

Unbounded Laplacians on Graphs: Basic Spectral Properties and the Heat Equation

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Abstract. We discuss Laplacians on graphs in a framework of regular Dirichlet forms. We focus on phenomena related to unboundedness of the Laplacians. This includes (failure of) essential selfadjointness, absence of essential spectrum and stochastic incompleteness.

Key words: Dirichlet forms, graphs, essential self adjointness, essential spectrum, isoperimetric inequalities, stochastic completeness

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Introduction

The study of Laplacians on graphs is a well established topic of research (see e.g. the monographs [4, 6] and references therein). Such operators can be seen as discrete analogues to Schrödinger operators. Accordingly their spectral theory has received quite some attention. Such operators also arise as generators of symmetric Markov processes and they appear in the study of heat equations on discrete structures. Recently, certain themes related to unboundedness properties of such operators have become a focus of attention. These themes include

- definition of the operators and essential selfadjointness,
- absence of essential spectrum,
- stochastic incompleteness.

In this paper we want to survey recent developments and provide some new results. Our principle goal is to make these topics accessible to non-specialists by providing a somewhat gentle and introductory discussion.

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Let us be more precise. We consider a graph with weights on edges and vertices. The weights can be seen to give a generalized vertex degree.

There is an obvious way to formally associate a symmetric nonnegative operator to such a graph. If the generalized vertex degrees are uniformly bounded this operator is bounded and all formal expressions make sense. If the generalized vertex degrees are not uniformly bounded already the definition of a self adjoint operator is an issue. This issue can be tackled by proving essential self-adjointness of the formal operator on the set of functions with compact support. This was done for locally finite weighted graphs by Jorgensen in [20] and for locally finite graphs by Wojciechowski in [33] (see [34] as well) and Weber in [32]. These results require local finiteness and do not allow for weights on the corresponding ℓ^2 space. As discussed by Keller/Lenz in [22] it is possible to get rid of the local finiteness requirement and to allow for weighted spaces by using Dirichlet forms. The corresponding results give a nonnegative selfadjoint (but not necessarily essentially selfadjoint) operator in quite some generality and provide a criterion for essential selfadjointness covering the earlier results of [20, 32, 33]. These topics are discussed in Section 1.

Having an unbounded nonnegative operator at ones disposal one may then wonder about its basic spectral features. These basic features include the position of the infimum of the spectrum and the existence of essential spectrum. Both issues can be approached via isoperimetric inequalities. In fact, lower bounds for the spectrum have been considered by Dodziuk [9] and Dodziuk/Kendall [11]. For planar graphs explicit estimates for the isoperimetric constant and hence for the spectrum can be found for instance in [18, 19, 23, 24, 31]. Triviality of the essential spectrum for general graphs has been considered by Fujiwara [14]. The corresponding results deal with bounded operators only. (They allow for unbounded vertex degree but then force boundedness of the operators by introducing weights on the corresponding ℓ^2 space.) Still, the methods can be used to provide lower bounds on the spectrum and prove emptiness of the essential spectrum for unbounded Laplacians as well. For locally finite graphs this has been done by Keller in [21]. Here, we present a generalization of the results of [21] to the general setting of regular Dirichlet forms. This generalization also extends the results of [14, 11] to our setting. This is discussed in Section 5.

Finally, we turn to a (possible) consequence of unboundedness in the study of the heat equation viz stochastic incompleteness. Stochastic incompleteness describes the phenomenon that mass vanishes in a diffusion process. While this may a priori not seem to be connected to unboundedness, it turns out to be connected. This has already been observed by Dodziuk/Matthai [12] and Dodziuk [10] in that they show stochastic completeness for certain bounded operators on graphs. A somewhat more structural connection is provided by our discussion below. For locally finite graphs stochastic completeness has recently been investigated by Weber in [32] and Wojciechowski [33]. In fact, Weber presents sufficient conditions and Wojciechowski gives a characterization of stochastic incompleteness. This characterization is inspired by corresponding work of Grigor'yan on manifolds [16] (see work of Sturm [28] for related results as well). As shown in [22] this characterization can be extended to regular Dirichlet forms. Details are discussed in Section 8. There, we also provide some further background extending [22]. Let us mention that this circle of ideas is strongly connected to questions concerning uniqueness of Markov process with given generator as discussed by Feller in [13] and Reuter in [27]. We take this opportunity to mention the very

recent survey [35] of Wojciechowski giving a thorough discussion of stochastic incompleteness for manifolds and graphs (with edge weight constant to one).

While our basic aim is to study unbounded Laplacians we complement our results by characterizing boundedness of the Laplacians in question in Section 3.

For a related study of basic spectral properties in terms of generalized solutions we refer the reader to [17].

The paper is organized as follows. In Section 1 we introduce our operators and discuss basic properties. Section 2 contains a useful minimum principle and some of its consequences. Boundedness of the Laplacians in question is characterized in Section 3. A useful tool, the so called co-area formulae are investigated in Section 4. They are used in Section 5 to provide an isoperimetric inequality which is then used to study bounds on the infimum of the (essential) spectrum. This allows us in particular to characterize emptiness of the essential spectrum. The connection to Markov processes is discussed in Section 7. A characterization of stochastic incompleteness is given in Section 8.

1. Graph Laplacians and Dirichlet forms

Throughout V will be a countably infinite set.

1.1. Weighted graphs

We will deal with weighted graphs with vertex set V . A symmetric weighted graph over V is a pair (b, c) consisting of a map $b : V \times V \rightarrow [0, \infty)$ with $b(x, x) = 0$ for all $x \in V$ and a map $c : V \rightarrow [0, \infty)$ satisfying the following two properties:

$$(b1) \quad b(x, y) = b(y, x) \text{ for all } x, y \in V.$$

$$(b2) \quad \sum_{y \in V} b(x, y) < \infty \text{ for all } x \in V.$$

Then b is called the edge weight and c is called killing term.

We consider (b, c) or rather the triple (V, b, c) as a weighted graph with vertex set V in the following way: An $x \in V$ with $c(x) \neq 0$ is thought to be connected to the point ∞ by an edge with weight $c(x)$. Moreover, $x, y \in V$ with $b(x, y) > 0$ are thought to be connected by an edge with weight $b(x, y)$. Vertices $x, y \in V$ with $b(x, y) > 0$ are called neighbors. More generally, $x, y \in V$ are called connected if there exist $x_0, x_1, \dots, x_n \in V$ with $b(x_i, x_{i+1}) > 0$, $i = 0, \dots, n$ and $x_0 = x$, $x_n = y$. This allows us to define connected components of V in the obvious way.

Two examples have attracted particular attention.

Example (Locally finite graphs): Let (V, b, c) be a weighted graph with $c \equiv 0$ and $b(x, y) \in \{0, 1\}$ for all $x, y \in V$. We can then think of the $(x, y) \in V \times V$ with $b(x, y) = 1$ as connected by an edge with weight 1. The condition (b2) then implies that any $x \in V$ is connected to only

finitely many $y \in V$. Such graphs are known as locally finite graphs. This is the class of examples studied in [21, 14, 33, 32].

Example (Locally finite weighted graphs): Let (V, b, c) be a weighted graph with $c \equiv 0$ and b satisfying

$$\#\{y : b(x, y) \neq 0\} < \infty$$

for all $x \in V$. Then, (V, b, c) is called a locally finite weighted graph. This is the class of examples studied in [10, 20].

1.2. Dirichlet forms on countable sets

Let m be a measure on V with full support (i.e., m is a map $m : V \rightarrow (0, \infty)$). Then, (V, m) is a measure space. A particular example is given by $m \equiv 1$. We will deal exclusively with real valued functions. Thus, $\ell^p(V, m)$, $0 < p < \infty$, is defined by

$$\{u : V \rightarrow \mathbb{R} : \sum_{x \in V} m(x) |u(x)|^p < \infty\}.$$

Obviously, $\ell^2(V, m)$ is a Hilbert space with inner product $\langle \cdot, \cdot \rangle = \langle \cdot, \cdot \rangle_m$ given by

$$\langle u, v \rangle := \sum_{x \in V} m(x) u(x) v(x) \quad \text{and norm} \quad \|u\| := \langle u, u \rangle^{\frac{1}{2}}.$$

Moreover we denote by $\ell^\infty(V)$ the space of bounded functions on V . Note that this space does not depend on the choice of m . It is equipped with the supremum norm $\|\cdot\|_\infty$ defined by

$$\|u\|_\infty := \sup_{x \in V} |u(x)|.$$

A symmetric nonnegative form on (V, m) is given by a dense subspace D of $\ell^2(V, m)$ called the domain of the form and a map

$$Q : D \times D \rightarrow \mathbb{R}$$

with $Q(u, v) = Q(v, u)$ and $Q(u, u) \geq 0$ for all $u, v \in D$.

Such a map is already determined by its values on the diagonal $\{(u, u) : u \in D\} \subseteq D \times D$. This motivates to consider the restriction of Q to the diagonal as an object on its own right. Thus, for $u \in \ell^2(V, m)$ we then define $Q(u)$ by

$$Q(u) := \begin{cases} Q(u, u) & : u \in D, \\ \infty & : u \notin D. \end{cases}$$

If $\ell^2(V, m) \rightarrow [0, \infty]$, $u \mapsto Q(u)$ is lower semicontinuous Q is called closed. If Q has a closed extension it is called closable and the smallest closed extension is called the closure of Q .

A map $C : \mathbb{R} \rightarrow \mathbb{R}$ with $C(0) = 0$ and $|C(x) - C(y)| \leq |x - y|$ is called a normal contraction. If Q is both closed and satisfies $Q(Cu) \leq Q(u)$ for all normal contractions C and all $u \in \ell^2(V, m)$ it is called a Dirichlet form on (V, m) (see [3, 7, 15, 25] for background on Dirichlet forms).

Let $C_c(V)$ be the space of finitely supported functions on V . A Dirichlet form on (V, m) is called regular if its domain contains $C_c(V)$ and the form is the closure of its restriction to the subspace $C_c(V)$. (The standard definition of regularity for Dirichlet forms would require that $D(Q) \cap C_c(V)$ is dense in both $C_c(V)$ and $D(Q)$. As discussed in [22] this is equivalent to our definition.)

1.3. From weighted graphs to Dirichlet forms

There is a one-to-one correspondence between weighed graphs and regular Dirichlet forms. This is discussed next.

To the weighted graph (V, b, c) we can then associate the form $Q^{\max} = Q_{b,c,m}^{\max} : \ell^2(V, m) \rightarrow [0, \infty]$ with diagonal given by

$$Q^{\max}(u) = \frac{1}{2} \sum_{x,y \in V} b(x, y)(u(x) - u(y))^2 + \sum_{x \in V} c(x)u(x)^2.$$

Here, the value ∞ is allowed. Let $Q^{\text{comp}} = Q_{b,c}^{\text{comp}}$ be the restriction of Q^{\max} to $C_c(V)$. It is not hard to see that Q^{\max} is closed. Hence Q^{comp} is closable on $\ell^2(V, m)$ and the closure will be denoted by $Q = Q_{b,c,m}$ and its domain by $D(Q)$.

As discussed in [22] (see [15] as well) the following holds.

Theorem 1. *The regular Dirichlet forms on (V, m) are exactly given by the forms $Q_{b,c,m}$ with weighted graphs (b, c) over V .*

Remark. One may wonder whether the regularity assumption is necessary in the above theorem. It turns out that not every Dirichlet form $Q_{b,c,m}^{\max}$ is regular. A counterexample is provided in [22].

For a given a weighted graph (V, b, c) the different choices of measure m will produce different Dirichlet forms. Two particular choices have attracted attention. One is the choice of $m \equiv 1$. Obviously, this choice does not depend on b and c . Another possibility is to use $n = m = m_{b,c}$ given by

$$n(x) := \sum_{y \in V} b(x, y) + c(x).$$

The advantage of this measure is that it produces a bounded form (see below for details).

1.4. Graph Laplacians

Let m be a measure on V of full support, (b, c) a weighted graph over V and $Q_{b,c,m}$ the associated regular Dirichlet form. Then, there exists a unique selfadjoint operator $L = L_{b,c,m}$ on $\ell^2(V, m)$ such that

$$D(Q) := \{u \in \ell^2(V, m) : Q(u) < \infty\} = \text{Domain of definition of } L^{1/2}$$

and

$$Q(u) = \langle L^{1/2}u, L^{1/2}u \rangle$$

for $u \in D(Q)$ (see e.g. Theorem 1.2.1 in [7]). As Q is nonnegative so is L .

Definition 2. Let V be a countable set and m a measure on V with full support. A graph Laplacian on V is an operator L associated to a form $Q_{b,c,m}$.

Our next aim is to describe the operator L more explicitly: Define the formal Laplacian $\tilde{L} = \tilde{L}_{b,c,m}$ on the vector space

$$\tilde{F} := \{u : V \longrightarrow \mathbb{R} : \sum_{y \in V} |b(x,y)u(y)| < \infty \text{ for all } x \in V\} \quad (1.1)$$

by

$$\tilde{L}u(x) := \frac{1}{m(x)} \sum_{y \in V} b(x,y)(u(x) - u(y)) + \frac{c(x)}{m(x)}u(x),$$

where, for each $x \in V$, the sum exists by assumption on u . The operator \tilde{L} describes the action of L in the following sense.

Proposition 3. Let (V, b, c) be a weighted graph and m a measure on V of full support. Then, the operator L is a restriction of \tilde{L} i.e.,

$$D(L) \subseteq \{u \in \ell^2(V, m) : \tilde{L}u \in \ell^2(V, m)\} \text{ and } Lu = \tilde{L}u$$

for all $u \in D(L)$.

In order to obtain further information we need a stronger condition. We define condition (A) as follows:

(A) For any sequence (x_n) of vertices in V such that $b(x_n, x_{n+1}) > 0$ for all $n \in \mathbb{N}$, the equality $\sum_{n \in \mathbb{N}} m(x_n) = \infty$ holds.

Let us emphasize that in general (A) is a condition on (V, m) and b together. However, if

$$\inf_{x \in V} m_x > 0$$

holds, then obviously (A) is satisfied for all graphs (b, c) over V . This applies in particular to the case that $m \equiv 1$.

Given (A) we can say more about the generators [22].

Theorem 4. Let (V, b, c) be a weighted graph and m a measure on V of full support such that (A) holds. Then, the operator L is the restriction of \tilde{L} to

$$D(L) = \{u \in \ell^2(V, m) : \tilde{L}u \in \ell^2(V, m)\}.$$

Remark. The theory of Jacobi matrices already provides examples showing that without (A) the statement becomes false [22].

The condition (A) does not imply that $\tilde{L}f$ belongs to $\ell^2(V, m)$ for all $f \in C_c(V)$. However, if this is the case, then (A) does imply essential selfadjointness. In this case, Q is the “maximal” form associated to the graph (b, c) . More precisely, the following holds [22].

Theorem 5. *Let V be a set, m a measure on V with full support, (b, c) a graph over V and Q the associated regular Dirichlet form. Assume $\tilde{L}C_c(V) \subseteq \ell^2(V, m)$. Then, $D(L)$ contains $C_c(V)$. If furthermore (A) holds, then the restriction of L to $C_c(V)$ is essentially selfadjoint and the domain of L is given by*

$$D(L) = \{u \in \ell^2(V, m) : \tilde{L}u \in \ell^2(V, m)\}$$

and the associated form Q satisfies $Q = Q^{\max}$ i.e.,

$$Q(u) = \frac{1}{2} \sum_{x, y \in V} b(x, y)(u(x) - u(y))^2 + \sum_{x \in V} c(x)u(x)^2$$

for all $u \in \ell^2(V, m)$.

Remark. Essential selfadjointness may fail if (A) does not hold as can be seen by examples [22].

If $\inf_{x \in V} m_x > 0$ then both (A) and $\tilde{L}C_c(V) \subseteq \ell^2(V, m)$ hold for any graph (b, c) over V . We therefore obtain the following corollary.

Corollary 6. *Let V be a set and m a measure on V with $\inf_{x \in V} m_x > 0$. Then, $D(L)$ contains $C_c(V)$, the restriction of L to $C_c(V)$ is essentially selfadjoint and the domain of L is given by*

$$D(L) = \{u \in \ell^2(V, m) : \tilde{L}u \in \ell^2(V, m)\}$$

and the associated form Q satisfies $Q = Q^{\max}$.

Remark. The corollary includes the case that $m \equiv 1$ and we recover the corresponding results of [11, 33, 32] on essential selfadjointness. (In fact, the cited works also have additional restrictions on b but this is not relevant here.)

2. Minimum principle and consequences

An important tool in the proofs of the results of the previous section is a minimum principle. This minimum principle shows in particular the relevance of (A) in our considerations. This is discussed in this section.

The following result is a variant and in fact a slight generalization of the minimum principle from [22].

Theorem 7. (Minimum principle) *Let (V, b, c) be a weighted graph and m a measure on V of full support. Let $U \subseteq V$ be connected. Assume that the function u on V satisfies*

- $(\tilde{L} + \alpha)u \geq 0$ on U for some $\alpha > 0$,
- $u \geq 0$ on $V \setminus U$.

Then, the value of u is nonnegative in any local minimum of u .

Proof. Let u attain a local minimum on U in x_m . Assume $u(x_m) < 0$. Then, $u(x_m) \leq u(y)$ for all $y \in U$ with $b(x_m, y) > 0$. As $u(y) \geq 0$ for $y \in V \setminus U$, we obtain $u(x_m) - u(y) \leq 0$ for all $y \in V$ with $b(x_m, y) \geq 0$. By the super-solution assumption we find

$$0 \leq \sum b(x_m, y)(u(x_m) - u(y)) + c(x_m)u(x_m) + m(x_m)\alpha u(x_m) \leq 0.$$

As b and c are nonnegative, m is positive and $\alpha > 0$, we obtain the contradiction $0 = u(x_m)$. \square

The relevance of (A) comes from the following consequence of the minimum principle first discussed in [22].

Proposition 8. (*Uniqueness of solutions on ℓ^p*) Assume (A). Let $\alpha > 0$, $p \in [1, \infty)$ and $u \in \ell^p(V, m)$ with $(\tilde{L} + \alpha)u \geq 0$ be given. Then, $u \geq 0$. In particular, any $u \in \ell^p(V, m)$ with $(\tilde{L} + \alpha)u = 0$ satisfies $u \equiv 0$.

Proof. We first show the first statement: Assume the contrary. Then, there exists an $x_0 \in V$ with $u(x_0) < 0$. By the previous minimum principle, x_0 is not a local minimum of u . Thus, there exists an x_1 connected to x_0 with $u(x_1) < u(x_0) < 0$. Continuing in this way we obtain a sequence (x_n) of connected points with $u(x_n) < u(x_0) < 0$. Combining this with (A) we obtain a contradiction to $u \in \ell^p(V, m)$.

As for the 'In particular' part we note that both u and $-u$ satisfy the assumptions of the first statement. Thus, $u \equiv 0$. \square

Remark. The situation for $p = \infty$ is substantially more complicated as can be seen by our discussion of stochastic completeness in Section 8. and in particular part (ii) of Theorem 25.

Using the previous minimum principle it is not hard to prove the following result. The result is in fact true for general Dirichlet forms as can be inferred from [29, 30]. For $U \subseteq V$ we denote by Q_U the closure of the Q restricted to $C_c(U)$ and by L_U the associated operator.

Proposition 9. (*Domain monotonicity*) Let (V, b, c) be a symmetric graph. Let $K_1 \subseteq V$ be finite and $K_2 \subseteq V$ with $K_1 \subseteq K_2$ be given. Then, for any $x \in K_1$

$$(L_{K_1} + \alpha)^{-1}f(x) \leq (L_{K_2} + \alpha)^{-1}f(x)$$

for all $f \in \ell^2(V, m)$ with $f \geq 0$ and $\text{supp } f \subseteq K_1$. A similar statement holds for the semigroups.

Proposition 10. (*Convergence of resolvents/semigroups*) Let (V, b, c) be a symmetric graph, m a measure on V with full support and Q the associated regular Dirichlet form. Let (K_n) be an increasing sequence of finite subsets of V with $V = \bigcup K_n$. Then, $(L_{K_n} + \alpha)^{-1}f \rightarrow (L + \alpha)^{-1}f$, $n \rightarrow \infty$ for any $f \in \ell^2(K_1, m_{K_1})$. (Here, $(L_{K_n} + \alpha)^{-1}f$ is extended by zero to all of V .) The corresponding statement also holds for the semigroups.

3. Boundedness of the Laplacian

Our main topic in this paper are the consequences of unboundedness of the Laplacian. In order to understand this unboundedness it is desirable to characterize boundedness of this operator. This is discussed in this section. We start with a little trick on how to get rid of the c in certain situations.

Let \dot{V} be the union of V and a point at infinity ∞ . We extend a function on V to \dot{V} by zero and let $b(\infty, x) = b(x, \infty) = c(x)$ for all $x \in V$. We then have

$$\sum_{y \in \dot{V}} b(x, y) = \sum_{y \in V} b(x, y) + c(x)$$

for all $x \in V$ and

$$Q(u) = \frac{1}{2} \sum_{x, y \in \dot{V}} b(x, y) (u(x) - u(y))^2$$

for all functions u in $D(Q)$.

We define an averaged vertex degree $d = d_{b,c,m}$ by

$$d(x) := \frac{1}{m(x)} \left(\sum_{y \in V} b(x, y) + c(x) \right).$$

Note that $d(x) = n(x)/m(x)$, where n was defined at the end of Section 1.3.

Theorem 11. *Let (V, b, c) be a weighted graph and $m : V \rightarrow (0, \infty)$ a measure on V and \tilde{L} the associated formal operator. Then, the following assertions are equivalent:*

- (i) *There exists a $C \geq 0$ with $d(x) \leq C$ for all $x \in V$.*
- (ii) *The form Q is bounded on $\ell^2(V, m)$.*
- (iii) *The restriction of \tilde{L} to $\ell^2(V, m)$ is bounded.*
- (iv) *The restriction of \tilde{L} to $\ell^\infty(V)$ is bounded.*

In this case the restriction of \tilde{L} to $\ell^p(V, m)$ is a bounded operator for all $p \in [1, \infty]$ and a bound is given by $2C$ with C from (i).

Proof. By the considerations at the beginning of the section we can assume $c \equiv 0$. For $x \in V$ we let δ_x be the function on V which is zero everywhere except in x , where it takes the value 1.

The equivalence between (ii) and (iii) is obvious as the operator associated to Q is a densely defined restriction of \tilde{L} .

Obviously (i) implies (iv) (with the bound $2C$). The implication (iv) \implies (i) follows by considering the vectors δ_x , $x \in V$.

(i) \implies (ii): As $(a - b)^2 \leq 2a^2 + 2b^2$ we obtain

$$\begin{aligned} Q(u, u) &= \frac{1}{2} \sum_{x, y \in V} b(x, y) (u(x) - u(y))^2 \\ &\leq \sum_{x, y \in V} b(x, y) u(x)^2 + \sum_{x, y \in V} b(x, y) u(y)^2 \\ &\leq C \sum_{x \in V} m(x) u(x)^2 + C \sum_{y \in V} m(y) u(y)^2 \\ &= 2C \|u\|^2. \end{aligned}$$

Here, we used the symmetry of b and the bound (i) in the previous to the last step.

(ii) \implies (i): This follows easily as $Q(\delta_x, \delta_x) = \sum_{y \in V} b(x, y)$ for all $x \in V$.

It remains to show the last statement: By interpolation between ℓ^2 and ℓ^∞ , we obtain boundedness of the operators on $\ell^p(V, m)$ for $p \in [2, \infty]$. Using symmetry we obtain the boundedness for $p \in [1, 2)$. Alternatively, we can directly establish that (i) implies the boundedness of the restriction of \tilde{L} on $\ell^1(V, m)$. As a bound for the operator norm on ℓ^∞ and on ℓ^2 is $2C$, we obtain this same bound on all ℓ^p . \square

Remark. The theorem can be seen as a generalization of the well known fact that a stochastic matrix generates an operator which is bounded on all ℓ^p .

Note that the theorem gives in particular that boundedness of the operator \tilde{L} on $\ell^2(V, m)$ is equivalent to boundedness on $\ell^\infty(V)$. This is far from being true for all symmetric operators on $\ell^2(V, m)$. For example, let A be the operator on $\ell^2(\mathbb{N}, 1)$ with matrix given by $a_{x,y} = 1/x$ if $y = 1$ and $a_{x,y} = 1/y$ if $x = 1$ and $a_{x,y} = 0$ otherwise. Then, A is bounded on ℓ^2 but not on ℓ^∞ . Conversely, using e.g. the measure $m(x) = x^{-4}$ on \mathbb{N} and suitable operators with only one or two ones in each row it is not hard to construct a bounded operator on $\ell^\infty(\mathbb{N})$ which is symmetric but not bounded on $\ell^2(V, m)$. Of course, if m is such that $\ell^2(V, m)$ is contained in $\ell^\infty(V)$ then any bounded operator on ℓ^∞ which is symmetric (and hence closed) on ℓ^2 must be bounded as well.

4. Co-area formulae

In this section we discuss some co-area type formulae. These formulae are well known for locally finite graphs e.g. [5] and carry over easily to our setting. They are useful in many contexts as e.g. the estimation of eigenvalues via isoperimetric inequalities. We use them in this spirit as well.

We start with some notation. Let (V, b, c) be a weighted graph with $c \equiv 0$, (which can assume without loss of generality by the trick mentioned in the beginning of Section 3.). For a subset $\Omega \subseteq V$ we define

$$\partial\Omega := \{(x, y) : \{x, y\} \cap \Omega \neq \emptyset \text{ and } \{x, y\} \cap V \setminus \Omega \neq \emptyset\}$$

and

$$|\partial\Omega| := \frac{1}{2} \sum_{(x,y) \in \partial\Omega} b(x,y).$$

We can now come to the so called co-area formula.

Theorem 12. (Co-area formula) *Let (V, b, c) be a weighted graph with $c \equiv 0$. Let $f : V \rightarrow \mathbb{R}$ be given and define for $t \in \mathbb{R}$ the set $\Omega_t := \{x \in V : f(x) > t\}$. Then,*

$$\frac{1}{2} \sum_{x,y \in V} b(x,y) |f(x) - f(y)| = \int_0^\infty |\partial\Omega_t| dt.$$

Proof. For $x, y \in V$ with $x \neq y$ we define the interval $I_{x,y}$ by

$$I_{x,y} := [\min\{f(x), f(y)\}, \max\{f(x), f(y)\}]$$

and let $|I_{x,y}|$ be the length of the interval. Let $1_{x,y}$ be the characteristic function of $I_{x,y}$. Then, $(x, y) \in \partial\Omega_t$ if and only if $t \in I_{x,y}$. Thus,

$$|\partial\Omega_t| = \frac{1}{2} \sum_{x,y \in V} b(x,y) 1_{x,y}(t).$$

Thus, we can calculate

$$\begin{aligned} \int_0^\infty |\partial\Omega_t| dt &= \frac{1}{2} \int_0^\infty \sum_{x,y \in V} b(x,y) 1_{x,y}(t) dt \\ &= \frac{1}{2} \sum_{x,y \in V} b(x,y) \int_0^\infty 1_{x,y}(t) dt \\ &= \frac{1}{2} \sum_{x,y \in V} b(x,y) |f(y) - f(x)|. \end{aligned}$$

This finishes the proof. □

Remark. *Note that the proof is essentially a Fubini type argument.*

The preceding formula can be seen as a first order co-area formula as it deals with differences of functions. There is also a zeroth order co-area type formula dealing with functions themselves. This is discussed next.

Theorem 13. *Let V be a countable set and $m : V \rightarrow (0, \infty)$ a measure on V . Let $f : V \rightarrow [0, \infty)$ be given and define for $t \in \mathbb{R}$ the set $\Omega_t := \{x \in V : f(x) > t\}$. Then,*

$$\sum_{x \in V} m(x) f(x) = \int_0^\infty m(\Omega_t) dt.$$

Proof. We have $x \in \Omega_t$ if and only if $1_{(t,\infty)}(f(x)) = 1$. Thus, we can calculate

$$\begin{aligned} \int_0^\infty m(\Omega_t) dt &= \int_0^\infty \sum_{x \in \Omega_t} m(x) dt \\ &= \int_0^\infty \sum_{x \in V} m(x) 1_{(t,\infty)}(f(x)) dt \\ &= \sum_{x \in V} m(x) \int_0^\infty 1_{(t,\infty)}(f(x)) dt \\ &= \sum_{x \in V} m(x) f(x). \end{aligned}$$

This finishes the proof. □

5. Isoperimetric inequalities and lower bounds on the (essential) spectrum

In this section we will provide lower bound on the infimum of the (essential) spectrum using an isoperimetric inequality. This will allow us in particular to provide criteria for emptiness of the essential spectrum. Our considerations extend the corresponding parts of [9, 11, 14, 21] (as discussed in more detail below).

We start with some notation used throughout this section. Let a weighted graph (V, b, c) with a measure $m : V \rightarrow (0, \infty)$ and the associated Dirichlet form Q be given. In this setting we define the constant $\alpha(U) = \alpha_{b,c,m}(U)$ for a subset $U \subseteq V$ by

$$\alpha(U) = \inf_{W \subseteq U, |W| < \infty} \frac{|\partial W|}{m(W)},$$

where as introduced in the previous section

$$|\partial W| = \sum_{x \in W, y \notin W} b(x, y) + \sum_{x \in W} c(x).$$

Note that for a finite set W and the characteristic function 1_W of W one has

$$\frac{|\partial W|}{m(W)} = \frac{Q(1_W)}{\|1_W\|^2}. \quad (5.1)$$

Recall the definition of the normalizing measure n on V

$$n(x) = \sum_{y \in V} b(x, y) + c(x).$$

Thus, we have two measures and thus two Hilbert spaces at our disposal. To avoid confusion, we will write $\|\cdot\|_m$ and $\|\cdot\|_n$ for the corresponding norms whenever necessary.

Note that $d(x) = n(x)/m(x)$. Define maximal and minimal averaged vertex degree by

$$d_U = d_{b,c,m}(U) = \inf_{x \in U} d(x)$$

and

$$D_U = D_{b,c,m}(U) = \sup_{x \in U} d(x),$$

where d is the averaged vertex degree, which was defined in Section 3. Recall $d(x) = n(x)/m(x)$ for $x \in V$.

We will also need the restrictions of operators on V to subsets of V . As in the end of Section 2 denote the closure of the restriction of a closed semibounded form Q with domain containing $C_c(V)$ to $C_c(U)$ by Q_U and its associated operator by L_U (for $U \subseteq V$ arbitrary).

For later use we also note that for the Dirichlet form Q associated to a graph (V, b, c) with measure m on V we have

$$\inf \sigma(L_U) = \inf_{u \in C_c(U)} \frac{Q(u)}{\|u\|^2} \leq \alpha(U) \leq \inf_{x \in U} d(x) = d_U$$

for any $U \subseteq V$. Here, the first equality is just the variational principle for forms, the second step follows from the definition of α and the last estimate follows by choosing $W = \{x\}$ for $x \in U$. In particular, α gives upper bound on the infimum of the spectrum. It is a remarkable (and well known) fact that $\alpha > 0$ implies also a lower bounds on the infimum of spectra. This is the core of the present section.

5.1. An isoperimetric inequality

In this subsection we provide an isoperimetric inequality in our setting. This inequality (and its proof) are generalizations of the corresponding considerations of [11, 14, 21] to our setting.

Proposition 14. *Let (V, b, c) be a weighted graph, $m : V \rightarrow (0, \infty)$ a measure on V and Q the associated regular Dirichlet form. Let $U \subseteq V$ and $\phi \in C_c(U)$. Then*

$$Q(\phi)^2 - 2\|\phi\|_n^2 Q(\phi) + \alpha_{b,c,m}(U)^2 \|\phi\|_m^4 \leq 0.$$

Proof. By the trick introduced at the beginning of Section we can assume without loss of generality that $c \equiv 0$. Define now \mathcal{A} by

$$\mathcal{A} = \frac{1}{2} \sum_{x,y \in \dot{V}} b(x,y) |\varphi(x)^2 - \varphi(y)^2| = \sum_{x,y \in \dot{V}} b(x,y) |\varphi(x) - \varphi(y)| |\varphi(x) + \varphi(y)|.$$

Following ideas of [11] for locally finite graphs (see [14, 21] as well) we now proceed as follows: By Cauchy-Schwarz inequality and a direct computation we have

$$\mathcal{A}^2 \leq Q(\phi) \left(\frac{1}{2} \sum_{x,y \in \dot{V}} b(x,y) |\varphi(x) + \varphi(y)|^2 \right) = Q(\phi) (2\|\phi\|_n^2 - Q(\phi)).$$

On the other hand we can use the first co-area formula (with $f = \varphi^2$), the definition of α and the second co-area formula to estimate

$$\mathcal{A} = \int_0^\infty |\partial\Omega_t| dt \geq \alpha \int_0^\infty m(\Omega_t) dt = \alpha \sum_{x \in V} m(x) \varphi^2(x) = \alpha \|\varphi\|_m^2.$$

Combining the two estimates on \mathcal{A} we obtain

$$Q(\varphi) (2 \|\varphi\|_n^2 - Q(\varphi)) \geq \|\varphi\|_m^4.$$

This yields the desired result. □

5.2. Lower bounds for the infimum of the spectrum

In this section we use the isoperimetric inequality of the previous section to derive bounds on the form Q . This is in the spirit of [11, 14, 21]. As usual we write

$$a \leq Q \leq b$$

(for $a, b \in \mathbb{R}$) whenever

$$a \|u\|^2 \leq Q(u) \leq b \|u\|^2$$

for all $u \in D(Q)$.

Proposition 15. *Let (V, b, c) be a weighted graph, $m : V \rightarrow (0, \infty)$ a measure on V and Q the associated regular Dirichlet form. Let $U \subseteq V$ be given and Q_U the restriction of Q to U . Then,*

$$d_U \left(1 - \sqrt{1 - \alpha_{b,c,n}(U)^2} \right) \leq Q_U \leq D_U \left(1 + \sqrt{1 - \alpha_{b,c,n}(U)^2} \right).$$

If $D_U < \infty$ then furthermore

$$D_U - \sqrt{D_U^2 - \alpha_{b,c,m}(U)^2} \leq Q_U \leq D_U + \sqrt{D_U^2 - \alpha_{b,c,m}(U)^2}.$$

Proof. We start by proving the first statement. Consider an arbitrary $\varphi \in C_c(U)$ with $\|\varphi\|_n = 1$. Then, Proposition 14 (applied with $m = n$) gives

$$Q(\varphi)^2 - 2Q(\varphi) + \alpha_{b,c,n}(U)^2 \leq 0$$

and hence

$$1 - \sqrt{1 - \alpha_{b,c,n}(U)^2} \leq Q(\varphi) \leq 1 + \sqrt{1 - \alpha_{b,c,n}(U)^2}.$$

As this holds for all $\varphi \in C_c(U)$ with $\|\varphi\|_n = 1$ and

$$d_U \|\varphi\|_m \leq \|\varphi\|_n \leq D_U \|\varphi\|_m$$

by definition of d_U and D_U , we obtain the first statement.

We now turn to the last statement. By definition of D_U we have $\|\varphi\|_n \leq D_U \|\varphi\|_m$. Thus, Proposition 14 gives

$$Q(\varphi)^2 - 2D_U \|\varphi\|_m^2 Q(\varphi) + \alpha_{b,c,m}(U)^2 \|\varphi\|_m^4 \leq 0.$$

Considering now $\varphi \in C_c(U)$ with $\|\varphi\|_m = 1$ we find that

$$D_U - \sqrt{D_U^2 - \alpha_{b,c,m}(U)} \leq Q(\varphi) \leq D_U + \sqrt{D_U^2 - \alpha_{b,c,m}(U)}$$

for all such φ . This finishes the proof. \square

As a first consequence of the previous proposition we obtain the following corollary first proven for $m = n$, and locally finite graphs in [14].

Corollary 16. *For a weighted graph (V, b, c) and $m = n$ we obtain*

$$1 - \sqrt{1 - \alpha_{b,c,n}^2} \leq Q \leq 1 + \sqrt{1 - \alpha_{b,c,n}^2}.$$

A second consequence of the above proposition is that the bottom of the spectrum being zero can be characterized by the constant α in the case of bounded operators. This is our version of the well known result that a graph with finite vertex degree is amenable if and only if zero belongs to the spectrum of the corresponding Laplacian.

Corollary 17. *Let (V, b, c) be a weighted graph and $D_U < \infty$ for $U \subseteq V$. Then $\inf \sigma(L_U) = 0$ if and only if $\alpha_{b,c,m}(U) = 0$.*

Proof. The direction ' \implies ' follows from Proposition 15 and the other direction ' \impliedby ' follows directly from equation 5.1. \square

Remark. The direction ' \impliedby ' in the previous corollary does not depend on the assumption $D_U < \infty$ for $U \subseteq V$ and is true in general.

5.3. Absence of essential spectrum

In this subsection we use the results of the previous subsection to study absence of essential spectrum. The key idea is that the essential spectrum of an operator is a suitable limit of the spectra of restrictions 'going to infinity'. This reduces the problem of proving absence of essential spectrum to proving lower bounds on the spectrum 'at infinity'. For locally finite graphs this has been done in [14, 21].

Let (V, b, c) be a weighted graph. Let \mathcal{K} be the set of finite sets in V . This set is directed with respect to inclusion and hence a net. Limits along this net will be denoted by $\lim_{K \in \mathcal{K}}$ and we will say that K tends to V . We then define

$$\alpha_{b,c,m}(\partial V) = \lim_{K \in \mathcal{K}} \alpha_{b,c,m}(V \setminus K).$$

Likewise let

$$\begin{aligned} d_{\partial V} &= d_{b,c,m}(\partial V) = \lim_{K \in \mathcal{K}} d_{b,c,m}(V \setminus K), \\ D_{\partial V} &= D_{b,c,m}(\partial V) = \lim_{K \in \mathcal{K}} D_{b,c,m}(V \setminus K). \end{aligned}$$

The following proposition is certainly well known and has in fact already been used in the past (see e.g. [21]). We include a proof as we could not find one in the literature. Note also that our result is more general than the result mentioned e.g. in [21] as we deal with forms.

Proposition 18. *Let Q be a closed form on $\ell^2(V, m)$, whose domain of definition contains $C_c(V)$. If Q is bounded below then*

$$\inf \sigma_{\text{ess}}(B) = \lim_{K \in \mathcal{K}} \inf \sigma(B_{V \setminus K}).$$

and if Q is bounded above then

$$\sup \sigma_{\text{ess}}(B) = \lim_{K \in \mathcal{K}} \sup \sigma(B_{V \setminus K})$$

holds, where B is the operator associated to Q and $B_{V \setminus K}$ the operator associated to $Q_{V \setminus K}$ for finite $K \subseteq V$.

Proof. It suffices to show the statement for Q which are bounded below (as the other statement then follows after replacing Q by $-Q$).

Without loss of generality we can assume $Q \geq 0$. Let $\lambda_0 := \inf \sigma_{\text{ess}}(B)$.

As the essential spectrum does not change by finite rank perturbations we have $\sigma_{\text{ess}}(B) = \sigma_{\text{ess}}(B_{V \setminus K}) \subseteq \sigma(B_{V \setminus K})$ and hence

$$\lambda_0 \in \sigma(B_{V \setminus K})$$

for any finite $K \subseteq V$. This gives

$$\inf \sigma_{\text{ess}}(B) \geq \lim_{K \in \mathcal{K}} \inf \sigma(B_{V \setminus K}).$$

To show the opposite inequality it suffices to prove that for arbitrary $\lambda < \lambda_0$ we have $\inf \sigma(B_{V \setminus K}) > \lambda$ for all sufficiently large finite K . Fix λ_1 with

$$\lambda < \lambda_1 < \lambda_0$$

and choose $\delta > 0$ such that $\lambda + \delta < \lambda_1$. Moreover let

$$\varepsilon = \frac{\lambda_1 - (\lambda + \delta)}{\lambda_1 + 1}.$$

The spectral projection $E_{(-\infty, \lambda_1]}$ of B to the interval $(-\infty, \lambda_1]$ is a finite rank operator since $B \geq 0$. This easily implies

$$\lim_{K \in \mathcal{K}} \|E_{(-\infty, \lambda_1]} P_K\| = 0,$$

where P_K is the projection onto $\ell^2(V \setminus K, m)$. Thus, there is K_ε finite with

$$\|E_{(-\infty, \lambda_1]} P_K\|^2 \leq \varepsilon$$

for all $K \supseteq K_\varepsilon$ finite. In particular, we have

$$\|E_{(-\infty, \lambda_1]} \psi\|^2 \leq \varepsilon \tag{5.2}$$

for all $\psi \in \ell^2(V \setminus K_\varepsilon, m)$ with $\|\psi\| = 1$ (as for such ψ we have $\psi = P_{K_\varepsilon} \psi$).

Consider now a finite K with $K \supseteq K_\varepsilon$ and let $\psi \in \ell^2(V \setminus K, m)$ be given with $\|\psi\| = 1$ such that

$$Q(\psi) = Q_{V \setminus K}(\psi) \leq (\inf \sigma(B_{V \setminus K}) + \varepsilon).$$

Let $\rho_\psi(\cdot)$ be the spectral measure associated to B and ψ . Then

$$\begin{aligned} Q(\psi) &= \int_0^\infty t d\rho_\psi(t) \\ &\geq \int_{\lambda_1}^\infty t d\rho_\psi(t) \\ &\geq \lambda_1 \int_{\lambda_1}^\infty d\rho_\psi(t) \\ &= \lambda_1 (\langle \psi, \psi \rangle - \langle E_{(-\infty, \lambda_1]} \psi, E_{(-\infty, \lambda_1]} \psi \rangle) \\ &\geq \lambda_1 (1 - \varepsilon). \end{aligned}$$

In the first step we used that B is positive and in the last step we used (5.2). By our choice of ψ and ε we get

$$\inf \sigma(B_{V \setminus K}) \geq Q(\psi) - \varepsilon \geq \lambda_1 (1 - \varepsilon) - \varepsilon = \lambda + \delta > \lambda.$$

This finishes the proof. □

Combining this proposition with Proposition 15 one gets estimates for the essential spectrum of the operator L .

The following provides a generalization of a main result of Fujiwara's theorem [14] to our setting. Fujiwara's result deals with locally finite graphs and $m = n$.

Theorem 19. *Let (V, b, c) be a weighted graph, $m : V \rightarrow (0, \infty)$ a measure on V and Q the associated regular Dirichlet form. Assume $D_{\partial V} = D_{b,c,m}(\partial V) < \infty$. Then, $\sigma_{\text{ess}}(L) = \{D_{\partial V}\}$ if and only if $\alpha_{b,c,m}(\partial V) = D_{\partial V}$.*

Proof. One direction ' \Leftarrow ' follows directly from Proposition 15 and Proposition 18. The other direction ' \Rightarrow ' follows from

$$\inf \sigma(L_U) \leq \alpha_{b,c,m}(U) \leq D_{b,c,m}(U)$$

for $U \subseteq V$ and Proposition 18 by taking $U = V \setminus K$ for K finite and considering the limit for K tending to V . □

Remark. The assumption $D_{b,c,m}(\partial V) < \infty$ implies boundedness of the operator (see Section 3.). Thus, $\sigma_{\text{ess}}(L)$ must be non-empty in this case. Proposition 18 shows that $\inf \sigma(L_{V \setminus K})$ and $\sup \sigma(L_{V \setminus K})$ converge necessarily to points in the essential spectrum of L (for K tending to V). The only way how the essential spectrum can consist of only one point is then that both limits agree. As $\inf \sigma(L_{V \setminus K}) \leq \alpha(V \setminus K)$ and $\sup \sigma(L_{V \setminus K}) \geq D_{b,c,m}$ this is only possible for $\alpha_{b,c,m}(\partial V) = D_{\partial V}$. In this way the theorem characterizes the only way how essential spectrum can consist of only one point.

The next theorem is a generalization to our setting of Theorem 2 in [21], which deals with locally finite graphs and $m \equiv 1$.

Theorem 20. *Let (V, b, c) be a weighted graph, $m : V \rightarrow (0, \infty)$ a measure on V and Q the associated regular Dirichlet form. Assume $\alpha_{b,c,n} > 0$. Then $\sigma_{\text{ess}}(L) = \emptyset$ if and only if $d_{\partial V} = \infty$.*

Proof. One direction ' \Leftarrow ' follows directly from Proposition 15 and 18. The other direction ' \Rightarrow ' follows from the fact that for all $U \subseteq V$ we have $\inf \sigma(L_U) \leq d_{b,c,m}(U)$ and Proposition 18. \square

6. An application

In this section we consider a locally finite graph i.e., $(V, b, 0)$ with b taking values in $\{0, 1\}$ with the measure $m \equiv 1$. Let Q_0 be the associated form and Δ the associated operator. Let $c : V \rightarrow [0, \infty)$ be given and define L to be the operator associated to $Q_{b,c,m}$. Thus,

$$L = \Delta + c$$

(at least on the formal level). This decomposition of L leads to a similar decomposition of the parameters α . In this way, both the geometry (encoded by b) and the potential (encoded by c) can lead to absence of essential spectrum according to the preceding considerations. This is discussed in further details next.

The Cheeger constant β_U of a subset $U \subseteq V$ is the smallest number such that for all finite $W \subseteq U$

$$|\partial W| \geq \beta_U \text{vol}(W),$$

where $|\partial W| = \langle \Delta 1_W, 1_W \rangle = \sum_{x \in W, y \notin W} b(x, y)$ is defined as above and $\text{vol}(W) = \|1_W\|_n^2 = \sum_{x \in W} n(x)$. If $\beta_V > 0$ one says that the graph is hyperbolic. Furthermore, let γ_U be given as the smallest number such that for all finite $W \subseteq U$

$$c(W) \geq \gamma_U \text{vol}(W),$$

where $c(W) = \langle c 1_W, 1_W \rangle = \sum_{x \in W} c(x)$.

For example $\gamma_V > 0$, if there is $C > 0$ such that $c(x) \geq C d(x)$, where $d(x)$ is the vertex degree. Finally let

$$\beta_{\partial V} = \lim_{K \in \mathcal{K}} \beta_{V \setminus K} \quad \text{and} \quad \gamma_{\partial V} = \lim_{V \in \mathcal{K}} \gamma_{V \setminus K}.$$

Hence the preceding section immediately gives the following corollary of Theorem 20.

Corollary 21. *Let $\beta_{\partial V} > 0$ or $\gamma_{\partial V} > 0$. Then $\sigma_{\text{ess}}(H) = \emptyset$ if and only if $d(x_n) + c(x_n) \rightarrow \infty$ along any infinite path (x_n) with pairwise distinct vertices.*

7. Graph Laplacians and Markov processes

We have already discussed that our Laplacians come from Dirichlet forms. Now, Dirichlet forms and symmetric Markov processes are intimately connected. The crucial link is given by the semigroup generated by a Dirichlet form. The connection to Markov processes means that

- there is a wealth of results on the semigroup associated to a graph Laplacian,
- there is a good interpretation of properties of the semigroup in terms of a stochastic process.

Details are discussed in this section.

7.1. Graph Laplacians, their semigroup and the heat equation

Let a measure m on V with full support and a graph (b, c) over V be given. Let Q be the associated form and L its generator.

Standard theory [8, 15, 25] implies that the operators of the associated semigroup e^{-tL} , $t \geq 0$, and the associated resolvent $\alpha(L+\alpha)^{-1}$, $\alpha > 0$ are positivity preserving and even markovian. Positivity preserving means that they map nonnegative functions to nonnegative functions. Markovian means that they map nonnegative functions bounded by one to nonnegative functions bounded by one.

This can be used to show that semigroup and resolvent extend to all $\ell^p(V, m)$, $1 \leq p \leq \infty$. These extensions are consistent i.e., two of them agree on their common domain [7]. The corresponding generators are denoted by L_p , in particular $L = L_2$. We can describe the action of the operator L_p explicitly. More precisely, the situation on ℓ^2 (see Proposition 3 and Theorem 4) holds here as well:

Theorem 22. *Let (V, b, c) be a weighted graph and m a measure on V of full support. Then, the operator L_p is a restriction of \tilde{L} for any $p \in [1, \infty]$. If furthermore (A) holds, then the operator L is the restriction of \tilde{L} to*

$$\{u \in \ell^p(V, m) : \tilde{L}u \in \ell^p(V, m)\}.$$

A function $N : [0, \infty) \times V \rightarrow \mathbb{R}$ is called a solution of the heat equation if for each $x \in V$ the function $t \mapsto N_t(x)$ is continuous on $[0, \infty)$ and differentiable on $(0, \infty)$ and for each $t > 0$ the function N_t belongs to the domain of \tilde{L} , i.e., the vector space \tilde{F} and the equality

$$\frac{d}{dt}N_t(x) = -\tilde{L}N_t(x)$$

holds for all $t > 0$ and $x \in V$. For a bounded solution N validity of this equation can easily be seen to automatically extend to $t = 0$ i.e., $t \mapsto N_t(x)$ is differentiable on $[0, \infty)$ and $\frac{d}{dt}N_t(x) = -\tilde{L}N_t(x)$ holds for any $t \geq 0$.

The following theorem is a standard result in the theory of semigroups. A proof in our context can be found in [22] (see [33, 34, 32] for related material on special graphs).

Theorem 23. *Let L be a selfadjoint restriction of \tilde{L} , which is the generator of a Dirichlet form on $\ell^2(V, m)$. Let v be a bounded function on V and define $N : [0, \infty) \times V \rightarrow \mathbb{R}$ by $N_t(x) := e^{-tL}v(x)$. Then, the function $N(x) : [0, \infty) \rightarrow \mathbb{R}$, $t \mapsto N_t(x)$, is differentiable and satisfies*

$$\frac{d}{dt}N_t(x) = -\tilde{L}N_t(x)$$

for all $x \in V$ and $t \geq 0$.

Let us conclude this section by noting that the semigroups are positivity improving for connected graphs. This has been shown in [22] in our setting after earlier results in [8, 32, 33] for locally finite graphs.

Theorem 24. *(Positivity improving) Let (V, b, c) be a connected graph and L be the associated operator. Then, both the semigroup e^{-tL} , $t > 0$, and the resolvent $(L + \alpha)^{-1}$, $\alpha > 0$, are positivity improving (i.e., they map nonnegative nontrivial ℓ^2 -functions to strictly positive functions).*

7.2. Connection to Markov processes

In this section we discuss the relationship between Dirichlet forms and Markov processes in our context. Let Q be the Dirichlet form associated to a weighted graph (V, b, c) with measure m . For convenience we assume $m \equiv 1$. Let L be the associated operator and e^{-tL} , $t > 0$, the associated semigroup. We will take the point of view that we already know that e^{-tL} is a semigroup of transition properties of a Markov process. We will then show how we can identify the key quantities of the Markov process in terms of the graph (V, b, c) .

A (time homogenous) Markov process on V consists of a particle moving in time without memory between the points of V . It is characterized by two sets of quantities: These are

- a function $a : V \rightarrow [0, \infty)$ such that e^{-ta_x} is the probability that a particle in x at time 0 is still in x at time t .
- a function $q : V \times V \rightarrow [0, \infty)$ such that $q_x(y)$ is the probability that the particle jumps to y from x .

Given such a Markov process we can define

$$P_t(x, y) := \text{Probability that the particle is in } y \text{ at time } t \text{ if it starts in } x \text{ at time } 0$$

for $t \geq 0$, $x, y \in V$ and the operators P_t provide a semigroup of operators. It is then possible to infer the quantities a and q from the behavior of P_t for small t in the following way:

$P_t(x, x)$ is the probability to find the particle at x at time t (for a particle starting at x at time 0). This means that the particle has either stayed at x for the whole time between 0 and t or has jumped from x away and come back by the time t . The probability that the particle stayed in x (i.e., did not move away) is e^{-ta_x} . The event that the particle left x and returned by the time t means that the

particle left x , which occurs with probability $1 - e^{-ta_x}$, and then returned from $V \setminus \{x\}$ to x in the remaining time, which occurs with probability $r(t)$ going to zero for $t \rightarrow 0$. Accordingly we have

$$P_t(x, x) = e^{-ta_x} + \phi_x(t),$$

where ϕ_x summarizes the probability of returning to x , is therefore bounded by $(1 - e^{-ta_x})r(t)$ and hence has derivative equal to zero at $t = 0$. We therefore obtain

$$\left. \frac{d}{dt} \right|_{t=0} P_t(x, x) = -a_x + \phi'_x(0) = -a_x.$$

By a similar reasoning the probability $P_t(x, y)$ is governed by the event that the particle starts at x at time 0 and has done one jump to y and then stayed in y up to the time t . The probability p_t for this event satisfies

$$(1 - e^{-ta_x})q_x(y)e^{-ta_y} \leq p_t \leq (1 - e^{-ta_x})q_x(y).$$

Here, the term e^{-ta_y} serves to take into account that the particle did not leave y . Accordingly,

$$P_t(x, y) = p_t + \psi(t),$$

where the derivative of ψ at 0 is zero and we obtain

$$\left. \frac{d}{dt} \right|_{t=0} P_t(x, y) = a_x q_x(y) + \psi'(0) = a_x q_x(y).$$

We now return to the Dirichlet form setting. As e^{-tL} describes a Markov process we can now set

$$P_t(x, y) = \langle e^{-tL} \delta_x, \delta_y \rangle$$

for $t \geq 0$, $x, y \in V$ and use this to calculate the a 's and q 's in terms of b and c as follows:

$$\sum_{y \in V} b(x, y) + c(x) = Q(\delta_x, \delta_x) = \left. \frac{d}{dt} \right|_{t=0} \langle e^{-tL} \delta_x, \delta_x \rangle = \left. \frac{d}{dt} \right|_{t=0} P_t(x, x) = -a_x$$

and

$$-b(x, y) = Q(\delta_x, \delta_y) = \left. \frac{d}{dt} \right|_{t=0} \langle e^{-tL} \delta_x, \delta_y \rangle = \left. \frac{d}{dt} \right|_{t=0} P_t(x, y) = q_x(y) a_x.$$

This gives

$$q_x(y) = \frac{b(x, y)}{\sum_{z \in V} b(x, z) + c(x)}, \quad a_x = \sum_{z \in V} b(x, z) + c(x)$$

for all $x, y \in V$. Note that symmetry of b does not imply symmetry of q but rather

$$a_x q_x(y) = a_y q_y(x).$$

If m is not identically equal to one, we will have to normalize the formula for P above by setting

$$P_t(x, y) = \frac{1}{m(x)m(y)} \langle e^{-tL} \delta_x, \delta_y \rangle$$

and change the emerging formulae accordingly.

8. Stochastic completeness

We consider a Dirichlet form Q on a weighted graph (V, b, c) with associated operator L and semigroup e^{-tL} . The preceding considerations show that

$$0 \leq e^{-tL}1(x) \leq 1$$

for all $t \geq 0$ and $x \in V$. The question, whether the second inequality is actually an equality has received quite some attention. In the case of vanishing killing term, this is discussed under the name of stochastic completeness or conservativeness. In fact, for $c \equiv 0$ and $b(x, y) \in \{0, 1\}$ for all $x, y \in V$, there is a characterization of stochastic completeness of Wojciechowski [33] (see the introduction for discussion of related results of Feller [13] and Reuter [27] as well). This characterization is an analogue to corresponding results on manifolds of Grigor'yan [16] and results of Sturm for general strongly local Dirichlet forms [28].

Our first main result concerns a version of this result for arbitrary regular Dirichlet forms on graphs. As we allow for a killing term c we have to replace $e^{-tL}1$ by the function

$$M_t(x) := e^{-tL}1(x) + \int_0^t (e^{-sL} \frac{c}{m})(x) ds, \quad x \in V.$$

It is possible (and necessary) to show that this quantity is well defined. In fact, it can be proven that it satisfies $0 \leq M \leq 1$ and that for each $x \in V$, the function $t \mapsto M_t(x)$ is continuous and even differentiable [22]. Note that for $c \equiv 0$, $M = e^{-tL}1$ whereas for $c \neq 0$ the inequality $M_t > e^{-tL}1$ holds on any connected component of V on which c does not vanish identically (as the semigroup is positivity improving).

We can give an interpretation of M in terms of a diffusion process on V as follows: For $x \in V$, let δ_x be the characteristic function of $\{x\}$. A diffusion on V starting in x with normalized measure is then given by $\delta_x/m(x)$ at time $t = 0$. It will yield to the amount of heat

$$\langle e^{-tL} \frac{\delta_x}{m(x)}, 1 \rangle = \langle \frac{\delta_x}{m(x)}, e^{-tL}1 \rangle = \sum_{y \in V} e^{-tL}(x, y) = e^{-tL}1(x)$$

within V at the time t . Thus, the first term of M describes the amount of heat within the graph at a given time.

Moreover, at each time s the rate of heat killed at the vertex y by the killing term c is given by $e^{-sL}(x, y)c(y)/m(y)$. The total amount of heat killed at y till the time t is then given by $\int_0^t e^{-sL}(x, y)c(y)/m(y) ds$. The total amount of heat killed at all vertices by c till the time t is accordingly given by

$$\sum_{y \in V} \int_0^t e^{-sL}(x, y) \frac{c(y)}{m(y)} ds = \int_0^t \sum_{y \in V} e^{-sL}(x, y) \frac{c(y)}{m(y)} ds = \int_0^t (e^{-sL} \frac{c}{m})(x) ds.$$

Thus, the second term of M describes the total amount of heat killed up to time t within the graph. Altogether, $1 - M_t$ is then the amount of heat transported to the 'boundary' of the graph by the time

t and M_t can be interpreted as the amount of heat, which has not been transported to the boundary of the graph at time t .

Our question concerning stochastic completeness then becomes whether the quantity

$$1 - M_t$$

vanishes identically or not. Our result reads (see [22] for a proof):

Theorem 25. (*Characterization of heat transfer to the boundary*) *Let (V, b, c) be a weighted graph and m a measure on V of full support. Then, for any $\alpha > 0$, the function*

$$w := \int_0^\infty \alpha e^{-t\alpha} (1 - M_t) dt$$

satisfies $0 \leq w \leq 1$, solves $(\tilde{L} + \alpha)w = 0$, and is the largest nonnegative function $l \leq 1$ with $(\tilde{L} + \alpha)l \leq 0$. In particular, the following assertions are equivalent:

- (i) *For any $\alpha > 0$ there exists a nontrivial, nonnegative, bounded l with $(\tilde{L} + \alpha)l \leq 0$.*
- (ii) *For any $\alpha > 0$ there exists a nontrivial, bounded l with $(\tilde{L} + \alpha)l = 0$.*
- (iii) *For any $\alpha > 0$ there exists a nontrivial, nonnegative, bounded l with $(\tilde{L} + \alpha)l = 0$.*
- (iv) *The function w is nontrivial.*
- (v) *$M_t(x) < 1$ for some $x \in V$ and some $t > 0$.*
- (vi) *There exists a nontrivial, bounded, nonnegative $N : V \times [0, \infty) \rightarrow [0, \infty)$ satisfying $\tilde{L}N + \frac{d}{dt}N = 0$ and $N_0 \equiv 0$.*

Let us give a short interpretation of the conditions appearing in the theorem. Conditions (i), (ii) and (iii) deal with eigenvalues of \tilde{L} considered as an operator on $\ell^\infty(V)$. Thus, they concern spectral theory in $\ell^\infty(V)$. Condition (v) refers to loss of mass at infinity. Finally condition (vi) is about unique solutions of a partial difference equation. Thus, the result connects properties from stochastic processes, spectral theory and partial difference equations.

Sketch of proof. We refrain from giving a complete proof of the theorem but rather discuss three key elements of the proof and how they fit together. These are the following three steps:

- (S1) If $N : [0, \infty) \times V \rightarrow \mathbb{R}$ is a bounded solution of $\frac{d}{dt}N = -\tilde{L}N$, then $v = \int_0^\infty \alpha e^{-t\alpha} N_t dt$ is a solution to $(\tilde{L} + \alpha)v = 0$ for any $\alpha > 0$.
- (S2) The function $N = 1 - M$ satisfies $0 \leq N \leq 1$ and $\frac{d}{dt}N = -\tilde{L}N$.
- (S3) The function $w = \int_0^\infty \alpha e^{-t\alpha} (1 - M_t) dt$ is the largest solution of $(\tilde{L} + \alpha)v = 0$ with $0 \leq v \leq 1$.

The proof of the first step is a direct calculation via partial integration. The second step is a direct calculation but requires quite some care as the quantities are defined via sums and integrals whose convergence is not clear. The fact that w of the last step is a solution follows from the second step. The minimality of the solution requires some care. It follows by approximating the graph via finite graphs. Here, a nontrivial issue is that this approximation may actually cut infinitely many edges (as we do not have locally finite edge degree).

Given the three steps, the proof of the theorem goes along the following line: The implication (v) \implies (i) follows from Step (S1) and (S2). The implication (i) \implies (v) follows from the maximality property in Step (3). The implication (v) \implies (vi) follows from Step (S2). The implication (vi) \implies (v) follows from Step (S1). The equivalence between (iv) and (v) is immediate from Step (S3). The equivalence between (i), (ii) and (iii) follows by taking suitable minima of (super-) solutions. \square

Definition 26. *The weighted graph (V, b, c) is said to satisfy (SI_∞) if one of the equivalent assertions of the theorem holds. If the graph is not (SI_∞) it is said to satisfy (SC_∞) .*

In the case of vanishing killing term (i.e. $c \equiv 0$) (SC_∞) and (SI_∞) are just the standard definitions of stochastic completeness and stochastic incompleteness. By a slight abuse of language we will call any graph satisfying (SC_∞) stochastically complete and any graph satisfying (SI_∞) stochastically incomplete.

Corollary 27. *Assume the situation of the previous theorem. Let \tilde{L} be the operator associated to the graph (V, b, c) . If \tilde{L} gives rise to a bounded operator on $\ell^2(V)$, then (V, b, c) satisfies (SC_∞) .*

Proof. If \tilde{L} is bounded on $\ell^2(V, m)$ it is bounded on $\ell^\infty(V)$ by Theorem 11. Then, the spectrum of \tilde{L} on ℓ^∞ is bounded and hence its set of eigenvalues is bounded as well. Thus, (ii) of the theorem must fail (for large α). \square

Remark. (a) The corollary shows that stochastic completeness is a phenomenon for unbounded operators.

(b) The corollary generalizes the results of Dodziuk/Matthai [12] and Wojciechowski [33]. It is furthermore relevant as its proof gives an abstract i.e., spectral theoretic reason for stochastic completeness in the case of bounded operators.

Let us finish this section by discussing how the existence of $\alpha > 0$ and $t > 0$ and $x \in V$ with certain properties in the above theorem is actually equivalent to the fact that all $\alpha > 0$, $t > 0$ and $x \in V$ have these properties. We first discuss the situation concerning the α 's.

Proposition 28. *Let (V, b, c) be a weighted graph and m a measure on V of full support. Then, the following are equivalent:*

(i) *For any $\alpha > 0$ there exists a nontrivial, nonnegative, bounded l with $(\tilde{L} + \alpha)l \leq 0$.*

(ii) *For some $\alpha > 0$ there exists a nontrivial, nonnegative, bounded l with $(\tilde{L} + \alpha)l \leq 0$.*

Proof. It suffices to show the implication (ii) \implies (i): By the maximality property of the function $w = \int_0^\infty \alpha e^{-t\alpha}(1 - M_t)dt$ discussed in the third step of the proof of the main result, (ii) implies that $M_t(x) < 1$ for some $x \in V$ and $t > 0$. Now, the claim (i) follows from the second step discussed in the proof of the main result. \square

We now show that loss of mass in one point at one time is equivalent to loss of mass in all points at all times (if the graph is connected). For locally finite graphs this is discussed in [33].

Proposition 29. *Let (V, b, c) be a connected weighted graph and m a measure on V of full support. Let M be defined as above. Then, the following assertions are equivalent:*

(i) *There exist $x \in V$ and $t > 0$ with $M_t(x) < 1$.*

(ii) *$M_t(x) < 1$ for all $x \in V$ and all $t > 0$*

Proof. The implication (ii) \implies (i) is clear. It remains to show the reverse implication. A direct calculation (invoking $\int_0^{t+s} \dots dr = \int_0^s \dots dr + \int_s^{t+s} \dots dr$) shows that

$$M_{t+s} = e^{-sL} M_t + \int_0^s e^{-rL} \frac{c}{m} dr.$$

This easily gives that

(1) $M_t \equiv 1$ for some $t > 0$ implies $M_{nt} \equiv 1$ for all $n \in \mathbb{N}$.

(2) $M_t \neq 1$ for some $t > 0$ implies that $M_{t+s} < 1$ for all $s > 0$.

(Here (1) follows by induction and (2) follows as $M_t \neq 1$ implies $M_t \leq 1$ and $M_t(x) < 1$ for some $x \in V$. As the graph is connected this implies $e^{-sL} M_t < e^{sL} 1$ and the statement follows.)

Assume now that $M_t(x) < 1$ for some $x \in V$ and $t > 0$. We consider M_r for $r > t$ and for $r < t$ separately: By (2), $M_r < 1$ for all $r > t$. Assume that $M_r = 1$ for some $0 < r < t$, then $M_s = 1$ for all $s \leq r$ by (2). Hence, by (1) $M_{ns} = 1$ for all $n \in \mathbb{N}$ and $0 < s \leq r$. This gives $M_r = 1$ for all $r > 0$ which contradicts $M_t \neq 1$. Thus, $M_r \neq 1$ for all $0 < r < t$. Hence, by (2) $M_r < 1$ for all $0 < r \leq t$. \square

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