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Unboundedness of solutions of time-dependent differential systems of parabolic type

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Abstract. Unboundedness of matrix solutions of time-dependent differential systems of parabolic type is studied. The key tool is to use the Picone-type identity for strongly elliptic systems. The results about oscillations of solutions are also derived.

Beginning with the work of McNabb [10], unboundedness of solutions has been investigated by numerous authors. We refer the reader to Dunninger [4], Jaroš, Kusano and Yoshida [5, 6] for scalar parabolic equations, and to Chan [1], Chan and Young [2, 3], Kuks [7], Kusano and Narita [8] for parabolic systems.

The purpose of this paper is to modify the results of Chan [1], Chan and Young [2] and obtain the results about the oscillations of matrix solutions.

We are concerned with the matrix solutions of the time-dependent differential system of parabolic type

$$\frac{\partial W}{\partial t} - P[W] = 0 \quad \text{in } \Omega \equiv G \times (0, \infty), \tag{1}$$

where G is a bounded domain in $\mathbb{R}^n$ with piecewise smooth boundary ∂G and

$$P[W] \equiv \sum_{i,j=1}^{n} \frac{\partial}{\partial x_i}\left(G_{ij}(x,t)\frac{\partial W}{\partial x_j}\right) + H(x,t)W.$$

It is assumed that :

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(H_1) $G_{ij}(x,t)$ $(i,j = 1,2,...,n)$ and $H(x,t)$ are $m \times m$ real symmetric matrix functions ;

(H_2) $G_{ij}(x,t) \in C^1(\overline{\Omega})$ $(i,j = 1,2,...,n)$ and $H(x,t) \in C(\overline{\Omega})$;

(H_3) $G_{ij}(x,t) = G_{ji}(x,t)$ $(i,j = 1,2,...,n)$ and the $mn \times mn$ matrix $\mathscr{G} = (G_{ij}(x,t))$ is positive definite in Ω.

The domain $\mathfrak{D}_P(\Omega)$ of P is defined to be the set of all $m \times m$ matrix functions $W \in C^2(\Omega) \cap C^1(\overline{\Omega})$.

Definition 1. A function $v : \overline{\Omega} \longrightarrow \mathbb{R}$ is said to be *oscillatory* on $\overline{\Omega}$ if v has a zero on $\overline{G} \times [t,\infty)$ for any $t > 0$. Otherwise, v is called *nonoscillatory* on $\overline{\Omega}$.

Definition 2. An $m \times m$ matrix $W(x,t) \in C^1(\tilde{\Omega})$, $\tilde{\Omega} \subset \Omega$, is said to be *prepared* in $\tilde{\Omega}$ with respect to P if the matrices

$$\sum_{j=1}^{n} W^T(x,t) G_{ij}(x,t) \frac{\partial W}{\partial x_j}(x,t) \quad (i = 1,2,...,n)$$

are symmetric in $\tilde{\Omega}$, where the superscript T denotes the transpose.

Theorem 1. *Let $W \in \mathfrak{D}_P(\Omega)$ and let $\det W \neq 0$ in $G \times I$, where I is any interval in $\mathbb{R}$. If W is prepared in $G \times I$ with respect to P, then the following identity holds for any m-column vector $u \in C^1(G)$:*

$$\begin{aligned} &\sum_{i,j=1}^{n} \left(W \frac{\partial}{\partial x_i} \left(W^{-1} u \right) \right)^T G_{ij}(x,t) \left(W \frac{\partial}{\partial x_j} \left(W^{-1} u \right) \right) \\ &+ \sum_{i,j=1}^{n} \frac{\partial}{\partial x_i} \left(u^T G_{ij}(x,t) \frac{\partial W}{\partial x_j} W^{-1} u \right) \\ = &\sum_{i,j=1}^{n} \left(\frac{\partial u}{\partial x_i} \right)^T G_{ij}(x,t) \frac{\partial u}{\partial x_j} - u^T H(x,t) u + u^T P[W] W^{-1} u. \end{aligned} \quad (2)$$

Proof. In the case where $G_{ij}(x,t) = G_{ij}(x)$, the identity (2) was established (see, e.g., Kusano and Yoshida [9, p.172]). The differentiations appearing in (2) are only partial differentiations with respect to x_i, and so we can consider t as a parameter. Hence, the identity (2) holds. □

Lemma. *Assume that $W \in \mathfrak{D}_P(\Omega)$ is symmetric and nonsingular on $G\times[t_0,\infty)$ for some $t_0 > 0$. If $\frac{\partial}{\partial t}\log W$ commutes with $\log W$ on $G\times[t_0,\infty)$, then we obtain*

$$\frac{\partial W}{\partial t}W^{-1} = \frac{\partial}{\partial t}\log W = \frac{\partial}{\partial t}\Big(\mathrm{Re}\,\log W\Big) \quad on \quad G\times[t_0,\infty), \tag{3}$$

where $\log W$ is the principal value and Re *means the real part.*

Proof. Since

$$W = \exp(\log W) = \sum_{j=0}^{\infty}\frac{1}{j!}(\log W)^j\,,$$

we have

$$\begin{aligned}\frac{\partial W}{\partial t} &= \sum_{j=0}^{\infty}\frac{1}{j!}\frac{\partial}{\partial t}(\log W)^j \\ &= \sum_{j=1}^{\infty}\frac{1}{j!}\,j\left(\frac{\partial}{\partial t}\log W\right)(\log W)^{j-1} \\ &= \left(\frac{\partial}{\partial t}\log W\right)\exp(\log W) \\ &= \left(\frac{\partial}{\partial t}\log W\right)W.\end{aligned}$$

Hence, we obtain

$$\frac{\partial W}{\partial t}W^{-1} = \frac{\partial}{\partial t}\log W.$$

Since W is a real symmetric matrix, there exists an orthogonal matrix S such that $S^{-1}WS = J$, where

$$J = \begin{pmatrix}\lambda_1 & & & 0\\ & \lambda_2 & & \\ & & \ddots & \\ 0 & & & \lambda_m\end{pmatrix},$$

λ_i $(i = 1,2,...,n)$ being the eigenvalues of W. It can be shown that

$$\begin{aligned}\log W &= \log(SJS^{-1}) = S(\log J)S^{-1} \\ &= S\begin{pmatrix}\log\lambda_1 & & & 0\\ & \log\lambda_2 & & \\ & & \ddots & \\ 0 & & & \log\lambda_m\end{pmatrix}S^{-1},\end{aligned}$$

where

$$\begin{aligned}\log \lambda_i &= \log|\lambda_i| + \sqrt{-1}\arg\lambda_i \quad (0 \le \arg\lambda_i < 2\pi) \\ &= \begin{cases} \log|\lambda_i| & \text{if } \lambda_i > 0 \\ \log|\lambda_i| + \sqrt{-1}\pi & \text{if } \lambda_i < 0. \end{cases}\end{aligned}$$

Hence, we obtain

$$\begin{aligned}\frac{\partial}{\partial t}\log W &= \frac{\partial}{\partial t}\left(S \begin{pmatrix} \log|\lambda_1| & & & 0 \\ & \log|\lambda_2| & & \\ & & \ddots & \\ 0 & & & \log|\lambda_m| \end{pmatrix} S^{-1} \right) \\ &= \frac{\partial}{\partial t}\Big(\mathrm{Re}\,\log W\Big). \qquad \square\end{aligned}$$

Theorem 2. *Assume that* (H_1)–(H_3) *hold, and that there is a nontrivial* m*-column vector function* $u \in C^1(\overline{G})$ *such that* $u = 0$ *on* ∂G *and*

$$\lim_{t\to\infty}\int_T^t M[u](s)ds = -\infty \quad \text{for any } T > 0, \tag{4}$$

where

$$M[u](t) \equiv \int_G \left[\sum_{i,j=1}^n \left(\frac{\partial u}{\partial x_i}\right)^T G_{ij}(x,t)\frac{\partial u}{\partial x_j} - u^T H(x,t)u \right] dx.$$

Let $W \in \mathfrak{D}_P(\Omega)$ *be a solution of* (1) *such that :*

(i) W *is symmetric in* Ω ;

(ii) W *is prepared in* Ω *with respect to* P ;

(iii) $\det W$ *is nonoscillatory on* $\overline{\Omega}$, *that is,* $\det W \neq 0$ *on* $\overline{G} \times [t_0, \infty)$ *for some* $t_0 > 0$;

(iv) $\frac{\partial}{\partial t}\log W$ *commutes with* $\log W$ *on* $G \times [t_0, \infty)$.

Then the following condition holds :

$$\lim_{t\to\infty}\int_G u^T\Big(\mathrm{Re}\,\log W\Big)u\,dx = \infty. \tag{5}$$

Proof. The hypotheses (i) and (ii) imply that the identity (2) holds in $G \times [t_0, \infty)$. Integrating (2) over G and taking account of (H_3) yield

$$\begin{aligned} 0 &\le M[u](t) + \int_G u^T P[W] W^{-1} u \, dx \\ &= M[u](t) + \int_G u^T \frac{\partial W}{\partial t} W^{-1} u \, dx, \quad t \ge t_0, \end{aligned}$$

which implies

$$-M[u](t) \le \int_G u^T \frac{\partial W}{\partial t} W^{-1} u \, dx, \quad t \ge t_0.$$

Using Lemma, we obtain

$$-M[u](t) \le \frac{d}{dt} \int_G u^T \Big(\mathrm{Re} \log W \Big) u \, dx, \quad t \ge t_0. \tag{6}$$

Integrating (6) on $[t_0, t]$, we have

$$-\int_{t_0}^{t} M[u](s) ds \le z(t) - z(t_0),$$

where

$$z(t) = \int_G u^T \Big(\mathrm{Re} \log W \Big) u \, dx.$$

It follows from the hypothesis (4) that

$$\lim_{t \to \infty} z(t) = \infty,$$

which is equivalent to (5). □

Theorem 3. *Assume that* (H_1)–(H_3) *hold, and that there is a nontrivial m-column vector function* $u \in C^1(\overline{G})$ *satisfying* (4) *and the boundary condition* $u = 0$ *on* ∂G. *Let* $W \in \mathfrak{D}_P(\Omega)$ *be a solution of* (1) *which satisfies the hypotheses* (i)–(iv) *of Theorem* 2. *Then,* $\|W\|$ *is unbounded in* Ω, *where*

$$\|W\| = \left(\mathrm{tr}\ W^T W \right)^{1/2},$$

tr W *being the trace of* W.

Proof. It is easy to see that

$$\left| \int_G u^T \Big(\mathrm{Re}\,\log W \Big) u \, dx \right| \le K \| \mathrm{Re}\,\log W \| \tag{7}$$

for some positive constant K. As was shown in the proof of Lemma, $\mathrm{Re}\,\log W$ can be written in the form

$$\mathrm{Re}\,\log W = S(\log \tilde{J}) S^{-1},$$

where S is an orthogonal matrix such that $S^{-1} W S = J$ and

$$\tilde{J} = \begin{pmatrix} |\lambda_1| & & & 0 \\ & |\lambda_2| & & \\ & & \ddots & \\ 0 & & & |\lambda_m| \end{pmatrix}.$$

Hence, we obtain

$$\| \mathrm{Re}\,\log W \| \le \|S\| \cdot \| \log \tilde{J} \| \cdot \|S^{-1}\| = m \, \| \log \tilde{J} \|. \tag{8}$$

We easily see that

$$\| \tilde{J} \| = \|J\| \le \|S^{-1}\| \cdot \|W\| \cdot \|S\| = m \, \|W\|. \tag{9}$$

Assume that $\|W\|$ is bounded. Then, $\|\tilde{J}\|$ is bounded from (9), and therefore $\| \log \tilde{J} \|$ is also bounded. The inequality (8) implies that $\| \mathrm{Re}\,\log W \|$ is bounded. In view of (7), we find that $\int_G u^T (\mathrm{Re}\,\log W) u \, dx$ is bounded. Hence, the condition (5) means that $\|W\|$ is unbounded. □

The following corollary is an immediate consequence of Theorem 3.

Corollary 1. *Assume that* (H_1)–(H_3) *hold, and that there is a nontrivial m-column vector function $u \in C^1(\overline{G})$ satisfying* (4) *and the boundary condition $u = 0$ on ∂G. Let $W \in \mathfrak{D}_P(\Omega)$ be a solution of* (1) *which satisfies the hypotheses* (i)–(ii) *of Theorem* 2. *If $\|W\|$ is bounded in Ω, then either that* $\det W$ *is oscillatory on* $\overline{\Omega}$, *or* (*if* $\det W$ *is nonoscillatory on* $\overline{\Omega}$) *that* (iv) *of Theorem* 2 *does not hold.*

We now consider the comparison equation

$$L[u] \equiv \sum_{i,j=1}^{n} \frac{\partial}{\partial x_i}\left(A_{ij}(x)\frac{\partial u}{\partial x_j}\right) + C(x)u, \tag{10}$$

where $A_{ij}(x)$ $(i, j = 1, 2, ..., n)$ and $C(x)$ satisfy the following hypotheses :

(H$_4$) $A_{ij}(x)$ $(i, j = 1, 2, ..., n)$ and $C(x)$ are $m \times m$ real symmetric matrix functions ;

(H$_5$) $A_{ij}(x) \in C^1(\overline{G})$ $(i, j = 1, 2, ..., n)$ and $C(x) \in C(\overline{G})$;

(H$_6$) $A_{ij}(x) = A_{ji}(x)$ $(i, j = 1, 2, ..., n)$ and the $mn \times mn$ matrix $\mathscr{A} = (A_{ij}(x))$ is positive definite in G.

The domain $\mathfrak{D}_L(G)$ of L is defined to be the set of all $m-$column vector functions $u \in C^2(G) \cap C^1(\overline{G})$.

Theorem 4. *Assume that* (H$_1$)–(H$_6$) *hold, and that there is a nontrivial m-column vector function $u \in \mathfrak{D}_L(G)$ such that :*

$$L[u] = 0 \quad in\ G, \tag{11}$$

$$u = 0 \quad on\ \partial G, \tag{12}$$

$$\lim_{t\to\infty} \int_T^t V[u](s)ds = \infty \quad for\ any\ T > 0, \tag{13}$$

where

$$V[u](t) \equiv \int_G \Bigg[\sum_{i,j=1}^{n} \left(\frac{\partial u}{\partial x_i}\right)^T \Big(A_{ij}(x) - G_{ij}(x,t)\Big)\frac{\partial u}{\partial x_j} + u^T\Big(H(x,t) - C(x)\Big)u\Bigg]dx.$$

Let $W \in \mathfrak{D}_P(\Omega)$ be a solution of (1) *which satisfies the hypotheses* (i)–(iv) *of Theorem* 2. *Then, the condition* (5) *holds.*

Proof. Proceeding as in the proof of Theorem 1, we observe that the following Picone-type identity

$$\sum_{i,j=1}^{n} \frac{\partial}{\partial x_i}\left(u^T A_{ij}(x)\frac{\partial u}{\partial x_j} - u^T G_{ij}(x,t)\frac{\partial W}{\partial x_j}W^{-1}u\right)$$

$$= \sum_{i,j=1}^{n} \left(\frac{\partial u}{\partial x_i}\right)^T \Big(A_{ij}(x) - G_{ij}(x,t)\Big) \frac{\partial u}{\partial x_j} + u^T \Big(H(x,t) - C(x)\Big) u$$
$$+ \sum_{i,j=1}^{n} \left(W \frac{\partial}{\partial x_i} \left(W^{-1} u\right)\right)^T G_{ij}(x,t) \left(W \frac{\partial}{\partial x_j} \left(W^{-1} u\right)\right)$$
$$+ u^T L[u] - u^T P[W] W^{-1} u$$

holds in $G \times [t_0, \infty)$. Integrating the above identity over G and taking account of the hypotheses yield the inequality

$$0 \geq V[u](t) - \int_G u^T P[W] W^{-1} u \, dx, \quad t \geq t_0$$

or

$$V[u](t) \leq \int_G u^T \frac{\partial W}{\partial t} W^{-1} u \, dx, \quad t \geq t_0.$$

Arguing as in the proof of Theorem 2, we conclude that the condition (5) holds. □

Theorem 5. *Assume that* (H_1)–(H_6) *hold, and that there is a nontrivial* m*-column vector function* $u \in \mathfrak{D}_L(G)$ *satisfying* (11)–(13). *Let* $W \in \mathfrak{D}_P(\Omega)$ *be a solution of* (1) *which satisfies the hypotheses* (i)–(iv) *of Theorem* 2. *Then,* $\|W\|$ *is unbounded in* Ω.

Proof. By the same arguments as were used in Theorem 3, we conclude that the conclusion follows from Theorem 4. □

We can obtain the analogue of Corollary 1.

Corollary 2. *Assume that* (H_1)–(H_6) *hold, and that there is a nontrivial* m*-column vector function* $u \in \mathfrak{D}_L(G)$ *satisfying* (11)–(13). *Let* $W \in \mathfrak{D}_P(\Omega)$ *be a solution of* (1) *which satisfies the hypotheses* (i)–(ii) *of Theorem* 2. *If* $\|W\|$ *is bounded in* Ω, *then either that* $\det W$ *is oscillatory on* $\overline{\Omega}$, *or* (*if* $\det W$ *is nonoscillatory on* $\overline{\Omega}$) *that* (iv) *of Theorem* 2 *does not hold.*

Example 1. We consider the matrix differential system

$$\frac{\partial W}{\partial t} - \left(\alpha \frac{\partial^2 W}{\partial x^2} + \beta W\right) = 0, \quad (x,t) \in (0,\pi) \times (0,\infty), \tag{14}$$

where α and β are positive constants with $\alpha < \beta$. Here $n = 1$, $G_{11}(x,t) = \alpha \mathrm{I}_m$ ($\mathrm{I}_m : m \times m$ identity matrix), $H = \beta\,\mathrm{I}_m$, $G = (0,\pi)$ and $\Omega = (0,\pi) \times (0,\infty)$. Letting

$$u = \begin{pmatrix} \sin x \\ \sin x \\ \vdots \\ \sin x \end{pmatrix},$$

we see that $u(0) = u(\pi) = 0$ and

$$\begin{aligned} M[u](t) &= \int_0^\pi \left[\alpha \left(\frac{\partial u}{\partial x} \right)^T \frac{\partial u}{\partial x} - \beta\, u^T u \right] dx \\ &= \int_0^\pi \left[\alpha m \cos^2 x - \beta m \sin^2 x \right] dx \\ &= \frac{\pi}{2}\, m(\alpha - \beta) < 0. \end{aligned}$$

Hence, we find that

$$\lim_{t\to\infty} \int_T^t M[u](s) ds = -\infty$$

for any $T > 0$. It follows from Theorem 2 that if W is a solution of (14) satisfying (i)–(iv) of Theorem 2, then (5) holds. One such solution is $W = e^{\beta t}\mathrm{I}_m$. In fact, it is clear that (i)–(iv) hold for $W = e^{\beta t}\mathrm{I}_m$, and that

$$\begin{aligned} &\lim_{t\to\infty} \int_0^\pi u^T \Big(\mathrm{Re}\, \log W \Big) u\, dx \\ = &\lim_{t\to\infty} \int_0^\pi \beta t m\, \sin^2 x\, dx \\ = &\lim_{t\to\infty} \frac{\pi}{2}\, \beta m t = \infty. \end{aligned}$$

Example 2. We consider the matrix differential system

$$\frac{\partial W}{\partial t} - \left(\alpha\, \frac{\partial^2 W}{\partial x^2} + \beta\, W \right) = 0, \quad (x,t) \in (-1,1) \times (0,\infty), \tag{15}$$

where α and β are positive constants satisfying $\alpha < (5/2)\beta$. Here $n = 1$, $G_{11}(x,t) = \alpha \mathrm{I}_m$, $H = \beta\,\mathrm{I}_m$, $G = (-1,1)$ and $\Omega = (-1,1) \times (0,\infty)$. We let

$$u = \begin{pmatrix} 1-x^2 \\ 1-x^2 \\ \vdots \\ 1-x^2 \end{pmatrix}$$

and find that $u(-1) = u(1) = 0$ and

$$\begin{aligned} M[u](t) &= \int_{-1}^{1} \left[\alpha \left(\frac{\partial u}{\partial x} \right)^T \frac{\partial u}{\partial x} - \beta\, u^T u \right] dx \\ &= \int_{-1}^{1} \left[4\alpha m x^2 - \beta m (1-x^2)^2 \right] dx \\ &= m \left(\frac{8}{3}\alpha - \frac{16}{15}\beta \right) < 0. \end{aligned}$$

Hence, it is easily seen that

$$\lim_{t\to\infty} \int_T^t M[u](s)ds = -\infty$$

for any $T > 0$. Theorem 3 implies that if W is a solution of (15) satisfying (i)–(iv) of Theorem 2, then $\|W\|$ is unbounded in $(-1,1) \times (0,\infty)$. For example, $W = e^{\beta t}\mathrm{I}_m$ is such a solution. In fact, we have that $\|W\| = \sqrt{m}\, e^{\beta t}$.

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