Unbounded Quantum Advantage in One-Way Strong Communication Complexity of a Distributed Clique Labelling Relation

Sumit Rout [1], Nitica Sakharwade [1], Some Sankar Bhattacharya [1], Ravishankar Ramanathan [2] and Paweł Horodecki [1]

[1] *International Centre for Theory of Quantum Technologies (ICTQT), University of Gdańsk, Jana Bażynskiego 8, 80-309 Gdańsk, Poland*

[2] *Department of Computer Science, The University of Hong Kong, Pokfulam Road, Hong Kong*

We investigate the one-way zero-error classical and quantum communication complexities for a class of relations induced by a distributed clique labelling problem. We consider two variants: 1) the receiver outputs an answer satisfying the relation - the traditional communication complexity of relations (CCR) and 2) the receiver has non-zero probabilities of outputting every valid answer satisfying the relation (equivalently, the relation can be fully reconstructed), that we denote the strong communication complexity of the relation (S-CCR). We prove that for the specific class of relations considered here when the players do not share any resources, there is no quantum advantage in the CCR task for any graph. On the other hand, we show that there exist, classes of graphs for which the separation between one-way classical and quantum communication in the S-CCR task grow with the order of the graph m, specifically, the quantum complexity is $O(1)$ while the classical complexity is $\Omega(\log m)$. Secondly, we prove a lower bound (that is linear in the number of cliques) on the amount of shared randomness necessary to overcome the separation in the scenario of fixed restricted communication and connect this to the existence of Orthogonal Arrays. Finally, we highlight some applications of this task to semi-device-independent dimension witnessing as well as to the detection of Mutually Unbiased Bases.

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I. INTRODUCTION

Quantum Shannon theory replaces the classical carrier of information with quantum systems in Shannon's model of communication [1]. This initiated a tide of attempts to understand the advantage of encoding classical information in a quantum system. Over the past few decades, there have been numerous works probing the advantage of quantum resources over classical counterparts in various information-theoretic scenarios. Many of these works provide a deeper insight into quantum theory. Some of these quantum advantages have found practical applications in the field of quantum cryptography [2, 3], quantum communication [4–8] and quantum computing [9–11] to name a few. In a prepare and measure scenario, the major share of effort has been devoted to showing an advantage in quantum communication complexity [12, 13], which involves computing the minimum communication required between distant parties in order to perform a distributed computation of functions [14].

Karchmer and Wigderson [15] initiated the study of the communication complexity of relations and established a connection between the communication complexity of certain types of relations and the complexity of Boolean circuits. In [16] Raz provided an example of an unbounded gap between the classical and quantum communication complexity for a relation. Another closely related line of study has been to explore the advantage of quantum communication in tasks based on orthogonality graphs. In most cases, orthogonality graphs that lead to quantum advantage are not Kochen-Specker colourable (KS-colourable) [17], thus connecting this set of tasks to the feature of quantum contextuality [18].

In this article, we introduce a new task based on the communication complexity of relations. For this task, we identify a class of relations based on graphs, such that there is an exponential gap between one-way zero-error classical and quantum communication. Two important points in which this work significantly differs from others is that, firstly, unlike in [17, 19] the quantum advantage in our proposed task is independent of the graphs being KS-colourable (or not). Secondly, the exponential advantage in [16] requires an infinite set of inputs, whereas the work presented in this article requires only a finite set of inputs to establish an unbounded gap.

In particular, we consider the one-way zero-error communication complexity of a relation (CCR) induced by rules of a distributed *clique labelling problem (CLP)* over a graph. For this CCR task where any valid answer belonging to the relation is accepted, we show that there is no advantage in using quantum systems as carriers of information. However, another version of the CCR task where Bob's output in different runs should span over the relation, called Strong Communication Complexity of Relation (S-CCR) entails a non-trivial quantum advantage. This new task of S-CCR is equivalent to the possibility of reconstructing the relation from the complete observed input-output statistics. Demanding reconstruction is a stronger form of communication complexity of relations since a function (a special case of relations) can always be reconstructed from the observed statistics while in general for a relation this does not hold.

Our main results consider two distinct scenarios depending on the availability of pre-shared correlations and direct communication resources between the two parties: (i) the spatially separated parties do not share any correlation, (ii) the communication channels can transmit systems of a fixed operational dimension. In the first scenario, we find that there exist communication tasks which entail an unbounded separation between the operational dimension of the classical and quantum message systems. We also demonstrate a quantum advantage for a relation induced by *Payley graphs* which are a class of vertex-transitive self-complementary graphs. In the second scenario, we show that there exist communication tasks which imply classical channels require to be assisted by unbounded amounts of pre-shared classical correlations while the quantum channel does not require any pre-shared resources. Additionally, we show that there exist graphs for which the task with a classical channel requires shared randomness linear in cliques whereas with shared entanglement it can be performed by a 1-ebit-assisted classical channel.

A. Outline of the paper

The article is organised as follows: in Sec. II we discuss the preliminaries about the orthogonal representation of graphs, and binary/KS colouring of a graph that we require in our communication task. In Sec. III we introduce the setup for the task of communication complexity of relations induced by graphs and subsequent variations. In Subsec. III B we introduce the clique labelling problem induced by a graph and show that considering the conventional one-way zero-error communication complexity of this relation does not lead to a quantum advantage. Next in Sec. III C we consider a stronger communication complexity scenario which implies that relations can be reconstructed from the observed input-output statistics. Sec. IV contains the bulk of our results where we show that the gap between classical and quantum communication required to compute and subsequently reconstruct the clique labelling relation is unbounded for a class of graphs. In Sec. V we list some applications of the proposed communication scenario and finally in Sec. VI we discuss an interpretation of the payoff of the proposed task in terms of a property of the graph used to execute the task and also list the open questions.

II. PRELIMINARIES

In this section, we briefly go over known concepts relevant to the article, including notions of orthogonal representation and binary colouring of graphs widely used in the study of contextuality [20] and finally, the notion of operational dimension that helps compare classical and quantum (communication) resources.

A. Graphs, Orthogonal Representation, and Binary Colouring

A graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ consists of a set of vertices $\mathcal{V} := (v_1, v_2, \ldots, v_n)$ and a set of edges $\mathcal{E} := (e_1, e_2, \ldots, e_m)$ between the vertices. Additionally, the edges may also have a directional property and a weight, which gives rise to further classifications of directed or undirected graphs and weighted or unweighted graphs. In this work, we consider simple undirected unweighted graphs. A subgraph of a graph $\mathcal{G}$ is a graph $\mathcal{G}' = (\mathcal{V}', \mathcal{E}')$ where $\mathcal{E}' \subseteq \mathcal{E}$ such that $\forall e_i \in \mathcal{E}'$ the vertices connected by e_i belong in $\mathcal{V}' \subseteq \mathcal{V}$. For any graph $\mathcal{G}$, a clique is a fully connected subgraph of $\mathcal{G}$. The size of the clique is given by the number of vertices in the subgraph.

Among many different representations of an arbitrary graph, orthogonal representation over complex fields has been shown to be useful in demonstrating the nonexistence of a non-contextual hidden variable model for quantum mechanics [20, 21]. Here we make use of a general definition of orthogonal representation. We describe the orthogonal representation of a graph over arbitrary fields as the following [22]:

Definition 1. *Given a graph $\mathcal{G} := (\mathcal{V}, \mathcal{E})$, an orthogonal representation of G over field $\mathbb{F}$ is described by the function $\phi : \mathcal{V} \to \mathbb{F}^d$, such that*
(i) for any two adjacent vertices v_i and v_j, $\langle \phi(v_i), \phi(v_j) \rangle = 0$,
(ii) $\phi(v_i) \neq \phi(v_j)$, for all $i \neq j$
where, d is the dimension of the vector space over field $\mathbb{F}$ and $\langle\ ,\ \rangle$ denotes the scalar product (bilinear form) over field $\mathbb{F}$.
This representation is faithful, if $\langle \phi(v_i), \phi(v_j) \rangle = 0$ implies that v_i and v_j are adjacent; and is orthonormal, if $|\phi(v_i)| = 1$ for all $v_i \in \mathcal{V}$.

An important problem regarding this representation is to find minimum d, such that the definition holds true. For such an optimal orthogonal representation we denote the *faithful orthogonal range* of the graph $\mathcal{G}$ over field $\mathbb{F}$ as $d_{\mathbb{F}}$ (for example, $d_{\mathbb{R}}$, $d_{\mathbb{C}}$ etc.). A lower and an upper bound to the faithful orthogonal range $d_{\mathbb{F}}$ satisfied over an arbitrary field $\mathbb{F}$, are given as follows:

$$\omega \leq d_{\mathbb{F}} \leq d_{\mathbb{F}'} \leq |\mathcal{V}| \quad (1)$$

where, $\mathbb{F}' \subseteq \mathbb{F}$, ω is the maximum clique size and $|\mathcal{V}|$ is the number of vertices in the graph $\mathcal{G}$ also known as the order of the graph. The lower bound follows from the fact that in the faithful orthogonal representation, there should be at least ω number of orthogonal vectors and the upper bound says that it is always trivially possible to provide an orthogonal representation with $|\mathcal{V}|$ number of mutually orthogonal vectors.

Lovász *et al*. [23] provided a necessary and sufficient condition for finding minimal d over the real field $\mathbb{R}$ for a class of orthogonal representations called *general position*, for which any set of d representing real vectors are linearly independent.

Proposition 1. *[Lovász et al. '89 [23]] Any graph $\mathcal{G} := (\mathcal{V}, \mathcal{E})$, has a general position faithful orthogonal representation in $\mathbb{R}^d$ if and only if at least $(|\mathcal{V}| - d)$ vertices are required to be removed to make the complementary graph $\bar{\mathcal{G}}$ disconnected.*

Later we will refer to this result to provide an upper bound on the faithful orthogonal range for a class of graphs.

Given a graph $\mathcal{G}$, the problems concerning the colouring of its vertices with one of two possible colours have been widely studied and share deep connections with quantum non-contextuality. We will refer to a graph along with a faithful orthogonal representation in minimum dimension as an orthogonality graph. In the following, we define the binary colouring of an orthogonality graph.

Definition 2. *Binary colouring of a graph $\mathcal{G} := (\mathcal{V}, \mathcal{E})$, is a binary function $\mathsf{f} : \mathcal{V} \to \{0, 1\}$, such that*
(i) for any two adjacent vertices v_i and v_j, $\mathsf{f}(v_i)\mathsf{f}(v_j) = 0$,
(ii) for any maximum clique C_k of the graph $\mathcal{G}$, there is exactly one vertex $v_* \in \mathsf{C}_k$, such that $\mathsf{f}(v_*) = 1$.

A point to note here is that not all graphs are binary colourable. A *Binary Colouring* of graph $\mathcal{G}$ with n vertices, if possible, can be thought of as a binary string of length n. On the other hand, the set of the binary strings corresponding to all different binary colourings uniquely describes the graph $\mathcal{G}$. In the subsequent sections, we will use the term "colouring of a graph" to refer to the binary colouring of the graph. A *uniquely* binary colourable graph is a graph that has only one possible binary colouring up to the permutation of the colours. For example, all *bipartite* graphs are uniquely binary colourable.

B. Operational dimension

In any communication protocol, the carrier of the message, as well as the sources of private or public coins, are physical systems, which may be described as classical or quantum (or more generally but outside the purview of this work by a post-quantum theory). In order to be able to compare resources an important notion from the study of Generalised Probability Theories (GPT) [24] is that of the concept of *Operational dimension*.

Definition 3. *The operational dimension of a system is the largest cardinality of the subset of states that can be perfectly distinguished by a single measurement.*

Importantly, the operational dimension of a theory is different from the dimension of the vector space V in which the state space Ω is embedded. For instance, for qubit the state space, the set of density operators $\mathcal{D}(\mathbb{C}^2)$ acting on $\mathbb{C}^2$ is embedded in $\mathbb{R}^3$. However, the operational dimension of this system is 2, as at most two-qubit states can be perfectly distinguished by a single measurement. Thus, the operation dimension is equivalent to the Hilbert space dimension for a quantum system. We will refer to this notion when comparing communication resources between the quantum and classical scenarios.

III. COMMUNICATION COMPLEXITY OF RELATIONS

In this Section, we will introduce the extension of bipartite communication complexity of functions to relations. A relation over a bipartite prepare and measure scenario is defined as a subset $\mathcal{R} \subseteq X \times Y \times B$, where X and Y are the set of possible input values of Alice and Bob, respectively and B is the set of possible output values that can be produced by Bob. A simple example is the relation $\mathcal{R}$ where X and Y are sets of Parents and the set B is the set of Children and a valid tuple (x, y, b) is when b is a child of x and y. Clearly, there might be multiple *correct answers* if x and y have multiple children. There is also the possibility of no valid output for a given x and y if they have no children. We will consider relations that have a valid output b for any valid input (x, y). Let us now define what is meant by the Communication Complexity of a Relation (or CCR).

Definition 4. ***CCR*** *The communication complexity of a relation $\mathcal{R} \subseteq X \times Y \times B$ is the minimum communication that Alice requires to make with Bob for any input variables $x \in X$ and $y \in Y$ such that Bob's output b gives the tuple (x, y, b) which belongs to $\mathcal{R}$. Note that Alice and Bob should know the relation $\mathcal{R}$ before the task commences.*

A protocol P to perform this task may involve one-way or two-way communications with single or multiple rounds. In this work, we will be interested in *one-way* communication protocols. The cost of a protocol P is defined as the minimum amount of communication required to perform the computation for any input (x, y). The communication complexity of the relation $\mathcal{R}$

is defined as the minimum cost over all protocols that can compute $\mathcal{R}$. In a generalised setting, the computation may allow for some small errors to lower the cost. Throughout this article we consider only *zero-error* protocols, *i.e.* $P(b|x,y) = 0$ whenever $(x,y,b) \notin \mathcal{R}$ for all $(x,y) \in X \times Y$. In most cases rather than finding the optimal protocol, which is a difficult task, one is interested in providing a lower bound for the communication complexity task. A trivial *zero-error* protocol has a cost of $\log|X|$ bits, which requires that Alice sends all information about her input to Bob and is also a trivial upper bound for communication complexity.

The protocols for classical communication complexity of relations depending on encoding and decoding strategies have the following types:

1. **Deterministic Protocol:** A classical one-way deterministic protocol is an encoding-decoding tuple $(\mathbb{E}, \mathbb{D})$, where $\mathbb{E}$ is a '$\log|X|$-bit to m-bit' deterministic function and $\mathbb{D}$ is a '$m \log|Y|$-bit to $\log|Z|$-bit' deterministic function, *i.e.* $\mathbb{E} : \{1, \cdots, |X|\} \mapsto \{0, \ldots, m-1\}$ and $\mathbb{D} : \{0, \ldots, m-1\} \times \{1, \cdots, |Y|\} \mapsto \{1, \cdots, |Z|\}$. The communication cost of such a protocol is defined as the length of the message in bits sent by Alice on the worst choice of inputs x and y. The one-way deterministic zero-error communication complexity of relation $\mathcal{R}$, denoted by $\mathbf{D}(\mathcal{R})$ is the cost of the best protocol (*i.e* protocol with minimum communication cost) that allows computation of relation $\mathcal{R}$ without any error.

2. **Protocol with Private Coins:** A classical one-way protocol with private coins is a tuple $(P_{\mathbb{E}}, P_{\mathbb{D}})$, where $P_{\mathbb{E}}$ and $P_{\mathbb{D}}$ are probability distributions over the space of deterministic encodings ($\mathbb{E}$) and decodings ($\mathbb{D}$), respectively. We denote the private coin-assisted communication complexity of relation $\mathcal{R}$ as $\mathbf{R}_{priv}(\mathcal{R})$, where $P_{\mathbb{E}}$ and $P_{\mathbb{D}}$ are probability distributions over the space of deterministic encodings ($\mathbb{E}$) and decodings ($\mathbb{D}$), respectively

3. **Protocol with Public Coins:** A classical one-way protocol with the public coin is $P_{\mathbb{E}\times\mathbb{D}}$, where $P_{\mathbb{E}\times\mathbb{D}}$ is a probability distribution over the space of Cartesian product of deterministic encodings and decodings. We denote the public coin-assisted communication complexity of relation $\mathcal{R}$ as $\mathbf{R}_{pub}(\mathcal{R})$, where $P_{\mathbb{E}\times\mathbb{D}}$ is a probability distribution over the space of Cartesian product of deterministic encodings and decodings

The communication complexities of a relation $\mathcal{R}$ satisfy the following ordering:

$$\mathbf{R}_{pub}(\mathcal{R}) \leq \mathbf{R}_{priv}(\mathcal{R}) \leq \mathbf{D}(\mathcal{R}) \tag{2}$$

In the communication complexity of functions, there is only a single *correct answer* that Bob may output. The task of communication complexity of relations differs from that of functions since there may be more than one correct answer for Bob. This allows us to define a stronger variation of CCR that enforces that Bob outputs all correct answers over different rounds of the prepare and measure scenario, which we call Strong Communication Complexity of Relations (S-CCR). Naturally then, when the relation is a function (a subclass of relations) S-CCR and CCR reduce to the same task.

Definition 5. ***S-CCR*** *The strong communication complexity of a relation $\mathcal{R} \subseteq X \times Y \times B$ is the minimum communication that Alice requires to make with Bob for any input variables $x \in X$ and $y \in Y$ such that Bob's output b gives the tuple (x,y,b) which belongs to $\mathcal{R}$ and that Bob's output b in different rounds of the prepare and measure scenario spans all valid b for each (x,y) input. Same as CCR Alice and Bob should know the relation $\mathcal{R}$ before the task commences.*

The aim of this task is to be able to decipher or *reconstruct* the relation $\mathcal{R}$ from the observed statistics $\{(x_i, y_i, b_i)|i = runs\}$. In the limit of $runs \to \infty$ the observed statistics can be used to get the conditional output probability distribution $\{P(b|x,y)\}_{x,y,b}$. Note that for S-CCR, the necessary information to guess or reconstruct $\mathcal{R}$ correctly is given by the non-zero value of the observed conditional probabilities when $(x,y,b) \in \mathcal{R}$ (and zero otherwise) rather than the exact probabilities. We can define a natural (but not convex) payoff for S-CCR as follows:

$$\mathcal{P}_{\mathcal{R}} = \min_{(x,y,b)\in\mathcal{R}} P(b|x,y). \tag{3}$$

When optimised over all protocols with or without public coins involving them, the best strategy yields the maximum achievable payoff for the given relation which we will refer to as algebraic upper bound $\mathcal{P}^*_{\mathcal{R}}$. This is trivially achieved if Alice communicates her input to Bob and Bob in turn uses this message and his input to give a randomly chosen output from the set of all correct answers in each run.

$$\mathcal{P}^*_{\mathcal{R}} = \max_{\mathrm{P}} \mathcal{P}_{\mathcal{R}} \tag{4}$$

One way to interpret the payoff $\mathcal{P}_{\mathcal{R}}$ is to think of it as related to the probability of success of reconstructing the relation $\mathcal{R}$ (See Appendix A) for the given protocol, the higher the value of $\mathcal{P}_{\mathcal{R}}$, the lesser runs one needs to reconstruct the relation. Note that for the success of reconstruction, we necessarily require $\mathcal{P}_{\mathcal{R}} > 0$. It is worth mentioning that in S-CCR we are interested the minimum communication that does the task. However, the optimal strategy using this amount of communication may not yield $\mathcal{P}^*_{\mathcal{R}}$. Further, two different sets of resources (communication and/or shared) of the same operational dimension, such as quantum $\mathcal{P}_{\mathcal{R}}^{Q}$ and classical $\mathcal{P}_{\mathcal{R}}^{Cl}$, that individually perform the S-CCR task may also yield different payoff when optimised over all the strategies given the mentioned resources.

In this work, we consider some specific relations induced by orthogonality graphs. These relations are specified by a distributed clique labelling problem. Before we explain the setup let us introduce clique labelling.

A. Binary colouring to clique labelling

To find communication complexity, we require the minimum amount of communication between parties. While working with orthogonal graphs we have the binary colouring of each vertex of a graph which can be compressed/encoded. Therefore, we require a mapping that takes one from a binary colouring of vertices to the label of a clique and vice-versa. Consider an orthogonal graph $\mathcal{G}$ with the set of vertices $\mathcal{V}$. The binary colour, denoted by $f(.)$ is defined over each vertex (Def. 2). Now additionally consider some indexing of the vertices $\{1, \cdots, |\mathcal{V}|\}$ of the graph. Let us define the set of vertices belonging to a maximum clique C_i as $\mathcal{V}_{C_i} \subseteq \mathcal{V}$. Observe that the binary colour takes value 1 for only one vertex of each maximum clique of say size ω (with largest ω and assume that each vertex belongs to at least one such clique), that is

$$\forall\ v \in \mathcal{V}_{C_i},\ f(v) = \delta_{v,v'} \text{ for a } v' \in \mathcal{V}_{C_i} \tag{5}$$

We can now define *clique labelling*.

Definition 6. *The clique labelling, g_{C_i}, for some clique C_i is a mapping from $f(v)$ for each vertex in a clique $v \in \mathcal{V}_{C_i}$ to a ω-valued label in $\Omega = \{0, ..., \omega - 1\}$. The vertices in $\mathcal{V}_{C_i}$ are ordered in increasing indices and the clique label is assigned from $\{0, ..., \omega - 1\}$ to the vertex whose binary colouring is 1 such that the clique label's position matches the vertex position in the index ordered set $\mathcal{V}_{C_i}$. More concretely, the clique label is assigned from $\{0, ..., \omega - 1\}$ such that the lowest clique label 0 is assigned if the vertex with the lowest index in $\mathcal{V}_{C_i}$ has a binary colouring 1, the second lowest clique label 1 is assigned if the vertex with the second lowest index in $\mathcal{V}_{C_i}$ has a binary colouring 1 and so on.*

For example, for $\omega = 3$ if a clique has vertices $\mathcal{V}_{C_i} = \{v_3, v_6, v_7\}$ then if $f(v_3) = 1$ then $g_{C_i} = 0$, if $f(v_6) = 1$ then $g_{C_i} = 1$ and if $f(v_7) = 1$ then $g_{C_i} = 2$, where g_{C_i} is the ω-valued clique label of clique C_i. Note that given the index of vertices, a clique and its clique label one can always map it back to the binary labelling of each vertex of a clique, that is the mapping is convertible, necessary for decoding.

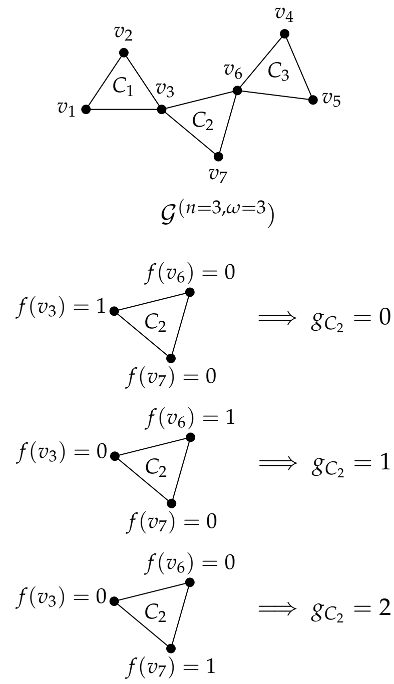


Figure 1: As an example for the graph given above the clique labelling of C_2 which has vertices $\mathcal{V}_{C_2} = \{v_3, v_6, v_7\}$ for some binary colourings is provided in this figure

B. Clique Labelling Problem (CLP)

Now, we present the class of relations for which we study CCR and S-CCR in this work. We are interested in relations based on the distributed *clique labelling problem* over a class of graphs. Here we consider graphs along with some faithful orthogonal representation in minimum dimension. We will refer to this graph along with this orthogonal representation together as an orthogonality graph. We consider an orthogonality graph $\mathcal{G}$ with n maximum cliques labelled as C_i where $i \in \{1, ..., n\}$ and the highest clique size of the graph be ω. We also assume that each vertex belongs to some ω-sized clique. We denote such graphs by $\mathcal{G}^{(n,\omega)}$. Let us define the set of maximum cliques as $\mathcal{C} = \{C_1, C_2, \cdots, C_n\}$ and the set of (input and output) clique labels as $\Omega = \{0, \cdots, \omega - 1\}$ for the graph $\mathcal{G}^{(n,\omega)}$. Note that the clique labels are related to the binary colouring of vertices through the definition 6.

The setup (given in Fig. 2) for our Clique Labelling Problem (CLP) is a prepare-and-measure scenario involving a referee and two spatially separated players, Alice and Bob. The referee shares the orthogonal graph $\mathcal{G}^{(n,\omega)}$ with some vertex indexing and a faithful orthogonal representation in minimum dimension with Alice and Bob at the beginning. The referee gives Alice the pair (C_x, a) as input: a clique of size ω randomly chosen from $\mathcal{G}^{(n,\omega)}$ and a random possible labelling of the same clique, i.e., $(C_x, a) \in X = \mathcal{C} \times \Omega$. The referee gives a clique of size ω randomly chosen from $\mathcal{G}^{(n,\omega)}$ to Bob as input, $C_y \in Y = \mathcal{C}$. We will consider the inputs to be uniformly distributed in the sense that C_x and C_y are both randomly chosen from $\mathcal{C}$ and a is uniformly chosen from Ω. Alice is allowed to send some communication (either classical or quantum depending on the scenario) to Bob which we will optimise to find the communication complexity. (We will also consider situations with classical and quantum public coins later.)

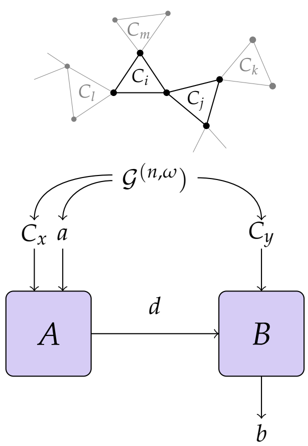


Figure 2: **Setup** Given an orthogonal graph $\mathcal{G}^{(n,\omega)}$ and Alice's input is a maximum clique and a clique label, *i.e.* (C_x, a) and Bob's input is some maximum clique C_y. Bob must output a valid clique labelling b for his input clique such that $(C_x, a, C_y, b) \in \mathcal{R}_{CLP}(\mathcal{G}^{(n,\omega)})$. Alice can send a physical system of operational dimension d to Bob.

Bob must output a valid labelling $b \in B = \Omega$ for C_y which satisfies the constraints provided below coming from the rules of the binary colouring of the orthogonal graph $\mathcal{G}^{(n,\omega)}$, that will define the relation. We call this *Consistent Labelling of Pairwise Cliques*:

1. If Alice and Bob receive the same clique Bob's clique labelling should be identical to Alice's input clique labelling, i.e. the binary colouring of the vertices of the maximal clique should be same.

2. If Alice and Bob receive two different cliques sharing some vertices, the binary colouring of each shared vertex (0 or 1) by Bob should be identical to Alice's colouring of the vertex.

3. In all other cases Bob can label the cliques independently of Alice's input label.

The conditions for consistent labelling of pairwise cliques are defined w.r.t. binary colourings, which can be then translated to conditions on the input and output clique labelling in $\{0, \ldots, \omega - 1\} = \Omega$ (definition 6).

The relation $\mathcal{R} \subseteq X \times Y \times B$ for the prepare and measure distributed CLP over the input and output sets $X = \mathcal{C} \times \Omega$, $Y = \mathcal{C}$ and $B = \Omega$, obeys the following structure $\mathcal{R}_{CLP} \subseteq \mathcal{C} \times \Omega \times \mathcal{C} \times \Omega$. The game is successful if the Clique Labelling Problem (CLP) is satisfied, that is the tuple $(x, y, b) \equiv (C_x, a, C_y, b) \in \mathcal{R}_{CLP}(\mathcal{G}^{(n,\omega)})$, where $\mathcal{R}_{CLP}(\mathcal{G}^{(n,\omega)})$ is the relation defined by the constraints of consistent labelling of pairwise clique given above for some graph $\mathcal{G}^{(n,\omega)}$. Note that having the relation is equivalent to having the graph itself.

In Sec. IV, we show that considering CCR for $\mathcal{R}_{CLP}(\mathcal{G}^{(n,\omega)})$ there exists a protocol where $d = \omega$-valued one-way communication from Alice to Bob both in the classical and quantum case win the game, and we do not have any quantum advantage. However, it is possible to realise unbounded quantum advantage when we look at the S-CCR for $\mathcal{R}_{CLP}(\mathcal{G}^{(n,\omega)})$.

We add one observation here that will become relevant for some of the results in Sec. IV. For a graph $\mathcal{G}$ to have an orthogonal representation in dimension ω, any two distinct cliques for this graph can have at most $\omega - 2$ points in common. Equivalently, every vertex v in C_i that is not in a clique C_j can be orthogonal to at most $\omega - 2$ vertices in C_j.

C. Reconstruction of the Relation $\mathcal{R}_{CLP}$

In the setup above Bob's output must be consistently labelled for CCR. Let us now consider the stronger version – S-CCR, where Bob must span all correct answers. This can be formulated as a *reconstruction game* where at the end of every round, the inputs and outputs of Alice and Bob are listed. After sufficient runs of the game, this list is shared with a randomly chosen Reconstructor (Fig. 3), who at the beginning does not have any information about the graph and the induced relation thereof. After obtaining the list the Reconstructor becomes aware of the cardinality of the input and/or output sets of Alice and Bob from the list. The Reconstructor must reconstruct the relation $\mathcal{R}_{CLP}(\mathcal{G}^{(n,\omega)})$ and thus the graph $\mathcal{G}^{(n,\omega)}$.

For reconstruction to be possible, the outcomes of Bob b should be such that after many runs of the game, the set of tuples $\{(C_x, a, C_y, b)\}$ can be used to deduce all the (non-)orthogonality relations in the graph $\mathcal{G}^{(n,\omega)}$ by the Reconstructor, without any prior information about the relation $\mathcal{R}_{CLP}(\mathcal{G}^{(n,\omega)})$ or graph $\mathcal{G}^{(n,\omega)}$.

After many runs of the game, the following payoff is

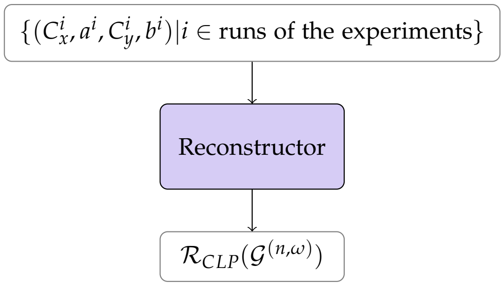


Figure 3: **Reconstruction of Relation** After many runs of the game, the statistics $\{(C_x^i, a^i, C_y^i, b^i)\}$ are sent to the Reconstructor, who attempts to recover $\mathcal{R}_{CLP}(\mathcal{G}^{(n,\omega)})$

calculated:

$$\mathcal{P}_{\mathcal{R}_{CLP}(\mathcal{G})} = \min_{(C_x,a,C_y,b)\in\mathcal{R}_{CLP}} P(b|C_x, C_y, a) \tag{6}$$

Here the minimisation is over the set of events in $\mathcal{R}_{CLP}$. The payoff $\mathcal{P}_{\mathcal{R}_{CLP}(\mathcal{G})}$ is necessarily non-zero if reconstruction is possible. The payoff can be interpreted as a measure of the efficiency of relation reconstruction over some number of runs.

Same as before, when optimised over all protocols with or without public coins, the best strategy yields the maximum achievable payoff for the given relation which we will refer to as algebraic upper bound $\mathcal{P}^*_{\mathcal{R}_{CLP}(\mathcal{G})}$.

D. Probability Table for CC$\mathcal{R}_{CLP}$ and S-CC$\mathcal{R}_{CLP}$

One can analyse the task of CCR for consistent labelling with relation $\mathcal{R}_{CLP}$ as well as the stronger task of distributed relation reconstruction problem (S-CCR) through a table of conditional probabilities $p(b|C_x, C_y, a)$, that Alice and Bob can write down (i.e their strategies) before the game begins. The rows of the table are given by Alice's possible inputs (C_x, a), and the columns are denoted by the tuple of inputs-outputs of Bob (C_y, b). This way of analysis will be important to understand some of the proofs. The favourable conditions of CC$\mathcal{R}_{CLP}$ (which is **(T0)**) and S-CC$\mathcal{R}_{CLP}$ (which are **(T0)-(T1)**) are provided in terms of the probability table below:

(T0): Consistent labelling If a tuple does not belong to the relation the corresponding conditional probability entry should be zero

$$\forall\ (C_x, a, C_y, b) \notin \mathcal{R}_{CLP}(\mathcal{G}^{(n,\omega)}) \implies P(b|C_x, a, C_y) = 0 \tag{7}$$

(T1): Relation Reconstruction If a tuple belongs to the relation the corresponding conditional probability entry should not be zero

$$\forall\ (C_x, a, C_y, b) \in \mathcal{R}_{CLP}(\mathcal{G}^{(n,\omega)}) \implies P(b|C_x, a, C_y) > 0 \tag{8}$$

Further, one can provide an algebraic upper bound $\mathcal{P}^*_{\mathcal{R}_{CLP}(\mathcal{G})}$ for a given graph $\mathcal{G}^{(n,\omega)}$ from the probability table in the following way. First, fix an input $(\tilde{C_x}, \tilde{a})$ for Alice and $\tilde{C_y}$ for Bob. Now count the number of events $(\tilde{C_x}, \tilde{a}, \tilde{C_y}, b) \in \mathcal{R}_{CLP}(\mathcal{G})$. Lets call this number $\eta(\tilde{C_x}, \tilde{a}, \tilde{C_y})$. Maximise $\eta(\tilde{C_x}, \tilde{a}, \tilde{C_y})$ over Alice's and Bob's input sets and call this number η. Given **(T0)-(T1)** is satisfied one has a non-zero payoff. The payoff satisfies the following inequality

$$0 < \mathcal{P}_{\mathcal{R}_{CLP}(\mathcal{G})} \leq \frac{1}{\eta} = \mathcal{P}^*_{\mathcal{R}_{CLP}(\mathcal{G})} \tag{9}$$

For example, in the case of a graph which has all maximum cliques of size ω that are all disconnected the upper bound for this graph for the payoff becomes $\mathcal{P}_{\mathcal{R}_{CLP}(\mathcal{G})} \leq \frac{1}{\omega} = \mathcal{P}^*_{\mathcal{R}_{CLP}(\mathcal{G})}$.

We can now provide the final condition:

(T2): Optimal Payoff When the payoff $\mathcal{P}_{\mathcal{R}_{CLP}(\mathcal{G})}$ achieves the algebraic upper bound, we say the payoff is optimal.

$$0 < \mathcal{P}_{\mathcal{R}_{CLP}(\mathcal{G})} = \frac{1}{\eta} = \mathcal{P}^*_{\mathcal{R}_{CLP}(\mathcal{G})} \tag{10}$$

It is worth highlighting that the payoff $\mathcal{P}_{\mathcal{R}_{CLP}(\mathcal{G})}$ is a *faithful* quantifier of the distributed relation reconstruction problem (S-CC$\mathcal{R}_{CLP}$), that is $\mathcal{P}_{\mathcal{R}_{CLP}(\mathcal{G})} > 0$ whenever relation reconstruction is possible and $\mathcal{P}_{\mathcal{R}_{CLP}(\mathcal{G})} = 0$ implies reconstruction is impossible. Moreover, winning the S-CC$\mathcal{R}_{CLP}$ game guarantees fulfilment of the condition in the CC$\mathcal{R}_{CLP}$ task. Our objective is to maximise the payoff $\mathcal{P}_{\mathcal{R}_{CLP}(\mathcal{G})}$ using only direct communication resources such as classical bits or qubits, and also using shared resources such as classical shared randomness or entanglement.

E. A concrete example

In this subsection, we provide an example of a particular simple graph to help solidify the ideas of (S-)CCR and the conditional probability table provided in this Section. Consider the graph $\mathcal{G}^{(n=2,\omega=3)}$ (see Fig. 4 for vertex indexing), with $n = 2$ maximum cliques of size $\omega = 3$ that share a common vertex.

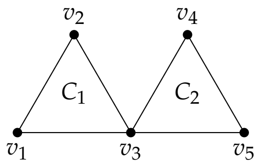


Figure 4: In this example, the graph $\mathcal{G}^{(2,3)}$ consists of two cliques C_1 and C_2 of size $\omega = 3$.

The mapping of binary colourings to clique labellings for clique C_1 can be given by:

$$\begin{aligned} f(v_1) = 1, f(v_2) = f(v_3) = 0 &\implies \quad g_{C_1} = 0 \\ f(v_2) = 1, f(v_1) = f(v_3) = 0 &\implies \quad g_{C_1} = 1 \\ f(v_3) = 1, f(v_1) = f(v_2) = 0 &\implies \quad g_{C_1} = 2 \end{aligned} \tag{11}$$

Similarly, the mapping of binary colourings to clique labellings for clique C_2 can be given by:

$$\begin{aligned} f(v_3) = 1, f(v_4) = f(v_5) = 0 &\implies \quad g_{C_2} = 0 \\ f(v_4) = 1, f(v_3) = f(v_5) = 0 &\implies \quad g_{C_2} = 1 \\ f(v_5) = 1, f(v_3) = f(v_4) = 0 &\implies \quad g_{C_2} = 2 \end{aligned} \tag{12}$$

Then relation $\mathcal{R}_{CLP}(\mathcal{G}^{(2,3)})$ induced by the clique labelling problem with tuples (C_x, a, C_y, b) can be concretely given by:

$$\begin{aligned} &\mathcal{R}_{CLP}(\mathcal{G}^{(2,3)}) = \{(C_1,0,C_2,1),(C_1,0,C_2,2),(C_1,1,C_2,1), \\ &(C_1,1,C_2,2),(C_1,2,C_2,0),(C_2,1,C_1,0),(C_2,1,C_1,1), \\ &(C_2,2,C_1,0),(C_2,2,C_1,1),(C_2,0,C_1,2),(C_1,0,C_1,0), \\ &(C_1,1,C_1,1),(C_1,2,C_1,2),(C_2,0,C_2,0),(C_2,1,C_2,1), \\ &(C_2,2,C_2,2)\} \end{aligned} \tag{13}$$

For this graph, the table of conditional probability $P(b|C_x, C_y, a)$ for all compatible labelling a, b and cliques C_x, C_y is the following:

The entries marked with * are the free non-negative entries up to normalisation and the entries with 0 or 1 are constrained from the consistency conditions for CCR. This will give a table **T0** satisfying condition (**T0**). A table satisfying condition (**T1**) must have positive numbers at all the entries marked with *. In this example, a table satisfying condition (**T2**) must have 0.5 at all the entries marked with *.

In the next section, we present the bulk of our key results first considering the scenario with only direct communication resources (section IV A - IV D) and later considering the scenario where the bounded amount of direct communication is assisted by shared resources (i.e. public coins).

		C_1			C_2		
		$b=0$	$b=1$	$b=2$	$b=0$	$b=1$	$b=2$
C_1	$a=0$	1	0	0	0	$*$	$*$
	$a=1$	0	1	0	0	$*$	$*$
	$a=2$	0	0	1	1	0	0
C_2	$a=0$	0	0	1	1	0	0
	$a=1$	$*$	$*$	0	0	1	0
	$a=2$	$*$	$*$	0	0	0	1

Table I: Example of a table of conditional probabilities $p(b|C_x, C_y, a)$ corresponding to the graph in Fig. 4 satisfying (**T0**), the necessary condition for CC$\mathcal{R}_{CLP}$, where $* \in [0,1]$ are free entries upto normalisation. For S-CC$\mathcal{R}_{CLP}$ the free entries marked by $*$ belong to $(0,1)$ upto normalisation. For achieving optimal payoff $\mathcal{P}^*_{\mathcal{R}_{CLP}(\mathcal{G}^{(2,3)})}$, all free elements are marked by $* = 0.5$.

IV. ONE-WAY ZERO-ERROR CLASSICAL AND QUANTUM CCR AND S-CCR

In the setup described in Section III B (Fig. 2), Alice and Bob have access to a noiseless one-way communication channel of limited capacity and arbitrary local sources of randomness (i.e. private coins) that are considered to be free resources. Another variation may have them, in addition, sharing some pre-shared correlations, i.e. public coin. First, we consider the scenario when no pre-shared randomness is allowed between Alice and Bob in subsections IV A-IV D. In the subsection IV A, we calculate the necessary and sufficient classical and quantum resource required to perfectly satisfy CCR for $\mathcal{R}_{CLP}(\mathcal{G}^{(n,\omega)})$ and show that there is no gap. Next we calculate the necessary and sufficient classical resource required to accomplish S-CCR for $\mathcal{R}_{CLP}(\mathcal{G}^{(n,\omega)})$ in subsection IV B, and we calculate the sufficient quantum resource required to accomplish S-CCR for $\mathcal{R}_{CLP}(\mathcal{G}^{(n,\omega)})$ when considering a class of orthogonality graphs in subsection IV C. We show that there is an unbounded separation between quantum and classical resources required to accomplish S-CCR for this class of graphs. In subsection IV D, we show that there still exists an advantage in using quantum communication resources compared to classical resources for an even larger class of graphs where the orthogonal range is less than the order of the graph, such as the Paley graphs, for which we explicitly show the advantage. Lastly, in subsection IV E, we consider the scenario when pre-shared resources (public coins) are allowed when the communication channel is bounded. First, we show that when considering some class of graphs, the assistance of shared randomness is necessary for bounded classical communication resources to accomplish S-CCR for $\mathcal{R}_{CLP}(\mathcal{G}^{(n,\omega)})$. The lower bound on this shared randomness increases linearly with the number of maximum cliques in the graph. Secondly, we also compare the resourcefulness of pre-shared entanglement with classical shared randomness when bounded *classical* communication is allowed between Alice and Bob to achieve non-zero payoff $\mathcal{P}_{\mathcal{R}_{CLP}(\mathcal{G})}$.

A. Classical and Quantum deterministic one-way Communication Complexity of $\mathcal{R}_{CLP}$

In this subsection, we calculate the necessary and sufficient 1) classical and 2) quantum communication required to perfectly satisfy CCR for $\mathcal{R}_{CLP}(\mathcal{G}^{(n,\omega)})$ without public coins for some graph $\mathcal{G}^{(n,\omega)}$ for the setup described in Sec III B. We will show that for CCR for the consistent labelling of pairwise cliques condition corresponding to the relation $\mathcal{R}_{CLP}(\mathcal{G}^{(n,\omega)})$ there is no advantage of quantum resource over its classical counterpart.

Alice can send an ω-level classical system using which Bob can choose a deterministic output b conditioned on his input C_y and Alice's message. We start by making an observation that both the classical and quantum one-way communication complexity for $\mathcal{R}_{CLP}(\mathcal{G}^{(n,\omega)})$ is bounded from below by the maximum clique size ω of the given graph. It follows from considering the scenario where both Alice and Bob are given the same clique $C_x = C_y$, Bob must know the input label of Alice (which has the size same as the maximum clique ω) to produce consistent labelling. Further, the quantum protocol can emulate any classical protocol through its coherent version. Therefore, the task reduces to showing that an ω-level communication is sufficient for the classical protocol for CCR.

Theorem 1. *Given a graph $\mathcal{G}^{(n,\omega)}$, the classical deterministic one way zero error communication complexity of $\mathcal{R}_{CLP}(\mathcal{G}^{(n,\omega)})$ is $\log_2 \omega$ bits.*

The essential idea of the proof is to show that there is a strategy or analogously a table of conditional probabilities $p(b|C_x, C_y, a)$ satisfying **(T0)** such that there are ω distinct rows. Thus, the aforementioned table of conditional probabilities can be compressed into another table with ω rows only. Alice upon communicating the row corresponding to her input enables Bob to output clique labelling consistently depending on his input clique. This strategy (table of conditional probability) involving the communication of $log_2(\omega)$ c-bits is necessarily of the following form. For every input of Alice (C_x, a) and Bob C_y, there is a deterministic b that Bob chooses to output. This specification is necessary in order for the probability table to satisfy **(T0)**.

Proof. For the complete proof see appendix B. ■

Thus, we observe no advantage in using quantum resources for communication over its classical analogue when considering CCR of $\mathcal{R}_{CLP}(\mathcal{G}^{(n,\omega)})$. Note that, for the graph $\mathcal{G}^{(n,\omega)}$, the faithful orthogonal range is $d_{\mathbb{C}} \geq \omega$ and therefore the classical strategy does not depend on the orthogonal representation of the graph. The orthogonal representation of the graph will be pertinent in the next two subsections (IV B, IV C), where we will consider the classical and quantum S-CCR to find an unbounded advantage.

B. Classical deterministic one-way Strong-Communication Complexity of $\mathcal{R}_{CLP}$

In this subsection, we calculate the amount of classical communication necessary and sufficient for accomplishing S-CCR of $\mathcal{R}_{CLP}$ when considering the class of orthogonality graphs $\mathcal{G}^{(n,\omega)}$ that satisfy the following conditions:

(G0): each vertex of the graph is part of at least one maximum clique of the graph,

(G1): $\forall v, v' \in \mathcal{V}$ belonging to two different maximum cliques $\exists\, u \in \mathcal{V}$ such that u is either adjacent to v or v' but not both.

Observation 1. *Given a graph $\mathcal{G}^{(n,\omega)}$ with maximum clique size ω, satisfying conditions (**G0**)-(**G1**), there exists induced subgraphs consisting of at least two maximum cliques of size ω, say C_i and C_j, such that there is at least one label of C_i for which there are at least two different consistent choices of labelling for the other clique C_j.*

Given an orthogonality graph $\mathcal{G}^{(n,\omega)}$ satisfying the properties listed above, we prove a tight lower bound for classical resources required for S-CCR of $\mathcal{R}_{CLP}(\mathcal{G}^{(n,\omega)})$. This bound is calculated for the *zero-error* scenario in which Bob never outputs an outcome b such that the tuple consisting of Alice's and Bob's input, (C_x, a) and C_y respectively, and Bob's output does not belong to the relation $\mathcal{R}_{CLP}(\mathcal{G}^{(n,\omega)})$, i.e., the case $(C_x, a, C_y, b) \notin \mathcal{R}_{CLP}(\mathcal{G}^{(n,\omega)})$ does not occur.

Lemma 1. *Given a graph $\mathcal{G}^{(n,\omega)}$ satisfying (**G0**)-(**G1**), it is necessary and sufficient to communicate a $|\mathcal{V}|$ level classical system, where $|\mathcal{V}|$ is the cardinality of the set of vertices in the graph, to perform S-CC$\mathcal{R}_{CLP}(\mathcal{G}^{(n,\omega)})$.*

Proof. Before the game begins, Alice and Bob construct the table M of conditional probabilities $p(b|C_x, C_y, a)$ which has $n\omega$ rows and $n\omega$ columns. We will refer to each of the rows and columns of this table as $(C_x, a)_{\mathsf{r}}$ and $(C_y, b)_{\mathsf{c}}$ respectively. Now upon receiving (C_x, a) if Alice communicates the relevant row to Bob, they can reconstruct the relation by Bob randomly reading off a valid non-zero entry from the given row by Alice and for his given clique. Therefore, we have a trivial upper bound on the dimension of the classical system which is required as $n\omega$. The deterministic classical strategy employed for Theorem 1 cannot reconstruct the graph since, the conditional probability table must contain nonzero entries corresponding to all events $(C_x, C_y, a) \in \mathcal{R}_{CLP}(\mathcal{G}^{(n,\omega)})$.

Therefore, we cannot use the strategy as used before. Nonetheless, observe that if two rows of the probability table can be made identical while satisfying the consistency condition, Alice and Bob can pre-assign them the same communication message. In the table, there is redundancy when the same vertex shows up in different cliques. For instance, let vertex v is in both maximum clique C_i and C_j. Then the rows in the conditional probability table corresponding to $(C_x = C_i, a)_\mathsf{r}$ and $(C_x = C_j, a')_\mathsf{r}$, where label a and a' for the clique C_i and C_j respectively colour the vertex v as 1, can be assigned the same entries. For such a vertex v, $(C_x = C_i, a, C_y = C_j, b = a'), (C_x = C_j, a', C_y = C_i, b = a), (C_x = C_j, a', C_y = C_j, b = a'), (C_x = C_i, a, C_y = C_i, b = a) \in \mathcal{R}_{CLP}(\mathcal{G}^{(n,\omega)})$. Also for any other $C_y (\neq C_i, C_j)$, $(C_x = C_i, a, C_y, b) \in \mathcal{R}_{CLP}(\mathcal{G}^{(n,\omega)}) \implies (C_x = C_j, a', C_y, b) \in \mathcal{R}_{CLP}(\mathcal{G}^{(n,\omega)})$. Thus, the entries in the table M corresponding to the rows $(C_x = C_i, a)_\mathsf{r}$ and $(C_x = C_j, a')_\mathsf{r}$ can be assigned identically (especially the non-zero entries) while guaranteeing perfect relationship reconstruction without violation of the consistency condition. The entries that are necessarily zero in one of the rows are also zero for the other row. Therefore, these two inputs $(C_x = C_i, a)_\mathsf{r}$ and $(C_x = C_j, a')_\mathsf{r}$ can be encoded in the same message.

Therefore, Alice and Bob can remove all redundant rows in this manner and end up with an encoding based on the compressed table which now has $|\mathcal{V}|$ distinct rows, where each row corresponds to a distinct vertex. Sufficiency of communicating $|\mathcal{V}|$-level classical system follows trivially since it allows Alice to send all information about her input. $|\mathcal{V}| \leq n\omega$ is saturated if all the maximum cliques are disconnected in the given graph.

Now we prove the necessity of $|\mathcal{V}|$-level classical system to achieve perfect S-CCR when considering $\mathcal{R}_{CLP}(\mathcal{G}^{(n,\omega)})$. For every vertex v in a clique C_i where $i \in \{1, \cdots, n\}$ there is an input corresponding to this vertex for Alice (C_i, a) where label a assigns colour 1 to v and rest of the vertices in the clique are assigned 0. This is due to condition **(G0)**. For any clique C_i, each of the Alice's input, (C_i, a) where $a \in \{0, \cdots, \omega - 1\}$, must be encoded with different message alphabet. This is because Bob needs to exactly guess the input clique label of Alice whenever his input is $C_y = C_i$. Now for any two vertices v, v' that belong to two different cliques, say $v \in C_i$ and $v' \in C_j$ (where $i \neq j$), there exists a clique C_k and a vertex $u \in C_k$ (where k maybe i, j or some other number) such that it is orthogonal exactly to one of these vertices (say v w.l.o.g.). This is due to condition **(G1)**. Let Alice's input corresponding to v and v' be $(C_x = C_i, a)$ and $(C_x = C_j, a')$ respectively and Bob has input $C_y = C_k$. For these rounds, $P(b|C_x = C_i, a, C_y = C_k) = 0$ and $P(b|C_x = C_j, a', C_y = C_k) > 0$ where Bob's output label b for clique $C_y = C_k$ assigns 1 exactly to vertex u. Thus, the inputs corresponding to every pair of vertices that do not belong to the same clique must be encoded with different message alphabets to obtain a non-zero payoff. Since there are $|\mathcal{V}|$ vertices, the classical message must be encoded in a system of dimension $\geq |\mathcal{V}|$.

We now argue that locally randomising over the deterministic encoding and decoding protocols (that is the usage of *Private Coins*) cannot lower the necessary classical communication, that is, using less than $\log_2 |\mathcal{V}|$ c-bits along with private coins cannot accomplish **(T0)**-**(T1)**. To see this, we will consider a convex combination of deterministic encoding of Alice for protocols with communication of a $(|\mathcal{V}| - 1)$-level classical system. Consider some maximum clique C_i. In any deterministic encoding, each of Alice's input (C_i, a) where $a \in \{0, \cdots, \omega - 1\}$, must be encoded with a different message alphabet. This is so because if Alice and Bob receive the same clique as input, then their labelling for the clique must match. Also, this deterministic encoding will encode some of the inputs corresponding to different vertices, say v and v', in the same message alphabet. There will be some inputs $(C_x = C_i, a)$ and $(C_x = C_{j(\neq i)}, a')$ encoded in the same message. Here labels a and a' assign binary colour 1 to v in C_i and $v' (\neq v)$ in C_j respectively while the rest of the vertices in these cliques are assigned 0. Individually, each of these encodings will be unsuccessful in relationship reconstruction. Furthermore, since, Alice and Bob do not have access to pre-shared randomness, therefore, Bob is not aware of Alice's choice of encoding in a given round. Thus, Bob cannot use decoding that is correlated with Alice's encoding strategy in a given round. If Bob tries to satisfy consistency conditions then he will not be able to have non-zero probability corresponding to all the events $(C_x, a, C_y, b) \in \mathcal{R}_{CLP}(\mathcal{G}^{(n,\omega)})$.

Thus, communication of $|\mathcal{V}|$-level classical system is necessary for reconstruction of the relation $\mathcal{R}_{CLP}(\mathcal{G}^{(n,\omega)})$. ■

As we show in this section, the amount of classical communication required for accomplishing zero error S-CCR, when considering $\mathcal{R}_{CLP}(\mathcal{G}^{(n,\omega)})$, scales linearly with the number of vertices in the graph. In the next section, we provide a sufficient amount of quantum communication that accomplishes S-CC$\mathcal{R}_{CLP}$ when no pre-shared randomness is allowed between Alice and Bob. We also show, that there exists an unbounded gap between quantum and classical resources when no public coin is pre-shared between the two players. This separation is observed for a sub-class of graphs considered in this section.

C. Unbounded quantum advantage in one-way Strong Communication Complexity of $\mathcal{R}_{CLP}$

In this subsection, we first calculate the amount of quantum communication sufficient for accomplishing S-CC$\mathcal{R}_{CLP}$ when considering the class of orthogonality graphs $\mathcal{G}^{(n,\omega)}$ that also satisfy **(G0)**-**(G1)**. Subsequently, we will show unbounded separation between classical and quantum communication resources for one way zero-error S-CC$\mathcal{R}_{CLP}(\mathcal{G}^{(n,\omega)})$ for some class of graphs.

Lemma 2. *Given a graph $\mathcal{G}^{(n,\omega)}$ satisfying* **(G0)**-**(G1)** *with faithful orthogonal range $d_{\mathbb{C}}$, it is sufficient to communicate a $d_{\mathbb{C}}$-level quantum system to perform S-CC$\mathcal{R}_{CLP}(\mathcal{G}^{(n,\omega)})$.*

Proof. The strategy to show that $d_{\mathbb{C}}$-level quantum system is sufficient to perform CCR and S-CCR when considering relation $\mathcal{R}_{CLP}(\mathcal{G}^{(n,\omega)})$, for a graph $\mathcal{G}^{(n,\omega)}$ which has orthogonal representation over the complex field in dimension $d_{\mathbb{C}}$ is straightforward, compared to the classical strategy. Alice and Bob are aware of the graph $\mathcal{G}^{(n,\omega)}$ and a faithful orthogonal representation of the graph in dimension $d_{\mathbb{C}}$ before the task. When Alice has access to her input (C_x, a), then she finds the vertex in the maximum clique C_x that is assigned value 1 by the input clique label a. Alice prepares a qudit in the state associated with the orthogonal representation of this vertex and sends the qudit to Bob. Bob then performs a measurement associated with his clique C_y. The projectors of the measurement correspond to the orthogonal representation of the vertices in the clique C_y. Based on the measurement outcome which corresponds to some vertex in the clique C_y Bob outputs as his label b that assigns this vertex binary colour 1. The quantum strategy guarantees consistent labelling of maximum cliques stated equivalently as $\mathcal{R}_{CLP}(\mathcal{G}^{(n,\omega)})$ due to the orthogonal representation. This concludes the proof. ∎

Now we are in a position to show that there are classes of graphs that give rise to quantum advantage. Lemma 1 and Lemma 2 lead us to the following theorem where we show the condition which guarantees quantum advantage in S-CC$\mathcal{R}_{CLP}$:

Theorem 2. *For any graph $\mathcal{G}^{(n,\omega)}$ satisfying* **(G0)**-**(G1)** *with faithful orthogonal range $d_{\mathbb{C}}$, there exists quantum advantage while performing S-CC$\mathcal{R}_{CLP}(\mathcal{G}^{(n,\omega)})$ successfully whenever $d_{\mathbb{C}} < |\mathcal{V}|$.*

Proof. The proof of this theorem follows directly from comparing Lemma 1 and Lemma 2. ∎

The problem of finding the smallest dimension in which a given graph $\mathcal{G}^{(n,\omega)}$ has a (faithful) orthogonal representation is known to be quite difficult [23, 25]. Since the existence of quantum advantage relies on the faithful orthogonal range being smaller than the order of the graph from the above theorem, it follows that the problem of defining the set of graphs that entail quantum advantage is at least as complex as providing a non-trivial upper bound to the faithful orthogonal range for any arbitrary graph. Despite this difficulty, one can identify some families of graphs that are useful for demonstrating an unbounded separation between classical and quantum communication for Strong Communication Complexity of $\mathcal{R}_{CLP}(\mathcal{G}^{(n,\omega)})$, which we shall identify. Let us consider graphs $\mathcal{G}^{(n,\omega)}$ that satisfy the following condition along with **(G0) - (G1)**:

(G2): At least $(V-k)$ vertices are required to be removed from the graph, where $k \in \mathbb{N}$ and $\omega \leq k < |\mathcal{V}|$, such that the complementary graph is fully disconnected.

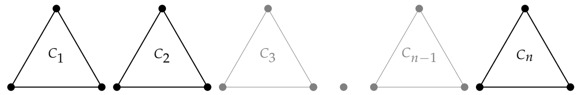
The following are the example of such graphs $\mathcal{G}^{(n,\omega)}$ satisfying the conditions **(G0) - (G2)**:

(1) **Disconnected graphs $\mathcal{G}^{(n,\omega)}_{disc.}$:** Graphs with n cliques of maximum cliques size ω all of which are disconnected from one another. Thus we have $|\mathcal{V}| = n\omega$. See Fig. 5 (a) for an example.

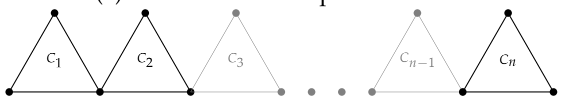
(2) **Nearest Neighbour Connected Cliques $\mathcal{G}^{(n,\omega)}_{NNCC(r)}$:** Graph with a chain of n cliques of maximum cliques size ω such that only clique C_i and C_{i+1} share r $(1 \leq r < \frac{\omega}{2})$ vertices where $i \in \{1, 2, \cdots, n-1\}$. The rest of the cliques do not share any additional vertices and edges. See Fig. 5 (b) for an example.

(3) **Paley graphs $\mathcal{G}_{Paley(q)}$:** The class of Paley graphs. (See subsection IV D)

Having shown that there is a class of graphs showing a quantum advantage in communication while considering the task of S-CC$\mathcal{R}_{CLP}$, we now address the question regarding the extent to which the separation between these resources can be extended.



(a) Disconnected Graphs for $\omega = 3$



(b) Graphs with Nearest Neighbour Connected Cliques for $\omega = 3$ and $r = 1$

Figure 5: An example for $\mathcal{G}^{(n,\omega=3)}$ of disconnected graphs (top (a)) and Graphs with nearest neighbour connected cliques (bottom (b))

Theorem 3. *For the class of graphs $\mathcal{G}^{(n,\omega)}$ satisfying conditions (**G0**)-(**G2**) with faithful orthogonal range ω, the separation between one-way classical and quantum communication, required for zero-error reconstruction of the given S-CC$\mathcal{R}_{CLP}$ induced by these graphs, is unbounded.*

Proof. Consider the class of graphs $\mathcal{G}^{(n,\omega)}$ satisfying conditions (**G0**)-(**G2**) with $k = \omega$. For instance, a graph $\mathcal{G}^{(n,\omega)}$ with a chain of n cliques of maximum cliques size ω such that only clique C_i and C_{i+1} share r ($0 \leq r < \frac{\omega}{2}$) vertices where $i \in \{1, 2, \cdots, n-1\}$. The rest of the cliques do not share any additional vertices or edges than defined above. Thus the number of vertices of the graph is $|\mathcal{V}| = n(\omega - r) + r$.

Lemma 1 tells us that a classical protocol that achieves zero-error S-CCR_{CLP} of this graph must communicate $\lceil \log_2\{n(\omega - r) + r\} \rceil$ bits. On the other hand, Lemma 2 implies that protocols using quantum resources can achieve the same by communicating $\lceil \log_2 \omega \rceil$ qubits, provided the graph $\mathcal{G}^{(n,\omega)}$ has a faithful orthogonal range $d_{\mathbb{C}} = \omega$.

According to Lovász's theorem (see Section II, Proposition 1) [23] a faithful orthogonal representation of the graph $\mathcal{G}^{(n,\omega)}$ exists in dimension $d_{\mathbb{R}} = \omega$, since it is necessary to remove at least $(n\omega - \omega)$ vertices from the complementary graph $\bar{\mathcal{G}}^{(n,\omega)}$ to make it completely disconnected. It also follows from Eq. (1) that for the graph $\mathcal{G}^{(n,\omega)}$, the faithful orthogonal range over complex field $d_{\mathbb{C}} = \omega$. As one can obtain such a faithful orthogonal representation of the graph $\mathcal{G}^{(n,\omega)}$ in dimension $d_{\mathbb{C}} = \omega$ and therefore the separation between classical ($\lceil \log_2\{n(\omega - r) + r\} \rceil$ bits) and quantum ($\lceil \log_2 \omega \rceil$ qubits) communication can be made unbounded by considering large n. ■

Given any graph $\mathcal{G}^{(n,\omega)}$, having an orthogonal range $d_{\mathbb{C}} = \omega$ and satisfying conditions (**G0**)-(**G1**), the maximum payoff $\mathcal{P}_{\mathcal{R}_{CLP}(\mathcal{G}^{(n,\omega)})}$ achievable by a direct quantum communication resource of operational dimension ω is connected to the optimal faithful orthogonal representation of the graph with orthogonal range $d_{\mathbb{C}}$. To see this, let us suppose that the given graph $\mathcal{G}^{(n,\omega)}$ does have a faithful orthogonal representation in dimension $d_{\mathbb{C}} = \omega$, then the maximum payoff for the quantum strategy is given by the maximisation of the minimum overlap of the vectors corresponding to any two disconnected vertices of the graph (following the same protocol as in Lemma 2). So, keeping in mind the correspondence between quantum strategy and faithful orthogonal representation of the graph $\mathcal{G}$ (satisfying (**G0**)-(**G1**) and having faithful orthogonal range $d_{\mathbb{C}} = \omega$), one can rephrase the payoff (Eqn 6) with communication of $d = \omega$-dimensional quantum system, as an optimisation over the faithful orthogonal representations of the graph $\mathcal{G}$ with orthogonal range ω on the complex field, *i.e.*

$$\mathcal{P}^{\mathbb{C}^{\omega}_{max}}_{\mathcal{R}_{CLP}(\mathcal{G})} = \min_{(C_x, a, C_y, b) \in \mathcal{R}_{CLP}(\mathcal{G})} P(b|C_x, C_y, a) \tag{14}$$

$$= \max_{FOR(\mathbb{C}^{\omega})} \left\{ \min_{(C_x, a, C_y, b) \in \mathcal{R}_{CLP}(\mathcal{G})} Tr[\Pi_a^{C_x} \Pi_b^{C_y}] \right\} \tag{15}$$

$$= \max_{FOR(\mathbb{C}^{\omega})} \min_{(i,j) \notin \mathcal{E}} |\langle v(i), v(j) \rangle|^2 \tag{16}$$

where, $FOR(\mathbb{C}^{\omega})$ denotes the set of all faithful orthogonal representations with range ω over complex field. This relation connects a property of the graph $\mathcal{G}$ (on the right) to an operational quantity (on the left).

D. Quantum advantage in one-way Strong-Communication Complexity of $\mathcal{R}_{CLP}$ for other graphs

In this section, we will consider a particular class of orthogonality graphs $(\mathcal{G}^{(n,\omega)}, \mathcal{V}, \mathcal{E})$ called Paley graphs. This class of graphs have been well studied in graph theory [26] and have found applications in quantum information [27, 28]. They satisfy the properties **(G0)-(G1)** (see observation 2). For the graphs $\mathcal{G}^{(n,\omega)}$ considered in this section, the faithful orthogonal range $d_{\mathbb{C}} < |\mathcal{V}|$. Note that we already know that for graphs satisfying (**G0**)-(**G1**), the classical strong communication complexity is the order of the graph, *i.e.* $\log_2 |\mathcal{V}|$ c-bit (Lemma 1). Thus, graphs having orthogonal range strictly less than the order of the graph entail an advantage of using quantum communication (following the same protocol described in the proof of Lemma 2) over classical communication when considering the one-way strong communication complexity of relation $\mathcal{R}_{CLP}$. For the class of well-known Paley graphs, we will show that it has a faithful orthogonal representation in a dimension slightly more than half of the order of the graph (see Theorem 4).

1. Paley graphs

Paley graphs $\mathcal{G}_{Paley(q)}$ are simple undirected graphs whose vertices denote the elements of a finite field $\mathbb{F}_q$ (of order prime power $q = 4k+1$ for positive integer k), and whose edges denote that the corresponding elements differ by a quadratic residue. Paley Graphs have the interesting property that they are vertex-transitive, self-complementary graphs which means that by Lovász's original result, the value of $\theta(\mathcal{G}_{Paley(q)})$ can be computed exactly to be $\theta(\mathcal{G}_{Paley(q)}) = |V(\mathcal{G}_{Paley(q)})|^{1/2} = \sqrt{q}$. Some simple Paley graphs are shown in figure 6. Next, we will show that the class of Paley graphs satisfy the condition **(G1)**.

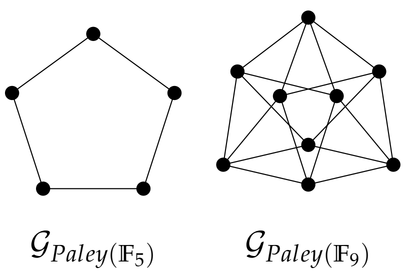


Figure 6: Example of the 5-Paley graph $\mathcal{G}_{Paley(\mathbb{F}_5)}$ (left) and the 9-Paley graph $\mathcal{G}_{Paley(\mathbb{F}_9)}$ (right)

Observation 2. *In the class of Paley graphs, any two vertices in the graph have the same degree, i.e. in a graph with q vertices, each vertex has $\frac{q-1}{2}$ neighbours. Every two adjacent vertices have $\frac{q-5}{4}$ common neighbours and every two non-adjacent vertices have $\frac{q-1}{4}$ common neighbours [26]. Thus, for every pair of different vertices v, v' there exists a third vertex u that is adjacent to exactly one of the vertex v or v'. This implies that condition (**G1**) is satisfied by Paley graphs.*

2. Quantum advantage in S-CCR for Paley graphs

We will show that there exists a *FOR* for Paley graphs with q vertices in $\frac{q+1}{2}$ dimension. Further, we show that the quantum protocol achieves the maximum payoff $\frac{2}{\sqrt{q}+1}$ when following the protocol mentioned in Lemma 2.

We note that $\theta(\mathcal{G}_{Paley(q)})$ can be computed using the semi-definite programming formulation given as

$$\theta(\mathcal{G}_{Paley(q)}) = \max_{M=(M_{i,j})_{i,j=1}^{q}} \sum_{i,j=1}^{q} M_{i,j}$$

$$\text{s.t. } M \succeq 0, \sum_i M_{i,i} = 1. \tag{17}$$

Let $\Gamma_{Paley(q)}$ denote the automorphism group of $\mathcal{G}_{Paley(q)}$, i.e., the set of all permutations σ that preserve the adjacency structure of the graph. Suppose M is an optimal solution point for the optimisation in (17), then $M^* = \frac{1}{|\Gamma_{Paley(q)}|} \sum_{\sigma \in \Gamma_{Paley(q)}} \sigma^T M \sigma$ also satisfies the constraints of positive semi-definiteness, trace one and the sum over entries being equal to $\theta(\mathcal{G}_{Paley(q)})$. Since $\mathcal{G}_{Paley(q)}$ is vertex-transitive, the sum over permutations in $\Gamma_{Paley(q)}$ goes over transpositions between every pair of vertices so that $M^*_{i,i} = 1/q$ for all $i \in [q]$. M^* is the Gram Matrix of a set of vectors (each of norm $1/\sqrt{q}$) forming an orthogonal representation of $\mathcal{G}_{Paley(q)}$. Let us denote by $S_{opt} = \{|u_1\rangle, \ldots, |u_q\rangle\}$ the corresponding set of normalised vectors forming the optimal solution to the Lovász-theta optimisation, and by $M_{opt} = qM^*$ the corresponding Gram Matrix. We see that

$$\theta(\mathcal{G}_{Paley(q)}) = \sum_{i,j=1}^{q} \frac{1}{q} \langle u_i | u_j \rangle. \tag{18}$$

In other words, we have $\sum_{i,j=1}^{q} \langle u_i | u_j \rangle = q^{3/2}$. By symmetry and the fact that every vertex in $\mathcal{G}_{Paley(q)}$ has degree $(q-1)/2$ it also follows that $\langle u_i | u_j \rangle = (q^{3/2} - q)/(q(q-1)/2) = 2/(q^{1/2}+1)$ for $i \nsim j$.

Let us now compute the dimensionality of the vectors $|u_i\rangle$ in S_{opt} that form the optimal representation giving rise to $\theta(\mathcal{G}_{Paley(q)})$. This quantity is the dimension of the vectors giving rise to the faithful representation S_{opt} that is traditionally denoted as $\xi^*(\mathcal{G}_{Paley(q)})$.

Theorem 4. *The dimension of the optimal representation of $\mathcal{G}_{Paley(q)}$ that gives rise to $\theta(\mathcal{G}_{Paley(q)})$ is $(q+1)/2$.*

Proof. We are looking to compute the dimension of the faithful representation that gives the optimal solution to the Lovász-theta optimisation of $\mathcal{G}_{Paley(q)}$, i.e., we want to find the minimum $\xi^*(\mathcal{G}_{Paley(q)})$ such that $|u_i\rangle \in \mathbb{R}^{\xi^*(\mathcal{G}_{Paley(q)})}$ for the vectors $|u_i\rangle \in S_{opt}$. This quantity is given by the rank of the Gram Matrix M_{opt} of the set of (normalised) vectors S_{opt}. We have that

$$(M_{opt})_{k,l} = \begin{cases} 1 & k = l \\ 0 & k \sim l \\ 2/(q^{1/2}+1) & (k \nsim l) \wedge (k \neq l). \end{cases}$$

In other words, $M_{opt} = I + \frac{2}{q^{1/2}+1} A(\overline{\mathcal{G}}_{Paley(q)})$, where $A(\overline{\mathcal{G}}_{Paley(q)})$ denotes the adjacency matrix of the complement of $\mathcal{G}_{Paley(q)}$ (which is isomorphic to $\mathcal{G}_{Paley(q)}$ since the graph is self-complementary).

To compute $rank(M_{opt})$, we calculate its spectrum and show that it has exactly $(q+1)/2$ non-zero eigenvalues, so that $rank(M_{opt}) = (q+1)/2$.

To do this, we compute the spectrum of $A(\overline{\mathcal{G}}_{Paley(q)}) = A(\mathcal{G}_{Paley(q)})$. Following [29], let us define a matrix K based on the quadratic characters $\chi(k-l)$

$$\chi(k-l) = \begin{cases} 1 & (k-l) \text{ is quadratic residue modulo } q \\ 0 & k=l \\ -1 & \text{else.} \end{cases} \tag{19}$$

by $K_{k,l} = \chi(k-l)$. By the property of the characters that $\chi(xy) = \chi(x)\chi(y)$ and $\sum_{x=0}^{q-1}\chi(x) = 0$, we have the following result.

Lemma 3. *$K^2 = qI - J$, where J denotes the all-ones matrix.*

Proof. We want to prove that the diagonal entries of K^2 are equal to $(q-1)$ and the off-diagonal entries are equal to -1. The diagonal entries are given by the squared norms of the columns of K, which have one zero entry, $(q-1)/2$ entries of value 1 (corresponding to the quadratic residues modulo q and the degree of each vertex in $\mathcal{G}_{Paley(q)}$) and $(q-1)/2$ entries of value -1. Therefore, the squared norms of the columns and hence the diagonal entries of K^2 are equal to $q-1$.

The off-diagonal entries $(K^2)_{k,l}$ are given by $\sum_{j=0}^{q-1}\chi(k-j)\chi(l-j) = \sum_{j'=0}^{q-1}\chi(j')\chi((l-k)+j')$. Since $\chi(0) = 0$, the term for $j' = 0$ vanishes and we have $\sum_{j'=1}^{q-1}\chi(j')\chi((l-k)+j')$. Since $\chi(j') \in \{\pm 1\}$ for $j' \neq 0$, the sum reduces to $\sum_{j'=1}^{q-1}\chi((l-k)+j')/\chi(j') = \sum_{j'=1}^{q-1}\chi((l-k)/j'+1)$ where we used the property of the characters that $\chi(xy) = \chi(x)\chi(y)$. We finally obtain $\sum_{j'=1}^{q-1}\chi((l-k)/j'+1) = \left[\sum_{j''=0}^{q-1}\chi(j'')\right] - \chi(1) = 0 - 1 = -1$ where we used the property that as j' ranges over $[q-1]$, the argument $(l-k)/j'+1$ ranges over elements $\{0,\dots,q-1\}\setminus\{1\}$. Therefore, we obtain the off-diagonal entries to be -1 thus showing that $K^2 = qI - J$. ∎

We also see by direct term-by-term comparison that the adjacency matrix of the Paley graph can be written as

$$A(\mathcal{G}_{Paley(q)}) = \frac{1}{2}(K + J - I). \tag{20}$$

We, therefore, obtain that

$$\left(A(\mathcal{G}_{Paley(q)})\right)^2 = \frac{q-1}{4}(J+I) - A(\mathcal{G}_{Paley(q)}). \tag{21}$$

Now observe that the all-ones vector $|j\rangle$ is an eigenvector of $A(\mathcal{G}_{Paley(q)})$ and consider another eigenvector $|e_\lambda\rangle$ corresponding to eigenvalue $\lambda \neq 0$. Since $|e_\lambda\rangle$ is orthogonal to $|j\rangle$, we have that $J|e_\lambda\rangle = 0$, so that

$$\left(A(\mathcal{G}_{Paley(q)})\right)^2 |e_\lambda\rangle = \lambda^2 |e_\lambda\rangle = \left(\frac{q-1}{4} - \lambda\right)|e_\lambda\rangle, \tag{22}$$

or in other words that

$$\lambda^2 + \lambda - \frac{q-1}{4} = 0, \quad \implies \lambda = \frac{1}{2}\left(-1 \pm q^{1/2}\right). \tag{23}$$

Thus, the spectrum and corresponding degeneracies of $A\left(\overline{\mathcal{G}}_{Paley(q)}\right)$ are found to be

$$\operatorname{spec}\left(A\left(\overline{\mathcal{G}}_{Paley(q)}\right)\right) = \begin{cases} (q-1)/2 & 1 \\ \frac{1}{2}\left(-1+q^{1/2}\right) & (q-1)/2 \\ \frac{1}{2}\left(-1-q^{1/2}\right) & (q-1)/2. \end{cases}$$

As we have seen, the Gram Matrix M_{opt} from the optimal representation giving rise to $\theta(\mathcal{G}_{Paley(q)})$ is given by

$$M_{opt} = I + \frac{2}{q^{1/2}+1} A\left(\overline{\mathcal{G}}_{Paley(q)}\right). \tag{24}$$

Therefore, the spectrum of M_{opt} consists of exactly $(q+1)/2$ non-zero eigenvalues given by

$$\operatorname{spec}(M_{opt}) = \begin{cases} \sqrt{q} & 1 \\ 2\sqrt{q}/(1+\sqrt{q}) & (q-1)/2 \\ 0 & (q-1)/2. \end{cases}$$

Therefore, we obtain that $rank(M_{opt}) = \xi^*(\mathcal{G}_{Paley(q)}) = (q+1)/2$.

We can even go further and note that since the adjacency matrix of the Paley graph is a circulant matrix (the k-th row is a cyclic permutation of the 1-st row with offset k), the *eigenvectors* of the adjacency matrix $A(\overline{\mathcal{G}}_{Paley(q)})$ (and therefore the Gram Matrix M_{opt}) are the Fourier vectors

$$|e_\lambda\rangle = \frac{1}{q}\left(1, \omega^\lambda, \omega^{2\lambda}, \dots, \omega^{(q-1)\lambda}\right), \tag{25}$$

with $\lambda = 0, 1, \dots, q-1$, where $\omega = \exp\left(\frac{2\pi i}{q}\right)$ is a primitive q-th root of unity. Note that $|e_0\rangle = |j\rangle$ is the all-ones vector. We can then explicitly calculate that

$$M_{opt}|e_\lambda\rangle = \left[\frac{\sqrt{q}-1}{\sqrt{q}+1} + \frac{1}{\sqrt{q}+1}\sum_{\substack{l:l\neq 1 \\ (1-l) \text{ is a quad. res. mod } q}} \omega^{(l-1)\lambda} - \frac{1}{\sqrt{q}+1}\sum_{\substack{l:l\neq 1 \\ (1-l) \text{ is not a quad. res. mod } q}} \omega^{(l-1)\lambda}\right]|e_\lambda\rangle \tag{26}$$

We can then explicitly compute for prime q not only the eigenvalues of M_{opt} as above but also see that the eigenvalue $\sqrt{q}$ corresponds to the eigenvector $|j\rangle = |e_0\rangle$, the eigenvalues $2\sqrt{q}/(1+\sqrt{q})$ correspond to the eigenvectors $|e_\lambda\rangle$ for λ being the remaining quadratic residues modulo q, and the zero eigenvalues correspond to the eigenvectors $|e_\lambda\rangle$ for λ being the quadratic non-residues modulo q.

■

From equation 6, when using the protocol mentioned in Lemma 2, the payoff function defined in for a graph $\mathcal{G}$ assumes the form shown below:

$$\mathcal{P}^{\mathrm{C}^d}_{\mathcal{R}_{CLP}(\mathcal{G})} = \max_{FOR_d(\mathcal{G})} \min_{(i,j)\notin E(\mathcal{G})} |\langle v_i|v_j\rangle|^2, \quad (27)$$

where $FOR_d(\mathcal{G})$ denotes the set of faithful orthogonal representations in dimension d of $\mathcal{G}$. Let us compute this function for the class of Paley graphs. Firstly, we consider

$$\mathcal{P}^{\mathrm{C}^d}_{\mathcal{R}_{CLP}(\mathcal{G}_{Paley(q)})} \leq \max_{FOR(\mathcal{G}_{Paley(q)})} \min_{(i,j)\notin E(\mathcal{G}_{Paley(q)})} |\langle v(i)|v(j)\rangle|^2, \quad (28)$$

where $FOR(\mathcal{G}_{Paley(q)})$ denotes the set of faithful orthogonal representations of $\mathcal{G}_{Paley(q)}$ in any dimension.

For $(k,l) \notin E(\mathcal{G}_{Paley(q)})$, let $S^{(k,l)}$ denote a point in $FOR(\mathcal{G}_{Paley(q)})$ that achieves the maximum for the optimisation problem in (28) with the minimum being realised at $(k,l) \notin E(\mathcal{G})$. That is, $S^{(k,l)} = \{|v_1^{(k,l)}\rangle, \dots, v_q^{(k,l)}\rangle\}$ with $\langle v_i^{(k,l)}|v_j^{(k,l)}\rangle = 0$ for $(i,j) \in E(\mathcal{G}_{Paley(q)})$ and $|\langle v_k^{(k,l)}|v_l^{(k,l)}\rangle|^2 \leq |\langle v_{k'}^{(k,l)}|v_{l'}^{(k,l)}\rangle|^2$ for any $(k',l') \in E(\overline{\mathcal{G}}_{Paley(q)})$, $(k',l') \neq (k,l)$. We claim that $S^{(k,l)} = S_{opt}$, that is the set of vectors realising the optimal value in the Lovász-theta optimisation. To this end, we claim that

$$|\langle v_k^{(k,l)}|v_l^{(k,l)}\rangle| \leq \frac{2}{\sqrt{q}+1}. \quad (29)$$

For suppose that $|\langle v_k^{(k,l)}|v_l^{(k,l)}\rangle| > \frac{2}{\sqrt{q}+1}$. Then consider the Gram Matrix $M^{(k,l)}$ formed by the set of normalised vectors in $S^{(k,l)}$. We see that $(1/q)M^{(k,l)}$ also satisfies the constraints of positive semi-definiteness and trace one for the Lovász-theta optimisation in Eq.(17). But if the minimum non-zero off-diagonal entry of $(1/q)M^{(k,l)}$ is larger than the minimum non-zero off-diagonal entry of the optimal matrix M^* (with both matrices having diagonal entries all equal to $(1/q)$) then we obtain that $\sum_{i,j=1}^{q}(1/q)\left(M^{(k,l)}\right)_{i,j} > \sum_{i,j=1}^{q}(M^*)_{i,j} = \theta(\mathcal{G}_{Paley(q)})$ which is a contradiction. Therefore, we must have that the quantum maximum value of the payoff function is at most

$$|\langle v_k^{(k,l)}|v_l^{(k,l)}\rangle|^2 = \left(\frac{2}{\sqrt{q}+1}\right)^2, \quad (30)$$

with the maximum achieved by the set of vectors S_{opt} in $\mathbb{R}^{(q+1)/2}$ that also incidentally achieve the optimum value of Lovász-theta for the graph $\mathcal{G}_{Paley(q)}$.

E. Public coins

In the previous subsections, we considered strong communication complexity of relation when public coins or pre-shared randomness between Alice and Bob was not allowed. Here, we consider that the parties have access to pre-shared correlations along with one-way direct communication resources. In public coin-assisted communication complexity problems, usually, the amount of communication necessary and/or sufficient is studied. In these problems, an unbounded amount of public coin is allowed to be shared between the players. However, here we allow for restricted direct communication, either quantum or classical, and compare the amount of shared randomness required to accomplish S-CCR when considering relation $\mathcal{R}_{CLP}(\mathcal{G}^{(n,\omega)})$. We find that there exist graphs for which non-zero payoff while using restricted classical communication implies the presence of shared correlation.

For a class of graphs satisfying **(G0)-(G2)** and having faithful orthogonal representation in dimension ω, we provide a lower bound on the amount of public coin/ shared correlation required for accomplishing the task of S-CC$\mathcal{R}_{CLP}$ when communicating $\log_2 \omega$ c-bit. We show that this lower bound grows as $\log_2 n$ with the number of maximum cliques n. Later on, we also show the lower bound on the amount of public coin which is necessary to achieve optimal payoff $\mathcal{P}^*_{\mathcal{R}_{CLP}(\mathcal{G}^{(n,\omega)})}$ for S-CC$\mathcal{R}_{CLP}$ is connected to the existence of Orthogonal Arrays (OA). On another note, we also show that there are graphs for which both quantum and classical communication using a ω-dimensional system require the assistance of public coins in order to achieve optimal payoff for the S-CC$\mathcal{R}_{CLP}$ task. In the end, we also compare the amount of quantum shared correlation with classical shared randomness that is required when only a restricted amount of one-way classical communication is allowed in order to perform relation reconstruction for some specific graphs. In these cases, we show there is an unbounded gap between quantum and classical shared randomness.

1. *Classical communication assisted by shared randomness*

In Theorem 1, we showed that a ω-level classical message is necessary and sufficient for accomplishing CC$\mathcal{R}_{CLP}$, *i.e.* satisfying (**T0**), while in Lemma 1 we showed that a $|\mathcal{V}|$-level classical message is necessary and sufficient for S-CC$\mathcal{R}_{CLP}$, *i.e.* simultaneously satisfying (**T0**)-(**T1**), when considering graph $\mathcal{G}^{(n,\omega)}$ that satisfies **(G0)-(G1)**. Here we consider the class of graphs $\mathcal{G}^{(n,\omega)}$ which satisfies the constraint (**G0**)-(**G2**) and has faithful orthogonal representation in minimum dimension ω. We first show that if we restrict classical communication to an ω-level classical message and allow shared randomness then one can satisfy (**T0**)-(**T1**) and achieve optimal payoff $\mathcal{P}^*_{\mathcal{R}_{CLP}(\mathcal{G}^{(n,\omega)})}$ for S-CC$\mathcal{R}_{CLP}(\mathcal{G}^{(n,\omega)})$ (See Obs. 3). We then ask what would be the lower bound on the shared randomness to satisfy (**T0**)-(**T1**) and achieve the optimal payoff for S-CC$\mathcal{R}_{CLP}(\mathcal{G}^{(n,\omega)})$ (**T2**).

Observation 3. *Given a graph $\mathcal{G}^{(n,\omega)}$, the strategy with only $\log_2 \omega$ c-bit classical communication for satisfying (**T0**) is based on Alice and Bob finding a suitable deterministic strategy, i.e table M, before the beginning of the game expressed through a $\omega \times n\omega$ table compressed from the $n\omega \times n\omega$ table of conditional probabilities $p(b|C_x, C_y, a)$. In the shared randomness scenario, Alice and Bob prepare all such deterministic strategies (or tables) each of which satisfy consistent labelling of cliques (**T0**) before the game begins and they index these tables. They use shared randomness to choose which table to use for a particular run of the game. Over multiple runs, they can satisfy (**T1**). Trivially, they could use shared randomness of the order of the total number of such deterministic strategies where each satisfies consistent labelling of the cliques (**T0**).*

For example consider the graph shown in Fig. 4 or the left graph of Fig. 8, we saw that one classical deterministic strategy was represented through Table VII in Appendix B. Similarly, Alice and Bob could prepare another Table II

		C_1			C_2		
		$b=0$	$b=1$	$b=2$	$b=0$	$b=1$	$b=2$
C_1	$a=0$	1	0	0	0	1	0
	$a=1$	0	1	0	0	0	1
	$a=2$	0	0	1	1	0	0
C_2	$a=0$	0	0	1	1	0	0
	$a=1$	1	0	0	0	1	0
	$a=2$	0	1	0	0	0	1

Table II: Another classical deterministic strategy for Graph in Fig. 4

If Alice and Bob use a bit of unbiased shared randomness to choose between Table VII and Table II, they effectively are using the strategy given in Table III which satisfies (**T0**) as well as (**T1**) and obtain optimal payoff since they fill all the entries $*$ with 0.5 since the optimal payoff for this graph is $\mathcal{P}^*_{\mathcal{R}_{CLP}(\mathcal{G}^{(2,3)})} = 0.5$. Now, we provide a lower bound on the amount of shared correlation required by Alice and Bob, when they are allowed to communicate $\log_2 \omega$ c-bit, in order to accomplish S-CC$\mathcal{R}_{CLP}(\mathcal{G}^{(n,\omega)})$.

		C_1			C_2		
		$b=0$	$b=1$	$b=2$	$b=0$	$b=1$	$b=2$
C_1	$a=0$	1	0	0	0	0.5	0.5
	$a=1$	0	1	0	0	0.5	0.5
	$a=2$	0	0	1	1	0	0
C_2	$a=0$	0	0	1	1	0	0
	$a=1$	0.5	0.5	0	0	1	0
	$a=2$	0.5	0.5	0	0	0	1

Table III: Effective classical strategy with shared randomness for Graph in Fig. 4

Theorem 5. *Given a graph $\mathcal{G}^{(n,\omega)}$ satisfying conditions (**G0**)-(**G2**) with faithful orthogonal range ω, for a protocol using communication of ω-level classical system, the lower bound on the amount of shared randomness required to perfectly accomplish S-CC$\mathcal{R}_{CLP}(\mathcal{G}^{(n,\omega)})$ is equal to the minimum amount of shared randomness required for the same task when considering another graph $\mathcal{G}^{(n,\omega=2)}$ with n disconnected maximum cliques.*

Proof. The amount of shared randomness, while communicating ω-level classical system, depends on the graph and can be upper bounded by the total number of different classical deterministic encoding and decoding strategies (or the total number of different tables of conditional probabilities that Alice and Bob can prepare while satisfying the constraints mentioned in Appendix B). We observe that for different graphs $\mathcal{G}$ with the same number of maximum cliques n of clique size ω, the graph in which all maximum cliques are disconnected requires the most amount of shared randomness. On the other hand, graphs in which every clique shares the most number of its vertices with other cliques, require the least amount of shared randomness due to the least number of $*$ entries in their conditional probability table (for example in Table IV). We also know that the most number of vertices that any two cliques can share is $\omega - 2$ to have an orthogonal representation in $\mathbb{C}^{\omega}$ (Proposition 1). An example is provided in Fig. 7 for $\omega = 5$ and $n = 2$.

To find the lower bound on shared randomness for a graph with n cliques with maximum clique size ω, we can calculate the shared randomness necessary for a graph where all the maximum size cliques share $\omega - 2$ vertices. Such a graph will saturate the lower bound. Such a graph has the order $|\mathcal{V}| = \omega + 2(n-1)$.

We also observe that for such a graph the associated conditional probability $n\omega \times n\omega$ table with entries $p(b|C_x, C_y, a)$, the number and structure of the free

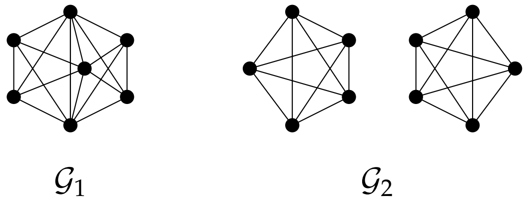


Figure 7: Two Graphs with ($\omega = 5, n = 2$), the graph on the left $\mathcal{G}_1$ has two cliques of size $\omega = 5$ and $\omega - 2 = 3$ vertices common between these cliques. The graph on the right $\mathcal{G}_2$ consists of two disconnected cliques of size $\omega = 5$.

entries $*$ is equivalent to that of a graph with n disconnected cliques of size $\omega = 2$. Thus the number of classical deterministic strategies and therefore shared randomness required for these two graphs are the same. For example, in the case of the graph shown in Fig. 4 (or left side of Fig. Fig. 8) we see that Table IV is the conditional probability table which is also equivalent (in terms of $*$) to the conditional probability table for the graph on the right $\mathcal{G}^{(n=2,\omega=2)}$ in Fig. 8. Therefore we

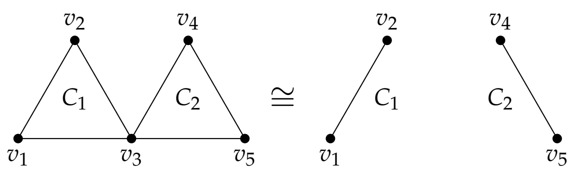


Figure 8: Calculating the lower bound of shared randomness for a graph with ω sized n cliques with $\omega - 2$ common vertices is equivalent (in terms of the number of classical deterministic strategies) to a graph with $\omega = 2$ sized n disconnected cliques, as shown in this example continuing the simple example provided before in Fig 4.

have shown that we can calculate the lower bound for shared randomness required for a graph with ω-sized n maximum cliques by calculating the shared randomness required for a graph with $\omega = 2$-sized n maximum cliques that are disconnected.

■

We now provide the explicit lower bounds on shared randomness required for the task S-CC$\mathcal{R}_{CLP}$ as a function of the number of maximum cliques in the graph.

Corollary 1. *Given a graph $\mathcal{G}^{(n,\omega)}$ satisfying* **(G0)**-**(G2)** *with faithful orthogonal range ω, it is necessary (but may not be sufficient) to share randomness with n-inputs (i.e. $\frac{1}{n}\sum_{i=1}^{n}(|ii\rangle\langle ii|)$) while communicating an ω-level classical system in order to accomplish S-CC$\mathcal{R}_{CLP}(\mathcal{G}^{(n,\omega)})$.*

Proof. By Theorem 5, to find the lower bound of shared randomness required for a graph $\mathcal{G}^{(n,\omega)}$ to accomplish

		C_1			C_2		
		$b=0$	$b=1$	$b=2$	$b=0$	$b=1$	$b=2$
C_1	$a=0$	1	0	0	0	$*$	$*$
	$a=1$	0	1	0	0	$*$	$*$
	$a=2$	0	0	1	1	0	0
C_2	$a=0$	0	0	1	1	o	0
	$a=1$	$*$	$*$	0	0	1	0
	$a=2$	$*$	$*$	0	0	0	1

$\cong$

		C_1		C_2	
		$b=0$	$b=1$	$b=0$	$b=1$
C_1	$a=0$	1	0	$*$	$*$
	$a=1$	0	1	$*$	$*$
C_2	$a=0$	$*$	$*$	1	0
	$a=1$	$*$	$*$	0	1

Table IV: The conditional probability table version of the equivalence based on the two graphs in Fig. 8, show that the Table for $\omega = 3$ sized $n = 2$ cliques with $\omega - 2 = 1$ common vertices is equivalent (in terms of number of classical deterministic strategies) to Table with $\omega = 2$ sized $n = 2$ disconnected cliques

S-CC$\mathcal{R}_{CLP}$, *i.e.* satisfy (**T0**) and (**T1**), we calculate the shared randomness required for the same task when considering a graph $\mathcal{G}^{(n,\omega=2)}$ where any two maximum cliques are disconnected.

For S-CC$\mathcal{R}_{CLP}$ of $\mathcal{G}^{(n,\omega=2)}$, we require a convex combination of deterministic strategies while communicating a classical system of ω dimension. The conditional probability table M resulting from the convex combination of these strategies must have positive entries in the off-diagonal block matrix $(C_x, C_{y\neq x})$ while the diagonal block matrices $(C_x, C_{y\neq x})$ must be equal to the identity matrix $\mathbf{I}_2$. Thus, in the conditional probability table M, for every $(C_x, a, C_{y\neq x})$ there must be a pair of deterministic table/ strategy such that one has $P(b|C_x, a, C_y) = 0$ and the other table has $P(b|C_x, a, C_y) = 1$ as its entry. Any classical deterministic strategy constitutes of filling the table of conditional probability such that every off-diagonal block matrix of this table $(C_x, C_{y\neq x})$ is either $\mathbf{I}_2$ or σ_x, where σ_x is the Pauli-x operator (or the NOT operator). The set of n classical strategies to achieve reconstruction are the following. The i^{th} strategy corresponds to the table where only off-diagonal block matrix $(C_1, C_i) = \sigma_x$ and rest $(C_1, C_{j(\neq i)}) = \mathbf{I}_2$ $\forall i \in \{2, \cdots, n\}$. Note that fixing the block matrices in the first row alone fixes the entire table if the amount of classical communication is restricted to 1-bit (See Appendix B).

Note that taking each such n deterministic classical strategies discussed earlier and their convex combinations yields a table of conditional probabilities $P(b|C_x, a, C_y)$, M, that leads to some non-zero payoff. It is worth mentioning that the payoff for the above strategy is $\mathcal{P}_{\mathcal{R}_{CLP}(\mathcal{G}^{(n,2)})} = \frac{1}{n} > 0$ and since a non-zero payoff ensures S-CC$\mathcal{R}_{CLP}$ or relation reconstruction, thus we satisfy **(T0)** and **(T1)**. However, this is not always the optimal payoff $\mathcal{P}^*_{\mathcal{R}_{CLP}(\mathcal{G}^{(n,2)})}$ for the graph under consideration.

■

Now, we will show that the lower bound on the amount of share randomness required for achieving optimal payoff for S-CC$\mathcal{R}_{CLP}$ while communicating ω-level classical system is related to the existence of some specific kinds of Orthogonal Arrays. Before moving forward, we first introduce Orthogonal Arrays.

Definition 7. *An $N \times k$ array A with entries from set S is called an orthogonal array $OA(N, k, s, t)$ with s levels, strength $t(\in \{0, 1, \cdots, k\})$ and index λ if every $n \times t$ sub-array of A contains each t-tuples based on S appearing exactly λ times as a row [30].*

Orthogonal Arrays have found interesting connections with absolutely maximally entangled states [31], multipartite entanglement [32, 33], quantum error-correcting codes[34] etc. Here, we will consider orthogonal arrays $OA(N, k, s, t)$ where $t = 2$ and $s = 2$ and $S = \{0, 1\}$. Let T_k be the minimum N for a fixed k such that $OA(N = T_k, k, s = 2, t = 2)$ is an orthogonal array with $S = \{0, 1\}$. Thus, in $OA(N = T_k, k, s = 2, t = 2)$ every $T_k \times 2$ sub-array has the tuples $\{(0, 0), (0, 1), (1, 0), (1, 1)\}$ appearing equal number of times as rows.

T_n is related to the amount of shared randomness necessary and sufficient for accomplishing the S-CCR of $\mathcal{R}_{CLP}(\mathcal{G}^{(n,\omega=2)})$ with optimal payoff $\mathcal{P}^*_{\mathcal{R}_{CLP}(\mathcal{G}^{(n,2)})}$ when $\log \omega$ c-bit classical communication is allowed from Alice to Bob.

Corollary 2. *Given a graph $\mathcal{G}^{(n,\omega)}$ satisfying **(G0)**-**(G2)** with faithful orthogonal range ω, it is necessary (but may not be sufficient) to share randomness with 2-inputs (for $n = 2$) and $\log_2 T_{n-1}$-inputs (for $n > 2$) while communicating an ω-level classical system to accomplish S-CC$\mathcal{R}_{CLP}(\mathcal{G}^{(n,\omega)})$ with optimal payoff $\mathcal{P}^*_{\mathcal{R}_{CLP}(\mathcal{G}^{(n,\omega)})}$.*

Consider a graph $\mathcal{G}^{(n,\omega)}$ satisfying **(G0)**-**(G2)** with faithful orthogonal range ω. From Theorem 5, to find the lower bound on shared randomness required for a graph $\mathcal{G}^{(n,\omega)}$ to satisfy optimal payoff **(T2)**, we calculate the shared randomness required to achieve $\mathcal{P}^*_{\mathcal{R}_{CLP}(\mathcal{G}^{(n,2)})} = 0.5$ for a graph $\mathcal{G}^{(n,\omega=2)}$ with n disconnected maximum size cliques.

Similar to the proof of Corollary 1 we will again consider a convex combination of deterministic strategies while communicating a classical system of ω dimension. For any such deterministic strategy, the associated conditional probability table has every off-diagonal block (C_x, C_y) to be either $\mathbf{I}_2$ or σ_x, where $\mathbf{I}_2$ is 2×2 identity matrix. Also, for any such deterministic strategy $(C_x, C_y) = (C_1, C_x) \oplus_2 (C_1, C_y)$ where $\mathbf{I}_2 \to 0$ and $\sigma_x \to 1$. Note that fixing the block matrices in the first row alone fixes the entire table if the amount of classical communication is restricted to 1-bit (See Appendix B).

In the final table of conditional probability M, we want each entry in every off-diagonal block (C_x, C_y) to be 0.5. This is possible if we have a uniform convex mixture of deterministic tables where half of them have $(C_x, C_y) = \sigma_x$ and the rest have $(C_x, C_y) = \mathbf{I}_2$ such that the effective weight for each free entry $*$ is 0.5. For n=2, convex combination of two deterministic tables, one with $(C_1, C_2) = \mathbf{I}_2$ and other with $(C_1, C_2) = \sigma_x$, give payoff $\mathcal{P}_{\mathcal{R}_{CLP}(\mathcal{G}^{(2,2)})} = 0.5$. For $n = 3$, we need four tables i.e. $(C_1, C_2) = \mathbf{I}_2$ or σ_x and $(C_1, C_3) = \mathbf{I}_2$ or σ_x. Not all convex combinations of any subset of these four deterministic strategies/tables will lead to a table M where some of the off-diagonal block matrix (C_x, C_y) have a contribution from an unequal number of $\mathbf{I}_2$ and σ_x. For $n \geq 2$, by the similar argument we need a minimal collection of deterministic tables such that corresponding to every two-block matrix of the form $(C_1, C_{j\neq 1})$ and $(C_1, C_{j'\neq 1})$, there are an equal number of tables where $(C_1, C_j) = \mathbf{I}_2$ and $(C_1, C_{j'}) = \mathbf{I}_2$, $(C_1, C_j) = \mathbf{I}_2$ and $(C_1, C_{j'}) = \sigma_x$, $(C_1, C_j) = \sigma_x$ and $(C_1, C_{j'}) = \mathbf{I}_2$, and $(C_1, C_j) = \sigma_x$ and $(C_1, C_{j'}) = \sigma_x$. This is exactly the problem for orthogonal arrays that have been discussed above if we substitute $\mathbf{I}_2 \to 0$ and $\sigma_x \to 1$. In other words, this corresponds to the minimum number of rows required so that any pair of columns have in their rows the entries $\{(0, 0), (0, 1), (1, 0), (1, 1)\}$ occurring an equal number of times. Thus, for a graph $\mathcal{G}^{(n,\omega=2)}$ with $n(> 2)$ the players Alice and Bob need shared randomness with T_{n-1} input to get optimal payoff $\mathcal{P}^*_{\mathcal{R}_{CLP}(\mathcal{G}^{(n,2)})} = 0.5$ when they are allowed to communicate ω level classical system. For any graph $\mathcal{G}^{(n,\omega>2)}$ considered here, Alice and Bob need shared randomness with at least T_{n-1} input to get optimal payoff $\mathcal{P}^*_{\mathcal{R}_{CLP}(\mathcal{G}^{(n,\omega)})}$ for the relation $\mathcal{R}_{CLP}(\mathcal{G}^{(n,\omega>2)})$ when they are allowed to communicate ω level classical system. This completes the proof.

Now we show that there exist some graphs $\mathcal{G}^{(n,\omega)}$ for which Alice and Bob need pre-shared correlation while communicating ω level quantum or classical system to accomplish S-CC$\mathcal{R}_{CLP}$ with optimal payoff. As a consequence of this result, there are graphs for which 1 c-bit classical communication when assisted by a finite amount of shared randomness can be powerful compared to 1 qubit quantum direct communication resources when considering this particular task and payoff.

Theorem 6. *There exist graphs $\mathcal{G}^{(n,\omega)}$ satisfying **(G0)**-**(G2)***

and faithful orthogonal range ω, such that while using ω dimensional classical or quantum channel, the assistance of public coins is necessary to perform S-CC$\mathcal{R}_{CLP}(\mathcal{G}^{(n,\omega)})$ with optimal payoff $\mathcal{P}^*_{\mathcal{R}_{CLP}(\mathcal{G}^{(n,\omega)})}$.

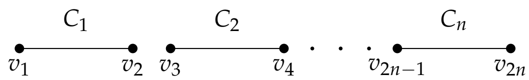


Figure 9: Example for a graph $\mathcal{G}^{(n,\omega)}$ satisfying Theorem 6 with n disconnected cliques of size $\omega = 2$.

Proof. Assume that Alice is allowed to communicate an ω dimensional system to Bob. We prove the above-mentioned theorem by showing the existence of a graph that satisfies the claim. Let us consider the graph $\mathcal{G}^{(n=\omega+2,\omega)}$ satisfying **(G0)**-**(G2)** and having faithful orthogonal representation in minimum dimension ω where any two the maximum size cliques are disconnected. For an example, see Fig. 9 where $\omega = 2$.

Note that for such a graph, the maximum payoff achievable by communicating $\log \omega$ qubit, $\mathcal{P}_{\mathcal{R}_{CLP}(\mathcal{G}^{(\omega+2,\omega)})}$, is always less than the optimal payoff $\mathcal{P}^*_{\mathcal{R}_{CLP}(\mathcal{G}^{(n,\omega)})} = \frac{1}{\omega}$. This is because, only $\omega + 1$ mutually unbiased bases (MUBs) are possible in $\mathbb{C}^\omega$, which can be used to encode and decode in an unbiased way, a maximum of $\omega + 1$ cliques in the considered graph. If Alice is allowed to send $\log_2 \omega$ c-bits without having access to shared randomness then the payoff obtained is zero (see Lemma 1). On the other hand, by using finite shared randomness, all the deterministic strategies using $\log_2 \omega$ c-bits which satisfy (**T0**) (which are finite in number) can be mixed to obtain the optimal payoff $\mathcal{P}^*_{\mathcal{R}_{CLP}(\mathcal{G}^{(n,\omega)})} = \frac{1}{\omega}$. ■

For the graph in Fig. 9, the necessary and sufficient amount of shared randomness to achieve $\mathcal{P}_{\mathcal{R}_{CLP}(\mathcal{G}^{(n,2)})} = \frac{1}{2}$ while communicating 1 c-bit is given in Corollary 2. Also, the maximum payoff achieved when 1 qubit is communicated from Alice to Bob is upper bounded by $\frac{1}{2}$ (the optimal payoff $\mathcal{P}^*_{\mathcal{R}_{CLP}(\mathcal{G}^{(n,2)})}$ can be achieved only for $n \leq 3$). Thus, 1 c-bit classical communication when assisted by a finite amount of shared randomness can outperform 1 qubit quantum direct communication resources when considering this task.

2. *Classical communication assisted by quantum entanglement*

At this point, a natural question is whether quantum correlations (quantum public coin) can enhance classical communication more than classical public coin. In the following theorem, we mention an instance where this is the case.

Theorem 7. *For classical communication with assistance from public coins, there exist graphs $\mathcal{G}^{(n,\omega)}$ satisfying conditions (**G0**)-(**G2**), such that the separation between classical and quantum public coins required for perfect S-CCR of relation $\mathcal{R}_{CLP}(\mathcal{G}^{(n,\omega)})$ is unbounded.*

Proof. Let us consider the graph $\mathcal{G}^{(n,\omega)}_{disc.}$ given by n disjoint cliques of size 2. 1 c-bit classical communication assisted by $n-1$ input shared randomness gives payoff 0 (see Corollary 1). On the other hand, when assisted by 1 bit of entanglement (a two-qubit maximally entangled state), Alice chooses n distinct orthogonal pairs of states from the equatorial circle of the Bloch sphere corresponding to the n possible input cliques. Now Alice and Bob perform the protocol the same as remote state preparation [35, 36], which allows perfect transmission of the states from an equatorial circle of the Bloch sphere with 1 bit of shared entanglement and 1 bit of classical communication. After successful transmission of the state, Bob performs qubit projective measurement based on his input C_y along one of the bases chosen by Alice. This makes the payoff $\mathcal{P}_{\mathcal{R}_{CLP}(\mathcal{G}^{(n,2)}_{disc.})} > 0$. Thus increasing n will require an increasing amount of shared randomness, while 1 bit of entanglement ensures quantum protocol to achieve a non-zero payoff. ■

For example, the symmetric choice of $n = 4$ directions on the Bloch sphere implies that this protocol can achieve $\mathcal{P}_{\mathcal{R}_{CLP}(\mathcal{G}^{(4,2)}_{disc.})} = \sin^2(\frac{\pi}{8}) \approx 0.1464$.

F. Summary of results

In this Section IV we have presented several results. Here, we quickly highlight the main results, summarised in the form of the following two tables V and VI. First, in Table V we have summarised the classical and quantum CC$\mathcal{R}_{CLP}$ and S-CC$\mathcal{R}_{CLP}$ without public coin assistance for different graphs. This table summarises the main results of this work where we quantify an unbounded quantum advantage for S-CC$\mathcal{R}_{CLP}$ for some class of graphs. Next, in Table VI, we summarise our result on the amount of public coin assistance required for restricted quantum and classical direct communication resources. Here, we have also mentioned our result on the unbounded advantage of sharing quantum public coins over sharing classical public coins when using only 1 c-bit direct communication.

Communication Task	Resource Comparison Classical	Quantum	Quantum Advantage	Ref.
$\text{CC}\mathcal{R}_{CLP}(\mathcal{G}^{(n,\omega)})$	$\log_2 \omega$ cbits	$\log_2 \omega$ qubits	$\Omega(1)$	Section IV A, Theorem 1
$\text{S-CC}\mathcal{R}_{CLP}(\mathcal{G}^{(n,\omega)})$	$\log_2 \|\mathcal{V}\|$ cbits	$\log_2 d_{\mathbb{C}}$ qubits	$\Omega(\log \|\mathcal{V}\|)$	Section IV C, Theorem 2
$\text{S-CC}\mathcal{R}_{CLP}(\mathcal{G}^{(n,\omega)}_{disc.})$	$\log_2 n\omega$ cbits	$\log_2 \omega$ qubits	$\Omega(\log n)$	Section IV C, Theorem 3
$\text{S-CC}\mathcal{R}_{CLP}(\mathcal{G}^{(n,\omega)}_{NNCC(r)})$	$\log_2(n(\omega - r) + r)$ cbits	$\log_2 \omega$ qubits	$\Omega(\log n)$	Section IV C, Theorem 3
$\text{S-CC}\mathcal{R}_{CLP}(\mathcal{G}_{Paley(q)})$	$\log_2 q$ cbits	$\log_2 \frac{q+1}{2}$ qubits	$\Omega(\log q)$	Theorem 2 & Section IV D

Table V: Resource comparison for classical vs quantum one-way communication tasks, $\text{CC}\mathcal{R}_{CLP}$ and $\text{S-CC}\mathcal{R}_{CLP}$, with some examples of quantum advantage for $\text{S-CC}\mathcal{R}_{CLP}$ for certain families of graphs considered in Section IV C

Resource Constraint	Communication Task	Resource Comparison Only Classical	Quantum allowed	Ref.
One-way Communication + Shared Randomness	$\text{S-CC}\mathcal{R}_{CLP}(\mathcal{G}^{(n,\omega)})$	$\log_2 \omega$ cbits + $\log_2 n$ bits SR*	$\log_2 \omega$ qubits + No SR required	Section IV E, Corollary 2
One-way Communication + Public coins	$\text{S-CC}\mathcal{R}_{CLP}(\mathcal{G}^{(n,\omega=2)})$	1 cbit + $\log n$ bit SR	1 cbit + 1 EPR pair	Section IV E, Theorem 7

Table VI: Resource Comparison for the communication task of $\text{S-CC}\mathcal{R}_{CLP}(\mathcal{G}^{(n,\omega)})$ considered in Section IV E where we allow public coins and compare purely classical protocols with hybrid protocols allowing some quantum resource — communication (first row) or entanglement (second row).

*Here the $\log_2 n$ bits SR allow $\text{S-CC}\mathcal{R}_{CLP}(\mathcal{G}^{(n,\omega)})$ but does not always achieve the optimal payoff $\mathcal{P}^*_{\mathcal{R}_{CLP}(\mathcal{G}^{(n,\omega)})}$. The SR necessary for achieving $\mathcal{P}^*_{\mathcal{R}_{CLP}(\mathcal{G}^{(n,\omega)})}$ is connected to the problem of orthogonal arrays.

V. APPLICATIONS

In this section, we discuss a number of useful applications of $\text{S-CC}\mathcal{R}_{CLP}$ task. The first application, in Section V A, is the operational detection of MUBs from the observation of the statistics. We consider some specific type of graph $\mathcal{G}$ with both maximum clique size and orthogonal representation in minimum dimension ω. If a quantum strategy using a ω level quantum system can achieve the upper bound of the optimal payoff (that is $\mathcal{P}^Q_{\mathcal{R}_{CLP}} = \mathcal{P}^*_{\mathcal{R}_{CLP}}$) for such a graph $\mathcal{G}$, then Bob must have used measurements corresponding to MUBs for decoding. In the next application, in Section V B, we consider the problem of detecting the non-classical resources in both direct communication and in the shared correlation (black-box) scenario. Finally, we consider a larger class of graphs that do not have orthogonal representation in dimension ω where ω is the size of a maximum clique and show that these graphs can be used to detect whether the dimension of the direct communication resource is greater than ω or otherwise. In the following, we discuss each of the applications in greater detail.

A. Detecting Mutually Unbiased Bases

We show the operational detection of MUBs from the observation of the statistics of our communication task, showing that quantumly achieving (**T2**) for some graphs implies the detection of MUBs.

A pair of projective measurements for a d-dimensional Hilbert space are mutually unbiased if the squared length of the projection of any basis element from the first onto any basis element of the second is exactly 1/d. Mutually unbiased bases (MUBs) are found to be optimal in several information-theoretic tasks and also in quantum cryptography [37–42].

Observation 4. *Consider a graph consisting of n maximum cliques of size ω that are completely disconnected from each other — $\mathcal{G}^{(n,\omega)}_{disc.}$. This graph has faithful orthogonal representation in dimension $d_{\mathbb{R}} = d_{\mathbb{C}} = \omega$. If a quantum strategy with direct communication of an ω-level system can achieve the optimal payoff i.e. $\mathcal{P}_{\mathcal{R}_{CLP}(\mathcal{G})} = (\frac{1}{\omega} = \mathcal{P}^*_{\mathcal{R}_{CLP}(\mathcal{G})})$, then the measurements performed by Bob must be those corresponding to MUBs.*

For example, let us consider one such graph, which allows for the detection of qubit-MUBs. The simplest graph consists of three maximum cliques of size $\omega = 2$ that are disconnected from each other, $\mathcal{G}^{(n=3,\omega=2)}_{disc.}$.

Upon receiving her input clique and clique label, Alice prepares her state in one of the pairs of the eigenstates of three qubit-MUBs corresponding to the disjoint cliques of this graph and sends the qubit to Bob. Bob performs his measurement corresponding to one of the above three MUBs based on his input clique. Evidently in this case, the payoff turns out to be $\mathcal{P}_{\mathcal{R}_{CLP}(\mathcal{G}^{(3,2)})} = \frac{1}{2}$. Conversely, one can see that in order to achieve the optimal payoff it is required to produce the prepare and measure probabilities corresponding to the disconnected pairs of vertices of the graph completely unbiased.

B. Semi-Device Independent Detection of Non-Classical Resources and Dimension Witness

In a *prepare and measure* setup, which underlies a number of information-theoretic tasks, two prime questions of practical interest are- (i) is the transmitted system (alternatively, are the prepare and measure devices) *non-classical*? and (ii) what is the operational dimension of the transmitted system? For quantum systems the second question reduces to finding a lower bound of the Hilbert space dimension, i.e. to find a *dimension witness* [43–46]. If these questions are answered based on the *input-output* probability distribution $\{P(b|x,y)\}$, where $x \in X$ and $y \in Y$ are inputs and $b \in B$ is the output, without referring to any information about the encoding and decoding devices, the protocol is *device independent*. If partial information about the devices is available, the scenario is called *semi-device independent*. In the following, we show that the proposed S-CC$\mathcal{R}_{CLP}$ task can be used as a semi-device independent witness of non-classicality as well as dimension.

While answering the first question we will consider two scenarios- first, where no public coin is available. This scenario allows us to determine the non-classicality of the transmitted system. Second, where only a finite amount of public coins are available and a classical bit has been transmitted, allows us to answer whether the public coin is non-classical or not. For both cases let us consider the two distant parties executing the S-CC$\mathcal{R}_{CLP}$ task with a class of graphs satisfying conditions condition (**G0**)-(**G1**). Now, in the first case let us also assume that it is known that the operational dimension of the transmitted system is strictly upper bounded by $|\mathcal{V}|$, the number of vertices of the graph. If the distant parties can achieve a non-zero payoff (calculated from $P(b|x,y)$ according to the definition in Eq.3), it follows from Theorem 1 that the transmitted system is non-classical. In the second case with a finite public coin and a classical bit communication, let us consider the graph $\mathcal{G}^{(n,\omega=2)}$ with all disconnected maximum cliques. This graph has a faithful orthogonal range $\omega = 2$. If the local dimension of the public coin is strictly upper-bounded by n, the number of maximum cliques in the graph. Then (see the example in the proof of Theorem 7) payoff $\mathcal{P}_{\mathcal{R}_{CLP}(\mathcal{G}^{(n,2)})} > 0$ implies that the public coin is non-classical. To answer the question about dimension witness we first observe the following: it follows from Theorem 1 that given a graph with n number of maximum cliques of size ω, achieving the CC$\mathcal{R}_{CLP}$ task requires at least ω-level system needs to be communicated from Alice to Bob. This fact applies to any arbitrary graph. Even in the presence of Public coins, if Alice's encoding and Bob's decoding can achieve CC$\mathcal{R}_{CLP}$ without any error, it will imply that the communicated system must have operational dimension at least greater than ω.

VI. SUMMARY & DISCUSSIONS

In a non-asymptotic *prepare and measure* scenario, the problem of efficient encoding of classical information in a quantum system has been a topic of interest in recent times [47–52]. Communication complexity, a prototype of distributed computing, measures the efficiency of such an encoding by the separation between the operational dimension of the classical and quantum message. A large separation for some distributed computation tasks demonstrates the advantage of quantum communication resources over classical ones. The present work proposes one such task, called strong-communication complexity of relations induced by the clique labelling problem.

In this S-CCR problem, we show that there exists a class of graphs for which the separation between the dimension of quantum and classical systems necessary can be made unbounded in the absence of public coins or shared randomness between the players. In the presence of public coins, however, this separation disappears. While quantum communication does not require public coins, the amount of public coin assistance that is necessary (but may not be sufficient) for classical communication for accomplishing the task scales linearly with the number of cliques. Additionally, we also show that a $1-ebit$ assisted classical $1-cbit$ channel performs a task that would otherwise require the assistance of a $1-cbit$ channel and an unbounded amount of classical public coin.

The present work can be seen as an addition to the earlier attempts to demonstrate the separation of classical and quantum communication complexity with relations [14, 16, 53, 54]. For example, Buhrman, Cleve, and Wigderson in [14] considered an exponential gap in classical and quantum communication for one-way and interactive protocols for a promise problem with zero error probability in the absence of Public coins. Later Raz in [16] showed that an exponential gap in communication exists for a relation in bounded-error interactive protocols. Bar-Yossef *et al.* in [53] showed an exponential separation for one-way protocols as well as simultaneous protocols with public coins for a relational problem, called the Hidden Matching Problem.

An important aspect of the present work is that the relations considered here are given by orthogonality graphs. A similar approach while demonstrating the advantage of quantum communication over classical was taken by Saha *et al.* in [17]. The authors in [17] considered a graph colouring task, called vertex equality problem, executed by two spatially separated parties. They showed that quantum advantage in one-way communication appeared whenever a class of graphs, called state-independent contextuality graphs (SIC graphs) are considered. Whereas quantum advantage in the communication task proposed in this article can be observed independent of the usefulness of the graphs in demonstrating state-independent contextuality. Therefore in our case, the quantum advantage in one-way communication can not be attributed to contextuality. Interestingly in [19] the authors showed that the quantum separation for computation of a partial-function via communication task based on state-independent contextuality witnesses can be unbounded, whereas our task, independent of contextuality can obtain a similar separation for the computation of a relation via communication.

In a practical setting, one may not always have the same input sets for both parties. One straightforward direction for a generalisation of S-CCR would entail, one party (Alice) receiving inputs of cliques and their label over some graph $\mathcal{G}_{\mathcal{A}} \subset \tilde{\mathcal{G}}$ while the other party (Bob) receives inputs of cliques to be labelled on some other graph $\mathcal{G}_{\mathcal{B}} \subset \tilde{\mathcal{G}}$ only to be consistent with the label of Alice if $\mathcal{G}_{\mathcal{A}} \cap \mathcal{G}_{\mathcal{B}} = \mathcal{G} \neq \emptyset$. As long as $\mathcal{G}$ allows for a quantum advantage over this restricted S-CC$\mathcal{R}(\mathcal{G})$, one could still find usefulness in this setting of different inputs.

This work leaves a number of questions open. For example, could there be a task such that the scaling of classical vs quantum communication with binary colourable graphs be exponential in the presence/absence of public coins (possibly for two-way communication complexity)? Could one obtain a linear scaling when the two parties compute a function instead of a relation? Besides these general questions, there are some particular points about the present study that remain unresolved. First, does the unbounded separation between classical and quantum communication persist when one departs from the zero-error scenario considered in this work and considers a degree of error in the computation? In Section IV E 1, the connection between a lower bound to the amount of classical public coin in the bounded communication setting and orthogonal arrays, shows that given arbitrary graphs with a large number of cliques, finding such a lower bound is a hard problem. In Section IV E 2 the advantage of using entanglement instead of a classical public coin (shared randomness) to assist a bounded classical communication has been demonstrated by achieving a higher payoff function. But it remains unknown what is the optimal payoff for the entanglement-assisted case. A monotonically decreasing payoff with the increasing number of maximum cliques (n) might suggest a limit of this advantage. Finally in the applications section (Section V A) the robustness of the scheme to detect MUBs is not known.

Finally, one can look at the present protocol from a foundational perspective. Namely, it can be seen, in a way, as a qualitative simulation of the quantum statistics on demand. In fact, the relation-reconstruction condition for the strong communication complexity proposed in this article could bridge the gap between conventional communication complexity and sampling problems with communication [55, 56]. Precisely, in our protocol, the spatially separated parties are given some set of favourable events and it is required that the events be quantitatively simulated by classical communication so that all of them occur with nonzero probability like it is in the quantum case. Considering the class of graphs in Fig.9, obtaining the non-zero value of the payoff function (Eq.27), reduces to the simulation of the set of correlations generated from pure qubit states and qubit projective measurements. In order to simulate the prepare-measure statistics of a qubit, the authors in [50] show that it is necessary and sufficient to communicate two classical bits when the parties are assisted by pre-shared randomness. In the same spirit Corollary 1 says that it is necessary to share an unbounded amount of randomness between the sender and receiver besides a finite amount of classical communication to simulate the statistics of qubits for the class of graphs as in Fig.9. Looking at the protocol from yet another angle, we can see it as a distribution of a (conditional) randomness with the help of a restricted communication channel. This raises the question of the possible relation of the present scheme to discrete analogues of bosonic sampling [57]. Quantum advantage in the later case relies on the hypotheses of the computational hardness of some classical tasks. It would be interesting to see whether additional graph structure and modification of the present protocol could imply exponential separation in sampling that would not rely on hypotheses of this type.

VII. ACKNOWLEDGEMENTS

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Appendix A: Success Probability for Reconstruction of Relations

Given a relation $\mathcal{R} \subseteq X \times Y \times B$ for the bipartite prepare and measure a scenario where X and Y are the set of inputs for Alice and Bob and B is the set of outputs for Bob, we are interested in success probability $P_k(\mathcal{R})$ of relation reconstruction after k number of rounds, where k is large. Additionally, Alice and Bob's protocol is agnostic to the number of rounds. Every tuple $(x, y, b) \in \mathcal{R}$ must occur at least once in these k rounds for the correct reconstruction of the relation $\mathcal{R}$. The cardinality $|\mathcal{R}| = \Gamma$ is the total number of all such events, which implies $k \geq \Gamma$ for reconstruction to be possible. Here we assume that the inputs are sampled from a uniform distribution. If Alice encodes her input $x \in X$ in the message τ_x in each round and Bob outputs $b \in B$ depending on his input $y \in Y$ and Alice's message,

$$P(x,y,b) = \sum_{\tau_x} P(x,y,b,\tau_x) \tag{A1}$$

$$= \sum_{\tau_x} P(b|y,\tau_x)P(\tau_x|x)P(x)P(y) \tag{A2}$$

$$P(x,y,b) = P(b|y,x)P(x)P(y) \tag{A3}$$

if $P(\tau_x|x) = 1 \forall x \in X$ (this is the situation in the scenario when a pre-shared public coin is not allowed). We can consider a strict ordering of the elements in $\mathcal{R}$. Given this ordered list, we can define $\alpha(k) = \{\alpha_1, \alpha_2, \cdots, \alpha_\Gamma\}$ where $\alpha_i \in \mathbb{N}$ is the frequency of occurrence of the i^{th} element (x_i, y_i, b_i) of ordered list $\mathcal{R}$ given k number of rounds have occurred and thus $\sum_{i=1}^{\Gamma} \alpha_i = k \;\forall\; \alpha$. The instances favourable for successful reconstruction of relation corresponds to the set of $\alpha(k)$ where each of the elements of $\mathcal{R}$ occur with non-zero frequency. The probability of reconstruction of $\mathcal{R}$ given k number of rounds is thus given by the total probability of occurrences of the $\alpha(k)$ with the aforementioned property.

$$\begin{aligned} P_k(\mathcal{R}) &= \sum_{\alpha(k)} P(\alpha(k)|k) \\ &= \sum_{\alpha} P(\{\alpha_1, \alpha_2, \cdots, \alpha_\Gamma\}|k) \\ &= \sum_{\alpha} (\prod_{i=1}^{\Gamma} P^{\alpha_i}(x_i, y_i, b_i)) \end{aligned} \tag{A4}$$

Since, $\forall\; \alpha \;\forall i \in \{1, 2 \cdots, \Gamma\}, \alpha_i > 0$, therefore,

$$P_k(\mathcal{R}) = \left(\prod_{i=1}^{\Gamma} P(x_i, y_i, b_i)\right) \left(\sum_{\alpha} (\prod_{i=1}^{\Gamma} P^{\alpha_i - 1}(x_i, y_i, b_i))\right) \tag{A5}$$

Notice that if any of the terms $P(x_i, y_i, b_i) = 0$ then the probability of successful reconstruction after k rounds $P_k(\mathcal{R})$ becomes zero as well. Therefore,

$$P_k(\mathcal{R}) \neq 0 \implies P(b|x,y) \neq 0 \;\; \forall (x,y,b) \in \mathcal{R} \tag{A6}$$

Remark: $P(b|x, y, \tau_x) = 1 \;\forall x \in X, y \in Y$ such that $\exists! b \in B$ satisfying $(x, y, b) \in \mathcal{R}$. For rest of the $(x, y, b) \in \mathcal{R}$, $P(b|x, y) \in (0, 1)$.

Now, we define $B_{x,y} = \{b \in B : (x, y, b) \in \mathcal{R}\}$ which is the set of all acceptable outputs for Bob given the input are x and y for Alice and Bob respectively. Then, $\sum_{b \in B_{x,y}} P(b|x, y, \tau_x) = 1 \;\;\forall\;\; B_{x,y}$.

We aim to maximise the success probability $P_k(\mathcal{R})$ in the scenario when Alice and Bob are not aware of the total number of rounds, say k_{max}, a priory and thus they should decide the probabilities of the events in $\mathcal{R}$ independent of k_{max}. To achieve this we start by using the Lagrange multiplier.

Now, in order to maximise the success probability of reconstruction for k number of rounds we define

$$L = P_k(\mathcal{R}) - \sum_{B_{x,y}} \lambda_{B_{x,y}} \left(1 - \sum_{b \in B_{x,y}} P(b|x,y)\right) \tag{A7}$$

(A8)

For j^{th} element (x_j, y_j, b_j) in ordered list of $\mathcal{R}$,

$$\begin{aligned} &\frac{\partial L}{\partial P(x_j, y_j, b_j)} = 0 \\ \implies &\sum_{\alpha(k)} \left(\alpha_j P(x_j, y_j, b_j)^{-1}\right) \left(\prod_{i=1}^{\Gamma} P^{\alpha_i}(x_i, y_i, b_i)\right) \\ &- \lambda_{B_{x_j, y_j}} \left(P(b_j|x_j, y_j)\right) \end{aligned} \tag{A9}$$

$$\implies \lambda_{B_{x_j,y_j}} = \frac{\sum_{\alpha(k)} \alpha_j \left(\prod_{i=1}^{\Gamma} P^{\alpha_i}(x_i, y_i, b_i)\right)}{P^2(x_j, y_j, b_j)} \tag{A10}$$

For a given k, the optimal probabilities $P(x_i, y_i, b_i) = P(b_i|x_i, y_i)P(x_i, y_i)$ can be calculated that yields maximum value of $P_k(\mathcal{R})$. However, for any arbitrary k, the expression of $\lambda_{B_{x_i,y_i}}$ is a function of k as $\alpha(k)$ and α_i are a function of k. Since Alice and Bob do not have prior information about k, thus they have to agree on values of probabilities $P(x_i, y_i, b_i)$ independent of k. Thus, the obvious solution is $P(b|x, y) = constant \ \forall\ b \in B_{x,y}$. Here we assume that the inputs are sampled from a uniform distribution. This shows the necessity of our payoff function. Maximising the Payoff guarantees that the $P_k(\mathcal{R})$ is maximised to some local maxima.

Appendix B: Proof of Theorem 1

Before we delve into the proof, let us introduce a few notations that we will frequently use to in the later sections. Prior to the game, Alice and Bob are given $\mathcal{G}^{(n,\omega)}$ and they a construct a table M whose entries are conditional probabilities $p(b|C_x, C_y, a)$ of compatible labels a, b, for all possible cliques $C_x, C_y \in \mathcal{G}^{(n,\omega)}$. In this table the probability $p(b|C_x, C_y, a) \equiv ((C_x, a, C_y, b))$ is the entry in the table M corresponding to the event (C_x, a, C_y, b) where $(C_x, a) \in X$, $C_y \in Y$ and $b \in B$. The rows and the columns of this table are indexed as $(C_x, a)_\mathsf{r}$ and $(C_y, b)_\mathsf{c}$ respectively. In this table, the index runs over all the a, b first and then updates the C_x, C_y. This table has $n\omega$ rows and $n\omega$ columns and may be perceived as a $n \times n$ block matrix with elements indexed (C_x, C_y). We have $\mathbf{I}_{\omega\times\omega}$ on the diagonal blocks of the table as Bob has to output the same label as Alice whenever they get the same cliques as input. The aforementioned game can be mapped to the following properties (**T0**) of the table M. We have equivalence between the communication game and the table M with the constraint (**T0**).

(T0): Consistent labelling of cliques: If the event $(C_x, a, C_y, b) \notin \mathcal{R}_{CLP}(\mathcal{G}^{(n,\omega)}) \implies P(b|C_x, a, C_y) = 0$.

Proof. If Alice and Bob manage to compress the nd rows of the table M (i.e., the set of all possible inputs for Alice) into at least ω partitions such that no two rows in the same partition have entries in any columns that are different (may be due to constraints imposed by property (**T0**) or by choice filling the probabilities corresponding to events outside $\mathcal{C}(\mathcal{G}^{(n,\omega)})$) then there exists a protocol such that Theorem 1 is satisfied. Alice will communicate with Bob the partition to which her input belongs and then Bob can suitably pick a label for her input clique C_y while satisfying the probability distribution table that players agreed upon at the start and thereby satisfying the consistency condition.

However, notice that there cannot be any less than ω number of partitions of the rows of the table M satisfying (**T0**) such that no two rows in the same partition have entries in any columns that are different. This can be easily shown as every two rows corresponding to each block diagonal entry of M, *i.e.* $(C_x = C_i, C_y = C_i) = \mathbf{I}_{\omega\times\omega}$, are distinct. Thus, each of the ω rows corresponding to Alice's input clique $C_x = C_i$ must belong to a different partition.

This implies that every disjoint partition $\tau(i)(i \in \{0, 1, \cdots, \omega - 1\})$ of the rows described above must have exactly one row of the form (C_x, a) for each clique C_x, i.e., a row corresponding to exactly one out of all the possible label a for every clique C_x. In the following, we argue that there are ω such disjoint partitions of rows. But before we proceed, we will list some properties of the table M when such partitioning is possible.

If there is an imposition that the rows of table M can be partitioned into at least ω disjoint partitions $\tau(i)$ while satisfying the constraints discussed above there are some additional restrictions regarding the structure of table M that can be decided by both Alice and Bob in order to win the CLP.

- If some row (say $(C_x, a')_\mathsf{r}$) of off-diagonal block matrix (C_x, C_y) has more than one non-zero entries (say$((C_x, a', C_y, \tilde{b})) \neq 0$ and $((C_x, a', C_y, \tilde{b}')) \neq 0$) then the corresponding row in M cannot belong to any partition that contains a row with index $(C_{x'(=y)}, a)_\mathsf{r}$ where $a \in \{0, \cdots, \omega - 1\}$ as there exist column $(C_y, b)_\mathsf{c}$ where these two rows have different entries. This is because the block matrix $(C_y, C_y) = \mathbf{I}_{\omega\times\omega}$ and thus none of the rows have non-zero entries in two different columns in this block. Thus, this row must belong to a new partition and thus increasing the total number of partitions to $\omega + 1$.
- If some column (say $(C_y, b')_\mathsf{c}$) of off-diagonal block matrix (C_x, C_y) has more than one non-zero row entries then the rows corresponding to these nonzero entries in M can only belong to the partition that contains the row $(C_{\tilde{x}(=y)}, \tilde{a}(= b'))$. However, as discussed above exactly one out of all the possible label a for every clique C_x can belong to a partition. Therefore, Alice and Bob will be forced to create at least $\omega + 1$ partitions. Therefore, if the number of partitions is restricted to ω then each row and column of every off-diagonal block matrix (C_x, C_y) is some permutation Π_{C_x,C_y} of $\mathbf{I}_{\omega\times\omega}$.
- The table must have the property $M = M^T$. If this does not hold then there exists an element

for which $((C_x, a, C_y, b)) = 1 \neq ((C_{x'(=y)}, a'(= b), C_{y'(=x)}, b'(= a)))$. The row $(C_x, a)_\mathsf{r}$ must belong to same partition as $(C_{x'(=y)}, \tilde{a}(= b))_\mathsf{r}$ as $((C_x, a, C_y, b)) = 1 = ((C_{x'(=y)}, \tilde{a}, C_y, b))$ only for $\tilde{a} = b$. For any other allowed value of $\tilde{a}$, $((C_{x'(=y)}, \tilde{a}, C_y, b)) = 0$. However, the row $(C_x, a)_\mathsf{r}$ and $(C_{x'(=y)}, \tilde{a}(= b))_\mathsf{r}$ have different entries in the column $(C_{y''(=x)}, b''(= a))_\mathsf{c}$. $((C_x, a, C_{y''(=x)}, b''(= a))) = 1 \neq ((C_{x'(=y)}, a'(= b), C_{y'(=x)}, b'(= a)))$. Thus, the row $(C_x, a)_\mathsf{r}$ cannot belong to any partition that contains a row indexed $(C_{x'(=y)}, \tilde{a})$ where $\tilde{a} \in \{0, \cdots, \omega - 1\}$.

Now, we will create a specific kind of ω disjoint partitions($\tau(i), i \in \{0, \cdots, \omega - 1\}$) of the input received by Alice considering a probability table having the form discussed above.

- **Step 1:** $\forall a \in \{0, 1, \cdots, \omega - 1\}, (C_1, a)_\mathsf{r} \in \tau(a)$.
- **Step 2:** $\forall j \in \{2, \cdots, n\}$, say the block matrix (C_1, C_j) is a permutation matrix Π_{1,C_j} then the row $(C_j, a')_\mathsf{r} \in \tau(a)$ where a' is the $(a)^{th}$ element of $\Pi_{1,C_j} * (0\ 1\ \cdots\ \omega - 1)^T$.

When Alice communicates the partition to which her input (C_x, a) belongs, Bob can pick the label for clique C_y that obeys the consistency condition $\mathcal{C}(\mathcal{G}^{(n,\omega)})$. It is important to note that each row associated with Alice's input clique C_x must belong to a distinct partition else Bob might not be able to assign a label obeying the consistency condition.

For example, consider the graph shown in Fig. 4. Alice and Bob adopt a deterministic strategy and fill the free entries marked with * in Table I with 0s and 1s as seen in Table VII.

		C_1			C_2		
		$b=0$	$b=1$	$b=2$	$b=0$	$b=1$	$b=2$
C_1	$a=0$	1	0	0	0	0	1
	$a=1$	0	1	0	0	1	0
	$a=2$	0	0	1	1	0	0
C_2	$a=0$	0	0	1	1	0	0
	$a=1$	0	1	0	0	1	0
	$a=2$	1	0	0	0	0	1

Table VII: Example of a table of conditional probabilities $p(b|C_x, C_y, a)$ for the graph in Fig. 4 satisfying (**T0**)⇔(**D0**)

For Table VII, we can make three partitions $\tau(0), \tau(1)$ and $\tau(2)$ for the rows such that exactly one row of each clique belongs to a partition. In this the partitions are $\tau(0) = \{(C_1, a = 0)_\mathsf{r}, (C_2, a = 2)_\mathsf{r}\}$, $\tau(1) = \{(C_1, a = 1)_\mathsf{r}, (C_2, a = 1)_\mathsf{r}\}$ and $\tau(2) = \{(C_1, a = 2)_\mathsf{r}, (C_2, a = 0)_\mathsf{r}\}$. Upon receiving C_x and a in each round Alice can send i corresponding to $\tau(i)$. After knowing the partition $\tau(i)$ Bob can always pick the label for his clique C_y that does not violate the consistency condition. Thus, a classical three-level system is sufficient for winning the game (**T0**) for the graph considered here. ■

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