample size which will insure its existence is $n = 2$. In that event EX_S^2 will not exist; in general for $n = k$, the first $k - 1$ moments of X_S will exist. It is clear that

$$E_0(X_S) = \int_0^\infty \left(1 - \frac{2}{\pi} \tan^{-1} x\right)^n dx .$$

From the table of Gröbner and Hofreiter ([8], page 156) we see that

$$\phi(X_S) = X_S - \frac{2}{\pi}\left(\frac{n}{n-1} + 2n \sum_{\nu=1}^{\infty} \frac{(-1)^\nu B_{2\nu}\pi^{2\nu}}{(n+2\nu-1)(2\nu)!}\right) \tag{6.14}$$

is an unbiased estimator of θ in the density $f(x - \theta)$.

EXAMPLE 6: Let X have the density $f(x) = \exp\{-(e^x - x - 1)\}$, $0 < x < \infty$, with distribution function $F(x) = 1 - \exp(1 - e^x)$.

$$E_0(X_S) = \int_0^\infty e^{-n(e^x-1)}\, dx ,$$

so for $n \geqq 1$ ·

$$\phi(X_S) = X_S + e^n Ei(-n) \tag{6.15}$$

is an unbiased estimator for θ in the density $f(x - \theta)$.

We shall now consider truncated versions of the densities $f(x - \theta)$: namely,

$$f_\theta(x) = \frac{f(x-\theta)}{\int_\theta^b f(t-\theta)\, dt}, \qquad -\infty < \theta < x < b .$$

Results are not analogous to those of the untruncated case, as the following theorem shows. All estimators occurring in the sequel will carry the subscript b.

THEOREM 7: *If X has the density $f_\theta(x)$, $\theta < x < b$, introduced above, then there exists no unbiased estimator for θ which has the translation property.*

PROOF: Let $\phi_b(X_1, X_2, \cdots, X_n)$ be such an estimator. We can write

$$\phi_b(X_1, X_2, \cdots, X_n) = X_1 + \phi_b(0, X_2 - X_1, \cdots, X_n - X_1).$$

Consequently,

$$\theta = E_\theta\phi_b(X_1, X_2, \cdots, X_n) = E_\theta(X_1) + c$$

for some c and all $\theta < b$, since the joint distribution of

$$(X_2 - X_1, X_3 - X_1, \cdots, X_n - X_1)$$

is independent of θ, and the expectation of the bounded random variable X_1 must exist. Thus, ϕ_b is unbiased if and only if $X_1 + c$ is unbiased. This quickly leads to

$$\int_0^{b-\theta} (x - c) f(x)\, dx = 0 \qquad -\infty < \theta < b,$$

which implies that $f(x) = 0$ for $-\infty < x < b$, contradicting the fact that $f(x)$ is a density.

It is difficult to construct a method leading directly to unbiased estimators based on the whole sample for the truncated case. There is a way to construct an estimator for θ based on a single observation X_1. An average could then of course be obtained from a sample $X_1, X_2, \cdots, X_n$. Since the construction is based on an integral equation of somewhat more interest than equations (I) and (II), in sections 4 and 5 respectively, it will be presented as

THEOREM 8: *If X has the density*

$$f_\theta(x) = \frac{f(x-\theta)}{\int_\theta^b f(t-\theta)\,dt}, \qquad \theta < x < b,$$

then, if there exists an unbiased estimator $\phi_b(X_1)$ of θ which has a Laplace transform, it will be determined uniquely by the relation[12]

$$\phi_b(X_1) = b - \mathfrak{L}^{-1}\left[\frac{1}{s^2} - \frac{\mathfrak{L}'[f(x)\,;s]}{s\mathfrak{L}[f(x)\,;s]}\,;b - X_1\right]$$

PROOF: The estimation equation can be written in the form

$$\int_\theta^b [b - \phi_b(x_1)]f(x_1 - \theta)\,dx_1 = (b-\theta)\int_\theta^b f(t-\theta)\,dt.$$

Now, let

$$b - \phi_b(b - x_1) = \psi_b(x_1), \qquad b - x_1 = y, \qquad b - \theta = \tau.$$

This produces the equation

$$\text{(III)} \qquad \int_0^\tau \psi_b(y)f(\tau - y)\,dy = \tau\int_0^\tau f(y)\,dy, \qquad 0 < \tau < \infty.$$

The transform $\mathfrak{L}[\psi_b(x);\,s]$ exists by hypothesis, and $\mathfrak{L}[f(x);\,s]$ exists because $f(x)$ is a density over $(0, \infty)$. Hence,

$$\text{(6.16)} \qquad \mathfrak{L}[\psi_b(x);s]\,\mathfrak{L}[f(x);\,s] = \mathfrak{L}\left[x\int_0^x f(y)\,dy;\,s\right],$$

and by virtue of the relation

$$\text{(6.17)} \qquad \mathfrak{L}\left[x\int_0^x f(y)\,dy;\,s\right] = -\mathfrak{L}'\left[\int_0^x f(y)\,dy;\,s\right] = -\frac{d}{ds}\left(\frac{\mathfrak{L}[f(x);\,s]}{s}\right),$$

we obtain

$$\mathfrak{L}[\psi_b(x);s] = \frac{1}{s^2} - \frac{\mathfrak{L}'[f(x);s]}{s\mathfrak{L}[f(x);s]}.$$

This proves the theorem.

[12] $\mathfrak{L}'[f(x);\,s] = (d/ds)\mathfrak{L}[f(x);\,s]$.

EXAMPLE 7: Consider the density

$$f_\theta(x) = \frac{1}{\sqrt{2\pi}} \frac{e^{-1/2(x-\theta)}}{(x-\theta)^{3/2} \int_\theta^b \frac{1}{\sqrt{2\pi}} \frac{e^{-1/2(t-\theta)}}{(t-\theta)^{3/2}} dt}, \qquad \theta < x < b.$$

From the Bateman Project Tables ([1], p. 246) we obtain $\mathfrak{L}[f(x); s] = \exp(-\sqrt{2s})$. Also, $\mathfrak{L}'[f(x); s] = -(2s)^{-1/2} \exp(-\sqrt{2s})$. Therefore,

$$(6.18) \qquad \phi_b(X_1) = b - \mathfrak{L}^{-1}\left[\frac{1}{s^2} + \frac{1}{s^{3/2}\sqrt{2}}; b - X_1\right] = X_1 - \sqrt{\frac{2(b - X_1)}{\pi}}$$

is an unbiased estimator for θ. We can also state that $\bar{X} - (2/\pi)^{1/2} n^{-1} \sum (b - X_k)^{1/2}$ is unbiased for θ. Note that the density obtained from $f_\theta(x)$ by putting $b = \infty$, has no first moment, and it is conjectured no unbiased estimator based on a single observation.

7. Applications to Life-Length Distributions. The purpose of this section is to derive minimum variance unbiased estimators for functions of the parameters of the important distributions in life-testing, namely the exponential, gamma, and Weibull distributions. The exponential distribution has been found to give a good fit to length-of-life data in many situations not involving fatigue: for example, lengths of telephone conversations, and lengths of life for electron tubes. The two main distributions which describe life-length under fatigue are the gamma and the Weibull distributions. Although the exponential distribution is a special case of both of the latter, it is not suitable since its use carries the implication that at any time future life-length is independent of past history. This appears untenable *per se* and has also been virtually disproved empirically by Freudenthal and Gumbel ([5], p. 579) in their work on the fatigue of metals. The model of Birnbaum and Saunders [2], which helps to explain the roles played by the gamma and Weibull distributions in fatigue testing, will be discussed briefly.

EXPONENTIAL DISTRIBUTION. Let X have the density

$$(7.1) \qquad f_{\theta,\rho}(x) = \rho e^{-\rho(x-\theta)}, \qquad x > \theta.$$

It is well known that $(X_S, \sum X_k)$ is a (vector) sufficient statistic for the (vector) parameter (θ, ρ). It will be convenient in what follows to consider instead $(X_S, \bar{X} - X_S)$ which of course is also sufficient for (θ, ρ). This statistic was extensively discussed by Epstein and Sobel [4]. They derived minimum variance unbiased estimators for θ and ρ after verifying completeness for $(X_S, \bar{X} - X_S)$. For our purposes the joint density of X_S and $\bar{X} - X_S$ is required; it is implicitly contained in the work of Epstein and Sobel, but was not written down. In view of the facts that X_S and $V = \sum(X_k - X_S)$ are independent, and that V is distributionally equivalent to the sum of $n - 1$ independent exponential random variables when $\theta = 0$ and $\rho = 1$, it is quickly shown that X_S and $Y = \bar{X} - X_S =$

V/n have joint density

(7.2) $$h_{\theta,\rho}(x_S, y) = \frac{(n\rho)^n}{\Gamma(n-1)} y^{n-2} e^{-n\rho(x_S+y-\theta)}, \qquad \theta < x_S < \infty, 0 < y < \infty.$$

All estimators given below will be functions of X_S and Y, and will have minimum variance.

For a general function $\xi(\theta, \rho)$ the estimation equation is

(IV) $$\int_\theta^\infty \int_0^\infty \phi(x_S, y) y^{n-2} e^{-n\rho(x_S+y)}\, dy\, dx_S = \frac{(n-2)!\,\xi(\theta, \rho) e^{-n\rho\theta}}{(n\rho)^n}$$

If $\xi(\theta, \rho)$ has one partial derivative with respect to θ, and if there exists an unbiased estimator $\phi(X_S, Y)$ which for almost all X_S is continuous and has a Laplace transform in Y, this estimator will be

(7.4) $$\phi(X_S, Y) = \frac{(n-2)!}{Y^{n-2}} \mathfrak{L}^{-1}\left[\frac{\xi(\theta, s/n)}{s^{n-1}} - \frac{\frac{\partial\xi}{\partial\theta}(\theta, s/n)}{s^n}; Y\right]_{\theta=X_S}$$

This formula is easily obtained from (IV) after differentiating both sides with respect to θ and replacing $n\rho$ by s.

Example 8: $\xi(\theta, \rho) = \theta^r$, $r \geqq 0$.

$$\phi(X_S, Y) = \frac{(n-2)!}{Y^{n-2}} \mathfrak{L}^{-1}\left[\frac{\theta^r}{s^{n-1}} - \frac{r\theta^{r-1}}{s^n}; Y\right]_{\theta=X_S}.$$

By virtue of $\mathfrak{L}^{-1}(s^{-\nu}) = Y^{\nu-1}/\Gamma(\nu)$, $\nu > 0$, we have

$$\phi(X_S, Y) = X_S^r - \frac{1}{n-1} rYX_S^{r-1}.$$

For $\xi(\theta, \rho) = \theta$, $\phi(X_S, Y) = X_S - Y/n - 1$ (Epstein and Sobel [4], Corollary 8).

Example 9: $\xi(\theta, \rho) = \rho^r$, $r < n - 1$.

$$\phi(X_S, Y) = \frac{(n-2)!}{Y^{n-2}} \mathfrak{L}^{-1}\left[\frac{s^{r+1-n}}{n^r}; Y\right] = \frac{1}{Y^r} \frac{\Gamma(n-1)}{n^r\Gamma(n-r-1)}$$

(See Epstein and Sobel [4], Corollary 6, for the case $r = -1$)

Example 10: (100 pth percentile) $\xi(\theta, \rho) = (1/\rho) \ln (1/1 - p) + \theta$.

$$\phi(X_S, Y) = X_S - \frac{1}{n-1} Y(1 + n \ln (1-p)).$$

Example 11: $\xi(\theta, \rho) = [P(X \leqq z \mid \theta, \rho)]^m$, $m = 1, 2, \cdots$.

We have

$$\xi(\theta, \rho) = [1 - e^{-\rho(z-\theta)}]^m, \qquad \theta < z,$$

and zero elsewhere.

$$\frac{\partial\xi}{\partial\theta}(\theta, s/n) = -\frac{ms}{n}[1 - e^{-s(z-\theta)/n}]^{m-1} e^{-s(z-\theta)/n}.$$

Thus,

$$\phi(\theta, y) = \frac{(n-2)!}{y^{n-1}} \mathfrak{L}^{-1}\left[\left\{\frac{1-\left(1-\frac{m}{n}\right)e^{-s(z-\theta)/n}}{s^{n-1}}\right\}\{1-e^{-s(z-\theta)/n}\}^{m-1}; y\right]$$

The expression in the first pair of braces inverts to the function

$$(7.5) \qquad \zeta(\theta, y) = \begin{cases} \dfrac{y^{n-2}}{(n-2)!}, & y < \dfrac{(z-\theta)}{n}, \\ \dfrac{y^{n-2}}{(n-2)!} - \left(1-\dfrac{m}{n}\right)\dfrac{[y-(z-\theta)/n]^{n-2}}{(n-2)!}, & y > \dfrac{z-\theta}{n}. \end{cases}$$

Now, a formula of the Bateman Project Tables ([1], p. 244) states that

$$\mathfrak{L}^{-1}[\delta(s)(1-e^{-as})^{\nu}; x] = \sum_{k=0}^{[xa^{-1}]} C_k^{\nu}(-1)^k \mathfrak{L}^{-1}[\delta(s); x - ak].$$

An application of this formula, together with (7.5), to the expression for $\phi(\theta, y)$ yields

$$(7.6) \qquad \phi(X_S, Y) = 1 + \sum\left[C_k^{m-1} + \left(1-\frac{m}{n}\right)C_{k-1}^{m-1}\right](-1)^k\left(1 - k\frac{(z-X_S)}{nY}\right)^{n-2}$$

with the summation extending over $k = 1, 2, \cdots$, Min $[m, nY(z - X_S)^{-1}]$. Note that $\xi(\theta, \rho)$ is also $P(X_L \leqq z \mid \theta, \rho)$ if X_L is the maximum of m observations. For $m = 1$, $\xi(\theta, \rho) = P(X \leqq z \mid \theta, \rho)$,

$$(7.7) \qquad \phi(X_S, Y) = \begin{cases} 0, & z < X_S, \\ 1 - \left(1-\dfrac{1}{n}\right)\left(1-\dfrac{z-X_S}{nY}\right)^{n-2}, & Y > (z-X_S)/n, \\ 1, & Y < (z-X_S)/n. \end{cases}$$

The estimator for $P(X \varepsilon A \mid \theta, \rho)$ is then

$$(7.8) \qquad \phi_A(X_S, Y) = \frac{1}{n} I_A(X_S) + \frac{(n-1)(n-2)}{n^2 Y}\int_{A(X_S, X_S+nY)}\left(1 - \frac{z-X_S}{nY}\right)^{n-3} dz.$$

Now consider the truncated exponential as an example of a Type I density (see Section 5) with

$$f_\theta(x) = k_1(\theta)h_1(x), \qquad \theta < x < b,$$

$$k_1(\theta) = \frac{1}{e^{-\theta} - e^{-b}}, \qquad h_1(x) = e^{-x}.$$

EXAMPLE 12: $\xi(\theta) = \theta^r, r \geqq 0.$

$$\xi'(X_S) = rX_S^{r-1}, \qquad f_{X_S}(X_S) = (e^{-X_S} - e^{-b})^{-1}e^{-X_S},$$

and by Theorem 3

$$\phi_b(X_S) = X_S^r - \frac{1}{n} r X_S^{r-1}(1 - e^{-(b-X_S)}).$$

Remark 2, following Theorem 3, applies and we have the estimator

$$\phi_\infty(X_S) = X_S^r - \frac{r}{n} X_S^{r-1}.$$

EXAMPLE 13: $\xi(\theta) = P(X \leqq z \mid \theta)$.

$$\phi^*(z \mid X_S) = \begin{cases} 0, & z < X_S < b, \\ \frac{1}{n} + \left(1 - \frac{1}{n}\right) \frac{1 - e^{-(z-X_S)}}{1 - e^{-(b-X_S)}}, & -\infty < X_S < z, \end{cases} \tag{7.9}$$

and

$$\phi_A(X_S) = \frac{1}{n} I_A(X_S) + \left(1 - \frac{1}{n}\right) \int_{A(X_S,b)} \frac{e^{-z}}{e^{-X_S} - e^{-b}}\, dz. \tag{7.10}$$

GAMMA DISTRIBUTION. The model of Birnbaum and Saunders [2] provides a framework for the discussion of results in this and the next subsection. They consider a structure consisting of m components which is subject to stress of some sort. Let S_λ be the length of life for the structure until $\lambda \leqq m$ components have failed. It is shown that S_λ has the density

$$f(x) = \frac{1}{\Gamma(\lambda)} \left(\int_0^x \gamma_\delta(t)\, dt\right)^{\lambda-1} \gamma_\delta(x) \exp\left\{-\int_0^x \gamma_\delta(t)\, dt\right\}. \tag{7.11}$$

They term $\gamma_\delta(t)$ the failure rate of a single component at time t under the damage function δ; it is assumed that

$$\gamma_\delta(t) = \omega(t)\delta(t), \tag{7.12}$$

where $\omega'(t)$ represents the deterioration of a component with time, and $\delta(t)$ represents the instantaneous damage at time t. The other assumption on which the above result depends is that the instantaneous damage to the remaining $m - j$ components after $j(< \lambda)$ have failed is inversely proportional to $m - j$.

The gamma distribution with scale parameter ρ and known parameter λ has the density

$$f_\rho(x) = \frac{\rho}{\Gamma(\lambda)} (\rho x)^{\lambda-1} e^{-\rho x}, \qquad 0 < x < \infty, \qquad 0 < \rho < \infty,$$

and is the distribution of the life-length of a structure which survives until λ components have failed, and is subject to constant instantaneous damage with no deterioration; that is, $\gamma_\delta(t) = \rho$. The statistic $\sum X_k$ is sufficient and complete for ρ; hence, all estimators in this subsection (except (7.19)) have the property of minimum variance. The density of $\sum X_k$ is known to be $\rho g(\rho x)$, where $g(x) = x^{n\lambda-1}e^{-x}/\Gamma(n\lambda)$, $0 < x < \infty$. Thus, by Theorem 1, Section 4

$$\phi\left(\sum X_k\right) = \frac{\Gamma(n\lambda)}{\Gamma(n\lambda - r)} \left(\frac{1}{\sum X_k}\right)^r \tag{7.13}$$

is a minimum variance unbiased estimator for $\xi(\rho) = \rho^r$, $r < n\lambda$. This result was also obtained for $0 < r < n\lambda$ by Washio et al [16].

Now consider

$$\xi(\rho) = \frac{\rho^{m\lambda}}{[\Gamma(\lambda)]^m}\left(\prod_{j=1}^m z_j\right)^{\lambda-1}\exp\left(-\rho\sum_{j=1}^m z_j\right), \tag{7.14}$$

the joint density of m gamma distributed random variables, with known parameter λ and unknown scale parameter ρ, evaluated at a fixed sample point $(z_1, z_2, \cdots, z_m)$. Recall two properties of Mellin transforms:

(i) $\mathfrak{M}[x^\gamma\zeta(x); s] = \mathfrak{M}[\zeta(x); s + \gamma]$,

(ii) $\mathfrak{M}[\zeta(ax); s] = (1/a^s)\,\mathfrak{M}[\zeta(x); s]$.

The corollary to Theorem 2 is to be used; accordingly, we employ properties (i) and (ii) to calculate

$$\mathfrak{M}\left[\frac{1}{x}\,\xi(x); s\right] = \frac{\left(\prod_{j=1}^m z_j\right)^{\lambda-1}\Gamma(m\lambda + s - 1)}{[\Gamma(\lambda)]^m\left(\sum_{j=1}^m z_j\right)^{m\lambda+s-1}},$$

$$\mathfrak{M}[g(x); s] = \frac{\Gamma(n\lambda + s - 1)}{\Gamma(n\lambda)};$$

whence,

$$\mathfrak{M}\left[\frac{1}{x}\,\phi\left(\frac{1}{x}\right); s\right] = \frac{\left(\prod_{j=1}^m z_j\right)^{\lambda-1}\Gamma(m\lambda + s - 1)\Gamma(n\lambda)}{[\Gamma(\lambda)]^m\left(\sum_{j=1}^m z_j\right)^{m\lambda+s-1}\Gamma(n\lambda + s - 1)}.$$

Let

$$\beta_{\mu,\nu}(x) = x^{\mu-1}(1 - x)^{\nu-1}/B(\mu, \nu)$$

for $0 < x < 1$, and 0 elsewhere. It can be shown that

$$\mathfrak{M}[\beta_{\mu,\nu}(x); s] = \frac{\Gamma(\mu + s - 1)\Gamma(\mu + \nu)}{\Gamma(\mu + \nu + s - 1)\Gamma(\mu)}: \tag{7.15}$$

consequently, from property (ii)

$$\frac{1}{x}\,\phi\left(\frac{1}{x}\right) = \frac{\Gamma(m\lambda)\left(\prod_{j=1}^m z_j\right)^{\lambda-1}}{[\Gamma(\lambda)]^m\left(\sum_{j=1}^m z_j\right)^{m\lambda-1}}\,\beta_{m\lambda,(n-m)\lambda}\left(x\sum_{j=1}^m z_j\right),$$

and finally for the estimator we have

$$\begin{aligned}\phi(z_1, z_2, \cdots, z_m \mid \textstyle\sum X_k) &= \frac{\Gamma(n\lambda)}{\Gamma[(n - m)\lambda][\Gamma(\lambda)]^m(\sum X_k)^{m\lambda}}\\ &\quad\cdot\left(\prod_{j=1}^m z_j\right)^{\lambda-1}\left(1 - \frac{\sum_{j=1}^m z_j}{\sum X_k}\right)^{(n-m)\lambda-}\end{aligned} \tag{7.16}$$

for $0 < \sum_{j=1}^{m} z_j < \sum X_k$, and 0 elsewhere. The special case $m = 1$ is easily handled. The estimators for $P(X \leqq z \mid \rho)$ and $P(X \epsilon A \mid \rho)$ are, respectively,

$$\phi^*(z \mid \sum X_k) = \begin{cases} \int_0^{z/\sum X_k} \beta_{\lambda,(n-1)\lambda}(t)\, dt & 0 < z < \sum X_k \\ 1, & z > \sum X_k, \end{cases} \tag{7.17}$$

and

$$\phi_A(\sum X_k) = \int_{A(0,\sum X_k)} \frac{1}{\sum X_k} \frac{\left(\frac{z}{\sum X_k}\right)^{\lambda-1}\left(1 - \frac{z}{\sum X_k}\right)^{(n-1)\lambda-1}}{B(\lambda, (n-1)\lambda)}\, dz \tag{7.18}$$

Numerical calculations can be carried out with the Table of the Incomplete Beta Function [13].

EXAMPLE 14: Consider the following sample of life-lengths in hours for a structure with 3 components:

592	198	458	780	132
1012	884	530	582	606

Suppose we wish to estimate $P(X \leqq 100 \mid \rho)$, the probability that the life of a given structure will not exceed 100 hours.

$$n = 10, \quad \lambda = 3, \quad (n-1)\lambda = 27, \quad \sum X_k = 5774.$$

The estimate is then (see (7.17))

$$\phi^*(100 \mid 5774) = 1 - I_{.9827}(27, 3) = .015.$$

One might also want to know one of the percentiles, say the 99th. We then estimate $b_{.99}/\rho$; where $b_{.99}$ is the 99th percentile of the unparametrized distribution, that is, for $\lambda = 3$, $\rho = 1$. The above formula (7.13) gives the estimate

$$\phi(5774) = \frac{1}{10(3)}(5774)(8.40) = 1680 \text{ hours}.$$

Let us consider for a moment the case of the gamma density $f(x - \theta) = (x - \theta)^{\lambda-1}e^{-(x-\theta)}/\Gamma(\lambda)$ with translation parameter. The following unbiased estimators, which do not have the minimum variance property, have been calculated from the formula (6.13) for $E_0(X_S)$:

$$\begin{aligned} & X_S - \frac{1}{n}\sum_{j=0}^{n}\binom{n}{j}\frac{j!}{n^j}, & \lambda = 2, \\ & X_S - \frac{1}{n}\sum_{j=0}^{n}\sum_{k=0}^{2j}\binom{n}{j}\binom{2j}{k}\frac{k!}{n^k}, & \lambda = 3. \end{aligned} \tag{7.19}$$

For $\lambda = 1$ we have the exponential case, and the estimator is known to be $X_S - (1/n)$. Each higher integral value of λ produces an expression with an additional summation sign.

WEIBULL DISTRIBUTION. This distribution fits the Birnbaum-Saunders model for the failure rate $\gamma_\delta(t) = \alpha\rho^{\alpha-1}t^{\alpha-1}$ which may arise in a variety of ways. For example, the instantaneous damage may vary as a power of the time, with no deterioration in the component or vice-versa; it is also possible for both $\omega(t)$ and $\delta(t)$ to vary as powers of the time. There appears to be no way to distinguish between these possibilities with their methods. We consider first the Weibull density with parameter ρ and fixed α,

$$f_\rho(x) = \alpha\rho(\rho x)^{\alpha-1}e^{-(\rho x)^\alpha}, \qquad x > 0,\ \rho > 0,\ \alpha \geqq 1. \tag{7.20}$$

The statistic $\sum X_k^\alpha$ is sufficient for ρ and has densiy $\rho^\alpha g(\rho^\alpha x)$, with $g(x) = x^{n-1}e^{-x}/\Gamma(n)$. This follows from the fact that $(\rho X)^\alpha$ has an exponential distribution with parameter 1 whenever X has the Weibull distribution with parameter ρ (see the first part of Section 4). The family of distributions for $\sum X_k^\alpha$ is known to be complete and hence all estimators (except (7.28)) for functions of ρ will have minimum variance.

By Theorem 1, Section 4,

$$\phi(\sum X_k^\alpha) = \frac{\Gamma(n)}{\Gamma(n - r/\alpha)}\left(\frac{1}{\sum X_k^\alpha}\right)^{r/\alpha}, \qquad r < n\alpha, \tag{7.21}$$

is the proper estimator for ρ^r. The joint density of m independently distributed observations, evaluated at a fixed point, can be estimated in a manner entirely analogous to the gamma case.

$$\xi(\rho) = \alpha^m\left(\prod_{j=1}^m z_j\right)^{\alpha-1}\rho^{m\alpha}\exp\left(-\sum_{j=1}^m(\rho z_j)^\alpha\right), \qquad m = 1, 2, \cdots, n-1. \tag{7.22}$$

Then,

$$\mathfrak{M}\left[\frac{1}{x}\phi\left(\frac{1}{x}\right); s\right] = \frac{\alpha^m\left(\prod_{j=1}^m z_j\right)^{\alpha-1}}{\left(\sum_{j=1}^m z_j^\alpha\right)^{m-1}}\,\frac{\Gamma(s+m-1)\Gamma(n)}{\left(\sum_{j=1}^m z_j^\alpha\right)^s\Gamma(s+n-1)},$$

and finally

$$\phi(\sum X_k^\alpha) = \frac{\Gamma(n)}{\Gamma(n-m)}\,\frac{\alpha^m\left(\prod_{j=1}^m z_j\right)^{\alpha-1}}{(\sum X_k^\alpha)^m}\left(1 - \frac{\sum_{j=1}^m z_j^\alpha}{\sum X_k^\alpha}\right)^{n-m-1} \tag{7.23}$$

for $0 < \sum_{j=1}^m z_j^\alpha < \sum X_k^\alpha$, and 0 otherwise. For the cases $\xi(\rho) = P(X \leqq z \mid \rho)$ and $P(X \varepsilon A \mid \rho)$, respectively, we then have

$$\phi^*(z \mid \sum X_k^\alpha) = \begin{cases} 1 - \left(1 - \dfrac{z^\alpha}{\sum X_k^\alpha}\right)^{n-1}, & 0 < z < (\sum X_k^\alpha)^{1/\alpha}, \\ 1, & z > (\sum X_k^\alpha)^{1/\alpha}, \end{cases} \tag{7.24}$$

$$\phi_A(\sum X_k^\alpha) = \int_{A(0,(\Sigma X_k^\alpha)^{1/\alpha}} \frac{(n-1)\alpha t^{\alpha-1}}{\sum X_k^\alpha}\left(1 - \frac{t^\alpha}{\sum X_k^\alpha}\right)^{n-2} dt. \tag{7.25}$$

The 100 pth percentile of the Weibull distribution is

$$\xi(\rho) = \frac{1}{\rho}\left[\ln\left(\frac{1}{1-p}\right)\right]^{1/\alpha}, \tag{7.26}$$

for which the estimator is

$$\phi(\sum X_k^\alpha) = \frac{\left[\ln\left(\frac{1}{1-p}\right)\right]^{1/\alpha}\Gamma(n)}{\Gamma(n+1/\alpha)}(\sum X_k^\alpha)^{1/\alpha} \tag{7.27}$$

For the location parameter case formula (6.13) of Section 6, previously used for the gamma distribution ($\lambda = 2, 3$), provides the estimator for θ,

$$\phi(X_S) = X_S - \frac{\Gamma(1/\alpha)}{\alpha n^{1/\alpha}}, \tag{7.28}$$

which does not possess the minimum variance property.

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