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UNCERTAIN PRIORITIZED OPERATORS AND THEIR APPLICATION TO MULTIPLE ATTRIBUTE GROUP DECISION MAKING

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Abstract. In this paper, we investigate the uncertain multiple attribute group decision making (MAGDM) problems in which the attributes and experts are in different priority level. Motivated by the idea of prioritized aggregation operators (Yager 2008), we develop some prioritized aggregation operators for aggregating uncertain information, and then apply them to develop some models for uncertain multiple attribute group decision making (MAGDM) problems in which the attributes and experts are in different priority level. Finally, a practical example about talent introduction is given to verify the developed approach and to demonstrate its practicality and effectiveness.

Keywords: multiple attribute group decision making (MAGDM), interval numbers, prioritized aggregation operators, uncertain prioritized weighted average (UPWA) operator, uncertain prioritized weighted geometric (UPWG) operator, uncertain prioritized weighted harmonic average (UPWHA) operator.

JEL Classification: C43, C61, D81.

Introduction

A multiple attribute decision making problem is to find a desirable solution from a finite number of feasible alternatives assessed on multiple attributes, both quantitative and qualitative (Chen, Lee 2010a, 2010b; Chen, Niou 2011a, 2011b; Merigo, Gil-Lafuente 2009, 2011; Liu 2009; Merigo 2011b; Xu, Cai 2012; Xu 2002; Yager 1988; Zhang, Liu 2010; Zhou *et al.* 2012) sion-making (i.e., multi-expert) is a typical decision-making activity where utilizing several experts alleviate some of the decision-making difficulties due to the problem's complexity and uncertainty (Wei 2010a, 2010b, 2011a, 2011b, 2012; Xu, Cai 2012; Ye 2011a, 2011b). However, under many conditions, for the real multiple attribute decision making

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problems, the decision information about alternatives is usually uncertain or fuzzy due to the increasing complexity of the socio-economic environment and the vagueness of inherent subjective nature of human think, thus, numerical values are inadequate or insufficient to model real-life decision problems.

In the literature, many aggregation operators and approaches have been developed to solve the multiple attribute group decision-making problems with interval numbers as follows.

Xu (2002) investigated the uncertain OWA operator in which the associated weighting parameters cannot be specified, but value ranges can be obtained and each input argument is given in the form of an interval of numerical values. The problem of ranking a set of interval numbers and obtaining the weights associated with the uncertain OWA operator is studied.

Xu (2010) developed some uncertain Bonferroni mean operators, and then combined them with the well-known ordered weighted averaging operator and Choquet integral respectively for aggregating uncertain information. They also gave their applications to multi-criteria decision making under uncertainty, and finally, some possible extensions for further research are discussed.

Merigo (2011a) presented the uncertain probabilistic weighted average (UPWA) which its main advantage is that it unifies the probability and the weighted average in the same formulation and considering the degree of importance that each case has in the analysis. Moreover, it is able to deal with uncertain environments represented in the form of interval numbers.

Merigo and Casanovas (2011a) presented the uncertain induced quasi-arithmetic OWA (Quasi-UIOWA) operator which is an extension of the OWA operator that uses the main characteristics of the induced OWA (IOWA), the quasi-arithmetic OWA (Quasi-OWA) and the uncertain OWA (UOWA) operator. Thus, this generalization uses quasi-arithmetic means, order inducing variables in the reordering process and uncertain information represented by interval numbers. A key feature of the Quasi-UIOWA operator is that it generalizes a wide range of aggregation operators such as the uncertain quasi-arithmetic mean, the uncertain weighted quasi-arithmetic mean, the UOWA, the uncertain weighted generalized mean, the uncertain induced generalized OWA (UIGOWA), the Quasi-UOWA, the uncertain IOWA, the uncertain induced ordered weighted geometric (UIOWG), and the uncertain induced ordered weighted quadratic averaging (UIOWQA) operator. They studied some of the main properties of this approach including how to obtain a wide range of particular cases. They further generalized the Quasi-UIOWA operator by using discrete Choquet integrals.

Merigo and Casanovas (2011b) analyzed in detail the ordered weighted averaging (OWA) operator and some of the extensions developed about it and developed some new extensions about the OWA operator such as the induced heavy OWA (IHOWA) operator, the uncertain heavy OWA (UHOWA) operator and the uncertain induced heavy OWA (UIHOWA) operator. For these three new extensions, they considered some of their main properties and a wide range of special cases found in the weighting vector such as the heavy weighted average (HWA) and the uncertain heavy weighted average (UHWA). They further generalized these models by using generalized and quasi-arithmetic means obtaining the generalized heavy weighted average (GHWA), the induced generalized HOWA (IGHOWA) and the uncertain IGHOWA (UIGHOWA) operator.

Merigo and Casanovas (2011c) introduced the uncertain generalized OWA (UGOWA) operator which is an extension of the OWA operator that uses generalized means and uncertain information represented as interval numbers. By using UGOWA, it is possible to obtain a wide range of uncertain aggregation operators such as the uncertain average (UA), the uncertain weighted average (UWA), the uncertain OWA (UOWA) operator, the uncertain ordered weighted geometric (UOWG) operator, the uncertain ordered weighted quadratic averaging (UOWQA) operator, the uncertain generalized mean (UGM), and many specialized operators. They studied some of its main properties, and they further generalized the UGOWA operator using quasi-arithmetic means.

Merigo and Wei (2011) presented the uncertain probabilistic ordered weighted averaging (UPOWA) operator. It is an aggregation operator that used probabilities and OWA operators in the same formulation considering the degree of importance of each concept in the analysis. Moreover, it also used uncertain information assessed with interval numbers in the aggregation process. The main advantage of this aggregation operator is that it is able to use the attitudinal character of the decision maker and the available probabilistic information in an environment where the information is very imprecise and can be assessed with interval numbers. They studied some of its main properties and particular cases such as the uncertain probabilistic aggregation (UPA) and the uncertain OWA (UOWA) operator.

Merigo *et al.* (2012) developed a new decision making approach for dealing with uncertain information and apply it in tourism management. They proposed the uncertain induced ordered weighted averaging – weighted averaging (UIOWAWA) operator and studied some of the main advantages and properties of the new aggregation such as the uncertain arithmetic UIOWA (UA-UIOWA) and the uncertain arithmetic UWA (UAUWA). They studied its applicability in a multi-person decision making problem concerning the selection of holiday trips.

Merigo and Casanovas (2011c) extended the generalized ordered weighted averaging operator and provide a new class of operators called the uncertain generalized ordered weighted averaging (UGOWA) operator. It provides a very general formulation that includes as special cases a wide range of aggregation operators and aggregates the input arguments taking the form of intervals rather than exact numbers. They further generalize the UGOWA operator to obtain the uncertain generalized hybrid averaging operator, the quasi uncertain ordered weighted averaging operator and the uncertain generalized Choquet integral aggregation operator.

From above analysis, we can see that interval numbers is a very useful tool to deal with uncertainty. More and more multiple attribute group decision making theories and methods under uncertain environment have been developed. Current methods are under the assumption that the attributes and the decision makers are at the same priority level. However, in many real and practical multiple attribute group decision making problems, attributes and decision makers have different priority level commonly. To overcome this drawback, Motivated by the ideal of prioritized aggregation operators (Yager 2008, 2009), in this paper, we propose some uncertain prioritized aggregation operators: uncertain prioritized weighted average (UPWA) operator, uncertain prioritized weighted geometric (UPWG) operator and uncertain prioritized weighted harmonic average (UPWHA) operator. The prominent characteristic of these proposed operators is that they take into account prioritization among the attributes and experts. Then, we have utilized these operators to develop some approaches to solve the uncertain multiple attribute group decision making problems in which the attributes and experts are in different priority level. To do so, the remainder of this paper is set out as follows. In the next section, we introduce some basic concepts related to interval numbers and some operational laws of interval numbers. In Section 2 we have developed some uncertain prioritized aggregation operators: uncertain prioritized weighted average (UPWA) operator, uncertain prioritized weighted geometric (UPWG) operator and uncertain prioritized weighted harmonic average (UPWHA) operator and studied some desirable properties of the proposed operator. The prominent characteristic of these proposed operators is that they take into account prioritization among the attributes and experts. In Section 3, we have applied these operators to develop some models for uncertain multiple attribute group decision making (MAGDM) problems in which the attributes and experts are in different priority level. In Section 4, a practical example about talent introduction is given to verify the developed approach and to demonstrate its practicality and effectiveness. In the last section, we conclude the paper and give some remarks.

1. Preliminaries

1.1. Interval numbers

In the following, we briefly describe some basic concepts and basic operational laws related to interval numbers.

Let $\tilde{a}=\left[a^L,a^U\right]=\left\{x\middle|a^L\le x\le a^U\right\}$, then $\tilde{a}$ is called an interval number. If $0\le a^L\le a^U$, then $\tilde{a}$ is called a positive interval number (Xu 2002).

Consider any two positive interval number $\tilde{a}=\left[a^L,a^U\right]$ and $\tilde{b}=\left[b^L,b^U\right]$ $\lambda\in\left[0,1\right]$, we define their operational laws as follows (Xu 2002):

1) $\tilde{a}+\tilde{b}=\left[a^L,a^U\right]+\left[b^L,b^U\right]=[a^L+b^L,a^U+b^U]$;

2) $\lambda\tilde{a}=\left[\lambda a^L,\lambda a^U\right]$;

3) $\tilde{a}^k=\left[\left(a^L\right)^k,\left(a^U\right)^k\right]$;

4) $\tilde{a}\cdot\tilde{b}=[a^L\cdot b^L,a^U\cdot b^U]$;

5) $\dfrac{\tilde{a}}{\tilde{b}}=\left[\dfrac{a^L}{b^U},\dfrac{a^U}{b^L}\right]$.

Definition 1 (Xu 2002). Let $\tilde{a}_1=\left[a_1^L,a_1^U\right]$ and $\tilde{a}_2=\left[a_2^L,a_2^U\right]$ be two interval numbers, and let $len\left(\tilde{a}_1\right)=a_1^U-a_1^L$, $len\left(\tilde{a}_2\right)=a_2^U-a_2^L$, then the degree of possibility of $\tilde{a}_1\ge\tilde{a}_2$ is defined as:

$$\rho\left(\tilde{a}_1\ge\tilde{a}_2\right)=\frac{\max\left(0,len\left(\tilde{a}_1\right)+len\left(\tilde{a}_2\right)-\max\left(0,a_2^U-a_1^L\right)\right)}{len\left(\tilde{a}_1\right)+len\left(\tilde{a}_2\right)}. \quad (1)$$

From Definition 1, we can easily get the following results easily:

1) $0 \le p(\tilde{a}_1 \ge \tilde{a}_2) \le 1, 0 \le p(\tilde{a}_2 \ge \tilde{a}_1) \le 1$;

2) $p(\tilde{a}_1 \ge \tilde{a}_2) + p(\tilde{a}_2 \ge \tilde{a}_1) = 1$. Especially, $p(\tilde{a}_1 \ge \tilde{a}_1) = p(\tilde{a}_2 \ge \tilde{a}_2) = 0.5$.

Definition 2. If $\tilde{a} = [a^L, a^U] = \{x | a^L \le x \le a^U\}$ is interval number, then the expected value of $\tilde{a}$ is:

$$E(\tilde{a}) = \frac{1}{2}(a^L + a^U). \tag{2}$$

1.2. Prioritized Average (PA) operator

The Prioritized Average (PA) operator was originally introduced by Yager (2008), which was defined as follows:

Definition 3 (Yager 2008)**.** Let $G = \{G_1, G_2, \cdots, G_n\}$ be a collection of attribute and that there is a prioritization between the attribute expressed by the linear ordering $G_1 \succ G_2 \succ G_3 \cdots \succ G_n$, indicate attribute G_j has a higher priority than G_k, if $j < k$. The value $G_j(x)$ is the performance of any alternative x under attribute G_j, and satisfies $G_j(x) \in [0,1]$. If:

$$PA(G_i(x)) = \sum_{j=1}^{n} w_j G_j(x), \tag{3}$$

where $w_j = \frac{T_j}{\sum_{j=1}^{n} T_j}$, $T_j = \prod_{k=1}^{j-1} G_k(x) (j = 2, \cdots, n)$, $T = 1$. Then PA is called the prioritized average (PA) operator.

2. Uncertain prioritized Aggregation operators

2.1. Uncertain prioritized weighted average (UPWA) operator

The prioritized average (Yager 2008) operators, however, have usually been used in situations where the input arguments are the exact values. We shall extend the PA operators to accommodate the situations where the input arguments are interval numbers information. In this section, we shall investigate the PA operator under interval number environments. Based on Definition 3, we give the definition of the uncertain prioritized weighted average (UPWA) operator as follows:

Definition 4. Let $\tilde{a}_j = [a_j^L, a_j^U] (j = 1, 2, \cdots, n)$ be a set of interval numbers, then we define the uncertain prioritized weighted average (UPWA) operator as follows:

$$UPWA(\tilde{a}_1, \tilde{a}_2, \cdots, \tilde{a}_n) = \frac{T_1}{\sum_{j=1}^{n} T_j} \tilde{a}_1 \oplus \frac{T_2}{\sum_{j=1}^{n} T_j} \tilde{a}_2 \oplus \cdots \oplus \frac{T_n}{\sum_{j=1}^{n} T_j} \tilde{a}_n, \tag{4}$$

where $T_j = \prod_{k=1}^{j-1} E(\tilde{a}_k) (j = 2, \cdots, n)$, $T_1 = 1$ and $E(\tilde{a}_k)$ is the expected value of $\tilde{a}_k = [a_k^L, a_k^U]$.

Based on operations of the interval numbers described in Section 2, we can drive the Theorem 1.

Theorem 1. Let $\tilde{a}_j = \left[a_j^L, a_j^U\right]\left(j = 1, 2, \cdots, n\right)$ be a set of interval numbers, then their aggregated value by using the UPWA operator is also an interval number, and

$$UPWA\left(\tilde{a}_1, \tilde{a}_2, \cdots, \tilde{a}_n\right) = \frac{T_1}{\sum_{j=1}^{n} T_j}\tilde{a}_1 \oplus \frac{T_2}{\sum_{j=1}^{n} T_j}\tilde{a}_2 \oplus \cdots \oplus \frac{T_n}{\sum_{j=1}^{n} T_j}\tilde{a}_n = \left[\sum_{j=1}^{n}\frac{T_j a_j^L}{\sum_{j=1}^{n} T_j}, \sum_{j=1}^{n}\frac{T_j a_j^U}{\sum_{j=1}^{n} T_j}\right], \tag{5}$$

where $T_j = \prod_{k=1}^{j-1} E\left(\tilde{a}_k\right)\left(j = 2, \cdots, n\right), T_1 = 1$ and $E\left(\tilde{a}_k\right)$ is the expected value of $\tilde{a}_k = \left[a_k^L, a_k^U\right]$.

It can be easily proved that the UPWA operator has the following properties.

Theorem 2. (Idempotency) Let $\tilde{a}_j = \left[a_j^L, a_j^U\right]\left(j = 1, 2, \cdots, n\right)$ be a set of interval numbers, where $T_j = \prod_{k=1}^{j-1} E\left(\tilde{a}_k\right)\left(j = 2, \cdots, n\right), T_1 = 1$ and $E\left(\tilde{a}_k\right)$ is the expected value of $\tilde{a}_k = \left[a_k^L, a_k^U\right]$. If all $\tilde{a}_j\left(j = 1, 2, \cdots, n\right)$ are equal, i.e. $\tilde{a}_j = \tilde{a}$ for all j, then:

$$UPWA\left(\tilde{a}_1, \tilde{a}_2, \cdots, \tilde{a}_n\right) = \tilde{a}. \tag{6}$$

Theorem 3. (Boundedness) Let $\tilde{a}_j = \left[a_j^L, a_j^U\right]\left(j = 1, 2, \cdots, n\right)$ be a set of interval numbers, where $T_j = \prod_{k=1}^{j-1} E\left(\tilde{a}_k\right)\left(j = 2, \cdots, n\right), T_1 = 1$ and $E\left(\tilde{a}_k\right)$ is the expected value of $\tilde{a}_k = \left[a_k^L, a_k^U\right]$, and let

$$\tilde{a}^- = \min_j \tilde{a}_j = \left[\min_j a_j^L, \min_j a_j^U\right];$$

$$\tilde{a}^+ = \max_j \tilde{a}_j = \left[\max_j a_j^L, \max_j a_j^U\right].$$

Then:

$$\tilde{a}^- \le UPWA\left(\tilde{a}_1, \tilde{a}_2, \cdots, \tilde{a}_n\right) \le \tilde{a}^+. \tag{7}$$

Theorem 4. (Monotonicity) Let $a_j^{(t+1)} \ge a_j^{(t)}$ and $\tilde{a}'_j = \left[a_j'^L, a_j'^U\right]\left(j = 1, 2, \cdots, n\right)$ be two set of interval numbers, where $T_j = \prod_{k=1}^{j-1} E\left(\tilde{a}_k\right), T'_j = \prod_{k=1}^{j-1} E\left(\tilde{a}'_k\right)\left(j = 2, \cdots, n\right), T_1 = T'_1 = 1$, $E\left(\tilde{a}_k\right)$

is the expected value of $\tilde{a}_k = \left[a_k^L, a_k^U\right]$, $E\left(\tilde{a}'_k\right)$ is the expected value of $\tilde{a}'_k = \left[a'^L_k, a'^U_k\right]$, if $\tilde{a}_j \le \tilde{a}'_j$, for all *j*, then:

$$UPWA\left(\tilde{a}_1, \tilde{a}_2, \cdots, \tilde{a}_n\right) \le UPWA\left(\tilde{a}'_1, \tilde{a}'_2, \cdots, \tilde{a}'_n\right). \tag{8}$$

2.2. Uncertain prioritized weighted geometric (UPWG) operator

Based on the UPWA operator and the geometric mean, here we define a uncertain prioritized weighted geometric (UPWG) operator:

Definition 5. Let $\tilde{a}_j = \left[a_j^L, a_j^U\right]\left(j = 1, 2, \cdots, n\right)$ be a set of interval numbers, then we define the uncertain prioritized weighted geometric (UPWG) operator as follows:

$$UPWG\left(\tilde{a}_1, \tilde{a}_2, \cdots, \tilde{a}_n\right) = \tilde{a}_1^{\frac{T_1}{\sum_{j=1}^n T_j}} \otimes \tilde{a}_2^{\frac{T_2}{\sum_{j=1}^n T_j}} \otimes \cdots \otimes \tilde{a}_n^{\frac{T_n}{\sum_{j=1}^n T_j}}, \tag{9}$$

where $T_j = \prod_{k=1}^{j-1} E\left(\tilde{a}_k\right)\left(j = 2, \cdots, n\right), T_1 = 1$ and $E\left(\tilde{a}_k\right)$ is the expected value of $\tilde{a}_k = \left[a_k^L, a_k^U\right]$.

Based on operations of the interval numbers described in Section 2, we can drive the Theorem 5.

Theorem 5. Let $\tilde{a}_j = \left[a_j^L, a_j^U\right]\left(j = 1, 2, \cdots, n\right)$ be a set of interval numbers, then their aggregated value by using the UPWG operator is also an interval number, and

$$\begin{aligned} &UPWG\left(\tilde{a}_1, \tilde{a}_2, \cdots, \tilde{a}_n\right) = \\ &\tilde{a}_1^{\frac{T_1}{\sum_{j=1}^n T_j}} \otimes \tilde{a}_2^{\frac{T_2}{\sum_{j=1}^n T_j}} \otimes \cdots \otimes \tilde{a}_1^{\frac{T_2}{\sum_{j=1}^n T_j}} = \\ &\left[\prod_{j=1}^n \left(a_j^L\right)^{\frac{T_j}{\sum_{j=1}^n T_j}}, \prod_{j=1}^n \left(a_j^U\right)^{\frac{T_j}{\sum_{j=1}^n T_j}}\right], \end{aligned} \tag{10}$$

where $T_j = \prod_{k=1}^{j-1} E\left(\tilde{a}_k\right)\left(j = 2, \cdots, n\right), T_1 = 1$ and $E\left(\tilde{a}_k\right)$ is the expected value of $\tilde{a}_k = \left[a_k^L, a_k^U\right]$.

It can be easily proved that the UPWG operator has the following properties.

Theorem 6. (Idempotency) Let $\tilde{a}_j = \left[a_j^L, a_j^U\right]\left(j = 1, 2, \cdots, n\right)$ be a set of interval numbers, where $T_j = \prod_{k=1}^{j-1} E\left(\tilde{a}_k\right)\left(j = 2, \cdots, n\right), T_1 = 1$ and $E\left(\tilde{a}_k\right)$ is the expected value of $\tilde{a}_k = \left[a_k^L, a_k^U\right]$. If all $\tilde{a}_j\left(j = 1, 2, \cdots, n\right)$ are equal, i.e. $\tilde{a}_j = \tilde{a}$ for all *j*, then:

$$UPWG\left(\tilde{a}_1, \tilde{a}_2, \cdots, \tilde{a}_n\right) = \tilde{a}. \tag{11}$$

Theorem 7. (Boundedness) Let $\tilde{a}_j = \left[a_j^L, a_j^U\right] (j = 1,2,\cdots,n)$ be a set of interval numbers, where $T_j = \prod_{k=1}^{j-1} E(\tilde{a}_k) (j = 2,\cdots,n), T_1 = 1$ and $E(\tilde{a}_k)$ is the expected value of $\tilde{a}_k = \left[a_k^L, a_k^U\right]$, and let

$$\tilde{a}^- = \min_j \tilde{a}_j = \left[\min_j a_j^L, \min_j a_j^U\right];$$

$$\tilde{a}^+ = \max_j \tilde{a}_j = \left[\max_j a_j^L, \max_j a_j^U\right].$$

Then:

$$\tilde{a}^- \le UPWG(\tilde{a}_1, \tilde{a}_2, \cdots, \tilde{a}_n) \le \tilde{a}^+ . \tag{12}$$

Theorem 8. (Monotonicity) Let $\tilde{a}_j = \left[a_j^L, a_j^U\right]$ and $\tilde{a}'_j = \left[a_j'^L, a_j'^U\right] (j = 1,2,\cdots,n)$ be two set of interval numbers, where $T_j = \prod_{k=1}^{j-1} E(\tilde{a}_k)$, $T'_j = \prod_{k=1}^{j-1} E(\tilde{a}'_k) (j = 2,\cdots,n)$, $T_1 = T'_1 = 1$, $E(\tilde{a}_k)$ is the expected value of $\tilde{a}_k = \left[a_k^L, a_k^U\right]$, $E(\tilde{a}'_k)$ is the expected value of $\tilde{a}'_k = \left[a_k'^L, a_k'^U\right]$, if $\tilde{a}_j \le \tilde{a}'_j$, for all *j*, then:

$$UPWG(\tilde{a}_1, \tilde{a}_2, \cdots, \tilde{a}_n) \le UPWG(\tilde{a}'_1, \tilde{a}'_2, \cdots, \tilde{a}'_n). \tag{13}$$

2.3. Uncertain prioritized weighted harmonic average (UPWHA) operator

Based on the UPWA operator and the harmonic average (Chen *et al.* 2004), here we define the uncertain prioritized weighted harmonic average (UPWHA) operator:

Definition 6. Let $\tilde{a}_j = \left[a_j^L, a_j^U\right] (j = 1,2,\cdots,n)$ be a set of interval numbers, then we define the uncertain prioritized weighted harmonic average (UPWHA) operator as follows:

$$UPWHA(\tilde{a}_1, \tilde{a}_2, \cdots, \tilde{a}_n) = \frac{1}{\dfrac{\frac{T_1}{\sum_{j=1}^n T_j}}{\tilde{a}_1} \oplus \dfrac{\frac{T_2}{\sum_{j=1}^n T_j}}{\tilde{a}_2} \oplus \cdots \oplus \dfrac{\frac{T_n}{\sum_{j=1}^n T_j}}{\tilde{a}_n}}, \tag{14}$$

where $T_j = \prod_{k=1}^{j-1} E(\tilde{a}_k) (j = 2,\cdots,n), T_1 = 1$ and $E(\tilde{a}_k)$ is the expected value of $\tilde{a}_k = \left[a_k^L, a_k^U\right]$.

Based on operations of the interval numbers described in Section 2, we can drive the Theorem 9.

Theorem 9. Let $\tilde{a}_j = \left[a_j^L, a_j^U\right](j=1,2,\cdots,n)$ be a set of interval numbers, then their aggregated value by using the UPWA operator is also an interval number, and

$$UPWHA(\tilde{a}_1,\tilde{a}_2,\cdots,\tilde{a}_n) = \frac{1}{\frac{\frac{T_1}{\sum_{j=1}^{n} T_j}}{\tilde{a}_1} \oplus \frac{\frac{T_2}{\sum_{j=1}^{n} T_j}}{\tilde{a}_2} \oplus \cdots \oplus \frac{\frac{T_n}{\sum_{j=1}^{n} T_j}}{\tilde{a}_n}} = \left[\frac{1}{\sum_{j=1}^{n} \frac{\frac{T_j}{\sum_{j=1}^{n} T_j}}{a_j^L}}, \frac{1}{\sum_{j=1}^{n} \frac{\frac{T_j}{\sum_{j=1}^{n} T_j}}{a_j^U}}\right], \tag{15}$$

where $T_j = \prod_{k=1}^{j-1} E(\tilde{a}_k)(j=2,\cdots,n), T_1 = 1$ and $E(\tilde{a}_k)$ is the expected value of $\tilde{a}_k = \left[a_k^L, a_k^U\right]$.

It can be easily proved that the UPWHA operator has the following properties.

Theorem 10. (Idempotency) Let $\tilde{a}_j = \left[a_j^L, a_j^U\right](j=1,2,\cdots,n)$ be a set of interval numbers, where $T_j = \prod_{k=1}^{j-1} E(\tilde{a}_k)(j=2,\cdots,n), T_1 = 1$ and $E(\tilde{a}_k)$ is the expected value of $\tilde{a}_k = \left[a_k^L, a_k^U\right]$. If all $\tilde{a}_j (j=1,2,\cdots,n)$ are equal, i.e. $\tilde{a}_j = \tilde{a}$ for all j, then:

$$UPWHA(\tilde{a}_1,\tilde{a}_2,\cdots,\tilde{a}_n) = \tilde{a}\,. \tag{16}$$

Theorem 11. (Boundedness) Let $\tilde{a}_j = \left[a_j^L, a_j^U\right](j=1,2,\cdots,n)$ be a set of interval numbers, where $T_j = \prod_{k=1}^{j-1} E(\tilde{a}_k)(j=2,\cdots,n)$, $T_1 = 1$ and $E(\tilde{a}_k)$ is the expected value of $\tilde{a}_k = \left[a_k^L, a_k^U\right]$, and let

$$\tilde{a}^- = \min_j \tilde{a}_j = \left[\min_j a_j^L, \min_j a_j^U\right];$$

$$\tilde{a}^+ = \max_j \tilde{a}_j = \left[\max_j a_j^L, \max_j a_j^U\right].$$

Then:

$$\tilde{a}^- \le UPWHA(\tilde{a}_1,\tilde{a}_2,\cdots,\tilde{a}_n) \le \tilde{a}^+\,. \tag{17}$$

Theorem 12. (Monotonicity) Let $\tilde{a}_j = \left[a_j^L, a_j^U\right]$ and $\tilde{a}'_j = \left[a_j'^L, a_j'^U\right]\left(j=1,2,\cdots,n\right)$ be two set of interval numbers, where $T_j = \prod_{k=1}^{j-1} E\left(\tilde{a}_k\right)$, $T'_j = \prod_{k=1}^{j-1} E\left(\tilde{a}'_k\right)\left(j=2,\cdots,n\right)$, $T_1 = T'_1 = 1$, $E\left(\tilde{a}_k\right)$ is the expected value of $\tilde{a}_k = \left[a_k^L, a_k^U\right]$, $E\left(\tilde{a}'_k\right)$ is the expected value of $\tilde{a}'_k = \left[a_k'^L, a_k'^U\right]$, if $\tilde{a}_j \le \tilde{a}'_j$, for all *j*, then:

$$UPWHA\left(\tilde{a}_1, \tilde{a}_2, \cdots, \tilde{a}_n\right) \le UPWHA\left(\tilde{a}'_1, \tilde{a}'_2, \cdots, \tilde{a}'_n\right). \tag{18}$$

3. An approach to multiple attribute group decision making with interval numbers information

In this section, we shall utilize the prioritized aggregation operators to multiple attribute group decision making.

For a multiple attribute group decision making problems with interval numbers information, let $X = \left\{X_1, X_2, \cdots, X_m\right\}$ be a discrete set of alternatives, Let $G = \left\{G_1, G_2, \cdots, G_n\right\}$ be a collection of attribute and that there is a prioritization between the attribute expressed by the linear ordering $G_1 \succ G_2 \succ G_3 \cdots \succ G_n$, indicate attribute G_j has a higher priority than G_s, if $j < s$, and let $D = \left\{D_1, D_2, \cdots, D_t\right\}$ be the set of decision makers and that there is a prioritization between the attribute expressed by the linear ordering $D_1 \succ D_2 \succ D_3 \cdots \succ D_t$, indicate attribute D_η has a higher priority than G_γ, if $\eta < \gamma$. Suppose that $\tilde{A}_k = \left(\tilde{a}_{ij}^{(k)}\right)_{m \times n} = \left[\left(a_{ij}^L\right)^{(k)}, \left(a_{ij}^U\right)^{(k)}\right]_{m \times n}$ is the multiple attribute group decision making matrix, where $\tilde{a}_{ij}^{(k)}$ is an attribute value, which take the form of interval number, given by the decision maker $D_k \in D$, for the alternative $X_i \in X$ with respect to the attribute $G_j \in G$.

Then, we utilize the UPWA (or UPWG, UPWHA) operator to develop an approach to multiple attribute group decision making problems with interval number information, which can be described as following:

Step 1. Normalize each attribute value $\tilde{a}_{ij}^{(k)}$ in the matrix $\tilde{A}_k$ into a corresponding element in the matrix $\tilde{R}_k = \left(\tilde{r}_{ij}^{(k)}\right)_{m \times n} \left(\tilde{r}_{ij}^{(k)} = \left[\left(r_{ij}^L\right)^{(k)}, \left(r_{ij}^U\right)^{(k)}\right]\right)$ using the following formulas:

$$\begin{cases} \left(r_{ij}^L\right)^{(k)} = \left(a_{ij}^L\right)^{(k)} \Big/ \sum_{i=1}^{m} \left(a_{ij}^U\right)^{(k)} \\ \left(r_{ij}^U\right)^{(k)} = \left(a_{ij}^U\right)^{(k)} \Big/ \sum_{i=1}^{m} \left(a_{ij}^L\right)^{(k)} \end{cases}, \text{ for benefit attribute } G_j,$$

$$i = 1,2,\cdots,m,\ j = 1,2,\cdots,n,\ k = 1,2,\cdots,t. \tag{19}$$

$$\begin{cases} \left(r_{ij}^L\right)^{(k)} = \left(1\Big/\left(a_{ij}^U\right)^{(k)}\right)\Big/\sum_{i=1}^{m} 1\Big/\left(a_{ij}^L\right)^{(k)} \\ \left(r_{ij}^U\right)^{(k)} = \left(1\Big/\left(a_{ij}^L\right)^{(k)}\right)\Big/\sum_{i=1}^{m} 1\Big/\left(a_{ij}^U\right)^{(k)} \end{cases}, \text{ for cost attribute } G_j,$$

$$i = 1,2,\cdots,m,\ j = 1,2,\cdots,n,\ k = 1,2,\cdots,t. \tag{20}$$

Step 2. Calculate the values of $T_{ij}^{(k)}\left(k = 1,2,\cdots,t\right)$ as follows:

$$T_{ij}^{(k)} = \prod_{\lambda=1}^{k-1} E\left(\tilde{r}_{ij}^{(\lambda)}\right)\left(k = 2,\cdots,t\right); \tag{21}$$

$$T_{ij}^{(1)} = 1. \tag{22}$$

Step 3. Utilize the decision information given in matrix $\tilde{R}_k$, and the UPWA operator:

$$\tilde{r}_{ij} = \left(r_{ij}^L, r_{ij}^U\right) = UPWA\left(\tilde{r}_{ij}^{(1)}, \tilde{r}_{ij}^{(2)}, \cdots, \tilde{r}_{ij}^{(t)}\right) =$$
$$\frac{T_{ij}^{(1)}}{\sum_{k=1}^{t} T_{ij}^{(k)}}\tilde{r}_{ij}^{(1)} \oplus \frac{T_{ij}^{(2)}}{\sum_{k=1}^{t} T_{ij}^{(k)}}\tilde{r}_{ij}^{(2)} \oplus \cdots \oplus \frac{T_{ij}^{(t)}}{\sum_{k=1}^{t} T_{ij}^{(k)}}\tilde{r}_{ij}^{(t)} =$$
$$\left[\sum_{k=1}^{t} \frac{T_{ij}^{(k)}\left(r_{ij}^L\right)^{(k)}}{\sum_{k=1}^{t} T_{ij}^{(k)}}, \sum_{k=1}^{t} \frac{T_{ij}^{(k)}\left(r_{ij}^U\right)^{(k)}}{\sum_{k=1}^{t} T_{ij}^{(k)}}\right],$$

$$i = 1,2,\cdots,m,\ j = 1,2,\cdots,n. \tag{23}$$

Or the UPWG operator:

$$\tilde{r}_{ij} = \left(r_{ij}^L, r_{ij}^U\right) = UPWG\left(\tilde{r}_{ij}^{(1)}, \tilde{r}_{ij}^{(2)}, \cdots, \tilde{r}_{ij}^{(t)}\right) =$$
$$\left(\tilde{r}_{ij}^{(1)}\right)^{\frac{T_{ij}^{(1)}}{\sum_{k=1}^{t} T_{ij}^{(k)}}} \otimes \left(\tilde{r}_{ij}^{(2)}\right)^{\frac{T_{ij}^{(2)}}{\sum_{k=1}^{t} T_{ij}^{(k)}}} \otimes \cdots \otimes \left(\tilde{r}_{ij}^{(t)}\right)^{\frac{T_{ij}^{(t)}}{\sum_{k=1}^{t} T_{ij}^{(k)}}} =$$
$$\left[\prod_{k=1}^{t}\left(\left(r_{ij}^L\right)^{(k)}\right)^{\frac{T_{ij}^{(k)}}{\sum_{k=1}^{t} T_{ij}^{(k)}}}, \prod_{k=1}^{t}\left(\left(r_{ij}^U\right)^{(k)}\right)^{\frac{T_{ij}^{(k)}}{\sum_{k=1}^{t} T_{ij}^{(k)}}}\right],$$

$$i = 1,2,\cdots,m,\ j = 1,2,\cdots,n. \tag{24}$$

Or the UPWHA operator:

$$\tilde{r}_{ij} = \left(r_{ij}^{L}, r_{ij}^{U}\right) = UPWHA\left(\tilde{r}_{ij}^{(1)}, \tilde{r}_{ij}^{(2)}, \cdots, \tilde{r}_{ij}^{(t)}\right) =$$

$$\frac{1}{\dfrac{T_{ij}^{(1)} \Big/ \sum_{k=1}^{t} T_{ij}^{(k)}}{\tilde{r}_{ij}^{(1)}} \oplus \dfrac{T_{ij}^{(2)} \Big/ \sum_{k=1}^{t} T_{ij}^{(k)}}{\tilde{r}_{ij}^{(2)}} \oplus \cdots \oplus \dfrac{T_{ij}^{(t)} \Big/ \sum_{k=1}^{t} T_{ij}^{(k)}}{\tilde{r}_{ij}^{(t)}}} =$$

$$\left[\frac{1}{\sum_{k=1}^{t} \left(\dfrac{T_{ij}^{(k)} \Big/ \sum_{k=1}^{t} T_{ij}^{(k)}}{\left(r_{ij}^{L}\right)^{(k)}} \right)}, \frac{1}{\sum_{k=1}^{t} \left(\dfrac{T_{ij}^{(k)} \Big/ \sum_{k=1}^{t} T_{ij}^{(k)}}{\left(r_{ij}^{U}\right)^{(k)}} \right)} \right],$$

$$i = 1,2,\cdots,m,\ j = 1,2,\cdots,n. \tag{25}$$

to aggregate all the individual decision matrices $\tilde{R}_k \left(k = 1,2,\cdots,t\right)$ into the collective decision matrix $\tilde{R} = \left(\tilde{r}_{ij}\right)_{m\times n} = \left[a_{ij}^{L}, a_{ij}^{U}\right]_{m\times n}$, $i = 1,2,\cdots,m,\ j = 1,2,\cdots,n$.

Step 4. Calculate the values of $T_{ij}\left(i = 1,2,\cdots,m, j = 2,\cdots,n\right)$ as follows:

$$T_{ij} = \prod_{\lambda=1}^{j-1} E\left(\tilde{r}_{i\lambda}\right)\left(i = 1,2,\cdots,m,\ j = 2,\cdots,n\right); \tag{26}$$

$$T_{i1} = 1, i = 1,2,\cdots,m. \tag{27}$$

Step 5. Aggregate all interval numbers into preference value $\tilde{r}_{ij}\left(j = 1,2,\cdots,n\right)$ by using the uncertain prioritized weighted average (UPWA) operator:

$$\tilde{r}_{i} = \left(r_{i}^{L}, r_{i}^{U}\right) = UPWA\left(\tilde{r}_{i1}, \tilde{r}_{i2}, \cdots, \tilde{r}_{in}\right) =$$

$$\frac{T_{i1}}{\sum_{j=1}^{n} T_{ij}} \tilde{r}_{i1} \oplus \frac{T_{i2}}{\sum_{j=1}^{n} T_{ij}} \tilde{r}_{i2} \oplus \cdots \oplus \frac{T_{in}}{\sum_{j=1}^{n} T_{ij}} \tilde{r}_{in} =$$

$$\left[\sum_{j=1}^{n} \frac{T_{ij} r_{ij}^{L}}{\sum_{j=1}^{n} T_{ij}}, \sum_{j=1}^{n} \frac{T_{ij} r_{ij}^{U}}{\sum_{j=1}^{n} T_{ij}} \right],$$

$$i = 1,2,\cdots,m. \tag{28}$$

Or the uncertain prioritized weighted geometric (UPWG) operator:

$$\tilde{r}_i = \left(r_i^L, r_i^U\right) = UPWG\left(\tilde{r}_{i1}, \tilde{r}_{i2}, \cdots, \tilde{r}_{in}\right) = \tilde{r}_{i1}^{\frac{T_{i1}}{\sum_{j=1}^{n} T_{ij}}} \otimes \tilde{r}_{i2}^{\frac{T_{i2}}{\sum_{j=1}^{n} T_{ij}}} \otimes \cdots \otimes \tilde{r}_{in}^{\frac{T_{in}}{\sum_{j=1}^{n} T_{ij}}} = \left[\prod_{j=1}^{n}\left(r_{ij}^L\right)^{\frac{T_{ij}}{\sum_{j=1}^{n} T_{ij}}}, \prod_{j=1}^{n}\left(r_{ij}^U\right)^{\frac{T_{ij}}{\sum_{j=1}^{n} T_{ij}}}\right],$$

$$i = 1, 2, \cdots, m. \tag{29}$$

Or the uncertain prioritized weighted harmonic average (UPWHA) operator:

$$\tilde{r}_i = \left(r_i^L, r_i^U\right) = UWHA_{\omega}\left(\tilde{r}_{i1}, \tilde{r}_{i2}, \cdots, \tilde{r}_{in}\right) = FPWHA\left(\tilde{a}_1, \tilde{a}_2, \cdots, \tilde{a}_n\right) \frac{1}{\frac{\frac{T_{i1}}{\sum_{j=1}^{n} T_{ij}}}{\tilde{r}_{i1}} \oplus \frac{\frac{T_{i2}}{\sum_{j=1}^{n} T_{ij}}}{\tilde{r}_{i2}} \oplus \cdots \oplus \frac{\frac{T_{in}}{\sum_{j=1}^{n} T_{ij}}}{\tilde{r}_{in}}} = \left[\frac{1}{\sum_{j=1}^{n} \frac{\frac{T_{ij}}{\sum_{j=1}^{n} T_{ij}}}{r_{ij}^L}}, \frac{1}{\sum_{j=1}^{n} \frac{\frac{T_{ij}}{\sum_{j=1}^{n} T_{ij}}}{r_{ij}^U}}\right],$$

$$i = 1, 2, \cdots, m. \tag{30}$$

to derive the overall interval numbers preference values $\tilde{r}_i \left(i = 1, 2, \cdots, m\right)$ of the alternative A_i.

Step 6. To rank these collective overall preference values $\tilde{r}_i \left(i = 1, 2, \cdots, m\right)$, we first compare each $\tilde{a}_i$ with all the $\tilde{r}_j \left(j = 1, 2, \cdots, m\right)$ by using Eq. (1). For simplicity, we let $p_{ij} = p\left(\tilde{r}_i \geq \tilde{r}_j\right)$, then we develop a complementary matrix as $P = \left(p_{ij}\right)_{m \times m}$, where $p_{ij} \geq 0$, $p_{ij} + p_{ji} = 1$, $p_{ii} = 0.5$, $i, j = 1, 2, \cdots, n$.

Summing all the elements in each line of matrix P, we have:

$$p_i = \sum_{j=1}^{m} p_{ij}, i = 1,2,\cdots,m .$$

Then we rank the collective overall preference values $\tilde{r}_i \left(i = 1,2,\cdots,m\right)$ in descending order in accordance with the values of $p_i \left(i = 1,2,\cdots,m\right)$.

Step 7. Rank all the alternatives $A_i \left(i = 1,2,\cdots,m\right)$ and select the best one(s) in accordance with the collective overall preference values $\tilde{r}_i \left(i = 1,2,\cdots,m\right)$.

Step 8. End.

4. Numerical example

In order to strengthen the construction projects selection, promote the building of construction projects selection, the construction companies of management in a city wants to select the prospect construction projects. This selection has been raised great attention from the construction project president D_1, dean of construction project D_2, and engineer office D_3 sets up the panel of decision makers which will take the whole responsibility for this selection. They made strict evaluation for 5 prospect construction projects $X_i \left(i = 1,2,3,4,5\right)$ according to the following four attributes: ① G_1 is the safety; ② G_2 is the product quality; ③ G_3 is the duration; ④ G_4 is the cost. Construction project president have the absolute priority for decision making, dean of the construction project comes next. Besides, this selection will be in strict accordance with the principle of combine ability with political integrity. The prioritization relationship for the criteria is as below, $G_1 \succ G_2 \succ G_3 \succ G_4$. The five prospect construction projects $X_i \left(i = 1,2,3,4,5\right)$ are to be evaluated using the 0-1 scale by the three decision makers $D_k \left(k = 1,2,3\right)$ under the above four attributes, and construct, respectively, the uncertain decision matrices are shown in Tables 1–3:

Table 1. Decision matrix $\tilde{A}_1$

	G_1	G_2	G_3	G_4
X_1	[0.80, 0.90]	[0.72, 0.80]	[0.91, 0.96]	[0.62, 0.68]
X_2	[0.88, 0.93]	[0.67, 0.83]	[0.60, 0.70]	[0.69, 0.75]
X_3	[0.95, 0.98]	[0.90, 0.95]	[0.77, 0.82]	[0.93, 0.96]
X_4	[0.82, 0.88]	[0.97, 1.00]	[0.98, 1.00]	[0.97, 1.00]
X_5	[0.78, 0.81]	[0.78, 0.81]	[0.83, 0.88]	[0.94, 0.99]

Table 2. Decision matrix $\tilde{A}_2$

	G_1	G_2	G_3	G_4
X_1	[0.75, 0.85]	[0.67, 0.75]	[0.86, 0.91]	[0.57, 0.63]
X_2	[0.83, 0.88]	[0.62, 0.78]	[0.55, 0.60]	[0.64, 0.70]
X_3	[0.90, 0.93]	[0.85, 0.90]	[0.72, 0.77]	[0.88, 0.91]
X_4	[0.77, 0.83]	[0.92, 0.95]	[0.93, 0.95]	[0.92, 0.95]
X_5	[0.73, 0.76]	[0.73, 0.76]	[0.78, 0.86]	[0.89, 0.90]

Table 3. Decision matrix $\tilde{A}_3$

	G_1	G_2	G_3	G_4
X_1	[0.72, 0.82]	[0.64, 0.72]	[0.83, 0.88]	[0.54, 0.60]
X_2	[0.80, 0.85]	[0.59, 0.75]	[0.52, 0.62]	[0.61, 0.67]
X_3	[0.87, 0.90]	[0.82, 0.87]	[0.69, 0.74]	[0.85, 0.88]
X_4	[0.74, 0.80]	[0.89, 0.92]	[0.90, 0.92]	[0.89, 0.92]
X_5	[0.70, 0.73]	[0.70, 0.73]	[0.75, 0.80]	[0.86, 0.91]

Taking into account all the attributes dimensional consistent, for convenience, we needn't normalize decision matrix $\tilde{R}_k$. Then, in order to select the most desirable candidate, we utilize the UPWA operator to develop an approach to multiple attribute group decision making problems with interval numbers information, which can be described as following:

Step 1. Because all the attribute values are ranged from 0–1, so we needn't normalize the decision matrices $\tilde{R}_k$.

Step 2. Utilize (21)–(22) to calculate the $T_{ij}^{(1)}$, $T_{ij}^{(2)}$, $T_{ij}^{(3)}$:

$$T_{ij}^{(1)} = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \end{bmatrix},$$

$$T_{ij}^{(2)} = \begin{bmatrix} 0.850 & 0.760 & 0.935 & 0.650 \\ 0.905 & 0.750 & 0.650 & 0.720 \\ 0.965 & 0.925 & 0.795 & 0.945 \\ 0.850 & 0.985 & 0.990 & 0.985 \\ 0.795 & 0.795 & 0.855 & 0.965 \end{bmatrix},$$

$$T_{ij}^{(3)} = \begin{bmatrix} 0.680 & 0.540 & 0.827 & 0.390 \\ 0.774 & 0.525 & 0.374 & 0.482 \\ 0.883 & 0.809 & 0.592 & 0.846 \\ 0.680 & 0.921 & 0.931 & 0.921 \\ 0.592 & 0.592 & 0.701 & 0.864 \end{bmatrix}.$$

Step 3. Utilize the decision information given in matrix $\tilde{R}_k$, and the UPWA operator to aggregate all the individual decision matrices $\tilde{R}_k\left(k=1,2,3\right)$ into the collective decision matrix $\tilde{R}=\left(\tilde{r}_{ij}\right)_{5\times 4}=\left[a_{ij}^L,a_{ij}^U\right]_{5\times 4}$. The aggregating results are shown in Table 4.

Table 4. Decision matrix $\tilde{R}$

	G_1	G_2	G_3	G_4
X_1	[0.762, 0.862]	[0.685, 0.765]	[0.869, 0.919]	[0.589, 0.649]
X_2	[0.840, 0.890]	[0.635, 0.795]	[0.569, 0.653]	[0.656, 0.716]
X_3	[0.908, 0.938]	[0.859, 0.909]	[0.734, 0.784]	[0.889, 0.919]
X_4	[0.782, 0.842]	[0.928, 0.958]	[0.938, 0.958]	[0.928, 0.958]
X_5	[0.744, 0.774]	[0.744, 0.774]	[0.791, 0.851]	[0.899, 0.935]

Step 4. Utilize (26)–(27) to calculate the values of $T_{ij}\left(i=1,2,\cdots,m,j=2,\cdots,n\right)$ as follows:

$$T_{ij}=\begin{bmatrix}1.000 & 0.812 & 0.725 & 0.894\\ 1.000 & 0.865 & 0.715 & 0.611\\ 1.000 & 0.923 & 0.884 & 0.759\\ 1.000 & 0.812 & 0.943 & 0.948\\ 1.000 & 0.759 & 0.759 & 0.821\end{bmatrix}.$$

Step 5. Aggregate all interval numbers $\tilde{r}_{ij}\left(j=1,2,\cdots,n\right)$ by using the uncertain prioritized weighted average (UPWA) operator to derive the overall interval numbers $\tilde{r}_i\left(i=1,2,\cdots,m\right)$ of the prospect construction projects A_i:

$$\tilde{r}_1=\left[0.731,0.808\right],\ \tilde{r}_2=\left[0.695,0.787\right],\ \tilde{r}_3=\left[0.849,0.889\right],$$
$$\tilde{r}_4=\left[0.886,0.923\right],\ \tilde{r}_5=\left[0.779,0.817\right].$$

Step 6. Rank these preference degrees $\tilde{r}_i\left(i=1,2,3,4,5\right)$, we first compare each $\tilde{r}_i$ with all the $\tilde{r}_j\left(j=1,2,3,4,5\right)$ by using Eq. (1), and then develop a complementary matrix:

$$P=\begin{bmatrix}0.500 & 0.667 & 0.000 & 0.000 & 0.250\\ 0.333 & 0.500 & 0.000 & 0.000 & 0.060\\ 1.000 & 1.000 & 0.500 & 0.044 & 1.000\\ 1.000 & 1.000 & 0.956 & 0.500 & 1.000\\ 0.750 & 0.940 & 0.000 & 0.000 & 0.500\end{bmatrix}.$$

Summing all the elements in each line of matrix P, we have:

$$p_1=1.148,\ p_2=0.893, p_3=3.544,\ p_4=4.456,\ p_5=2.190.$$

Then we rank the preference degree $\tilde{r}_i\left(i=1,2,3,4,5\right)$ in descending order in accordance with the values of $p_i\left(i=1,2,\cdots,5\right)$: $\tilde{r}_4 \succ \tilde{r}_3 \succ \tilde{r}_5 \succ \tilde{r}_1 \succ \tilde{r}_2$.

Step 7. Rank all the all the construction project $A_i\left(i=1,2,\cdots,5\right)$ in accordance with the preference degree $\tilde{r}_i\left(i=1,2,\cdots,5\right)$: $A_4 \succ A_3 \succ A_5 \succ A_1 \succ A_2$, and thus the most desirable all the construction project is A_4.

Based on the UPWG operator, then, in order to select the most desirable prospect construction project, we can develop an approach to multiple attribute group decision making problems with interval numbers information, which can be described as following:

Step 1′. *See Step 1.*

Step 2′. *See Step 2.*

Step 3′. Utilize the decision information given in matrix $\tilde{R}_k$, and the UPWG operator to aggregate all the individual decision matrices $\tilde{R}_k\left(k=1,2,3\right)$ into the collective decision matrix $\tilde{R}=\left(\tilde{r}_{ij}\right)_{5\times4}=\left[a_{ij}^L,a_{ij}^U\right]_{5\times4}$. The aggregating results are shown in Table 5.

Table 5. Decision matrix $\tilde{R}$

	G_1	G_2	G_3	G_4
X_1	[0.761, 0.861]	[0.684, 0.764]	[0.868, 0.919]	[0.588, 0.648]
X_2	[0.839, 0.889]	[0.634, 0.794]	[0.568, 0.651]	[0.655, 0.715]
X_3	[0.908, 0.938]	[0.859, 0.909]	[0.733, 0.783]	[0.888, 0.918]
X_4	[0.781, 0.841]	[0.927, 0.957]	[0.937, 0.957]	[0.927, 0.957]
X_5	[0.743, 0.773]	[0.743, 0.773]	[0.791, 0.851]	[0.898, 0.934]

Step 4′. Utilize (26)–(27) to calculate the values of $T_{ij}\left(i=1,2,\cdots,m,j=2,\cdots,n\right)$ as follows:

$$T_{ij}=\begin{bmatrix}1.000 & 0.811 & 0.724 & 0.894\\ 1.000 & 0.864 & 0.714 & 0.610\\ 1.000 & 0.923 & 0.884 & 0.758\\ 1.000 & 0.811 & 0.942 & 0.947\\ 1.000 & 0.758 & 0.758 & 0.821\end{bmatrix}.$$

Step 5′. Aggregate all interval numbers $\tilde{r}_{ij}\left(j=1,2,\cdots,n\right)$ by using the uncertain prioritized weighted geometric (UPWG) operator to derive the overall interval numbers $\tilde{r}_i\left(i=1,2,\cdots,m\right)$ of the construction projects A_i:

$$\tilde{r}_1=\left[0.724,0.802\right],\ \tilde{r}_2=\left[0.686,0.781\right],\ \tilde{r}_3=\left[0.845,0.886\right],$$
$$\tilde{r}_4=\left[0.882,0.920\right],\ \tilde{r}_5=\left[0.777,0.814\right].$$

Step 6′. Rank these preference degrees $\tilde{r}_i\left(i=1,2,3,4,5\right)$, we first compare each $\tilde{r}_i$ with all the $\tilde{r}_j\left(j=1,2,3,4,5\right)$ by using Eq. (1), and then develop a complementary matrix:

$$P=\begin{bmatrix}0.500 & 0.672 & 0.000 & 0.000 & 0.220\\ 0.328 & 0.500 & 0.000 & 0.000 & 0.032\\ 1.000 & 1.000 & 0.500 & 0.051 & 1.000\\ 1.000 & 1.000 & 0.949 & 0.500 & 1.000\\ 0.780 & 0.968 & 0.000 & 0.000 & 0.500\end{bmatrix}.$$

Summing all the elements in each line of matrix P, we have:

$$p_1 = 1.391,\ p_2 = 0.860,\ p_3 = 3.551,\ p_4 = 4.449,\ p_5 = 2.249.$$

Then we rank the preference degree $\tilde{r}_i\left(i = 1,2,3,4,5\right)$ in descending order in accordance with the values of $p_i\left(i = 1,2,\cdots,5\right)$: $\tilde{r}_4 \succ \tilde{r}_3 \succ \tilde{r}_5 \succ \tilde{r}_1 \succ \tilde{r}_2$.

Step 7′. Rank all the all the construction projects $A_i\left(i = 1,2,\cdots,5\right)$ in accordance with the preference degree $\tilde{r}_i\left(i = 1,2,\cdots,5\right)$: $A_4 \succ A_3 \succ A_5 \succ A_1 \succ A_2$, and thus the most desirable construction project is A_4.

Based on the UPWHA operator, then, in order to select the most desirable construction project, we can develop an approach to multiple attribute group decision making problems with interval numbers, which can be described as following:

Step 1″. *See Step 1.*

Step 2″. Utilize the decision information given in matrix $\tilde{R}_k$, and the UPWHA operator to aggregate all the individual decision matrices $\tilde{R}_k\left(k = 1,2,3\right)$ into the collective decision matrix $\tilde{R} = \left(\tilde{r}_{ij}\right)_{5\times 4} = \left[a_{ij}^L, a_{ij}^U\right]_{5\times 4}$. The aggregating results are shown in Table 6.

Table 6. Decision matrix $\tilde{R}$

	G_1	G_2	G_3	G_4
X_1	[0.760, 0.860]	[0.683, 0.763]	[0.868, 0.918]	[0.587, 0.647]
X_2	[0.839, 0.889]	[0.633, 0.793]	[0.567, 0.650]	[0.654, 0.715]
X_3	[0.907, 0.937]	[0.858, 0.908]	[0.736, 0.786]	[0.888, 0.918]
X_4	[0.780, 0.840]	[0.927, 0.957]	[0.936, 0.956]	[0.927, 0.957]
X_5	[0.742, 0.772]	[0.742, 0.772]	[0.790, 0.850]	[0.897, 0.933]

Step 3″. Utilize (26)–(27) to calculate the values of $T_{ij}\left(i = 1,2,\cdots,m, j = 2,\cdots,n\right)$ as follows:

$$T_{ij} = \begin{bmatrix} 1.000 & 0.810 & 0.723 & 0.893 \\ 1.000 & 0.864 & 0.714 & 0.609 \\ 1.000 & 0.922 & 0.883 & 0.757 \\ 1.000 & 0.810 & 0.942 & 0.946 \\ 1.000 & 0.757 & 0.757 & 0.820 \end{bmatrix}.$$

Step 4″. Aggregate all interval numbers $\tilde{r}_{ij}\left(j = 1,2,\cdots,n\right)$ by using the uncertain prioritized weighted harmonic average (UPWHA) operator to derive the overall interval numbers $\tilde{r}_i\left(i = 1,2,\cdots,m\right)$ of the prospect construction projects A_i:

$$\tilde{r}_1 = \left[0.718, 0.795\right],\ \tilde{r}_2 = \left[0.677, 0.774\right],\ \tilde{r}_3 = \left[0.842, 0.883\right],$$
$$\tilde{r}_4 = \left[0.879, 0.918\right],\ \tilde{r}_5 = \left[0.774, 0.811\right].$$

Step 5". Rank these preference degree $\tilde{r}_i\left(i=1,2,3,4,5\right)$, we first compare each $\tilde{r}_i$ with all the $\tilde{r}_j\left(j=1,2,3,4,5\right)$ by using Eq. (1), and then develop a complementary matrix:

$$P=\begin{bmatrix} 0.500 & 0.675 & 0.000 & 0.000 & 0.186 \\ 0.325 & 0.500 & 0.000 & 0.000 & 0.002 \\ 1.000 & 1.000 & 0.500 & 0.058 & 1.000 \\ 1.000 & 1.000 & 0.942 & 0.500 & 1.000 \\ 0.814 & 0.998 & 0.000 & 0.000 & 0.500 \end{bmatrix}.$$

Summing all the elements in each line of matrix *P*, we have:

$$p_1=1.131,\ p_2=0.827,\ p_3=3.558,\ p_4=4.442,\ p_5=2.312.$$

Then we rank the preference degree $\tilde{r}_i\left(i=1,2,3,4,5\right)$ in descending order in accordance with the values of $p_i\left(i=1,2,\cdots,5\right)$: $\tilde{r}_4 \succ \tilde{r}_3 \succ \tilde{r}_5 \succ \tilde{r}_1 \succ \tilde{r}_2$.

Step 6″. Rank all the construction projects $A_i\left(i=1,2,\cdots,5\right)$ in accordance with the preference degree $\tilde{r}_i\left(i=1,2,\cdots,5\right)$: $A_4 \succ A_3 \succ A_5 \succ A_1 \succ A_2$, and thus the most desirable construction project is A_4.

In this section, we have proposed three approaches to solve the uncertain multiple attribute group decision making problems in which the attributes and experts are in different priority level. From the above analysis, we can see that the main advantages of the proposed operators and approaches over the traditional uncertain aggregation operators and approaches are not only due to the fact that our operators accommodate the interval numbers but also due to the consideration of the prioritization among the attributes and the decision makers, which makes it more feasible and practical.

Conclusions

In this paper, we investigate the uncertain multiple attribute group decision making (MAGDM) problem in which the attributes and experts are in different priority level. Then, Motivated by the ideal of prioritized aggregation operators (Yager 2008), we have developed some prioritized aggregation operators for aggregating interval numbers information: uncertain prioritized weighted average (UPWA) operator, uncertain prioritized weighted geometric (UPWG) operator and uncertain prioritized weighted harmonic average (UPWHA) operator. The prominent characteristic of these proposed operators is that they take into account prioritization among the attributes and experts. Then, we have utilized these operators to develop some approaches to solve the uncertain multiple attribute group decision making problems in which the attributes and experts are in different priority level. Finally, a practical example about talent introduction is given to verify the developed approach and to demonstrate its practicality and effectiveness.

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