**Integral Equations
and Operator Theory**

Unbounded Jacobi Matrices with a Few Gaps in the Essential Spectrum: Constructive Examples

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Abstract. We give explicit examples of unbounded Jacobi operators with a few gaps in their essential spectrum. More precisely a class of Jacobi matrices whose absolutely continuous spectrum fills any finite number of bounded intervals is considered. Their point spectrum accumulates to $+\infty$ and $-\infty$. The asymptotics of large eigenvalues is also found.

1. Introduction

In this paper we look for examples of unbounded Jacobi matrices with several gaps in the essential spectrum. Let $\ell_0^2 = \ell_0^2(\mathbb{N})$ be the space of sequences $\{f_k\}_1^\infty$ with a finite number of nonzero coordinates. For given real sequences $\{a_k\}_1^\infty$ and $\{b_k\}_1^\infty$ the Jacobi operator $\mathcal{J}_0$ acts in ℓ_0^2 by the formula

$$(\mathcal{J}_0 f)_k = a_{k-1}f_{k-1} + b_k f_k + a_k f_{k+1} \tag{1.1}$$

where $k = 1, 2, \ldots$ and $a_0 = f_0 = 0$. The a_k's and b_k's are called the weights and diagonal terms, respectively.

In what follows we deal with positive sequences $\{a_k\}_1^\infty$. By Carleman's criterion [1] one can extend $\mathcal{J}_0$ to a self-adjoint operator $\mathcal{J}$ in $\ell^2 = \ell^2(\mathbb{N})$ provided that

$$\sum_k \frac{1}{a_k} = +\infty.$$

The above mentioned aim of finding an example of unbounded Jacobi operator $\mathcal{J}$ such that $\mathbb{R} \setminus \sigma_{\mathrm{ess}}(\mathcal{J})$ consists of the union of several intervals could have been solved by using a general theorem of Stone. This theorem asserts that for a finite Borel measure μ on $\mathbb{R}$ one can find sequences $\{a_k\}_1^\infty$ and $\{b_k\}_1^\infty$ (expressed in terms of some moments of μ) such that the operator of multiplication by x in $L^2(\mu)$ is unitarily equivalent to the Jacobi operator

associated to these sequences by (1.1) [23]. Therefore, one can easily prove the existence of unbounded Jacobi operators with arbitrary many gaps. However, we want to find *explicit* examples of $\{a_k\}_1^\infty$ and $\{b_k\}_1^\infty$ which define $\mathcal{J}$ with a few gaps in its essential spectrum.

The construction given below uses the Gilbert–Pearson subordination theory of Jacobi matrices [14] combined with the asymptotic analysis of formal solutions (generalized eigenvectors) of the system

$$a_{n-1}u_{n-1} + b_n u_n + a_n u_{n+1} = \lambda u_n \qquad (1.2)$$

where $n = 2, 3, \ldots$ and $\lambda \in \mathbb{R}$. One can rewrite (1.2) in the standard form

$$\vec{u}_{n+1} = B_n(\lambda)\vec{u}_n \qquad (1.3)$$

where $n = 2, 3, \ldots$ and

$$\vec{u}_n := \begin{pmatrix} u_{n-1} \\ u_n \end{pmatrix},$$

$$B_n(\lambda) := \begin{pmatrix} 0 & 1 \\ -\frac{a_{n-1}}{a_n} & \frac{\lambda - b_n}{a_n} \end{pmatrix}.$$

The matrix $B_n(\lambda)$ is called the "transfer matrix" and will be used below in our analysis of generalized eigenvectors.

It is well known that bounded Jacobi matrices with almost periodic entries have infinitely many gaps in the essential spectrum (in the generic case) [22]. We do not know any paper with similar results in the case of unbounded Jacobi matrices. However, we could expect that these results hold true for them. This is not our aim here because we want to keep control on the form of gaps in the essential spectrum in terms of some parameters which appear in the entries of Jacobi matrices we shall construct below. Moreover, this control might allow to construct examples of unbounded Jacobi matrices with a few so called "mobility edges". In the case of one gap such a construction has been done in our recent work [13]. Finally, we should mention a recent paper of Christiansen et al. [4] concerning finite gap bounded Jacobi matrices which describes the so called "isospectral torus".

Recall that to our best knowledge almost all explicit examples of unbounded Jacobi matrices considered in the last 30 years had essential spectra either empty or equal to the whole real line or a half line. Only several years ago appeared the first explicit examples of unbounded Jacobi matrices with one gap in the essential spectrum. These examples are given by the following expressions:

$$a_n = n^\alpha + c_n,$$
$$b_n \equiv 0,$$

where $\alpha \in (0, 1]$ and $\{c_n\}$ is a periodic sequencce of period two with $c_1 - c_2 \neq 0$. It turns out that the Jacobi operator $\mathcal{J}_1$ defined by these sequences has

$$\sigma_{\text{ess}}(\mathcal{J}_1) = \sigma_{\text{ac}}(\mathcal{J}_1) = (-\infty, -|c_1 - c_2|] \cup [|c_1 - c_2|, +\infty),$$

see [6,8,12,17]. Below we shall try to explain the origin of this phenomena from a general point of view.

It turns out that a similar strategy (local periodicity) is also crucial for the formation of gaps in the essential spectrum of the unbounded Jacobi matrices considered in the present work. In particular the characteristic polynomial $p(\lambda)$ [see (3.1)] resemble but is different from the one of bounded Jacobi matrices. By choosing (locally) the entries a_n as suitable powers of a small parameter ε, it is proved that the absolutely continuous spectrum E of J consists of *exactly* N disjoint intervals. However, this E is not the same as the corresponding one in the case of bounded periodic Jacobi matrices. In particular, the calculations in the last section show that we can construct examples with an arbitrary number of gaps in $\sigma_{\mathrm{ess}}(J)$. Note that the essential spectrum of J does not coincide with the absolutely continuous one of the corresponding periodic Jacobi matrix. Hence the results on the existence of gaps do not follow from the classical theory of periodic Jacobi matrices.

In this paper we concentrate on *explicit* examples of unbounded Jacobi matrices with finitely many (arbitrary many) intervals of absolutely continuous spectrum. However, in our examples the essential spectrum is bounded. The alternate case of finitely many bounded gaps in the essential spectrum will be considered in a separate paper by using a different technique.

The paper is organized as follows. In Sect. 2.1 we present the main idea of the construction in the simplest case of one gap, through analyzed in the above mentioned papers from an essentially different point of view. In Sect. 2.2 we describe the explicit construction of examples of unbounded Jacobi matrices with finitely many gaps. Its generalization via perturbation theory is considered in Sect. 2.3. The next Sect. 3 deals with the detailed analysis of the case of an arbitrary even "period" N [see (2.1)]. Calculation of the absolutely continuous spectrum of the constructed Jacobi operator (for general even N) is given in Sect. 5. Moreover, the nondegeneracy of the gaps structure is shown there. Finally, the asymptotics of the discrete spectrum is under consideration in Sect. 4.

2. Construction of Explicit Examples with Several Gaps

2.1. Heuristics

The idea of the construction of an unbounded Jacobi matrix $\mathcal{J}$ with several gaps in $\sigma_{\mathrm{ess}}(\mathcal{J})$ is based on the following heuristic reasoning.

Take a family of infinite Jacobi matrices (for simplicity all with zero diagonal, $b_n \equiv 0$) and periodic with respect to some parameters. The essential spectrum of any member of the family consists of a finite union of intervals which are symmetric with respect to zero

$$(-x_M, -y_M) \cup \cdots \cup (-x_1, -y_1) \cup (y_1, x_1) \cup \cdots \cup (y_M, x_M),$$

for an integer M. We choose some parameters of these Jacobi matrices in such a way that the "internal" intervals

$$(-x_{M-1}, -y_{M-1}), \ldots, (y_{M-1}, x_{M-1})$$

are almost fixed, and try to choose other parameters tending to infinity, with $(-x_M, -y_M)$, (y_M, x_M), tending to $-\infty$, $+\infty$ respectively. Next we construct

an unbounded Jacobi matrix $\mathcal{J}$ (with $b_n \equiv 0$) which looks similar (locally in index n) to the above ones from the family (but surely not a periodic one). Moreover, each copy of our periodic family appears "approximately" within the matrix $\mathcal{J}$ on some (eventually extended) sequence of matrix elements of index n. Additionally we shall try to choose the parameters smoothly or rather adiabatically to be able to track the asymptotics of solutions of (1.2) and therefore to control the a.c. spectrum. These heuristic arguments will be made precise below.

The One Gap Case

We illustrate the above idea in the one gap case, which was first found in [17]. However, the reasoning given in [17] was completely different. Let $\mathcal{J}_{\mathrm{p}}$ be the periodic Jacobi matrix given by $b_n \equiv 0$ and

$$a_{2n-1} = w_1, \quad a_{2n} = w_2, \quad w_1 > w_2 > 0.$$

Then the product $B_2 B_1$ of the transfer matrices is

$$B_2 B_1 = \begin{pmatrix} 0 & 1 \\ \frac{w_1}{w_2} & \frac{\lambda}{w_2} \end{pmatrix} \begin{pmatrix} 0 & 1 \\ \frac{w_2}{w_1} & \frac{\lambda}{w_1} \end{pmatrix} = \begin{pmatrix} -\frac{w_2}{w_1} & \frac{\lambda}{w_1} \\ -\frac{\lambda}{w_1} & -\frac{w_1}{w_2} + \frac{\lambda^2}{w_1 w_2} \end{pmatrix}.$$

Two bands of $\sigma_{\mathrm{ac}}(\mathcal{J}_{\mathrm{p}})$ are given by

$$\{\lambda \mid |\mathrm{tr}(B_2 B_1| \le 2\},$$

see [24]. It is easy to check that $\sigma_{\mathrm{ac}}(\mathcal{J}_{\mathrm{p}})$ is the union of two intervals

$$[-(w_1 + w_2), -(w_1 - w_2)] \cup [w_1 - w_2, w_1 + w_2].$$

We may keep $w_1 - w_2$ equal to a fixed positive constant c but with $w_1 + w_2$ tending to $+\infty$. For example we can choose

$$w_1 = n^\alpha + c_1, \quad \alpha \in (0, 1], \quad w_2 = n^\alpha + c_2, \quad c = |c_1 - c_2|.$$

In this way we obtain a 2-periodic perturbation of the Jacobi matrix with weights $a_n = n^\alpha$ and zero diagonal which was studied in [5,6,8,12,17,21]. Below we shall exploit a similar idea in the construction of our example.

2.2. Basic Examples

According to the above heuristics we define basic examples of Jacobi matrices as follows.

Definition 1 ($\mathcal{J}(\alpha)$). Let $z_1, \ldots, z_N$ be N positive numbers and $\alpha \in (0, 1]$. The Jacobi matrix $\mathcal{J}(\alpha)$ is defined by zero diagonal and weights

$$a_{l(N+2)+k} = \begin{cases} (l+1)^\alpha, & \text{if } k = 1, 2, & l = 0, 1, \ldots \\ z_{k-2}, & \text{if } k = 3, 4, \ldots, N+2, & l = 0, 1, \ldots \end{cases} \tag{2.1}$$

Remark. We must assume that all $z_i \ne 0$, otherwise $\mathcal{J}(\alpha) = \oplus M_s$, for some finite Jacobi matrices M_s.

One can prove that for $N = 4$ the spectral picture of $\mathcal{J}(\alpha)$ has the following form (in the generic case) (Fig. 1).
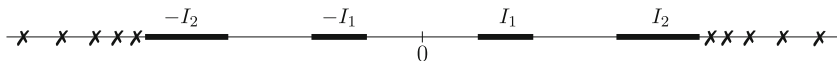
FIGURE 1. Spectrum of $\mathcal{J}(\alpha)$ for $N = 4$

Here

$$\sigma_{\mathrm{ac}}(\mathcal{J}(\alpha)) = -I_2 \cup -I_1 \cup I_1 \cup I_2$$

with known intervals I_1, I_2 and some eigenvalues of $\mathcal{J}(\alpha)$ denoted by $\times$ which tend to $\mp\infty$. This last assertion is obvious because $\mathcal{J}(\alpha)$ is an unbounded Jacobi matrix.

The above choice of the position of the weights l^α in the l-th block, $l = 1, 2, \ldots$:

$$(l^\alpha, l^\alpha, z_1, \ldots, z_N)$$

could surely be different but definitely not arbitrary. For another example with the l-th block of weights given by

$$(l^\alpha, z_1, \ldots, z_N, l^\alpha)$$

we obtain a similar spectral picture. What is essential for both examples is that the norm of $B_{(N+2)l+3}(\lambda)B_{(N+2)l+2}(\lambda)B_{(N+2)l+1}(\lambda)$ remains bounded and smooth in l, as $l \to \infty$. However, $\|B_{(N+2)l+2}(\lambda)B_{(N+2)l+1}(\lambda)\| \to \infty$, as $l \to \infty$.

In Sect. 5 we shall prove that indeed the spectral picture of $\mathcal{J}(\alpha)$ (described above for $N = 4$) looks similar for general N.

Theorem 2. *If $\mathcal{J}(\alpha)$ is the Jacobi operator defined by (2.1), then*

$$\sigma_{\mathrm{ac}}\left(\mathcal{J}(\alpha)\right) \supset \bar{E}$$

where E, defined by (2.5), is a finite collection of intervals filled by purely absolutely continuous spectrum of local multiplicity one a.e. with respect to the Lebesgue measure.

Proof. It turns out that the spectral analysis of $\mathcal{J}(\alpha)$ can be based on the asymptotic behaviour of generalized eigenvectors corresponding to all λ which do not belong to $\cup_{i=1}^{N} \partial I_i$, where ∂I_i is the boundary of I_i. Due to the irregular behaviour (jumps) of products of transfer matrices within blocks we shall study the products over the whole blocks, see [10]. These products behave regularly with respect to the indices s of blocks. Using (1.3) we have

$$\vec{u}_{(N+2)s+1} = B_{(N+2)s}(\lambda) \cdots B_{(N+2)(s-1)+1}(\lambda)\vec{u}_{(N+2)(s-1)+1}. \qquad (2.2)$$

We want to find the asymptotic behaviour of $\vec{u}_{(N+2)s+1}$ as $s \to +\infty$. By definition of the weights in the s-th block we have

$$B_{(N+2)s}(\lambda) \cdots B_{(N+2)(s-1)+1}(\lambda) \qquad (2.3)$$

$$= \underbrace{\begin{pmatrix} 0 & 1 \\ -\frac{z_{N-1}}{z_N} & \frac{\lambda}{z_N} \end{pmatrix} \cdots \begin{pmatrix} 0 & 1 \\ -\frac{z_1}{z_2} & \frac{\lambda}{z_2} \end{pmatrix}}_{M(\lambda)} \underbrace{\begin{pmatrix} 0 & 1 \\ -\frac{s^\alpha}{z_1} & \frac{\lambda}{z_1} \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & \frac{\lambda}{s^\alpha} \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -\frac{z_N}{s^\alpha} & \frac{\lambda}{s^\alpha} \end{pmatrix}}_{C}.$$

The product of the last three matrices above is

$$C = \begin{pmatrix} -\frac{\lambda z_N}{s^{2\alpha}} & -1 + \frac{\lambda^2}{s^{2\alpha}} \\ \frac{z_N}{z_1} - \frac{\lambda^2 z_N}{z_1 s^{2\alpha}} & -\frac{2\lambda}{z_1} + \frac{\lambda^3}{z_1 s^{2\alpha}} \end{pmatrix}. \tag{2.4}$$

If $M(\lambda)$ denotes the product of the first $(N-1)$ matrices to the left of C, then we can rewrite the left hand side of (2.3) as

$$A(\lambda) + s^{-2\alpha} B(\lambda),$$

where

$$A(\lambda) := M(\lambda) \begin{pmatrix} 0 & -1 \\ \frac{z_N}{z_1} & -\frac{2\lambda}{z_1} \end{pmatrix},$$

$$B(\lambda) := M(\lambda) \begin{pmatrix} -\lambda z_N & \lambda^2 \\ -\frac{\lambda^2 z_N}{z_1} & \frac{\lambda^3}{z_1} \end{pmatrix}.$$

Note that $A(\lambda)$, $B(\lambda)$ does not depend on s and $B(\lambda)$ is not the monodromy matrix generated by the periodic sequence $z_1, \ldots, z_N$.

Next we observe that

$$\det[A(\lambda) + s^{-2\alpha} B(\lambda)] = \frac{z_{N-1}}{z_N} \cdots \frac{z_1}{z_2} \cdot \frac{s^\alpha}{z_1} \cdot \frac{z_N}{s^\alpha} = 1.$$

If $\operatorname{disc} A := (\operatorname{tr} A)^2 - 4 \det A$ denotes the discriminant of a 2×2 matrix A, then

$$\operatorname{disc}[A(\lambda) + s^{-2\alpha} B(\lambda)] = \left(\operatorname{tr}[A(\lambda) + s^{-2\alpha} B(\lambda)] \right)^2 - 4.$$

Notation (elliptic part). *We denote*

$$E := \{ \lambda \in \mathbb{R} \mid |\operatorname{tr} A(\lambda)| < 2 \}. \tag{2.5}$$

Below we shall describe some properties of the function $\lambda \mapsto \operatorname{tr} A(\lambda)$. Now we only note that $\operatorname{tr} A(\lambda)$ is a polynomial in λ of degree N. Therefore, each of the equations $\operatorname{tr} A(\lambda) = \pm 2$ has at most N solutions. It follows that E consists of at most N intervals.

If $\lambda \in E$ then, for some s_0,

$$\operatorname{disc}[A(\lambda) + s^{-2\alpha} B(\lambda)] < 0 \text{ for any } s \geq s_0.$$

Consequently, the product in (2.3) is an elliptic matrix for any $s \geq s_0$. Thus, the matrix $A(\lambda) + s^{-2\alpha} B(\lambda)$ has two distinct eigenvalues

$$\mu_\lambda^{(1)}(s), \ \mu_\lambda^{(2)}(s) \text{ for } s \gg 1,$$

with

$$\left| \mu_\lambda^{(i)}(s) \right| = 1, \quad i = 1, 2 \text{ and } \mu_\lambda^{(2)}(s) = \overline{\mu_\lambda^{(1)}(s)}.$$

Consider the system of equations

$$\vec{y}_s = \left(A(\lambda) + s^{-2\alpha} B(\lambda) \right) \vec{y}_{s-1}, \quad \vec{y}_s := \vec{u}_{(N+2)s+1}, \quad s = 2, 3, \ldots$$

The above properties of the matrices $A(\lambda) + s^{-2\alpha}B(\lambda)$ allow to apply [19, Theorem 3.2] and to obtain a basis of solutions of (1.3) with the following asymptotic formula

$$\vec{u}^{(k)}_{(N+2)s}(\lambda) = \prod_{r=2}^{s-1} \mu^{(k)}_\lambda(r)\,(f_k + o(1)), \quad k = 1, 2, \tag{2.6}$$

where f_k are the eigenvectors of $A(\lambda)$. By definition of the weights and by direct computation we also obtain the asymptotic behaviour of bases of solutions of (1.3) for the remaining subsequences of indices:

$$\vec{u}^{(k)}_{(N+2)s+1}(\lambda) = B_{(N+2)s}(\lambda)\vec{u}^{(k)}_{(N+2)s}(\lambda), \tag{2.7}$$

$$\vec{u}^{(k)}_{(N+2)s+2}(\lambda) = \left[\begin{pmatrix} -\frac{z_{N-1}}{z_N} & \frac{\lambda}{z_N} \\ 0 & 0 \end{pmatrix} + (s+1)^{-\alpha} \begin{pmatrix} 0 & 0 \\ -\frac{z_{N-1}}{z_N} & -z_N + \frac{\lambda^2}{z_N} \end{pmatrix} \right]$$
$$\times \vec{u}^{(k)}_{(N+2)s}(\lambda), \tag{2.8}$$

$$\vec{u}^{(k)}_{(N+2)s+3}(\lambda) = \begin{pmatrix} 0 & 1 \\ -1 & \lambda(s+1)^{-\alpha} \end{pmatrix} \vec{u}^{(k)}_{(N+2)s+2}(\lambda), \tag{2.9}$$

$$\vec{u}^{(k)}_{(N+2)s+4}(\lambda) = \left[\begin{pmatrix} 0 & -1 \\ \frac{z_N}{z_1} & -\frac{2\lambda}{z_1} \end{pmatrix} + (s+1)^{-2\alpha} \begin{pmatrix} -\lambda z_N & \lambda^2 \\ -\lambda^2 z_N & \frac{\lambda^3}{z_1} \end{pmatrix} \right]$$
$$\times \vec{u}^{(k)}_{(N+2)s+1}(\lambda), \tag{2.10}$$

$$\vec{u}^{(k)}_{(N+2)s+l}(\lambda) = \prod_{i=2}^{l-3} A_i(\lambda)\vec{u}^{(k)}_{(N+2)s+4}(\lambda), \quad l = 5, \ldots, N+3, \tag{2.11}$$

where

$$A_i(\lambda) = \begin{pmatrix} 0 & 1 \\ -\frac{z_{i-1}}{z_i} & \frac{\lambda}{z_i} \end{pmatrix}.$$

Combining (2.6), (2.7), (2.8), (2.9), (2.10) and (2.11) we see that the system (1.2) has no subordinated solutions for $\lambda \in E$ [see (2.5)]. This completes the proof. $\square$

Notation (hyperbolic part). *Now we turn to the remaining case of*

$$S := \{\lambda \in \mathbb{R} \mid |\operatorname{tr} A(\lambda)| > 2\}. \tag{2.12}$$

It happens that in this case we again can find the asymptotic behaviour of generalized eigenvectors for $\lambda \in S$. This hyperbolic asymptotics will allow us to prove that the spectrum of $\mathcal{J}(\alpha)$ in S is pure point. In order to prove its discreteness we assume that $\alpha \in (\frac{1}{2}, 1)$. This restriction is not necessary. The results (we prove below) hold true for any $\alpha \in (0, 1]$ but their proofs are less elementary.

Theorem 3. *The spectrum of $\mathcal{J}(\alpha)$ in S is discrete.*

Proof. For $\vec{y}_s := \vec{u}_{(N+2)s+1}$ we consider the system

$$\vec{y}_s = \left(A(\lambda) + s^{-2\alpha}B(\lambda)\right)\vec{y}_{s-1}, \quad \lambda \in S, \quad s = 2, 3, \ldots$$

By diagonalization of $A(\lambda)$ we can write $A(\lambda) = V(\lambda)D(\lambda)V(\lambda)^{-1}$ with $D(\lambda)$ diagonal. Hence, for $\vec{z}_s := V(\lambda)^{-1}\vec{y}_s$ we have the new system

$$\vec{z}_s = \left(D(\lambda) + s^{-2\alpha}B_1(\lambda)\right)\vec{z}_{s-1}, \quad s = 2, 3, \ldots \tag{2.13}$$

where $B_1(\lambda) = V^{-1}(\lambda)B(\lambda)V(\lambda)$. All functions $V(\lambda)$, $V(\lambda)^{-1}$, $D(\lambda)$ of λ can be chosen continuous [8].

Fix a bounded interval $I \subset S$. Since $\alpha > \frac{1}{2}$ we can apply a uniform version of the Benzaid–Lutz result [19, Theorem 3.2] and obtain a basis $\vec{\phi}_n^{(k)}(\lambda)$ ($k = 1, 2$) of solutions of (2.13) such that

$$\sup_{\lambda \in I} \left\| \vec{\phi}_n^{(k)}(\lambda) \left(\prod_{i=N_0}^{n-1} \mu_i^{(k)}(\lambda) \right)^{-1} - e_k \right\| \xrightarrow[n \to +\infty]{} 0, \tag{2.14}$$

the spectrum

$$\sigma\left(A(\lambda) + s^{-2\alpha}B(\lambda)\right) = \left\{\mu_s^{(1)}(\lambda), \mu_s^{(2)}(\lambda)\right\}$$

and N_0 is so large that $\mathrm{tr}\left(A(\lambda) + s^{-2\alpha}B(\lambda)\right) < -2$ for all $s \geq N_0$. The case $\mathrm{tr}\left(A(\lambda) + s^{-2\alpha}B(\lambda)\right) > 2$ is similar. It follows that

$$0 < \mu_s^{(1)}(\lambda) := \frac{1}{2}\left(-\mathrm{tr}\,A_s - \sqrt{\mathrm{disc}\,A_s}\right) < 1, \tag{2.15}$$

where $A_s := A(\lambda) + s^{-2\alpha}B(\lambda)$. Using (2.14) and (2.15) we have

$$\left\{\sup_{\lambda \in I}\|\vec{\phi}_n^{(1)}(\lambda)\|\right\}_{n=1}^{\infty} \in \ell^2.$$

This can be easily seen, by repeating the reasoning given in the proofs of (2.7), (2.8), (2.9), (2.10) and (2.11). In particular, the "eigenvalue equation" (1.2) has a nontrivial subordinated solution.

Consequently, $S \cap \sigma_{\mathrm{ac}}\left(\mathcal{J}(\alpha)\right) = \varnothing$ and so $E \subset \sigma_{\mathrm{ac}}\left(\mathcal{J}(\alpha)\right) \subset \bar{E}$ (by Theorem 2). The subordination theory shows that $\sigma_{\mathrm{sc}}\left(\mathcal{J}(\alpha)\right)$ is empty [15]. Therefore, S contains only eigenvalues of $\mathcal{J}(\alpha)$. Using [19, Lemma 4.2] for a given $n \geq N_0$ and any $\varepsilon > 0$ one can find $\delta > 0$ such that

$$\left.\begin{array}{c}|\lambda - \mu| < \delta \\ \lambda, \mu \in I\end{array}\right\} \implies \|\vec{\phi}_n^{(1)}(\lambda) - \vec{\phi}_n^{(1)}(\mu)\| < \varepsilon. \tag{2.16}$$

Now we can repeat the argument of the proof of [19, Theorem 5.3]. Although this argument was given for a particular class of Jacobi matrices it can be applied in general situations. To prove discreteness of $\sigma\left(\mathcal{J}(\alpha)\right)$ in S assume that there exists a sequence of distinct eigenvalues $\{\varrho_m\}$ of $\mathcal{J}(\alpha)$ which accumulates at $\varrho \in S$. Altogether (2.14) provides a uniform tails estimate, and (2.16) guarantees the coordinate convergence

$$\vec{\phi}_n^{(1)}(\varrho_m) \xrightarrow[m \to \infty]{} \vec{\phi}_n^{(1)}(\varrho).$$

This allows to repeat the reasoning given in [19, p. 184].

Indeed, by orthogonality of the eigenvectors $\{\vec{\phi}_n^{(1)}(\varrho_s)\}$ of $\mathcal{J}(\alpha)$ (for different ϱ_s) we have

$$0 = \lim_{\substack{m\to\infty \\ k\to\infty \\ k\neq m}} \left\langle \{\vec{\phi}_n^{(1)}(\varrho_k)\}, \{\vec{\phi}_n^{(1)}(\varrho_m)\} \right\rangle_{\ell^2} = \sum_{s=1}^{+\infty} |\vec{\phi}_s^{(1)}(\varrho)|^2.$$

The last equality can be easily verified by using (2.14) and (2.16). Therefore, $\vec{\phi}_n^{(1)}(\varrho) = 0$, and this contradiction completes the proof of Theorem 3.   □

2.3. Generalization

Slight perturbations of the above examples allow to obtain a class of unbounded Jacobi matrices with the same spectral picture as before.

Notation. D^1 *denotes the space of bounded variation sequences.*

Definition 4 ($\widetilde{\mathcal{J}}(\alpha)$). The Jacobi matrix $\widetilde{\mathcal{J}}(\alpha)$ is defined by zero diagonal and by the new weights

$$\tilde{a}_{l(N+2)+k} = \begin{cases} (l+1)^\alpha + c_k(l), & \text{if } k = 1, 2, \ l = 0, 1, \ldots \\ z_{k-2} + t_k(l), & \text{if } k = 3, \ldots, N+2, \ l = 0, 1, \ldots \end{cases} \tag{2.17}$$

submitted to the following two assumptions:

$$\{t_k(l)\}_{l\geq 0} \in \ell^2 \cap D^1, \quad k = 3, 4, \ldots, N+2. \tag{2.18}$$

$$\{c_1(l)/l^\alpha\}_{l\geq 0} \ \& \ \{c_2(l)/l^\alpha\}_{l\geq 0} \in \ell^2 \cap D^1. \tag{2.19}$$

Remark. Note that (2.19) allows unbounded sequences $\{c_1(l)\}$ and $\{c_2(l)\}$.

By repeating the computations leading to (2.4) we find

$$B_{(N+2)s}(\lambda) \cdots B_{(N+2)(s-1)+1}(\lambda)$$
$$= \begin{pmatrix} 0 & 1 \\ -\frac{z_{N-1}+t_{N-1}(l)}{z_N+t_N(l)} & \frac{\lambda}{z_N+t_N(l)} \end{pmatrix} \cdots \begin{pmatrix} 0 & 1 \\ -\frac{z_1+t_1(l)}{z_2+t_2(l)} & \frac{\lambda}{z_2+t_2(l)} \end{pmatrix} \tag{2.20}$$
$$\times \begin{pmatrix} 0 & 1 \\ -\frac{l^\alpha+c_2(l)}{z_1+t_1(l)} & \frac{\lambda}{z_1+t_1(l)} \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -\frac{l^\alpha+c_1(l)}{l^\alpha+c_2(l)} & \frac{\lambda}{l^\alpha+c_2(l)} \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -\frac{z_N+t_N(l)}{l^\alpha+c_1(l)} & \frac{\lambda}{l^\alpha+c_1(l)} \end{pmatrix}.$$

Note that due to (2.18) we have the following decomposition

$$\begin{pmatrix} 0 & 1 \\ -\frac{z_{s-1}+t_{s-1}(l)}{z_s+t_s(l)} & \frac{\lambda}{z_s+t_s(l)} \end{pmatrix} = C_s + W_{sl}(\lambda) + R_{sl}(\lambda), \tag{2.21}$$

where

$$C_s := \begin{pmatrix} 0 & 1 \\ -\frac{z_{s-1}}{z_s} & \frac{\lambda}{z_s} \end{pmatrix},$$

$$W_{sl}(\lambda) := \begin{pmatrix} 0 & 0 \\ \frac{z_{s-1}t_s(l)-z_s t_{s-1}(l)}{z_s{}^2} & -\frac{\lambda t_s(l)}{z_s} \end{pmatrix},$$

$$\{\|R_{sl}(\lambda)\|\}_{l\geq 0} \in \ell^1, \quad s = 2, \ldots, N.$$

Using (2.21) and distributive rule in (2.20) we can rewrite the product of the first $N - 1$ matrices of the right side of (2.20) as follows:

$$M(\lambda) + \sum_{r=2}^{N-1} C_N \ldots C_{r+1} W_{rl}(\lambda) C_{r-1} \ldots C_2 + R_l{}^{(1)}(\lambda),$$

with $\{\|R_l^{(1)}(\lambda)\|\}_{l \geq 0} \in \ell^1$. Since $\{W_{rl}(\lambda)\}_{l \geq 0} \in \ell^2 \cap D^1$ the above product has the form:

$$M(\lambda) + T_l(\lambda) + R_l^{(1)}(\lambda), \tag{2.22}$$

where $\{T_l(\lambda)\}_{l \geq 0} \in \ell^2 \cap D^1$.

In turn, due to (2.19) the product of the three last matrices of the right side of (2.20) can be written

$$D_N + V_l + R_l^{(2)}(\lambda) \tag{2.23}$$

with

$$D_N = \begin{pmatrix} 0 & -1 \\ \frac{z_N}{z_1} & -2\frac{\lambda}{z_1} \end{pmatrix},$$

$$V_l = \begin{pmatrix} 0 & (c_2(l) - c_1(l))l^{-\alpha} \\ \frac{z_1^2 t_N(l-1) - z_N t_N(l)}{z_1{}^2} + \frac{z_N(c_2(l) - c_1(l))}{z_1 l^{-\alpha}} & 0 \end{pmatrix},$$

$\{\|R_l^{(2)}(\lambda)\|\}_{l \geq 0} \in \ell^1$.

Combining (2.22) and (2.23) we rewrite the right hand side of (2.20) as

$$A(\lambda) + S_l(\lambda) + R_l^{(3)}(\lambda), \tag{2.24}$$

where

$$A(\lambda) = M(\lambda) D_N(\lambda),$$

$$\{S_l(\lambda)\}_{l \geq 0} \in \ell^2 \cap D^1,$$

$$\{\|R_l^{(3)}(\lambda)\|\}_{l \geq 0} \in \ell^1.$$

By repeating the proofs of Theorems 2 and 3 we obtain

Theorem 5. *Let $\tilde{J}(\alpha)$ be the Jacobi operator defined by the weights (2.17) submitted to conditions (2.18) and (2.19) with $\alpha > 1/2$. Then the statements of Theorems 2 and 3 remain valid for $\tilde{J}(\alpha)$.*

Proof (sketch). We first note that due to the definitions of $R_{sl}(\lambda), T_l(\lambda), M(\lambda)$ and (2.24) the error terms $R_l^{(1)}(\lambda)$ and $R_l^{(2)}(\lambda)$ are continuous functions of λ. Moreover, both $R_l^{(1)}(\lambda)$ and $R_l^{(2)}(\lambda)$ have uniform tail estimates, i.e. for any compact subset $X \subset \mathbb{R}$ and for any $\varepsilon > 0$ there exists $K = K(\varepsilon, X)$ such that

$$\sup_{\lambda \in X} \sum_{l=K}^{\infty} \|R_l^{(s)}(\lambda)\| < \varepsilon, \quad s = 1, 2.$$

Consequently, $R_l^{(3)}(\lambda)$ satisfies similar tail estimates. Remember that $A(\lambda)$ and $S_l(\lambda)$ are also continuous functions of λ (to be more precise they are polynomials in λ).

Using a uniform Levinson type theorem (see [19] for the case of uniform ℓ^1 perturbations and [20] for the case of uniform D^1 ones) one can easily complete the proof of the analogues of Theorems 2 and 3. $\qquad\square$

Remark. By adding a nonzero main diagonal $\{b_k\} \in \ell^2 \cap D^1$ one obtains a still larger class of Jacobi matrices with the same spectral picture as $\mathcal{J}(\alpha)$. This surely introduces a compact perturbation of the Jacobi operator and by Weyl's theorem it preserves the essential spectrum, but our goal is to preserve $\sigma_{\mathrm{ac}}(\mathcal{J}(\alpha))$.

3. Analysis of Particular Examples

One of the aims of particular examples analysis is to verify the nondegeneracy of arbitrary many gaps in the essential spectrum.

Properties of the Characteristic Polynomial

We shall discuss some properties of the polynomial $p(\lambda) := \operatorname{tr} A(\lambda)$ which will be useful below.

We define the antidiagonal and diagonal matrices, respectively

$$A_k := \begin{pmatrix} 0 & 1 \\ -\frac{z_{k-1}}{z_k} & 0 \end{pmatrix}, \quad B_k := \begin{pmatrix} 0 & 0 \\ 0 & \frac{1}{z_k} \end{pmatrix}, \quad k = 1, \ldots, N$$

where $z_0 := z_N$. In what follows we assume that N is an *even* number and all z_j are positive (without loss of generality). We have

$$p(\lambda) = \operatorname{tr}[(A_N + \lambda B_N) \cdots (A_2 + \lambda B_2)(-A_1 - 2\lambda B_1)] =: \sum_{k=0}^{N} c_k \lambda^k. \quad (3.1)$$

Lemma 3.1. *Let N be an even number. Then the coefficients c_k of the characteristic polynomial $p(\lambda) := \operatorname{tr} A(\lambda)$ satisfy:*

(i) $c_{2m+1} = 0$, *for $m = 0, 1, 2, \ldots$*

(ii) *The sign of c_{2m} coincides with $-(-1)^{\frac{N-2m}{2}}$.*

Proof. We pick up the first n_1 antidiagonal terms A_i $(i = N, \ldots, N - n_1 + 1)$. After that in the next to the right matrix we take only the term λB_{N-n_1}. The same procedure can be repeated, i.e., we pick up the next n_2 antidiagonal matrices A_s $(s = N - n_1 - 1, \ldots, N - n_1 - n_2)$, etc.

Totally, we get a sequence $n_1, n_2, \ldots, n_{k+1}$, $0 \le n_s \le N$ such that $\sum_{s=1}^{k+1} n_s = N - k$. Observe that all the traces

$$\operatorname{tr}(A_N \cdots A_{N-n_1+1} B_{N-n_1} \cdots) = 0,$$

provided that k is odd. This is clear because then $N - k$ is an odd number and therefore the above product $A_N \cdots A_{N-n_1+1} B_{N-n_1} \cdots$ has an odd number of anti-diagonal matrices A_s and a certain number of diagonal matrices B_r. Hence the whole product gives an anti-diagonal matrix (therefore with zero

trace). Moreover, in all nonzero terms $\operatorname{tr}\left(A_N \cdots A_{N-n_1+1} B_{N-n_1} \cdots\right)$ all the numbers $n_1, n_2, \ldots, n_{k+1}$ must be even. Indeed, this fact is a consequence of the following elementary matrix algebra. First note that due to the invariance of the trace with respect to the cyclic permutations one can reduce the proof to the case that n_1 should be even. Assume for a moment that n_1 is an odd number. Then the first to the left product of n_1 anti-diagonal matrices $A_N \cdots A_{N-n_1+1}$ is again an anti-diagonal matrix. The next term B_{N-n_1} has the form $\begin{pmatrix} 0 & 0 \\ 0 & \star \end{pmatrix}$ with one nonzero entry in the right lower corner. The remaining product of matrices contains $k-1$ diagonal matrices B_s and $N - n_1 - k$ anti-diagonal matrices A_s. Since the numbers N and k are even and n_1 is odd, so $N - n_1 - k$ is odd, and therefore the remaining product must also contain an odd number n_r of matrices A_i's. We can assume that the remaining product begins from the left with an anti-diagonal matrix. In fact, all diagonal factors at the left of the remaining product multiplied by B_{N-n_1} would produce a matrix of the form $\begin{pmatrix} 0 & 0 \\ 0 & \star \end{pmatrix}$. Therefore, the whole product would have the following structure

$$\begin{pmatrix} 0 & \star \\ \star & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 \\ 0 & \star \end{pmatrix}\begin{pmatrix} 0 & \star \\ \star & 0 \end{pmatrix}\begin{pmatrix} * & \\ & * \end{pmatrix} \cdots \begin{pmatrix} * & \\ & * \end{pmatrix}$$

where all matrices $\begin{pmatrix} * & \\ & * \end{pmatrix}$ are diagonal. But the product of these diagonal matrices $\begin{pmatrix} * & \\ & * \end{pmatrix}$ contains B_i's and so the product of all of them has the form $\begin{pmatrix} 0 & 0 \\ 0 & \star \end{pmatrix}$. It follows that the whole product is equal to

$$\begin{pmatrix} 0 & \star \\ \star & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 \\ 0 & \star \end{pmatrix}\begin{pmatrix} 0 & \star \\ \star & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 \\ 0 & \star \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.$$

This contradiction proves that n_1 is even. A more detailed analysis of the product of n_i anti-diagonal A_s shows that it is a positive diagonal matrix muliplied by $(-1)^{\frac{n_i}{2}}$. This fact can be easily checked using the structure of

$$A_s = \begin{pmatrix} 0 & \alpha_s \\ \beta_s & 0 \end{pmatrix}, \quad \alpha_s > 0,\ \beta_s < 0.$$

Since all B_s have non-negative entries, the sign of c_k (for even k) coincides with

$$-(-1)^{\frac{n_1}{2}+\cdots+\frac{n_k}{2}} = -(-1)^{\frac{N-k}{2}}.$$

Therefore, the signs of $c_0, c_2, \ldots, c_N$ form an alternating sequence. In particular, the sign of c_0 is equal to $-(-1)^{\frac{N}{2}}$. The proof is complete. $\square$

Remark 3.2. Note that vanishing of c_{2m+1} is not surprising, due to the symmetry $\lambda \mapsto -\lambda$ of $\sigma\left(\mathcal{J}(\alpha)\right)$ with vanishing main diagonal of $\mathcal{J}(\alpha)$.

Remark 3.3. Let $N = 2M$. Denote $\mu := \lambda^2$. Then

$$P(\mu) := p(\sqrt{\mu}) = c_{2M}\mu^M + \cdots + c_2\mu + c_0,$$

with alternating signs of the coefficients. Observe that

$$c_{2M} = -\frac{2}{z_N z_{N-1} \ldots z_1}$$

and so $c_{2M} \neq 0$.

4. Asymptotics of the Discrete Spectrum at Infinity

In this section we shall find asymptotic formulae for the eigenvalues of $\mathcal{J}(\alpha)$ at $\pm\infty$. More precisely, we shall compute the main terms of these formulae (modulo bounded corrections). In recent years appeared several papers devoted to the asymptotic behaviour of the eigenvalues of unbounded, self-adjoint Jacobi matrices [3,7,11,16,25]. However, the results found in these works do no apply to $\mathcal{J}(\alpha)$. Therefore, we have decided to include asymptotic formulas for the eigenvalues of $\mathcal{J}(\alpha)$, although they are not the main concern of this work.

The idea of computing approximate values of $\sigma_{\mathrm{p}}\left(\mathcal{J}(\alpha)\right)$ (for sufficiently large eigenvalues) is based on the following decomposition

$$\mathcal{J}(\alpha) = \mathcal{J}_1 + \mathcal{J}_2$$

where $\mathcal{J}_1$, $\mathcal{J}_2$ are Jacobi matrices and the weights of $\mathcal{J}_1$ are obtained from the weights of $\mathcal{J}(\alpha)$ by replacing all z_i $(i = 1, \ldots, N)$ by zeros whereas the weights of $\mathcal{J}_2$ are defined by replacing all terms (k^α, k^α) $(k = 1, 2, \ldots)$ by zeros. The definition of $\mathcal{J}_1$ implies that

$$\mathcal{J}_1 = \bigoplus_{k=1}^{\infty}(M_k \oplus 0_N), \tag{4.1}$$

where 0_N is the $(N-1) \times (N-1)$ zero matrix and

$$M_k := \begin{pmatrix} 0 & k^\alpha & 0 \\ k^\alpha & 0 & k^\alpha \\ 0 & k^\alpha & 0 \end{pmatrix}.$$

By definition of $\mathcal{J}_2$ we have

$$\mathcal{J}_2 = \bigoplus_{k=1}^{\infty}(0_2 \oplus L_k), \tag{4.2}$$

where

$$L_k := L := \begin{pmatrix} 0 & z_1 & 0 & 0 & \ldots & \ldots & 0 \\ z_1 & 0 & z_2 & 0 & \ldots & \ldots & 0 \\ 0 & z_2 & 0 & z_3 & \ldots & \ldots & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots & z_{N-1} & 0 & z_N \\ 0 & 0 & \ldots & \ldots & 0 & z_N & 0 \end{pmatrix}.$$

Well known elementary estimates and (4.2) show that

$$\|\mathcal{J}_2\| \leq 2 \max_i |z_i| \cdot \cos \frac{\pi}{N+1}$$

and

$$\sigma_{\mathrm{ess}}(\mathcal{J}_2) = \{0\} \cup \sigma(L).$$

Applying [2, Theorem 4, Chap. 9, Sec. 4, p. 219] we have

$$\sigma_{\mathrm{ess}}(\mathcal{J}(\alpha)) \subset [-\|\mathcal{J}_2\|, \|\mathcal{J}_2\|] = [-\|L\|, \|L\|]. \tag{4.3}$$

In what follows we shall consider $\sigma_{\mathrm{p}}(\mathcal{J}(\alpha))$ outside the interval $[-\|L\|, \|L\|]$. Since the spectrum of $\mathcal{J}(\alpha)$ outside this interval is discrete and symmetric with respect to zero ($b_n \equiv 0$) it accumulates at $\mp\infty$.

Denote $\|L\| = a$, and take a large number $M \gg a$. For a self-adjoint operator T, let $\Pi_T(\alpha, \beta)$ be the number of eigenvalues of T in the interval (α, β).

At this point we use standard arguments of perturbation theory. Namely applying [2, Lemma 3, Chap. 9, Sec. 4, p. 218] twice (with $A = \mathcal{J}_1$, $V = \mathcal{J}_2$ or $A = \mathcal{J}(\alpha)$, $V = -\mathcal{J}_2$, respectively) we have the inequalities

$$\Pi_{\mathcal{J}_1}(2a, M) \leq \Pi_{\mathcal{J}(\alpha)}(a, M+a) \leq \Pi_{\mathcal{J}_1}(0, M+2a). \tag{4.4}$$

Note that

$$\sigma(M_k) = \left\{ -\sqrt{2}k^\alpha, 0, \sqrt{2}k^\alpha \right\}.$$

By definition of $\mathcal{J}_1$ and standard arguments we can write

$$\Pi_{\mathcal{J}_1}(2a, M) \sim \left(\frac{M}{\sqrt{2}} \right)^{\frac{1}{\alpha}}, \tag{4.5}$$

$$\Pi_{\mathcal{J}_1}(a, M+a) \sim \left(\frac{M}{\sqrt{2}} \right)^{\frac{1}{\alpha}} + \left(\frac{M}{\sqrt{2}} \right)^{\frac{1}{\alpha}} \frac{a}{\alpha M}, \tag{4.6}$$

as $M \to +\infty$. Combining (4.4), (4.5) and (4.6) we have

$$\Pi_{\mathcal{J}(\alpha)}(a, M+a) = \left(\frac{M}{\sqrt{2}} \right)^{\frac{1}{\alpha}} + r_M, \tag{4.7}$$

where $r_M = \mathrm{O}\left(M^{\frac{1}{\alpha}-1} \right)$, as $M \to +\infty$. Let

$$1 + a < \lambda_1(\mathcal{J}(\alpha)) < \lambda_2(\mathcal{J}(\alpha)) < \cdots$$

be the sequence of all eigenvalues of $\mathcal{J}(\alpha)$ greater than $a + 1$. The above reasoning and formula (4.7) prove

Theorem 6. *The large n asymptotic behaviour of $\lambda_n(\mathcal{J}(\alpha))$ is given by*

$$\lambda_n(\mathcal{J}(\alpha)) = \sqrt{2}\,n^\alpha\,(1 + \mathrm{O}(1)). \tag{4.8}$$

Open Problems. *Two open problems about $\sigma_\mathrm{p}(\mathcal{J}(\alpha))$.*

(i) *We expect that there is no concentration of $\sigma_\mathrm{p}(\mathcal{J}(\alpha))$ to the right and left of $\sigma_\mathrm{ess}(\mathcal{J}(\alpha))$.*

(ii) *Is $\sigma_\mathrm{p}(\mathcal{J}(\alpha))$ in the "interior" gaps finite?*

5. Approximate Lengths of Bands (N Even)

In this section we consider the Jacobi matrix $\mathcal{J}(\alpha)$ associated to a particular sequence $z_1, \ldots, z_N$ (all are powers ε^{γ_s} of a small positive parameter ε) which will allow to compute the small ε asymptotics of the lengths of the bands of E. This choice of the z_s will also allow to give sufficient conditions (in terms of the above exponents γ_s) for E being exactly the union of N disjoint intervals.

Most of the proofs will be sketchy in order to avoid lengthy but straightforward calculations.

Definition 7 ($\mathcal{J}^\varepsilon(\alpha)$). Let ε be a positive number. We denote $\mathcal{J}^\varepsilon(\alpha)$ the Jacobi matrix $\mathcal{J}(\alpha)$ associated to a sequence $z_1 = \varepsilon^{\gamma_1}, \ldots, z_N = \varepsilon^{\gamma_N}$ with $\gamma_1 < \cdots < \gamma_N$.

Lemma 8. *Assuming that the sequence $\{\gamma_k\}$ is convex, the coefficients c_{2k} (see (3.1)) admit the following asymptotics, as $\varepsilon \to 0$:*

$$c_{2k} \sim -(-1)^{\frac{N}{2}-k}(z_N z_{N-1} \cdots z_{N-2k+1})^{-1}\Phi(k), \tag{5.1}$$

where

$$\Phi(k) := \frac{z_{N-2k-1} z_{N-2k-3} \cdots z_1}{z_{N-2k} z_{N-2k-2} \cdots z_2}.$$

Proof. Indeed, the direct computation of

$$\mathrm{tr}\left(\sum_{\substack{n_1,\ldots,n_k \\ n_1+\cdots+n_k=N-k}} X_{n_1} \ldots X_{n_k} \right)$$

for a suitable choice of matrices X_i shows that the dominant term among all components of the above traces is given by

$$\mathrm{tr}(B_N B_{N-1} \cdots B_{N-2k+1} A_{N-2k} \cdots A_2 A_1) \tag{5.2}$$

for sufficiently small ε. This fact can be easily verified using the convexity of the sequence $\{\gamma_s\}$. Using our choice of $z_s = \varepsilon^{\gamma_s}$ and (5.2) we find

$$c_{2k} \sim \pm(-1)^k \varepsilon^{-\Delta_k}, \quad \text{as } \varepsilon \to 0, \tag{5.3}$$

where

$$\Delta_k := \gamma_N + \gamma_{N-1} + \cdots + \gamma_{N-2k+1}$$

$$+(\gamma_{N-2k} - \gamma_{N-2k-1}) + \cdots + (\gamma_2 - \gamma_1) \text{ for } k = 1, \ldots, \frac{N}{2}, \quad (5.4)$$

$$\Delta_0 := \gamma_N - \gamma_{N-1} + \gamma_{N-2} - \gamma_{N-3} + \cdots + \gamma_2 - \gamma_1,$$

with $\gamma_0 := 0$. Thus,

$$\Delta_{s+1} - \Delta_s = 2\gamma_{N-2s-1}, \quad s \geq 1. \quad (5.5)$$

Note that the Δ_s are positive and increasing. It follows that

$$\Delta_{s+1} - \Delta_s = 2\gamma_{N-2s-1} < 2\gamma_{N-2s+1} = \Delta_s - \Delta_{s-1}, \quad (5.6)$$

i.e., the sequence $\{\Delta_s\}$ is concave. □

Below we shall find approximate roots μ_{-s} (respectively μ_{+s}) of the equation $P(\mu) = -2$ (respectively $P(\mu) = 2$), $s = 1, \ldots, N = 2M$. Since

$$c_{2M} = -\frac{2}{z_N z_{N-1} \ldots z_1} < 0,$$

it is clear that $P(\mu) \to -\infty$, as $\mu \to +\infty$, independently on the parity of M. Hence the largest root coincides with μ_{-1} and corresponds to the equation $P(\mu) = -2$.

It turns out that a rough approximation of the roots can be found by the following simple procedure. Although simple this rough approximation dramatically reduces the complexity of finding the roots and is sufficient for our aim. Surely this procedure works due to the special form of the coefficients of the polynomial P.

First Step

We look for the solutions of the following simplified equations (just keeping two leading terms of the polynomial P):

$$c_{2k}\mu^k + c_{2k-2}\mu^{k-1} = 0, \quad k = 1, \ldots, \frac{N}{2}. \quad (5.7)$$

The solutions of (5.7) are:

$$\mu_k \equiv 0,$$

$$\tilde{\mu}_k := -\frac{c_{2k-2}}{c_{2k}} \sim \varepsilon^{\Delta_k - \Delta_{k-1}}, \text{ as } \varepsilon \to 0.$$

Below (the second step) by using a better approximation of the polynomial P we shall split the zero solution into smaller distinct roots.

Observe that all terms $c_{2s}\tilde{\mu}_k^s$ decay, for $s \neq k-1, k$, faster than $c_{2k}\tilde{\mu}_k^k$ and $c_{2k-2}\tilde{\mu}_k^{k-1}$ as $\varepsilon \to 0$. Indeed, we have two possibilities:

- $s < k - 1$. By concavity of $\{\Delta_r\}$ we have the following inequalities:

$$\Delta_k - \Delta_s = \Delta_k - \Delta_{k-1} + \cdots + \Delta_{s+1} - \Delta_s$$

$$> (k-s)(\Delta_k - \Delta_{k-1}).$$

But these inequalities are equivalent to

$$s(\Delta_k - \Delta_{k-1}) - \Delta_s > k(\Delta_k - \Delta_{k-1}) - \Delta_k$$

which by the form of c_{2s} implies faster decay of $c_{2s}\tilde{\mu}_k^s$.

- $s > k$. Similarly we have

$$\Delta_{s-1} - \Delta_{k-1} = \Delta_{s-1} - \Delta_{s-2} + \cdots + \Delta_k - \Delta_{k-1}$$

$$< (s-k)(\Delta_k - \Delta_{k-1}).$$

Thus,

$$\Delta_s - \Delta_k < \Delta_{s-1} - \Delta_{k-1} < (s-k)(\Delta_k - \Delta_{k-1}).$$

Again the last inequality is equivalent to

$$k(\Delta_k - \Delta_{k-1}) - \Delta_k < s(\Delta_k - \Delta_{k-1}) - \Delta_s$$

which implies the desired faster decay of $c_{2s}\tilde{\mu}_k^s$.

Second Step

We assume that one can find "better" approximate roots by choosing

$$\tilde{\mu}_{\pm k} = \tilde{\mu}_k \left(1 + \delta_\pm(k)\right), \quad k = 1, \ldots, \frac{N}{2} \tag{5.8}$$

as the solutions of the "better" equation

$$c_{2k}\tilde{\mu}_{\pm k}^k + c_{2k-2}\mu_{\pm k}^{k-1} = \pm 2 \tag{5.9}$$

(recall that we look for the solutions of $P(\mu) = \pm 2$).

Combining the asymptotic form for c_{2k} [see (5.3), (5.8) and (5.9)] we obtain

$$\delta_\pm(k) \sim \pm 2\varepsilon^{\Delta_k - k(\Delta_k - \Delta_{k-1})} \tag{5.10}$$

as $\varepsilon \to 0$. Since $\tilde{\mu}_{+s} \sim \tilde{\mu}_{-s} \gg \tilde{\mu}_{+,s+1} \sim \tilde{\mu}_{-,s+1}$ as $\varepsilon \to 0$ we have

$$\tilde{\mu}_{+s} - \tilde{\mu}_{-s} = \left(\delta_+(s) - \delta_-(s)\right)\tilde{\mu}_s \sim 4\varepsilon^{\Delta_s - (s-1)(\Delta_s - \Delta_{s-1})}. \tag{5.11}$$

This allows to calculate the lengths of bands $(\sqrt{\tilde{\mu}_{+s}}, \sqrt{\tilde{\mu}_{-s}})$ or $(\sqrt{\tilde{\mu}_{-s}}, \sqrt{\tilde{\mu}_{+s}})$. In fact,

$$\sqrt{\tilde{\mu}_{+s}} - \sqrt{\tilde{\mu}_{-s}} = \frac{\tilde{\mu}_{+s} - \tilde{\mu}_{-s}}{\sqrt{\tilde{\mu}_{+s}} + \sqrt{\tilde{\mu}_{-s}}} \sim 2\varepsilon^{\Delta_s - (s-\frac{1}{2})(\Delta_s - \Delta_{s-1})} \tag{5.12}$$

as $\varepsilon \to 0$.

The above arguments lead to

Theorem 9. *Let $\mathcal{J}^\varepsilon(\alpha)$ be the Jacobi matrix defined by a convex sequence $\gamma_1 < \cdots < \gamma_N$ as in Definition 7. The lengths of the bands $(\sqrt{\tilde{\mu}_{-s}}, \sqrt{\tilde{\mu}_{+s}})$ of the spectrum of $\mathcal{J}^\varepsilon(\alpha)$ lying in $\mathbb{R}_+$ (the spectrum is symmetric w.r.t. zero) have the following asymptotics, as $\varepsilon \to 0$:*

$$\sqrt{\tilde{\mu}_{+s}} - \sqrt{\tilde{\mu}_{-s}} \sim 2\varepsilon^{\Delta_s - (s-\frac{1}{2})(\Delta_s - \Delta_{s-1})}.$$

Remark. The above "approximate" lengths of the bands are decreasing as the parameter s increases. Moreover, $\tilde{\mu}_{\pm s}$ have different orders for different s. More precisely

$$\tilde{\mu}_{-1} > \tilde{\mu}_{+1} \gg \tilde{\mu}_{+2} > \tilde{\mu}_{-2} \gg \ldots$$

for ε sufficiently small.

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