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Uncertain chance-constrained programming model for project scheduling problem

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In this paper, we consider an uncertain project scheduling problem, in which activity durations, with no historical data generally, are estimated by belief degrees and assumed to be uncertain variables. To achieve different management goals, we build three uncertain chance-constrained programming models for project scheduling problem, in which the chance constraint must reach a predetermined confidence level. Moreover, these models can all be transformed to their crisp forms, and an intelligent algorithm is designed to search the optimal schedule. Finally, a numerical example is presented to illustrate the usefulness of the proposed model.

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1. Introduction

Project scheduling problem is to determine the schedule of allocating resources, especially the loan allocation times, so as to balance the total cost and the completion time of a project. Researchers have studied project scheduling problem more and more widely since 1960s. Kelley (1961, 1963) originally proposed the project scheduling problem under deterministic environment, in which the activity duration was assumed to deterministic. From then on, deterministic models were continuously discussed by other researchers (Demeulemeester, 1995; Vanhoucke, 2008).

However, the activity durations in the project scheduling problem are always variational owing to many indeterministic factors, such as the weather change, the fluctuations of machine prices and the absent workers. As a result, the schedules gained from deterministic models might perform very poor. Assume that activity duration was a random variable. Probability theory, which is based on probability measure, was first introduced to project scheduling problem by Freeman (1960). After that, Charnes *et al* (1964) employed stochastic chance-constrained programming philosophy into project scheduling problem. In 1984, Möhring (1984) modeled project scheduling problems into two aspects: minimize the total cost of the project under the constraints of the completion time, or minimize the completion time under the constraints of the total cost. From then on, Golenko-Ginzburg and Gonik (1997) established an expected value model for solving a simple type of project scheduling problem. In 2005, Ke and Liu (2005) built three stochastic models for PSP. Interested readers may refer to Loostma (1966), Parks and Ramsing (1969), and Özdamar and Alanya (2001), to study various types of stochastic project scheduling problem, in which different probability distributions were employed to describe the activity durations.

As far as we know, a premise of applying probability theory is that the available probability distribution is close enough to the true frequency. Thus, we have to rely on a great quantity of observed data and employ statistics to get the probability distribution. However, each project is somewhat unique and its activity durations are usually lack of historical data in real world. In this case, probability theory applied into project scheduling problem may be out of work, and belief degrees given by the experienced project managers or experts might be employed to estimate values or distributions. Possibility theory first attempted to model the belief degree based on possibility measure. In 1979, Prade (1979) applied fuzzy set to PSP. Then, Dubois *et al* (1995) built a chance constraint model considering both fuzzy duration times and the fuzzy due dates. In the next few years, Wang (2002) developed a fuzzy beam search approach for solving product development project scheduling, Long and Öhsato (2008) performed a fuzzy critical chain method for fuzzy resource-constrained project scheduling problem, and Ke and Liu (2010) established three models for fuzzy project scheduling problem.

Belief degree used to be considered as subjective probability or fuzzy concept. However, surveys showed that human beings usually estimate a much wider range of values than the object actually takes (Liu, 2015). This conservatism of human beings makes the belief degree deviate far from the frequency. As a

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result, the belief degree cannot be treated as a random variable, otherwise some counterintuitive phenomena may happen. On the other hand, for any events A and B no matter whether they are independent or not, possibility measure (Pos) satisfies $Pos\{A \cup B\} = Pos\{A\} \vee Pos\{B\}$. But, the measure of a union of events is usually greater than the maximum of the measures of individual events when they are not independent (Liu, 2015). Therefore, these indeterminacies described by human belief degrees cannot be treated as random variables or fuzzy variables. When indeterminacy behaves neither randomness nor fuzzyness, uncertainty theory based on uncertain measure, founded by Liu (2007) in 2007, can be applied. Different from probability measure satisfying normality, nonnegativity and additivity axioms, uncertain measure follows normality, duality, subadditivity and product axioms. Nowadays, uncertainty theory has become a branch of axiomatic mathematics and has been successfully applied in plenty of practical problems: inventory problem (Qin and Kar, 2013), currency option pricing problem (Wang and Ning, 2016), interest rate problem (Zhang *et al,* 2016), maximum flow problem (Han *et al,* 2014) among others.

With regard to project scheduling problem under uncertain environment, Liu (2009) first built an uncertain project scheduling model, of which the objective was to minimize the total cost under the constraints of the completion time. Zhang and Chen (2012) built expected time minimized model for PSP in 2012. Ji and Yao (2014) recently gave an uncertain multi-objective programming model for PSP. In this paper, we further study the uncertain project scheduling problem and build three models based on uncertain chance-constrained programming with the assumptions that the constraints will maintain at a predetermined confidence level. Meanwhile, we set the objective as minimizing the total cost or completion time at a preset confidence level.

To investigate the aforementioned problem carefully, the remainder of this paper is organized as follows. The next section is intended to introduce some basic concepts and theorems of uncertainty theory and uncertain chance-constrained programming. Section 3 describes the project scheduling problem under uncertain environment in detail. In Section 4, an α-cost model for project scheduling problem is presented and a genetic algorithm is introduced to solve the model. Then, a numerical example is given to illustrate the usefulness of the α-cost model in Section 5. To achieve different management goals, Section 6 presents other two extended optimization models for project scheduling problem. Finally, some conclusions are covered in Section 7.

For convenience, some notations used in the later sections are introduced as follows:

Λ^c: the complement of Λ;
$a \wedge b$: $a \wedge b = \min(a, b)$;
$a \vee b$: $a \vee b = \max(a, b)$;
$\lceil a \rceil$: the minimal integer greater than or equal to a;
$\mathcal{M}$: uncertain measure;
Pr: probability measure;
$(\Gamma, \mathcal{L}, \mathcal{M})$: uncertainty space.

2. Preliminary

In this section, we make a brief overview of some foundational concepts and theorems of uncertainty theory and introduce uncertain chance-constrained programming which is used in the later sections.

2.1. Uncertainty theory

Definition 2.1 (Liu, 2007) Let Γ be a nonempty set, and $\mathcal{L}$ be a σ-algebra on Γ. A set function $\mathcal{M}$ is called an *uncertain measure* if it satisfies the following axioms:

Axiom 1 (*Normality Axiom*) $\mathcal{M}\{\Gamma\} = 1$;

Axiom 2 (*Duality Axiom*) $\mathcal{M}\{\Lambda\} + \mathcal{M}\{\Lambda^c\} = 1$ for any $\Lambda \in \mathcal{L}$;

Axiom 3 (*Subadditivity Axiom*) For every countable sequence of events $\Lambda_1, \Lambda_2, \ldots$, we have

$$\mathcal{M}\left\{\bigcup_{i=1}^{\infty} \Lambda_i\right\} \leq \sum_{i=1}^{\infty} \mathcal{M}\{\Lambda_i\}.$$

Besides, the product uncertain measure on the product σ-algebre is defined by Liu (2009) as follows:

Axiom 4 (*Product Axiom*) Let $(\Gamma_k, \mathcal{L}_k, \mathcal{M}_k)$ be uncertainty spaces for $k = 1, 2, \ldots$ The product uncertain measure $\mathcal{M}$ is an uncertain measure satisfying

$$\mathcal{M}\left\{\prod_{k=1}^{\infty} \Lambda_k\right\} = \bigwedge_{k=1}^{\infty} \mathcal{M}_k\{\Lambda_k\}$$

where Λ_k are arbitrarily chosen events from $\mathcal{L}_k$ for $k = 1, 2, \ldots$, respectively.

Definition 2.2 (Liu, 2007) An *uncertain variable* ξ is a measurable function from an uncertainty space $(\Gamma, \mathcal{L}, \mathcal{M})$ to the set of real numbers, i.e., for any Borel set B of real numbers, the set

$$\{\xi \in B\} = \{\gamma \in \Gamma | \xi(\gamma) \in B\}$$

is an event.

Definition 2.3 (Liu, 2007) A k-dimensional *uncertain vector* is a function ξ from an uncertainty space $(\Gamma, \mathcal{L}, \mathcal{M})$ to the set of k-dimensional real vectors such that is an event for any Borel set B of k-dimensional real vectors.

Uncertain variable is used to represent quantities in uncertainty. Sometimes, to model the real-life uncertain optimization problem, it is sufficient to know the uncertainty distribution rather than the uncertain variable itself.

Definition 2.4 (Liu, 2007) The *uncertainty distribution* Φ of an uncertain variable ξ is defined by

$$\Phi(x) = \mathcal{M}\{\xi \le x\}, \quad \forall x \in \Re.$$

In addition, the inverse function Φ^{-1} of Φ is called the *inverse uncertainty distribution* of ξ.

Example 2.1 *An uncertain variable ξ is said to be linear if its uncertainty distribution is*

$$\Phi(x) = \begin{cases} 0, & \text{if } x < a \\ (x-a)/(b-a), & \text{if } a \le x \le b \\ 1, & \text{if } x > b \end{cases}$$

which is denoted by $\xi \sim \mathcal{L}(a,b)$. Apparently, the inverse uncertainty distribution of $\mathcal{L}(a,b)$ is

$$\Phi^{-1}(\alpha) = \alpha(b-a) + a, \quad \alpha \in [0,1].$$

Definition 2.5 (Liu, 2009) The uncertain variables $\xi_1, \xi_2, \ldots, \xi_n$ are said to be *independent* if

$$\mathcal{M}\left\{\bigcap_{i=1}^{n}(\xi_i \in B_i)\right\} = \bigwedge_{i=1}^{n}\mathcal{M}\{\xi_i \in B_i\}$$

for any Borel sets $B_1, B_2, \ldots, B_n$ of real numbers.

Theorem 2.1 (Liu, 2015) *Let $\xi_1, \xi_2, \ldots, \xi_n$ be independent uncertain variables with regular uncertainty distributions $\Phi_1, \Phi_2, \ldots, \Phi_n$, respectively. If the function $f(x_1, x_2, \ldots, x_n)$ is strictly increasing with respect to $x_1, x_2, \ldots, x_m$ $(m \le n)$ and strictly decreasing with respect to $x_{m+1}, x_{m+2}, \ldots, x_n$, $\xi = f(\xi_1, \xi_2, \ldots, \xi_n)$ has an inverse uncertainty distribution*

$$\Psi^{-1}(\alpha) = f\big(\Phi_1^{-1}(\alpha), \ldots, \Phi_m^{-1}(\alpha), \Phi_{m+1}^{-1}(1-\alpha), \ldots, \Phi_n^{-1}(1-\alpha)\big),$$

and

$$\mathcal{M}\{f(\xi_1, \xi_2, \ldots, \xi_n) \le 0\} \ge \alpha \quad (\alpha \in [0,1])$$

holds if and only if

$$f\big(\Phi_1^{-1}(\alpha), \Phi_2^{-1}(\alpha), \ldots, \Phi_m^{-1}(\alpha), \Phi_{m+1}^{-1}(1-\alpha), \Phi_{m+2}^{-1}(1-\alpha), \ldots, \Phi_n^{-1}(1-\alpha)\big) \le 0.$$

2.2. Uncertain chance-constrained programming

Assume that $\boldsymbol{x}$ is a decision vector, $\boldsymbol{\xi}$ is an uncertain vector, $f(\boldsymbol{x}, \boldsymbol{\xi})$ is a return function, and $g_j(\boldsymbol{x}, \boldsymbol{\xi})$ are constraint functions, $j = 1, 2, \ldots, p$. Since the uncertain constraints $g_j(\boldsymbol{x}, \boldsymbol{\xi}) \le 0$ do not define a deterministic feasible set, it is naturally desired that the uncertain constraints hold with chance β_j, where β_j is a predetermined confidence level. That is, a chance constraint can be expressed as $\mathcal{M}\{g_j(\boldsymbol{x}, \boldsymbol{\xi}) \le 0\} \ge \beta_j$, $j = 1, 2, \ldots, p$. Based on this idea, uncertain chance-constrained programming was proposed by Liu (2009) as the optimization theory in generally uncertain environments, and its general form can be written as:

$$\begin{cases} \max_{\boldsymbol{x}} \bar{f} \\ \textit{subject to}: \\ \quad \mathcal{M}\{f(\boldsymbol{x}, \boldsymbol{\xi}) \le \bar{f}\} \ge \alpha, \\ \quad \mathcal{M}\{g_j(\boldsymbol{x}, \boldsymbol{\xi}) \le 0\} \ge \beta_j, \quad j = 1, 2, \ldots, p \end{cases} \tag{1}$$

where α is also a predetermined confidence level, and $\alpha, \beta_j \in [0,1]$, $j = 1, 2, \ldots, p$.

3. Problem description

In general, a project can be described by a network $G = (\mathcal{V}, \mathcal{A})$, where $\mathcal{V} = \{1, 2, \ldots, n+1\}$ is the set of nodes and $\mathcal{A}$ is the set of arcs. In the network G, node 1 and $n+1$ mark the beginning and the end of the project, respectively. An element $k \in \mathcal{V}$ denotes a milestone of the project, and an element $(i,j) \in \mathcal{A}$ denotes an activity from the milestones i to j.

Project scheduling is a complex process involving many activities that require optimizing and can be regarded as a dynamic decision process. The manager makes decisions on which activity to start at each decision times subject to some constraints. In many projects, especially the large-scale ones, loan is always the source for the capital. Hence, how to make the schedule of loan time for the activity and economize the payout becomes significant to the decision-maker. That is, project scheduling problem considered in this paper is to schedule the loan allocation time for a project such that the total cost and the completion time of the project are balanced under some constraints. The solution of project scheduling problem is an activity order list which depicts the starting time of each activity.

In reality, a project usually undergoes a wide variety of disruptions such as fluctuant execution durations, unavailable resources, overdue materials and the schedules gained from deterministic models might perform very poor as a result. In order to deal with more realistic circumstances, where the variability in durations should not be neglected, stochastic project scheduling problem has been developed, in which activity durations are represented as random variables.

However, the actual activity duration of the majority of the projects can only be attained after it is completed. Hence, majority of the projects are unique in nature and the activity durations are usually lack of historical data in the real world. According to the information incompleteness, the manager can merely apply partial information which is available before or at the time he makes decision. In such cases, the manager has to rely on the belief degree given by the experienced project managers or experts. Since the indeterminacies described by the belief degrees cannot be treated as random variables, employing uncertain variables to denote the activity durations is feasible. Besides, in order to simplify, the other assumptions and some simplifications in the project scheduling problem are stated as follows:

1. each activity can be processed only when the loan needed is allocated and all the foregoing activities are finished;
2. each activity should be processed without interruption;
3. the costs of all the activities are obtained via loans with a given interest rate;
4. the cost needed for each activity is a constant;
5. all uncertain variables "activity durations" are assumed to be independent.

Hereafter, in order to model the project scheduling problem with uncertainty theory conveniently, we introduce the following indices and parameters, which will be used in the following formulas and models:

ξ_{ij}: the uncertain duration time of activity (i,j);

$\boldsymbol{\xi} = \{\xi_{ij} | (i,j) \in \mathcal{A}\}$: the vector of uncertain duration times;

x_{ij}: decision variable representing the loan time of activity (i,j), and we assume that all the decision variables are nonnegative integers;

$\boldsymbol{x} = \{x_{ij} | (i,j) \in \mathcal{A}\}$: the vector of decision variables;

Φ_{ij}: the uncertainty distribution of ξ_{ij};

$S_{ij}(\boldsymbol{x}, \boldsymbol{\xi})$: the starting time of activity (i,j), and the starting time of activity $(1,j) \in \mathcal{A}$ is defined as $S_{1j}(\boldsymbol{x}, \boldsymbol{\xi}) = 0$, which means that the starting time of the total project is assumed to be 0;

Ψ_{ij}: the uncertainty distribution of $S_{ij}(\boldsymbol{x}, \boldsymbol{\xi})$;

$T(\boldsymbol{x}, \boldsymbol{\xi})$: the completion time of the project G;

Ψ: the uncertainty distribution of $T(\boldsymbol{x}, \boldsymbol{\xi})$;

c_{ij}: a given loan amount of activity (i,j);

r: the loan interest rate;

$C(\boldsymbol{x}, \boldsymbol{\xi})$: the total cost of the project G;

Υ: the uncertainty distribution of $C(\boldsymbol{x}, \boldsymbol{\xi})$.

Thus, the project scheduling problem is to find the optimal schedule $\boldsymbol{x}$ so that the total cost $C(\boldsymbol{x}, \boldsymbol{\xi})$ or the completion time $T(\boldsymbol{x}, \boldsymbol{\xi})$ can be minimized. Based on the above assumptions, we then formulate the completion time and the total cost of uncertain project scheduling problem. The starting time of activity $(1,j) \in \mathcal{A}$ is its loan time, denoted by

$$S_{1j}(\boldsymbol{x}, \boldsymbol{\xi}) = x_{1j}, \quad (1,j) \in \mathcal{A}.$$

The starting time of the whole project is the minimal value of $S_{1j}(\boldsymbol{x}, \boldsymbol{\xi})$. And the starting time of activity (i,j) can be expressed by the following recursive formula

$$S_{ij}(\boldsymbol{x}, \boldsymbol{\xi}) = x_{ij} \vee \max_{(k,i) \in \mathcal{A}} (S_{ki}(\boldsymbol{x}, \boldsymbol{\xi}) + \xi_{ki}).$$

By a recursive process, we can derive the completion time $T(\boldsymbol{x}, \boldsymbol{\xi})$ of the total project by

$$T(\boldsymbol{x}, \boldsymbol{\xi}) = \max_{(k,n+1) \in \mathcal{A}} \left(S_{k,n+1}(\boldsymbol{x}, \boldsymbol{\xi}) + \xi_{k,n+1}\right). \tag{2}$$

Applying the compound interest formula to calculate the future value of all the loans, the total cost of the project can be written as

$$C(\boldsymbol{x}, \boldsymbol{\xi}) = \sum_{(i,j) \in \mathcal{A}} c_{ij} (1 + r)^{\lceil (T(\boldsymbol{x}, \boldsymbol{\xi}) - x_{ij}) \rceil}. \tag{3}$$

4. Uncertain chance-constrained programming model for PSP

After establishing the completion time and the total cost formulas in Section 3, we can model the project scheduling problem in many ways according to different management goals. For example, part of the decision-makers may expect to optimize expected objectives and establish the expected value model for PSP. Although expected value model is widely used for solving various types of practical problems, it does not take into account the uncertain measure of disobeying the constraints. In fact, sometimes the managers have to consider the risk, referred to as the uncertain measure that some unfavorable event will occur. That is, the management goal may include the condition of satisfying some chance constraints with at least some predetermined confidence levels. To achieve this management goal, we link with uncertain chance-constrained programming (UCCP) which is applied to solve these practical optimization problems. In this section, we first give a definition of α-cost and then establish an α-cost model based on UCCP.

Definition 4.1 The α-cost of a project is defined as

$$\min\{\bar{C} | \mathcal{M}\{C(\boldsymbol{x}, \boldsymbol{\xi}) \le \bar{C}\} \ge \alpha\}$$

where α is a predetermined confidence level and $\alpha \in [0, 1]$.

In practice, if the manager wants to seek maximum benefits before the due date T_0, we tend to minimize the α-cost of the project under the completion time chance constraint with a predetermined confidence level. Following the idea of UCCP, we can present an α-cost model

$$\begin{cases} \min_{\boldsymbol{x}} \bar{C} \\ subject \;\; to: \\ \quad \mathcal{M}\{C(\boldsymbol{x},\xi) \le \bar{C}\} \ge \alpha, \\ \quad \mathcal{M}\{T(\boldsymbol{x},\xi) \le T_0\} \ge \beta, \\ \quad \boldsymbol{x} \ge 0, \;\; integers \end{cases} \tag{4}$$

where $\alpha, \beta \in [0,1]$ are two predetermined confidence levels. For each fixed feasible solution $\boldsymbol{x}$, the objective value $\bar{C}$ should be the minimum value that the cost function $C(\boldsymbol{x},\xi)$ achieves with at least predetermined confidence level α.

Since the function $T(\boldsymbol{x},\xi)$ (see expression (2)) is increasing with respect to ξ_{ij} ($(i,j) \in \mathcal{A}$) and uncertain variables (ξ_{ij}) are independent, we have the inverse uncertainty distribution of the completion time $T(\boldsymbol{x},\xi)$ by Theorem 2.1

$$\Psi^{-1}(\boldsymbol{x},\alpha) = \max_{(k,n+1)\in\mathcal{A}} \left(\Psi^{-1}_{k,n+1}(\boldsymbol{x},\alpha) + \Phi^{-1}_{k,n+1}(\boldsymbol{x},\alpha) \right). \tag{5}$$

Note that the total cost of the project

$$C(\boldsymbol{x},\xi) = \sum_{(i,j)\in\mathcal{A}} c_{ij}(1+r)^{\lceil (T(\boldsymbol{x},\xi)-x_{ij}) \rceil}$$

is increasing with respect to $T(\boldsymbol{x},\xi)$, the total cost $C(\boldsymbol{x},\xi)$ has an inverse uncertainty distribution

$$\Upsilon^{-1}(\boldsymbol{x},\alpha) = \sum_{(i,j)\in\mathcal{A}} c_{ij}(1+r)^{\lceil (\Psi^{-1}(\boldsymbol{x},\alpha)-x_{ij}) \rceil}. \tag{6}$$

By Theorem 2.1, the constraints in model (4)

$$\mathcal{M}\{C(\boldsymbol{x},\xi) \le \bar{C}\} \ge \alpha \;\; \text{and} \;\; \mathcal{M}\{T(\boldsymbol{x},\xi) \le T_0\} \ge \beta$$

are, respectively, equivalent to

$$\Upsilon^{-1}(\boldsymbol{x},\alpha) \le \bar{C} \;\; \text{and} \;\; \Psi^{-1}(\boldsymbol{x},\beta) \le T_0$$

where $\Upsilon^{-1}(\boldsymbol{x},\alpha)$ and $\Psi^{-1}(\boldsymbol{x},\beta)$ are defined by (6) and (5), respectively.

Hence, model (4) can be transformed into an equivalent crisp programming model as below,

$$\begin{cases} \min_{\boldsymbol{x}} \; \Upsilon^{-1}(\boldsymbol{x},\alpha) \\ subject \;\; to: \\ \quad \Psi^{-1}(\boldsymbol{x},\beta) \le T_0, \\ \quad \boldsymbol{x} \ge 0, \;\; integers. \end{cases} \tag{7}$$

Hereafter, a genetic algorithm is introduced to solve model (7), which contains the following steps.

Step 1: Representation

Use $\boldsymbol{x} = \{x_{ij} | (i,j) \in \mathcal{A}\}$ as a chromosome to represent a solution, where x_{ij} denotes the allocation loan time of activity $(i,j) \in \mathcal{A}$.

Step 2: Initialization

Generate x_{ij} randomly for all $(i,j) \in \mathcal{A}$, and obtain a vector $\boldsymbol{x} = \{x_{ij} | (i,j) \in \mathcal{A}\}$.

Give the population size of one generation *pop_size*, a crossover probability P_c and a mutation probability P_m.

Give α and β, calculate $\Psi^{-1}(\boldsymbol{x},\beta)$, and verify whether it is less than T_0.

If so, the vector $\boldsymbol{x}$ is feasible, and we get a chromosome; otherwise, regenerate $\boldsymbol{x}$.

Repeat this process for *pop_size* times, and we get *pop_size* chromosomes $\boldsymbol{x}_1, \boldsymbol{x}_2, \ldots, \boldsymbol{x}_{pop\_size}$.

Step 3: Crossover

For each chromosome $\boldsymbol{x}_i$, generate a random number $u_i \in [0,1], i = 1,2,\ldots,pop\_size$.

If $u_i < P_c$, $\boldsymbol{x}_i$ is selected as a parent for crossover operation.

Divide all the selected chromosomes into some groups such that each group contains only two chromosomes, and perform crossover operations on these groups. Redo the crossover operation until obtain the feasible children.

Step 4: Mutation

For each chromosome $\boldsymbol{x}_i$, generate a random number $u_i \in [0,1], i = 1,2,\ldots,pop\_size$.

If $u_i \le P_m$, $\boldsymbol{x}_i$ is selected for mutation operation. Generate a random vector $\boldsymbol{d} = \{d_{ij} | (i,j) \in \mathcal{A}\}$.

If $\boldsymbol{x}_i + \boldsymbol{d}$ is feasible, replace $\boldsymbol{x}_i$ with $\boldsymbol{x}_i + \boldsymbol{d}$; otherwise, generate a random number $u \in [0,1]$, and verify whether the vector $\boldsymbol{x}_i + u \times \boldsymbol{d}$ is feasible.

If $\boldsymbol{x}_i + u \times \boldsymbol{d}$ is feasible, replace $\boldsymbol{x}_i$ with $\boldsymbol{x}_i + u \times \boldsymbol{d}$; otherwise, regenerate the random number u until $\boldsymbol{x}_i + u \times \boldsymbol{d}$ is feasible.

Step 5: Evaluation

Calculate $\Upsilon^{-1}(\boldsymbol{x}_i,\alpha), i = 1,2,\ldots,pop\_size$. According to $\Upsilon^{-1}(\boldsymbol{x}_i,\alpha)$ in an ascending order, rearrange $\boldsymbol{x}_1, \boldsymbol{x}_2, \ldots, \boldsymbol{x}_{pop\_size}$ and denote them by $\boldsymbol{x}'_1, \boldsymbol{x}'_2, \ldots, \boldsymbol{x}'_{pop\_size}$. Then, the fitness of $\boldsymbol{x}'_i$ is $Eval(\boldsymbol{x}'_i) = q(1-q)^{i-1}$ for $i = 1,2,\ldots,pop\_size$, where $q \in [0,1]$ is a given parameter.

Step 6: Selection

Calculate the cumulative fitness $q_k = \sum_{i=1}^{k} Eval(\boldsymbol{x}'_i), \;\; k = 1,2,\ldots,pop\_size$, and set $q_0 = 0$.

Repeat the following process for *pop_size* times to get a new generation of the chromosomes:

Generate a random number $u \in (0, q_{pop\_size}]$ and select the chromosome $\boldsymbol{x}_k$ if $q_{k-1} < u \le q_k$.

Step 7: Repetition

Repeat **Steps 3–6** for a number of cycles, and select the best chromosome as the optimal solution for model (7).

Remark 4.1 As we know, the fundamental assumption of applying probability theory is that the estimated probability is close enough to the long-run cumulative frequency. In practice, this means that we need sufficient statistical data. When the managers own the available and sufficient information of the activity duration in PSP, the activity durations can be represented as random variables and stochastic chance-constrained programming model for PSP is feasible. Ke and Liu (2005) proposed a stochastic chance-constrained programming model

$$\begin{cases} \min_{\boldsymbol{x}} \bar{C} \\ subject \quad to: \\ \quad Pr\{C(\boldsymbol{x},\xi) \le \bar{C}\} \ge \alpha, \\ \quad Pr\{T(\boldsymbol{x},\xi) \le T_0\} \ge \beta, \\ \quad \boldsymbol{x} \ge 0, \; integers. \end{cases} \tag{8}$$

Since model (8) cannot be converted into an equivalent crisp programming model, we need to carry out the additional stochastic simulation on the constraint conditions. Successively, we have to embed the stochastic simulation into genetic algorithm to design a hybrid intelligent algorithm to search for the optimal schedule for PSP. However, we can solve uncertain chance-constrained programming model (4) only by means of genetic algorithm. Compared with model (4), solving model (8) greatly increases the computational complexity.

5. Numerical example

In this section, we consider a project schedule problem containing 8 milestones and 10 activities as shown in Figure 1. For assumption of uncertainty, the activity durations are estimated by experts and characterized as uncertain variables. Interested readers can consult Liu (2015) for more details about how to use experts belief degrees to estimate uncertain distributions. The activity durations and the costs needed for the relevant activities in the project are presented in Table 1, and the loan interest rate is set to $r = 0.6\%$ according to some practical project cases.

If the project manager expects to finish this project in $T_0 = 26$ time units with the confidence level $\beta = 0.85$ and meanwhile seek the minimum α-cost ($\bar{C}$), model (4) is needed and applied. Assume the confidence level $\alpha = 0.9$, the 0.9-cost model for this project scheduling problem is:

$$\begin{cases} \min_{\boldsymbol{x}} \quad \bar{C} \\ subject \quad to: \\ \quad \mathcal{M}\{C(\boldsymbol{x},\xi) \le \bar{C}\} \ge 0.9, \\ \quad \mathcal{M}\{T(\boldsymbol{x},\xi) \le 26\} \ge 0.85, \\ \quad \boldsymbol{x} \ge 0, \; integers. \end{cases} \tag{9}$$

According to model (5), model (9) is equivalent to

$$\begin{cases} \min_{\boldsymbol{x}} \quad \Upsilon^{-1}(\boldsymbol{x}, 0.9) \\ subject \quad to: \\ \quad \Psi^{-1}(\boldsymbol{x}, 0.85) \le 26, \\ \quad \boldsymbol{x} \ge 0, \; integers. \end{cases} \tag{10}$$

In the following, a genetic algorithm is employed to search for the optimal solution for model (10). There exist three parameters given in advance in the genetic algorithm: the population size of one generation *pop_size*, the probability of crossover P_c and the probability of mutation P_m. To demonstrate the effectiveness of the algorithm, the above three parameters will be given in several values and different optimal results are compared and evaluated in Table 2.

From Table 2, it can be seen that all the optimal costs differ little from each other. Therefore, in order to further compare the difference among these costs, we introduce an evaluation index called relative error, which corresponds with the last column in Table 2. The relative error is calculated by the formula

$$|\text{actual value} - \text{optimal value}|/\text{optimal value} \times 100\%,$$

where the optimal value means the minimal one of all the costs in Table 2. It follows from Table 2 that the relative error does not exceed 0.47% when different parameters are selected, which actually implies that the intelligent algorithm is effective to solve model (4).

As can be seen from Table 2, the total cost reaches its minimum at $pop\_size = 80$, $P_m = 0.1$ and $P_c = 0.4$. At the same time, the optimal schedule for allocating the loan times of activities is shown in Table 3. In addition, the expected total cost of the project in Figure 1 is 7957.6, and the expected completion time is 22 time units.

Since there are two parameters α and β in model (4), we had better to analyze the relationship between the objective value ($\bar{C}$) and parameters (α, β). Next, we first consider one case as under different β and fixed α ($\alpha = 0.9$), and the result of the objective value $\bar{C}$ is displayed in Table 4; then, consider other case as under different α and fixed β ($\beta = 0.85$), and obtain the result of the objective value $\bar{C}$ in Table 5.

From Table 4, we can find that the result of the objective value $\bar{C}$ does not exhibit monotonic trend with the change of the parameter β. However, Table 5 shows that the result of the objective value $\bar{C}$ obviously exhibits monotone increasing trend with the increase in the parameter α. In other words, the objective value in model (4) is mainly associated with the predetermined confidence level α for the total cost. Besides, the parameter α in model (4) is usually fixed with subjective prior knowledge and it usually depends on the specific practical problem.

6. Extended project scheduling models

We have discussed the α-cost model for PSP in Section 4, in which the α-cost is to be minimized under the total cost chance constraint. However, the α-cost model cannot achieve all practical management goals. Different project managers may have different goals. To meet other goals, this section will propose another two optimization models for PSP based on

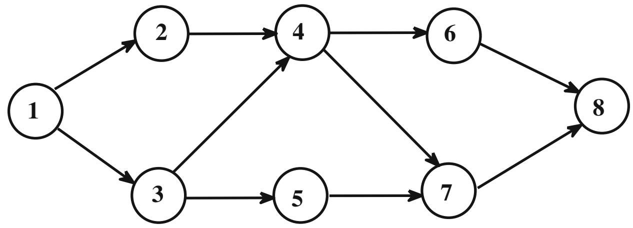

Figure 1 A project with 8 milestones and 10 activities.

UCCP. More specifically, if the manager wishes to finish the project as soon as possible, we optimize the schedule of allocating loans in order to minimize the total time; if the manager expects to pursue the maximum interests as well as to finish as soon as possible, we need to optimize the schedule in order to minimize the total cost and completion time under uncertain measure constraints. To satisfy these two requirements, we first define a concept of β-time and then establish a β-time model for PSP based on UCCP. Finally, connecting α-cost model and β-time model, we propose a multi-objective uncertain project scheduling model.

Definition 6.1 The β-time of a project is defined as

$$\min\{\bar{T}|\mathcal{M}\{T(\boldsymbol{x},\xi)\leq\bar{T}\}\geq\beta\}$$

where β is a predetermined confidence level and $\beta\in[0,1]$.

When the project managers hope to finish the project as soon as possible under the budget C_0, we tend to minimize the β-time of the project under the total cost chance constraint with a predetermined confidence level. In light of the idea of UCCP, we propose a β-time model

Table 1 Duration times and costs of all activities

Activity	(1,2)	(1,3)	(2,4)	(3,4)	(3,5)
Duration time	$\mathcal{L}(3,5)$	$\mathcal{L}(2,4)$	$\mathcal{L}(4,5)$	$\mathcal{L}(3,8)$	$\mathcal{L}(5,7)$
Cost	1000	800	900	1100	700
Activity	(4,6)	(4,7)	(5,7)	(6,8)	(7,8)
Duration time	$\mathcal{L}(3,6)$	$\mathcal{L}(4,6)$	$\mathcal{L}(5,6)$	$\mathcal{L}(3,7)$	$\mathcal{L}(4,7)$
Cost	900	1200	600	800	1100

Table 2 Comparison of optimal costs of model (10)

pop_size	50	50	50	80	80	80
P_c	0.4	0.5	0.5	0.4	0.5	0.4
P_m	0.05	0.05	0.1	0.05	0.05	0.1
Cost	7985.9	7988.3	7995.3	7982.2	7969.9	7957.6
Relative error (%)	0.36	0.39	0.47	0.31	0.15	0.00

Table 3 Optimal allocation times of the loans in model (10)

Activity	(1,2)	(1,3)	(2,4)	(3,4)	(3,5)
Allocation time	1	1	6	5	4
Activity	(4,6)	(4,7)	(5,7)	(6,8)	(7,8)
Allocation time	10	9	11	15	16

Table 4 The objective value $\bar{C}$ in model (4) under different β and fixed $\alpha\,(\alpha=0.9)$

β	0.7	0.75	0.8	0.85	0.9	0.95
$\bar{C}$	8069.3	8093.5	8074.9	8084.4	8067.5	8081.1

Table 5 The objective value $\bar{C}$ in model (4) under different α and fixed $\beta\,(\beta=0.85)$

α	0.7	0.75	0.8	0.85	0.9	0.95
$\bar{C}$	7991.0	8010.5	8042.6	8064.2	8081.9	8103.9

$$\begin{cases} \min_{\boldsymbol{x}} \bar{T} \\ subject \quad to: \\ \quad \mathcal{M}\{C(\boldsymbol{x},\xi) \leq C_0\} \geq \alpha, \\ \quad \mathcal{M}\{T(\boldsymbol{x},\xi) \leq \bar{T}\} \geq \beta, \\ \quad \boldsymbol{x} \geq 0, \ integers \end{cases} \tag{11}$$

where $\alpha, \beta \in [0,1]$ are two predetermined confidence levels. For each fixed feasible solution $\boldsymbol{x}$, the objective value $\bar{T}$ should be the minimum value that the time function $T(\boldsymbol{x},\xi)$ achieves with at least predetermined confidence level β.

Similar to model (4), the equivalent crisp programming model of model (11) is

$$\begin{cases} \min_{\boldsymbol{x}} \quad \Psi^{-1}(\boldsymbol{x},\beta) \\ subject \quad to: \\ \quad \Upsilon^{-1}(\boldsymbol{x},\alpha) \leq C_0, \\ \quad \boldsymbol{x} \geq 0, \ integers. \end{cases} \tag{12}$$

When the project managers not only expect to complete the project as quickly as possible, but also seek to minimize a cost value at the same time, we establish a multi-objective uncertain chance-constrained programming model

$$\begin{cases} \min_{\boldsymbol{x}} \quad \{\bar{C},\bar{T}\} \\ subject \quad to: \\ \quad \mathcal{M}\{C(\boldsymbol{x},\xi) \leq \bar{C}\} \geq \alpha, \\ \quad \mathcal{M}\{T(\boldsymbol{x},\xi) \leq \bar{T}\} \geq \beta, \\ \quad \boldsymbol{x} \geq 0, \ integers. \end{cases} \tag{13}$$

According to Theorem 2.1, model (13) is equivalent to

$$\begin{cases} \min_{\boldsymbol{x}} \quad \{\Upsilon^{-1}(\boldsymbol{x},\alpha), \Psi^{-1}(\boldsymbol{x},\beta)\} \\ subject \quad to: \\ \quad \boldsymbol{x} \geq 0, \ integers. \end{cases} \tag{14}$$

The most common approach to solve multi-objective optimization is the weighted sum method. Thus, employing the weighted sum method, model (14) could be transformed into the following programming model

$$\begin{cases} \min_{\boldsymbol{x}} \quad \omega_1 \Upsilon^{-1}(\boldsymbol{x},\alpha) + \omega_2 \Psi^{-1}(\boldsymbol{x},\beta) \\ subject \quad to: \\ \quad \boldsymbol{x} \geq 0, \ integers \end{cases} \tag{15}$$

where ω_1 and ω_2 $(\omega_1, \omega_2 > 0)$ are the weights of the first objective $\Upsilon^{-1}(\boldsymbol{x},\alpha)$ and the second objective $\Psi^{-1}(\boldsymbol{x},\beta)$, respectively.

Remark 6.1 Similar to model (4), a genetic algorithm can also be designed to solve models (12) and (15). Here we no longer elaborate it in detail.

7. Conclusions

The paper studied the project scheduling problem under uncertain environment, of which the activity duration was described as an uncertain variable. Based on uncertain chance-constrained programming, the α-cost model, β-time model and a multi-objective programming model were built to satisfy the different management goals. Moreover, the proposed uncertain project scheduling models can all be transformed into corresponding crisp equivalent form, respectively. For purpose of capturing optimal strategy, an intelligent algorithm was designed to solve the α-cost model, and a numerical example was provided to illustrate the effectiveness of this model and algorithm as well.

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