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UNBIASEDNESS OF TESTS FOR HOMOGENEITY OF VARIANCES

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0. Summary. Consider the classical homogeneity of variances model. That is, suppose we have a one way layout, with independent random samples of equal sizes in each column. We assume the samples are from normal populations with unknown means and unknown variances. We wish to test the hypothesis that all the variances are equal. R. V. Laue (1965) defined a two parameter family of statistics to test the homogeneity hypothesis. The family is defined as a function of the ratio of mean value functions of the sample variances. Included in the family are many of the well-known tests for homogeneity. In this paper we investigate which of the tests is unbiased. Although we are unable to resolve the question for every test in the family, we can demonstrate the unbiased character for several subfamilies, which include some of the better known tests.

1. Introduction. Consider the fixed one way layout. That is, let z_{ij}, $i = 1, 2, \cdots, k$; $j = 1, 2, \cdots, n+1$, be k independent random samples from normal distributions with unknown means and unknown variances σ_i. Assume $k \geqq 3$. We wish to test the hypothesis that $\sigma_1 = \sigma_2 = \cdots = \sigma_k$, against the alternative that not all σ's are equal. Let s_i denote the ith sample variance. R. V. Laue (1965) defined a two parameter family of statistics to test the homogeneity hypothesis. The family is defined as

$$T(\lambda, \eta) = [(kn)/(\lambda - \eta)] \log R(\lambda, \eta), \tag{1.1}$$

for all finite λ and η, where $R(\lambda, \eta) = M(\lambda)/M(\eta)$, and $M(t) = [(1/k)\sum_{j=1}^{k} s_j^{\,t}]^{1/t}$. Note that many well-known statistics for homogeneity are monotone functions of some member of the family. For example, $T(1, 0)$ is related to the likelihood ratio test; $T(2, 1)$ is related to a test proposed by Stevens (1936); $T(0, 0)$ is related to a statistic suggested by Bechhofer (1960) and Bartlett and Kendall (1946). Furthermore, $T(\infty, 1)$, defined as the limit, in some sense, as $\lambda \to \infty$, is related to Cochran's test (1941) and $T(\infty, -\infty)$ is related to Hartley's test (1950). Laue proved that tests in the family share the properties of the likelihood ratio test. That is, they are consistent, similar, and asymptotically distributed as chi-squared.

In this paper we study the question of which tests in the family are unbiased. By unbiased we mean that the power function of the test has a minimum when the hypothesis is true. We show that the tests corresponding to $\lambda \geqq 0$, $\eta \leqq 0$, are unbiased. Since $T(\lambda, \eta) = T(\eta, \lambda)$ it follows that unbiasedness is also established for $\lambda \leqq 0$, $\eta \geqq 0$. The cases for other λ, η are unresolved.

As far as the authors know, heretofore unbiasedness has been shown only for the likelihood ratio test by Pitman (1939) and Brown (1939), and Hartley's test by Ramachandran (1956).

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In the next section we prove the results. We conclude with some remarks in Section 3.

2. Unbiasedness of tests. In this section we prove that for $\lambda \geqq 0$, $\eta \leqq 0$, the tests in (1.1) are unbiased. The proof is achieved essentially as follows: The problem is reformulated in terms of testing the homogeneity of translation parameters. This is done by considering $x_i = \log s_i$, $i = 1, 2, \cdots, k$. Since the test statistics are permutation invariant and translation invariant in terms of the x_i, we will be able to apply a theorem of Mudholkar (1966). The application of this theorem reduces the proof of unbiasedness to demonstrating that the acceptance regions of the tests, expressed in terms of the x_i, are convex.

We start by stating a definition and the theorem proved by Mudholkar.

DEFINITION. A function $f(x)$ on k-dimensional Euclidean space L_k is said to be unimodal if the set $K_u = \{x \mid f(x) \geqq u\}$ is convex for each $u \geqq 0$.

Now let $G = \{g_i, i = 1, 2, \cdots, N\}$ be a finite group of Lebesgue measure-preserving linear transformations of L_k onto L_k. Let E be a convex set of k-space, invariant under G, or G invariant, i.e. $x \in E$ implies $g_i x \in E$, $i = 1, 2, \cdots, N$. Let $f(x) \geqq 0$ be a function on k-space satisfying.

(2.1)
(i) the unimodality condition
(ii) G-invariance condition: $f(g_i x) = f(x)$, $i = 1, 2, \cdots, N$, for each x in L_k, and
(iii) $\int_E f(x)\,dx < \infty$ in the Lebesgue sense.

For a set $\alpha = \{\alpha_1, \alpha_2, \cdots, \alpha_N\}$, $\alpha_i \geqq 0$, $\sum_{i=1}^N \alpha_i = 1$, and a vector y of k-space let us define

$$\alpha(y) = \sum_{i=1}^N \alpha_i g_i y. \tag{2.2}$$

Now we give

THEOREM 2.1. (Mudholkar (1966)). *Let $f(x)$ satisfy conditions* (2.1). *Let E be convex and G-invariant. For each set $\alpha = \{\alpha_1, \alpha_2, \cdots, \alpha_N\}$, $\alpha_i \geqq 0$, $\sum_{i=1}^N \alpha_i = 1$ and vector y of L_k we have*

$$\int_E f(x+\alpha(y))\,dx \geqq \int_E f(x+y)\,dx. \tag{2.3}$$

We will apply Theorem 2.1 when the group G is the permutation group in L_k. (See Mudholkar, page 1330). Also we choose a special set of α_i, namely $\alpha_i = 1/N$, $i = 1, 2, \cdots, N$ and a special set of vectors y. The y vectors we choose are those which can be written as

$$y = r\mathbf{1} + y^*, \tag{2.4}$$

where y^* is such that its coordinates sum to zero, r is a scalar, and $\mathbf{1}$ is the vector, all of whose components are 1. With such choices G, α, and y, it follows from (2.2) that (2.3) becomes

$$\int_E f(x+r\mathbf{1})\,dx \geqq \int_E f(x+r\mathbf{1}+y^*)\,dx. \tag{2.5}$$

Now let X be a $k \times 1$ random vector whose density is characterized by a $k \times 1$ translation parameter θ. That is, the density of X is $f(x-\theta)$. Suppose we wish to test the hypothesis that $\theta_1 = \theta_2 = \cdots = \theta_k$. In order to use Theorem 2.1 to yield the unbiasedness property of a test for such a hypothesis we prove

THEOREM 2.2. *Let f be a translation parameter family of densities satisfying* (2.1). *Then any test, whose acceptance region E is*

(2.6) (i) *convex*
(ii) *permutation invariant*
(iii) *translation invariant about the equiangular line, i.e.* $E + r\mathbf{1} = E$, *is unbiased.*

PROOF. We show that the probability of acceptance under the null hypothesis is a maximum. Let θ be any alternative. Clearly θ can be written as in (2.4) since any vector can be partitioned in this way. Furthermore, since θ is an alternative the components of y^* cannot all be equal. Now the probability of acceptance is

$$\int_E f(x-\theta)\,dx. \tag{2.7}$$

Since E is translation invariant, it follows from (2.7), upon a change of variables that the probability of acceptance under the null hypothesis can be written as

$$\int_E f(x+r\mathbf{1})\,dx. \tag{2.8}$$

From (2.5), (2.8) is greater than the right-hand side of (2.5), which by similar reasoning represents the probability of acceptance under an alternative. This completes the proof of the theorem.

At this point we may seek the unbiasedness results for the original model. We let $x_i = \log s_i$ and $\theta_i = \log \sigma_i$, $i = 1, 2, \cdots, k$. The hypothesis is equivalent to $\theta_1 = \theta_2 = \cdots = \theta_k$. Without loss of generality we let $\lambda \geqq \eta$. For λ, η finite, $\lambda > \eta$, $\eta \neq 0$, the test statistics in (1.1) yield acceptance regions

$$E(\lambda, \eta) = \{x : [\textstyle\sum_{i=1}^k e^{\lambda x_i}]^{1/\lambda} / [\sum_{i=1}^k e^{\eta x_i}]^{1/\eta} \leqq a\}, \tag{2.9}$$

where a is a positive constant. We now prove

THEOREM 2.3. *For $\lambda \geqq 0$, $\eta \leqq 0$, the tests in* (1.1) *are unbiased.*

PROOF. Let $f(x)$ denote the density of the x_i, $i = 1, 2, \cdots, k$. By virtue of Theorem 2.2, the present theorem is true if we can show that f is a translation parameter family satisfying (2.1) and if the acceptance regions determined by the tests satisfy (2.6).

Since the variables $(n s_i/\sigma_i)$, $i = 1, 2, \cdots, k$, are independent chi-squared variables with n degrees of freedom it follows that

$$\begin{aligned} f(x_1, x_2, \cdots, x_k; \theta_1, \theta_2, \cdots, \theta_k) \\ = C \exp[(n/2) \textstyle\sum (x_i - \theta_i)] \cdot \exp[-(\frac{1}{2}) \sum e^{(x_i - \theta_i)}], \end{aligned} \tag{2.10}$$

where C is a positive constant.

Clearly f is a translation parameter family satisfying conditions (ii) permutation invariance, and (iii) of (2.1). Furthermore f satisfies (i) of (2.1). For let

$$(2.11)\quad K_u = \{x \mid f(x) \geqq u\} = \{x : \log f(x) \geqq \log u\} = \{x : -(n/2)\textstyle\sum_{i=1}^k x_i + (\frac{1}{2})\sum_{i=1}^k e^{x_i} \leqq -\log u\}.$$

It is easily seen, by considering the matrix of second derivatives, that the term in the brackets on the right-hand side of (2.11) is a convex function of the k variables $(x_1, x_2, \cdots, x_k)$. This in turn implies that K_u is convex.

Finally we must show that the regions in (2.9), and those regions corresponding to tests in (1.1) for $\eta = 0$, or $\lambda = 0$, satisfy (2.6). (Note $T(\lambda, 0)$ or $T(0, 0)$ are limits of $T(\lambda, \eta)$ obtained using L'Hospital's rule.) Clearly conditions (ii) and (iii) of (2.6) are satisfied. It only remains to verify (i), that the regions are convex.

For the time being let $\lambda = -v\eta$, where v is a positive rational number. That is, $v = p/q$, where p and q are integers. Then for $\lambda > 0$, $\eta < 0$, from (2.9) we get

$$E(\lambda, \eta) = \{x : [\textstyle\sum e^{\lambda x_i}]^{1/\lambda}[\sum e^{\eta x_i}]^{v/\lambda} \leqq a\}$$

$$(2.12)\quad = \{x : [\textstyle\sum e^{\lambda x_i}]^q[\sum e^{\eta x_i}]^p \leqq a^{q\lambda}\}$$

$$= \{x : [\textstyle\sum_{i_1, i_2, \cdots i_q, i_{q+1}, \cdots i_{q+p}=1}^k \exp(\lambda \sum_{j=1}^q x_{i_j} + \eta \sum_{j=q+1}^{q+p} x_{i_j})] \leqq a^{q\lambda}\}$$

where, of course each variable x_i, $i = 1, 2, \cdots, k$, is treated symmetrically in the sum on the right-hand side of (2.12) and $\sum^k$ means all possible combinations of the x_{i_j}. Clearly the function in brackets in the term following the last equal sign in (2.12) is a convex function of the k variables $x_1, x_2, \cdots, x_k$. This is so since it is the sum of convex functions. This implies that $E(\lambda, \eta)$ is convex for the designated λ and η.

To establish convexity of the sets $E(\lambda, \eta)$ for the cases where v is not rational, or $\eta = 0$, or $\lambda = \eta = 0$, we use the following limiting argument. For v not rational, let v_n be a sequence of rational numbers such that $\lim_{n\to\infty} v_n = v$. Define $\eta_n = -\lambda/v_n$. Then by continuity of $T(\lambda, \eta)$ it follows that for every fixed x, $\lim_{n\to\infty} T(\lambda, \eta_n) = T(\lambda, \eta)$. For the case $\eta = 0$, define $\eta_n = -1/n$ and λ_n, a sequence of rationals such that $\lim_{n\to\infty} \lambda_n = \lambda$. For the case $\lambda = \eta = 0$, let $\lambda_n = 1/n$ and $\eta_n = -1/n$. By the definition of $T(\lambda, 0)$ and $T(0, 0)$, it follows $\lim_{n\to\infty} T(\lambda_n, -1/n) = T(\lambda, 0)$, and $\lim_{n\to\infty} T(1/n, -1/n) = T(0, 0)$.

Now let $\varphi(\lambda, \eta)$ denote the critical function corresponding to $E(\lambda, \eta)$. That is $\varphi(\lambda, \eta) = 0$ if $x \in E$, and $\varphi(\lambda, \eta) = 1$ otherwise. It is easy to verify that if $\lim_{n\to\infty} T(\lambda_n, \eta_n) = T(\lambda, \eta)$, then $\lim_{n\to\infty} \varphi(\lambda_n, \eta_n) = \varphi(\lambda, \eta)$ pointwise (that is, for every x). At this point we may apply a theorem of Matthes and Truax (1967), Theorem 2.1, page 684. That theorem says essentially that if φ_n is a sequence of critical functions converging weakly to a critical function φ, then if the acceptance regions corresponding to φ_n are closed and convex then the acceptance region corresponding to φ is closed and convex. Since we have already established convexity of the acceptance regions for the case v rational, it is clear that the sequences of tests defined above have acceptance regions which are convex and closed. Furthermore the corresponding sequences of critical functions converge pointwise, which implies they converge weakly to a critical function. Hence it follows that the acceptance regions corresponding to $T(\lambda, \eta)$ for v not rational, $T(\lambda, 0)$ or $T(0, 0)$ are convex. This completes the proof of Theorem 2.3.

3. Remarks.

REMARK 1. Note that Hartley's test is equivalent to $\max s_i/\min s_i$, which from (1.1) is a monotone function of $(\lambda-\eta)\, T(\infty, -\infty)$. We could let $\lambda = n$, and $\eta = -n$ and consider the sequence $2nT(n, -n)$, whose corresponding acceptance regions (in the x variables) are closed and convex. This would prove unbiasedness of Hartley's test by the same argument given to prove Theorem 2.3.

REMARK 2. Cochran's test is equivalent to $\max s_i/\bar{s}$, where $\bar{s}$ is the mean of the s_i, which is a monotone function of $(\lambda-\eta)\, T(\infty, 1)$. We show that the acceptance region (in the x variables) for this test is not convex. For consider the acceptance region

$$E(\infty, 1) = \{x : [e^{\max x_i}/\textstyle\sum e^{x_i}] \leqq a\}. \tag{3.1}$$

Choose vectors $x^{(1)}$ and $x^{(2)}$ such that their maximum coordinates are equal, their other coordinates are unequal but $\sum e^{x_i^{(1)}} = \sum e^{x_i^{(2)}}$. Also let $a = e^{\max x_i^{(1)}}/\sum e^{x_i^{(1)}}$. Note then that for $0 < \alpha < 1$,

$$\textstyle\sum \exp[\alpha x_i^{(1)} + (1-\alpha)x_i^{(2)}] < \alpha \sum e^{x_i^{(1)}} + (1-\alpha)\sum e^{x_i^{(2)}} = \sum e^{x_i^{(1)}}. \tag{3.2}$$

From (3.1) and (3.2) we get

$$\begin{aligned}&[\exp[\max(\alpha x_i^{(1)} + (1-\alpha)x_i^{(2)})]/\textstyle\sum \exp[\alpha x_i^{(1)} + (1-\alpha)x_i^{(2)}]]\\ &\qquad > e^{\max x_i^{(1)}}/\textstyle\sum e^{x_i^{(1)}} = a.\end{aligned} \tag{3.3}$$

Whereas $x^{(1)}$ and $x^{(2)} \in E$, (3.3) implies that $\alpha x^{(1)} + (1-\alpha)x^{(2)} \notin E$. This verifies that $E(\infty, 1)$ is not convex.

REMARK 3. By reasoning as in Remark 2 it follows that $E(\infty, \eta)$ for $\eta > 0$ is not convex. This, along with the argument of Theorem 2.3 proves that $E(\lambda, \eta)$, $\lambda > \eta > 0$, cannot be convex for all finite λ.

REMARK 4. The application of Theorem 2.1 given to prove Theorem 2.2 gives a bit more than unbiasedness. The proof indicates that the power function is monotone non-decreasing along rays orthogonal to the equiangular line.

REMARK 5. It would be a simple matter to generate more unbiased tests. This would merely involve defining acceptance regions (in the x_i) variables which are convex, translation invariant, and permutation invariant.

REFERENCES

[1] BARTLETT, M. S. and KENDALL, D. G. (1946). The statistical analysis of variance-heterogeneity and the logarithmic transformation. *J. Roy. Statist. Soc. Suppl.* **8** 128–138.

[2] BECHHOFER, R. E. (1960). A multiplicative model for analyzing variances which are affected by several factors. *J. Amer. Statist. Assoc.* **55** 245–264.

[3] BROWN, G. W. (1939). On the power of the L_1 test for equality of several variances. *Ann. Math. Statist.* **10** 119–129.

[4] COCHRAN, W. G. (1941). The distribution of the largest of a set of variances as a fraction of their total. *Ann. Eugenics* **11** 47.

[5] HARTLEY, H. O. (1950). The maximum F ratio as a shortcut test for heterogeneity of variances. *Biometrika* **37** 308–312.

[6] LAUE, R. V. (1965). Heterogeneity of variances in a one-way and two-way classification model. Ph.D. dissertation, Rutgers Univ.

[7] MATTHES, T. K. and TRUAX, D. R. (1967). Tests of composite hypotheses for the multivariate exponential family. *Ann. Math. Statist.* **38** 681–697.

[8] MUDHOLKAR, G. S. (1966). The integral of an invariant unimodal function over an invariant convex set—an inequality and applications. *Proc. Amer. Math. Soc.* **17** 1327–1333.

[9] PITMAN, E. J. G. (1939). Tests of hypotheses concerning location and scale parameters. *Biometrika* **31**, 200–215.

[10] RAMACHANDRAN, K. V. (1956). On the Tukey test for the equality of means and the Hartley test for equality of variances. *Ann. Math. Statist.* **27** 825–831.

[11] STEVENS, W. L. (1936). Heterogeneity of a set of variances. *J. Genetics* **33** 398–399.