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Uncertain hypothesis test for uncertain differential equations

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Abstract

Uncertain hypothesis test is a statistical tool that uses uncertainty theory to determine whether some hypotheses are correct or not based on observed data. As an application of uncertain hypothesis test, this paper proposes a method to test whether an uncertain differential equation fits the observed data or not. In order to demonstrate the test method, some numerical examples are provided. Finally, both uncertain currency model and stochastic currency model are used to model US Dollar to Chinese Yuan (USD–CNY) exchange rates. As a result, it is shown that the uncertain currency model fits the exchange rates well, but the stochastic currency model does not.

Keywords Uncertainty theory · Uncertain statistics · Uncertain hypothesis test · Uncertain differential equation · Uncertain currency model

1 Introduction

Uncertainty theory, founded by Liu (2007) and perfected by Liu (2009), has been successfully applied to many fields, such as science, engineering, finance, environment, etc. Among the different applications, uncertain statistics was first introduced by Liu (2010) as a set of mathematical methods to collect, analyze and interpret data based on uncertainty theory. As a part of uncertain statistics, uncertain hypothesis test is a statistical tool based on uncertainty theory to determine whether some hypotheses are correct or not according to observed data. This work was initialized by Ye and Liu (2022). Following that, uncertain hypothesis test has been applied to other research

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areas of uncertain statistics, including uncertain regression analysis and uncertain time series analysis.

Uncertain regression analysis is used to explore the relationship between explanatory variables and response variables with uncertainty theory. Yao and Liu (2018) first proposed that the disturbance term of a regression model is not a random variable but an uncertain variable, which marks the beginning of the uncertain regression analysis. On this basis, Lio and Liu (2018) explored to estimate the uncertain disturbance term and obtained an interval estimation for predicting the response variables. In order to test whether an uncertain regression model is a good fit to observed data, Ye and Liu (2022) proposed a test method based on uncertain hypothesis test. As applications of uncertain regression analysis and uncertain hypothesis test, Liu (2021b) used the uncertain growth model to study the number of COVID-19 infections in China, and Ye (2022a) used the linear uncertain regression model to explore the relationship between labour income share and four influence factors, including trade openness, financial development, government intervention and industrial structure.

Uncertain time series analysis is used to predict future values via the previously observed values based on uncertainty theory. Yang and Liu (2019) first presented that the disturbance term of a time series model is an uncertain variable instead of a random variable, which marks the beginning of the uncertain time series analysis. In order to test whether an uncertain time series model is a good fit to observed data, Ye (2022b) proposed a test method based on uncertain hypothesis test. As applications of uncertain time series analysis and uncertain hypothesis test, Ye and Yang (2021) studied the number of COVID-19 infections in China, Ye (2022c) discussed the birth rates in China, and Ye (2022d) investigated the grain yield in China.

As another part of uncertain statistics, uncertain differential equation (Liu 2008) is used to model the time evolution of a dynamic system. When using uncertain differential equation in practice, we first need to estimate unknown parameters in an uncertain differential equation to fit observed data as much as possible based on uncertainty theory. For that purpose, Yao and Liu (2020) first proposed the moment estimation based on the difference scheme of uncertain differential equation. Following that, Yang et al. (2020) presented the minimum cover estimation, Sheng et al. (2020) investigated the least squares estimation, and Liu and Liu (2022) introduced the maximum likelihood estimation. However, the above parameter estimation methods of uncertain differential equation based on difference scheme are not suitable for the case where the interval times between observations are not short enough. In order to deal with this problem, Liu and Liu (2022) presented the concept of residual to establish a connection between uncertain differential equation and observed data, and proposed a new method of parameter estimation in uncertain differential equation based on residuals. Up to now, uncertain differential equation has been widely applied in many fields such as finance (Liu 2013), chemical reaction (Tang and Yang 2021), electric circuit (Liu 2021a), pharmacokinetics (Liu and Yang 2021), software reliability (Liu and Kang 2022), COVID-19 (Lio and Liu 2021), Alibaba stock (Liu and Liu 2022) and so on.

With the help of residuals, this paper employs uncertain hypothesis test to determine whether an uncertain differential equation is a good fit to the observed data. The rest of the paper is organized as follows. Section 2 introduces some basic knowledge about uncertain hypothesis test. After that, uncertain hypothesis test is used to determine whether a set of observed data follow a given linear uncertainty distribution in Sect. 3. On this basis, Sect. 4 provides a method to test whether an uncertain differential equation fits the observed data, and gives two numerical examples to illustrate the test method. In Sect. 5, uncertain currency model is applied to USD–CNY exchange rates. Then, some conclusions are made in Sect. 6. Finally, as a comparison with uncertain currency model in Sect. 5, stochastic currency model is also applied to USD–CNY exchange rates in the appendix, and the results show that the uncertain currency model fits the exchange rates well, but the stochastic currency model does not.

2 Preliminaries

This section will introduce some basic knowledge of uncertain hypothesis test. Let ξ be an uncertain variable with uncertainty distribution Φ_θ where θ is an unknown parameter. Consider the following hypotheses:

$$H_0 : \theta = \theta_0 \quad \text{versus} \quad H_1 : \theta \neq \theta_0$$

where H_0 is called a null hypothesis, and H_1 is called an alternative hypothesis. Assume

$$z_1, z_2, \ldots, z_n$$

are a set of observed data of the uncertain variable ξ. A rejection region for the null hypothesis H_0 is a set $W \subset \Re^n$. If the vector of observed data belongs to the rejection region W, i.e.,

$$(z_1, z_2, \ldots, z_n) \in W,$$

then we reject H_0. Otherwise, we accept H_0. A core problem is how to choose a suitable rejection region W for the given hypothesis H_0.

Definition 1 (Ye and Liu 2022) Let ξ be a population with uncertainty distribution Φ_θ where θ is an unknown parameter. A rejection region $W \subset \Re^n$ is said to be a test for the two-sided hypotheses $H_0 : \theta = \theta_0$ versus $H_1 : \theta \neq \theta_0$ at significance level α if

(i) for any $(z_1, z_2, \ldots, z_n) \in W$, there are at least α of indexes i's with $1 \leq i \leq n$ such that

$$\mathcal{M}_{\theta_0}\{\xi > z_i\} \vee \mathcal{M}_{\theta_0}\{\xi < z_i\} > 1 - \frac{\alpha}{2},$$

(ii) for some $\theta \neq \theta_0$ and some $(z_1, z_2, \ldots, z_n) \in W$, there are more than $1 - \alpha$ of indexes i's with $1 \leq i \leq n$ and at least α of indexes j's with $1 \leq j \leq n$ such that

$$\mathcal{M}_\theta\{\xi > z_i\} \vee \mathcal{M}_\theta\{\xi < z_i\} < \mathcal{M}_{\theta_0}\{\xi > z_j\} \vee \mathcal{M}_{\theta_0}\{\xi < z_j\}.$$

In order to test whether a normal uncertainty distribution $\mathcal{N}(e_0, \sigma_0)$ is a good fit to a set of observed data $z_1, z_2, \ldots, z_n$, Ye and Liu (2022) proved by Definition 1 that a test at significance level α is

$$W = \left\{(z_1, z_2, \ldots, z_n) : \text{ there are at least } \alpha \text{ of indexes } i\text{'s with } 1 \leq i \leq n \right.$$
$$\left. \text{such that } z_i < \Phi_0^{-1}\left(\frac{\alpha}{2}\right) \text{ or } z_i > \Phi_0^{-1}\left(1 - \frac{\alpha}{2}\right)\right\}$$

where Φ_0^{-1} is the inverse uncertainty distribution of $\mathcal{N}(e_0, \sigma_0)$, i.e.,

$$\Phi_0^{-1}(\alpha) = e_0 + \frac{\sigma_0\sqrt{3}}{\pi} \ln \frac{\alpha}{1-\alpha}.$$

If the vector of observed data belongs to the test W, i.e.,

$$(z_1, z_2, \ldots, z_n) \in W,$$

then the normal uncertainty distribution $\mathcal{N}(e_0, \sigma_0)$ is not a good fit to the observed data. Otherwise, the normal uncertainty distribution $\mathcal{N}(e_0, \sigma_0)$ is a good fit to the observed data.

3 Uncertain hypothesis test for linear uncertainty distribution

In this section, we would like to use the uncertain hypothesis test to determine whether a set of observed data follow a given linear uncertainty distribution.

Theorem 1 *Let ξ be an uncertain variable that follows a linear uncertainty distribution $\mathcal{L}(a, b)$ with unknown parameters a and b. Then the test for the hypotheses*

$$H_0 : a = a_0 \textit{ and } b = b_0 \quad \textit{versus} \quad H_1 : a \neq a_0 \textit{ or } b \neq b_0$$

at significance level α is

$$W = \left\{(z_1, z_2, \ldots, z_n) : \textit{ there are at least } \alpha \textit{ of indexes } i\textit{'s with } 1 \leq i \leq n \right.$$
$$\left. \textit{such that } z_i < \Phi_0^{-1}\left(\frac{\alpha}{2}\right) \textit{ or } z_i > \Phi_0^{-1}\left(1 - \frac{\alpha}{2}\right)\right\}$$

where Φ_0^{-1} is the inverse uncertainty distribution of $\mathcal{L}(a_0, b_0)$, i.e.,

$$\Phi_0^{-1}(\alpha) = (1-\alpha)a_0 + \alpha b_0.$$

Proof In order to prove that W is a test, we need to verify that W simultaneously meets Conditions (i) and (ii) in Definition 1. For any $(z_1, z_2, \ldots, z_n) \in W$, it follows from the definition of W that there are at least α of indexes i's with $1 \le i \le n$ such that

$$z_i < \Phi_0^{-1}\left(\frac{\alpha}{2}\right) \text{ or } z_i > \Phi_0^{-1}\left(1 - \frac{\alpha}{2}\right),$$

i.e.,

$$\mathcal{M}_0\{\xi > z_i\} = 1 - \Phi_0(z_i) > 1 - \frac{\alpha}{2} \quad \text{or} \quad \mathcal{M}_0\{\xi < z_i\} = \Phi_0(z_i) > 1 - \frac{\alpha}{2}.$$

Thus W meets Condition (i) in Definition 1.

In order to prove Condition (ii), we take

$$a_1 = \frac{3a_0 - b_0}{2}, \quad b_1 = \frac{a_0 + b_0}{2}$$

and

$$z_i = a_0, \quad i = 1, 2, \ldots, n.$$

It is clear that $(z_1, z_2, \ldots, z_n) \in W$. Let Φ_1 denote the uncertainty distribution of $\mathcal{L}(a_1, b_1)$. On the one hand, we have

$$\mathcal{M}_1\{\xi > z_i\} = 1 - \Phi_1(z_i) = 0.5, \quad \mathcal{M}_1\{\xi < z_i\} = \Phi_1(z_i) = 0.5,$$

$i = 1, 2, \ldots, n$. Thus

$$\mathcal{M}_1\{\xi > z_i\} \vee \mathcal{M}_1\{\xi < z_i\} = 0.5 \le 1 - \frac{\alpha}{2}, \quad i = 1, 2, \ldots, n.$$

On the other hand, we have

$$\mathcal{M}_0\{\xi > z_j\} = 1 - \Phi_0(z_j) = 1 - 0 > 1 - \frac{\alpha}{2}, \quad j = 1, 2, \ldots, n.$$

Thus

$$\mathcal{M}_1\{\xi > z_i\} \vee \mathcal{M}_1\{\xi < z_i\} < \mathcal{M}_0\{\xi > z_i\} \vee \mathcal{M}_0\{\xi < z_i\},$$

$i, j = 1, 2, \ldots, n$. Therefore W meets Condition (ii) in Definition 1. □

Example 1 Let us use uncertain hypothesis test to determine whether the linear uncertainty distribution $\mathcal{L}(0, 1)$ fits the 30 observed data

0.607 0.808 0.909 0.989 0.830 0.938 0.737 0.676 0.838 0.690
0.627 0.652 0.708 0.616 0.379 0.594 0.542 0.079 0.118 0.330
0.156 0.235 0.039 0.052 0.255 0.039 0.365 0.213 0.480 0.500

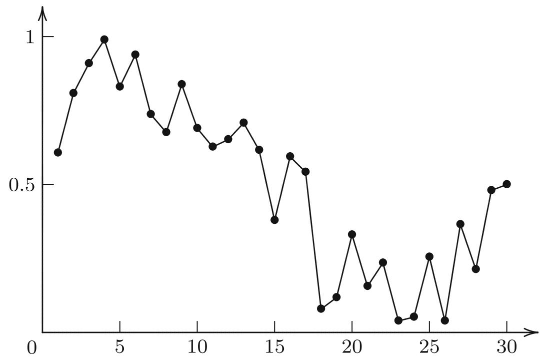


Fig. 1 Observed data of the linear uncertainty distribution $\mathcal{L}(0, 1)$ in chronological order

plotted in Fig. 1.

Suppose Φ^{-1} is the inverse uncertainty distribution of $\mathcal{L}(0, 1)$, i.e.,

$$\Phi^{-1}(\alpha) = \alpha.$$

Given a significance level $\alpha = 0.05$, we obtain

$$\Phi^{-1}\left(\frac{\alpha}{2}\right) = 0.025, \quad \Phi^{-1}\left(1 - \frac{\alpha}{2}\right) = 0.975.$$

It follows from $\alpha \times 30 = 1.5$ and Theorem 1 that the test is

$$W = \{(z_1, z_2, \ldots, z_{30}) : \text{ there are at least 2 of indexes } i\text{'s with } 1 \le i \le 30 \\ \text{such that } z_i < 0.025 \text{ or } z_i > 0.975\}.$$

Since only the 4th observed datum $0.989 \notin [0.025, 0.975]$, the vector of observed data does not belong to W. Thus the linear uncertainty distribution $\mathcal{L}(0, 1)$ is a good fit to the observed data.

Example 2 Let us use uncertain hypothesis test to determine whether the linear uncertainty distribution $\mathcal{L}(5, 6)$ fits the 30 observed data

5.149 5.307 5.227 5.290 5.203 5.014 5.240 5.104 5.238 5.374
5.322 5.397 5.692 5.311 5.364 5.617 5.351 5.240 5.731 5.472
5.552 5.960 5.834 5.831 5.944 5.985 5.754 5.908 5.909 5.680

plotted in Fig. 2.

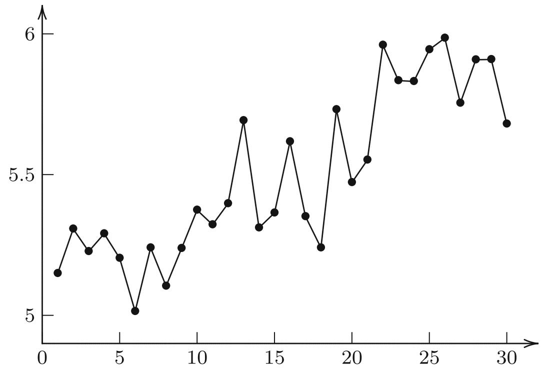
Suppose Φ^{-1} is the inverse uncertainty distribution of $\mathcal{L}(5, 6)$, i.e.,

$$\Phi^{-1}(\alpha) = \alpha + 5.$$

Given a significance level $\alpha = 0.05$, we obtain

$$\Phi^{-1}\left(\frac{\alpha}{2}\right) = 5.025, \quad \Phi^{-1}\left(1 - \frac{\alpha}{2}\right) = 5.975.$$

Fig. 2 Observed data of the linear uncertainty distribution $\mathcal{L}(5, 6)$ in chronological order



It follows from $\alpha \times 30 = 1.5$ and Theorem 1 that the test is

$$W = \{(z_1, z_2, \ldots, z_{30}) : \text{ there are at least 2 of indexes } i\text{'s with } 1 \leq i \leq 30 \\ \text{such that } z_i < 5.025 \text{ or } z_i > 5.975\}.$$

Since the 6th and 26th observed data $5.014, 5.985 \notin [5.025, 5.975]$, the vector of observed data belongs to W. Thus the linear uncertainty distribution $\mathcal{L}(5, 6)$ is not a good fit to the observed data.

4 Uncertain differential equation

In this section, we would like to employ the uncertain hypothesis test to determine whether an uncertain differential equation fits the observed data well. Consider an uncertain differential equation

$$\mathrm{d}X_t = f(t, X_t)\mathrm{d}t + g(t, X_t)\mathrm{d}C_t \tag{1}$$

where f and g are known continuous functions, and C_t is a Liu process. Assume

$$x_{t_1}, x_{t_2}, \ldots, x_{t_n} \tag{2}$$

are observed values of some uncertain process X_t at times $t_1, t_2, \ldots, t_n$ with $t_1 < t_2 < \cdots < t_n$, respectively. For each index i $(2 \leq i \leq n)$, let X_{t_i} be the solution of the updated uncertain differential equation

$$\mathrm{d}X_t = f(t, X_t)\mathrm{d}t + g(t, X_t)\mathrm{d}C_t, \quad X_{t_{i-1}} = x_{t_{i-1}} \tag{3}$$

at time t_i. Denote the uncertainty distribution of X_{t_i} by Φ_{t_i}. It is clear that

$$\Phi_{t_i}(X_{t_i}) \tag{4}$$

is a linear uncertain variable $\mathcal{L}(0, 1)$ whose uncertainty distribution is

$$\Psi(x) = \begin{cases} 0, & \text{if } x \le 0 \\ x, & \text{if } 0 < x \le 1 \\ 1, & \text{if } 1 < x, \end{cases}$$

and inverse uncertainty distribution is

$$\Psi^{-1}(\alpha) = \alpha.$$

Then, Liu and Liu (2022) called

$$\varepsilon_i = \Phi_{t_i}(x_{t_i}) \tag{5}$$

the ith residual of the uncertain differential equation (1) corresponding to the observed data (2) by substituting X_{t_i} with the observed value x_{t_i} in (4). Thus the residual ε_i may be regraded as a sample of the linear uncertainty distribution $\mathcal{L}(0, 1)$.

If the uncertain differential equation (1) does fit the observed data (2) well, then the $n-1$ residuals $\varepsilon_2, \varepsilon_3, \ldots, \varepsilon_n$ should follow the linear uncertainty distribution $\mathcal{L}(0, 1)$, i.e.,

$$\varepsilon_2, \varepsilon_3, \ldots, \varepsilon_n \sim \mathcal{L}(0, 1).$$

Thus, to test whether the uncertain differential equation (1) fits the observed data (2) well, we should test whether the linear uncertainty distribution $\mathcal{L}(0, 1)$ fits the residuals $\varepsilon_2, \varepsilon_3, \ldots, \varepsilon_n$ defined in (5), i.e.,

$$\varepsilon_2, \varepsilon_3, \ldots, \varepsilon_n \sim \mathcal{L}(0, 1).$$

To do so, it follows from Theorem 1 that the test at a given significance level α (e.g. 0.05) is

$$W = \left\{ (z_2, z_3, \ldots, z_n) : \text{ there are at least } \alpha \text{ of indexes } i\text{'s with } 2 \le i \le n \right.$$
$$\left. \text{such that } z_i < \frac{\alpha}{2} \text{ or } z_i > 1 - \frac{\alpha}{2} \right\}.$$

If the vector of the $n-1$ residuals $\varepsilon_2, \varepsilon_3, \ldots, \varepsilon_n$ belongs to the test W, i.e.,

$$(\varepsilon_2, \varepsilon_3, \ldots, \varepsilon_n) \in W,$$

then the uncertain differential equation (1) is not a good fit to the observed data (2). If

$$(\varepsilon_2, \varepsilon_3, \ldots, \varepsilon_n) \notin W,$$

Table 1 Observed data in Example 3

t	X_t	t	X_t	t	X_t	t	X_t	t	X_t
0.00	1.00	0.46	1.11	0.94	0.60	1.49	1.18	1.86	3.34
0.04	1.01	0.52	0.97	1.04	0.62	1.54	1.26	1.91	3.70
0.09	1.05	0.61	1.01	1.16	0.56	1.58	1.57	1.98	4.03
0.19	1.12	0.70	1.00	1.27	0.79	1.64	1.69	2.03	4.38
0.29	0.96	0.79	0.82	1.32	0.95	1.70	2.04	2.14	6.34
0.38	1.06	0.84	0.75	1.44	1.10	1.79	2.51	2.19	6.57

then the uncertain differential equation (1) is a good fit to the observed data (2).

Example 3 Let us employ the uncertain hypothesis test to determine whether the uncertain differential equation

$$\mathrm{d}X_t = X_t\mathrm{d}t + 2X_t\mathrm{d}C_t \tag{6}$$

fits the 30 observed data in Table 1 on the time horizon from 0 to 2.19.

In Table 1, denote the observed values of X_t at times $t_1, t_2, \ldots, t_{30}$ by

$$x_{t_1}, x_{t_2}, \ldots, x_{t_{30}},$$

respectively. For each index i $(2 \le i \le 30)$, we solve the updated uncertain differential equation

$$\mathrm{d}X_t = X_t\mathrm{d}t + 2X_t\mathrm{d}C_t, \quad X_{t_{i-1}} = x_{t_{i-1}}$$

and obtain the uncertainty distribution of X_{t_i} as follows,

$$\Phi_{t_i}(x) = \left(1 + \exp\left(\frac{\pi(t_i - t_{i-1} + \ln x_{t_{i-1}} - \ln x)}{2\sqrt{3}(t_i - t_{i-1})}\right)\right)^{-1}.$$

It follows from (5) that the ith residual is

$$\varepsilon_i = \left(1 + \exp\left(\frac{\pi(t_i - t_{i-1} + \ln x_{t_{i-1}} - \ln x_{t_i})}{2\sqrt{3}(t_i - t_{i-1})}\right)\right)^{-1}.$$

See Fig. 3. In order to test whether the uncertain differential equation (6) fits the observed data well, we should test whether the linear uncertainty distribution $\mathcal{L}(0, 1)$ fits the 29 residuals $\varepsilon_2, \varepsilon_3, \ldots, \varepsilon_{30}$. Given a significance level $\alpha = 0.05$, it follows from $\alpha \times 29 = 1.45$ and Theorem 1 that the test is

$$W = \{(z_2, z_3, \ldots, z_{30}) : \text{ there are at least 2 of indexes } i\text{'s with } 2 \le i \le 30 \text{ such that } z_i < 0.025 \text{ or } z_i > 0.975\}.$$

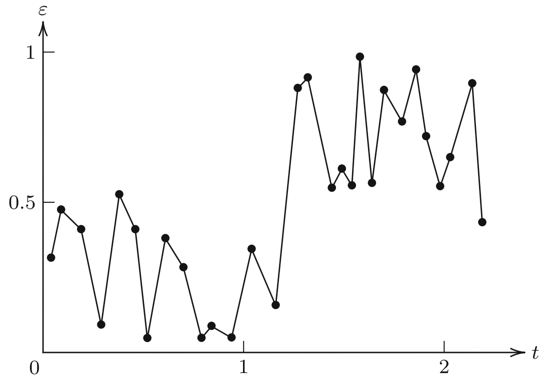


Fig. 3 Residual plot of uncertain differential equation (6)

Table 2 Observed data in Example 4

t	X_t	t	X_t	t	X_t	t	X_t	t	X_t
0.00	2.00	1.30	6.63	3.30	14.9	5.40	3.64	7.60	2.49
0.20	2.75	1.60	8.35	3.50	12.7	5.90	3.71	7.90	2.46
0.40	3.59	1.70	8.82	3.80	9.81	6.20	3.12	8.30	2.90
0.60	4.45	2.00	15.3	4.30	7.85	6.50	2.75	8.70	2.99
0.80	5.48	2.30	12.4	4.60	5.27	6.80	3.44	9.20	4.19
1.00	6.97	2.80	14.1	4.90	4.35	7.30	3.16	9.50	4.23

Since only $\varepsilon_{21} = 0.983 \notin [0.025, 0.975]$, we have $(\varepsilon_2, \varepsilon_3, \ldots, \varepsilon_{30}) \notin W$. Thus the uncertain differential equation (6) is a good fit to the observed data.

Example 4 Let us employ the uncertain hypothesis test to determine whether the uncertain differential equation

$$\mathrm{d}X_t = (5 - X_t)\mathrm{d}t + X_t\mathrm{d}C_t \tag{7}$$

fits the 30 observed data in Table 2 on the time horizon from 0 to 9.5.

In Table 2, denote the observed values of X_t at times $t_1, t_2, \ldots, t_{30}$ by

$$x_{t_1}, x_{t_2}, \ldots, x_{t_{30}},$$

respectively. For each i $(2 \le i \le 30)$, we can calculate the residual ε_i of the updated uncertain differential equation

$$\mathrm{d}X_t = (5 - X_t)\mathrm{d}t + X_t\mathrm{d}C_t, \quad X_{t_{i-1}} = x_{t_{i-1}}$$

with the help of an algorithm proposed by Liu and Liu (2022). See Fig. 4. In order to test whether the uncertain differential equation (7) fits the observed data well, we

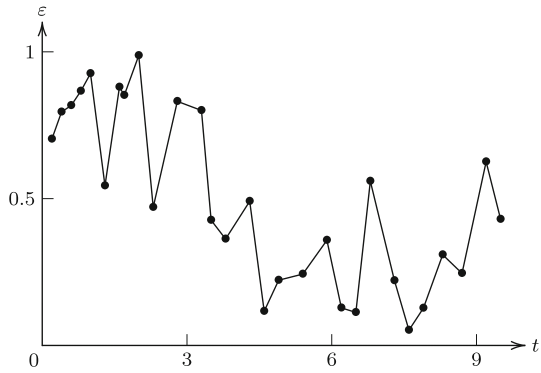


Fig. 4 Residual plot of uncertain differential equation (7)

should test whether the linear uncertainty distribution $\mathcal{L}(0, 1)$ fits the 29 residuals

$$\varepsilon_2, \varepsilon_3, \ldots, \varepsilon_{30}.$$

Given a significance level $\alpha = 0.05$, it follows from $\alpha \times 29 = 1.45$ and Theorem 1 that the test is

$$W = \{(z_2, z_3, \ldots, z_{30}) : \text{ there are at least 2 of indexes } i\text{'s with } 2 \le i \le 30 \\ \text{such that } z_i < 0.025 \text{ or } z_i > 0.975\}.$$

Since only $\varepsilon_{10} = 0.987 \notin [0.025, 0.975]$, we have $(\varepsilon_2, \varepsilon_3, \ldots, \varepsilon_{30}) \notin W$. Thus the uncertain differential equation (7) is a good fit to the observed data.

5 Uncertain currency model

Example 5 Table 3 shows US Dollar to Chinese Yuan (USD–CNY) exchange rates (weekly average) in Forex Capital Markets (FXCM) from October 2019 to June 2021, which are plotted in Fig. 5.

Let $i = 1, 2, \ldots, 91$ represent the weeks from October 1, 2019 to June 30, 2021, and denote the exchange rates by

$$x_1, x_2, \ldots, x_{91}.$$

Assume the exchange rate X_t follows the uncertain differential equation

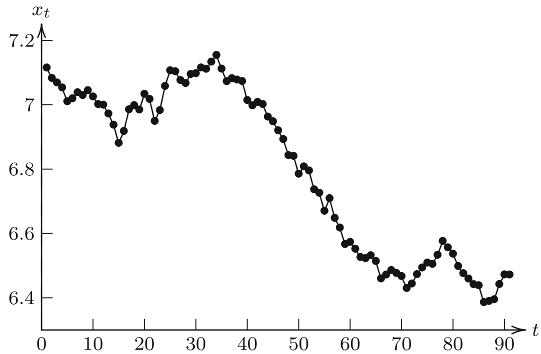
$$\mathrm{d}X_t = (m - aX_t)\mathrm{d}t + \sigma \mathrm{d}C_t \tag{8}$$

where m, a and σ are unknown parameters to be estimated, and C_t is a Liu process. Then, Liu and Liu (2022) suggested that the moment estimate (m, a, σ) is the solution of the system of equations,

Table 3 USD–CNY exchange rates (weekly average) from October 1, 2019 to June 30, 2021

7.1145	7.0821	7.0679	7.0526	7.0096	7.0188	7.0376	7.0289	7.0441
7.0245	7.0009	6.9990	6.9715	6.9368	6.8804	6.9173	6.9847	6.9975
6.9839	7.0327	7.0169	6.9486	6.9822	7.0573	7.1059	7.1030	7.0757
7.0663	7.0945	7.0969	7.1147	7.1106	7.1327	7.1536	7.1111	7.0721
7.0813	7.0770	7.0725	7.0135	6.9970	7.0075	7.0011	6.9618	6.9473
6.9194	6.8921	6.8422	6.8399	6.7845	6.8071	6.7946	6.7359	6.7253
6.6694	6.7085	6.6473	6.6171	6.5659	6.5726	6.5514	6.5257	6.5221
6.5310	6.5131	6.4592	6.4712	6.4851	6.4759	6.4664	6.4293	6.4434
6.4724	6.4931	6.5084	6.5043	6.5325	6.5756	6.5556	6.5360	6.4977
6.4753	6.4587	6.4413	6.4379	6.3855	6.3890	6.3943	6.4416	6.4716
6.4713								

Fig. 5 USD–CNY exchange rates (weekly average) from October 1, 2019 to June 30, 2021



$$\begin{cases} \dfrac{1}{90}\sum_{i=2}^{91} \varepsilon_i(m,a,\sigma) = \dfrac{1}{2} \\ \dfrac{1}{90}\sum_{i=2}^{91} \varepsilon_i^2(m,a,\sigma) = \dfrac{1}{3} \\ \dfrac{1}{90}\sum_{i=2}^{91} \varepsilon_i^3(m,a,\sigma) = \dfrac{1}{4} \end{cases}$$

where

$$\varepsilon_i(m,a,\sigma) = \left(1 + \exp\left(\frac{\pi((ax_{i-1} - m)\exp(-a) + m - ax_i)}{\sqrt{3}\sigma(1-\exp(-a))}\right)\right)^{-1},$$

$i = 2, 3, \ldots, 91$ are residuals of the uncertain differential equation (8). Solving the above system of equations, we get

$$m = 1.4448, \quad a = 0.2136, \quad \sigma = 0.0775.$$

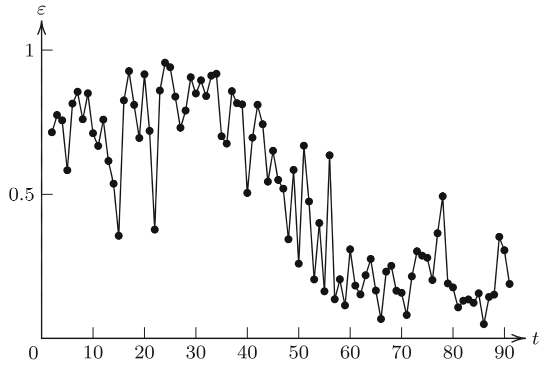


Fig. 6 Residual plot of the uncertain currency model (9) corresponding to USD–CNY exchange rates

Thus we obtain an uncertain currency model

$$\mathrm{d}X_t = (1.4448 - 0.2136X_t)\mathrm{d}t + 0.0775\mathrm{d}C_t \tag{9}$$

where X_t represents the exchange rate.

Finally, let us test whether the uncertain currency model (9) fits USD–CNY exchange rates. That is, we should test whether the linear uncertainty distribution $\mathcal{L}(0, 1)$ fits the 90 residuals

$$\varepsilon_i(1.4448, 0.2136, 0.0775), \ \ i = 2, 3, \ldots, 91.$$

See Fig. 6.

Given a significance level $\alpha = 0.05$, it follows from $\alpha \times 90 = 4.5$ and Theorem 1 that the test is

$$W = \{(z_2, z_3, \ldots, z_{91}) : \text{ there are at least 5 of indexes } i\text{'s with } 2 \le i \le 91 \\ \text{such that } z_i < 0.025 \text{ or } z_i > 0.975\}.$$

Since all residuals ε_i, $i = 2, 3, \ldots, 91$ are between 0.025 and 0.975, we have

$$(\varepsilon_2, \varepsilon_3, \ldots, \varepsilon_{91}) \notin W.$$

Thus the uncertain currency model (9) is a good fit to USD–CNY exchange rates.

6 Conclusion

In order to test whether an uncertain differential equation fits the observed data well, this paper presented the uncertain hypothesis test to determine whether the uncertain differential equation is a good fit to the observed data by testing whether the residuals of the uncertain differential equation follow the linear uncertainty distribution $\mathcal{L}(0, 1)$. Then, some numerical examples were given to demonstrate the test method. Finally, both uncertain currency model and stochastic currency model were applied to USD–CNY exchange rates. The results showed that the uncertain currency model fits the exchange rates well, but the stochastic currency model does not.

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Appendix: Stochastic currency model

Let us reconsider USD–CNY exchange rates (weekly average) from October 1, 2019 to June 30, 2021 in Example 5. See Table 3. Let $i = 1, 2, \ldots, 91$ represent the weeks from October 1, 2019 to June 30, 2021, and denote the exchange rates by

$$x_1, x_2, \ldots, x_{91}.$$

Assume the exchange rate X_t follows the stochastic differential equation

$$\mathrm{d}X_t = (m - aX_t)\mathrm{d}t + \sigma \mathrm{d}W_t \tag{10}$$

where m, a and σ are unknown parameters to be estimated, and W_t is a Wiener process. For any fixed parameters m, a, σ and index i $(2 \le i \le 91)$, we solve the updated stochastic differential equation

$$\mathrm{d}X_t = (m - aX_t)\mathrm{d}t + \sigma \mathrm{d}W_t, \quad X_{i-1} = x_{i-1}$$

and find that X_i is a normal random variable with expected value

$$e_i = \frac{m}{a} + \left(x_{i-1} - \frac{m}{a}\right)\exp(-a)$$

and variance

$$\nu^2 = \frac{\sigma^2}{2a}(1 - \exp(-2a)).$$

Thus the probability distribution of the normal random variable X_i is

$$\Phi_i(x) = \frac{1}{\nu\sqrt{2\pi}} \int_{-\infty}^{x} \exp\left(-\frac{(y - e_i)^2}{2\nu^2}\right) \mathrm{d}y,$$

and $\Phi_i(X_i)$ is always a uniform random variable $\mathcal{U}(0, 1)$ since

$$\Pr\{\Phi_i(X_i) \le x\} = \Pr\{X_i \le \Phi_i^{-1}(x)\} = \Phi_i(\Phi_i^{-1}(x)) = x$$

for any x with $0 < x < 1$. Substitute X_i with the corresponding observed value x_i, and write

$$\varepsilon_i(m, a, \sigma) = \Phi_i(x_i).$$

Then $\varepsilon_i(m, a, \sigma)$ is always a sample of uniform probability distribution $\mathcal{U}(0, 1)$ and called the ith residual of the stochastic differential equation (10). For each positive integer k, the kth sample moment of the 90 residuals $\varepsilon_2(m, a, \sigma), \varepsilon_3(m, a, \sigma), \ldots, \varepsilon_{91}(m, a, \sigma)$ is

$$\frac{1}{90}\sum_{i=2}^{91}\varepsilon_i^k(m, a, \sigma),$$

and the k-th population moment of the uniform probability distribution $\mathcal{U}(0, 1)$ is

$$\frac{1}{k+1}.$$

Since the number of unknown parameters in the stochastic differential equation (10) is 3, the moment estimate (m, a, σ) is obtained by equating the first 3 sample moments to the corresponding first 3 population moments. In other words, the moment estimate (m, a, σ) should solve the system of equations,

$$\begin{cases} \frac{1}{90}\sum_{i=2}^{91}\varepsilon_i(m, a, \sigma) = \frac{1}{2} \\ \frac{1}{90}\sum_{i=2}^{91}\varepsilon_i^2(m, a, \sigma) = \frac{1}{3} \\ \frac{1}{90}\sum_{i=2}^{91}\varepsilon_i^3(m, a, \sigma) = \frac{1}{4} \end{cases}$$

whose root is

$$m = 1.4896, \quad a = 0.2202, \quad \sigma = 0.0731.$$

Thus we obtain a stochastic currency model

$$\mathrm{d}X_t = (1.4896 - 0.2202X_t)\mathrm{d}t + 0.0731\mathrm{d}W_t \tag{11}$$

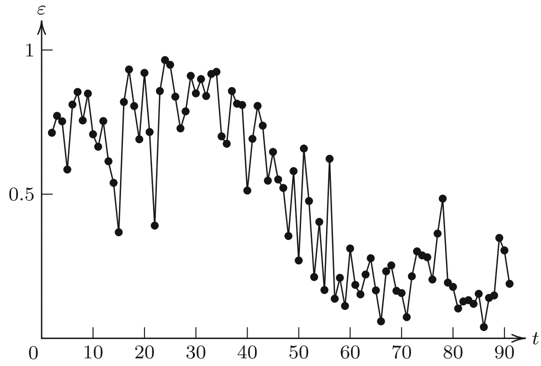
where X_t represents exchange rate.

Does the stochastic currency model (11) fit USD–CNY exchange rates? In order to answer this question, let us consider the 90 residuals

$$\varepsilon_i(1.4896, 0.2202, 0.0731), \quad i = 2, 3, \ldots, 91.$$

See Fig. 7. If the stochastic currency model (11) fits USD–CNY exchange rates well, then the residuals in Fig. 7 should be from the same population in the sense of probability theory. However, when we try to divide those residuals into two parts, i.e., the

Fig. 7 Residual plot of stochastic currency model (11) corresponding to USD–CNY exchange rates



first half and the last half, and use two-sample Kolmogorov-Smirnov test to determine whether the residuals of these two parts are from the same population, we get the p-value of 5×10^{-16} via the function "kstest2" in Matlab. This means the residuals neither come from the same population nor are white noise in the sense of probability theory, let alone follow the uniform probability distribution $\mathcal{U}(0, 1)$. Thus the stochastic currency model (11) does not fit USD–CNY exchange rates. Furthermore, we cannot find a better stochastic differential equation to fit the USD–CNY exchange rates. This is the reason why we use uncertain differential equation to model currency exchange rate rather than stochastic differential equation.

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