Uncertainty modelling and structured singular value computation applied to an electro-mechanical system

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Abstract: The investigation of closed-loop systems subject to model perturbations is an important issue to assure stability robustness of a control design. A large variety of model perturbations can be described by norm-bounded uncertainty models. A general approach for modelling structured complex and real-valued parametric perturbations is presented. The resulting robustness analysis problem is solved nonconservatively using real and complex-structured singular-value calculations. The uncertainty modelling and robustness analysis are shown for a high-accuracy 5D electromechanical positioning device to be used in optical (Compact Disc) recording.

1 Introduction

To ensure that a model-based control system design will work well with the actual system it is necessary to analyse the closed-loop robustness properties for model perturbations, such as unmodelled parasitic dynamics, linearisation errors and parametric uncertainties. In past years, much research effort has been spent to solve the multivariable robustness analysis problem. An important development is based on the description of model uncertainties as transfer functions which are norm-bounded but otherwise unknown, and using singular values as indicators [1]. Owing to the use of norms the singular-value analysis method is appropriate for all situations with little knowledge about the perturbations. Its major disadvantage is its conservatism, as indicated by Doyle and others [2] in the sense that the uncertainty model set is much larger than necessary and does not account for structure of perturbations. For that reasons, Doyle [3] introduced the structured singular-value analysis. Recently, Fan and others [4, 5] have given an extension to include real-valued uncertainties.

This paper presents a general procedure to model norm-bounded perturbations and some computational aspects of real and complex-structured singular-value analysis.

A general concept which is very useful for norm-bounded uncertainty modelling, and especially for robustness analysis with the structured singular value, is the linear fractional transformation (LFT). As an example, consider a system with uncertainty, Fig. 1. The

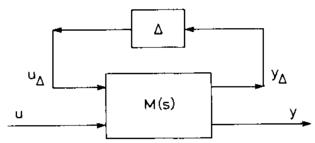

Fig. 1 *System with uncertainty feedback*

transfer function $M(s)$ represents the transfer function from the exogenous signals u (references, disturbances, control inputs, etc.) and the uncertainty outputs u_Δ, to the controlled variables y (tracking error, measured signals, etc.) and the uncertainty inputs y_Δ. The uncertainty is denoted in Fig. 1 as the transfer function $\Delta(s)$. The system $M(s)$ is partitioned according to the dimensions of the signal sets involved:

$$\begin{pmatrix} y_\Delta \\ y \end{pmatrix} = \begin{pmatrix} M_{11}(s) & M_{12}(s) \\ M_{21}(s) & M_{22}(s) \end{pmatrix} \begin{pmatrix} u_\Delta \\ u \end{pmatrix} \tag{1}$$

The upper linear fractional transformation on M and Δ is denoted as $F_u(M, \Delta)$ and is defined to be equal to the transfer function from u to y: $F_u(M, \Delta) = M_{22} + M_{21}(I - \Delta M_{11})^{-1} \Delta M_{12}$.

2 Complex norm-bounded uncertainty modelling

Complex-valued model uncertainties are often used to describe unmodelled dynamics, for instance actuator and sensor dynamics or parasitic system dynamics. Such uncertainties can be described as input/output transfer functions. Well known complex uncertainty descriptions are the multiplicative input and output uncertainty and the additive structure, Fig. 2.

Definition 2.1: A $p \times p$ complex-valued norm-bounded unstructured perturbation Δ_c is the set of $p \times p$ transfer functions $\Delta(s)$: $\mathbf{C} \to \mathbf{C}^{p \times p}$ which are analytic in the closed right half-plane and have a norm-bound less than or

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equal to some given positive function $\bar{\delta}_c(\omega) \in \mathscr{R}^+$:

$$\Delta_c = \{\Delta(s) \text{ stable} \mid \bar{\sigma}(\Delta(j\omega)), \leqslant \bar{\delta}_c(\omega) \quad \omega \in (-\infty, \infty)\} \quad (2)$$

with $\bar{\sigma}$ denoting the maximum singular value. The normalised uncertainty set is given by $B\Delta_c = \{\Delta(s) \in \Delta_c \mid \bar{\sigma}(\Delta(j\omega)) \leqslant 1, \omega \in (-\infty, \infty)\}$.

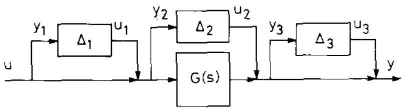

Fig. 2 *System with complex uncertainties*

Notice that this definition is restricted to square uncertainty matrices. Any nonsquare uncertainty can be made square by adding zero rows or columns.

If the uncertainty modelling results in structural zeros in entries of Δ_c, the uncertainty is called 'structured'. A well-known structure is the (block-)diagonal one. For such cases the uncertainty set can be described with its structure information and with the block-diagonal entries given as elements of Δ_c or $B\Delta_c$, see the following section.

To write the uncertainties in a unique format, these types of model perturbations are expressed in the linear fractional form. This can be done systematically by writing down the transfer function of the system connected to the uncertainties. Consider Fig. 2, for example. Label the inputs of each uncertainty (e.g. y_1, y_2, y_3) and also their outputs (u_1, u_2, u_3). Then write down the transfer function matrix $M(s)$ between (u_1, u_2, u_3, u) and (y_1, y_2, y_3, y). The upper linear fractional form of Fig. 1 results as the interconnection structure in which the uncertainty matrix Δ equals diag (Δ_1, Δ_2, Δ_3). In the sequel we call such an LFT a μ-interconnection structure. For the example, it can be easily verified that the matrix $M(s)$ of Fig. 1 is related to $G(s)$ of Fig. 2 as follows:

$$M = \left[\begin{array}{ccc|c} 0 & 0 & 0 & I \\ I & 0 & 0 & I \\ G & I & 0 & G \\ \hline G & I & I & G \end{array}\right] \quad (3)$$

To further generalise the procedure, the (block-diagonal) elements of $\Delta(s)$ should be normalised. This can be done using scaling-per-frequency or by weighting functions that are rational transfer functions [6]. In both cases the scaling can be absorbed in the interconnection matrix M.

In summary, the procedure to model perturbations using complex-valued uncertainties is to (i) describe the perturbations using norm-bounded input/output models at the locations which arise from the physical model, (ii) label the inputs and outputs of these uncertainties, (iii) write down the transfer functions between all inputs and outputs, and (iv) collect these transfer functions into an LFT of the form of Fig. 1, with the Δ-feedback loop block-diagonally structured. A last step is to scale all Δs and to absorb the scaling factors into the interconnection matrix M.

3 Parametric uncertainty modelling

3.1 Definitions and introduction

Real-valued norm-bounded perturbations can be used to describe a large class of uncertainties in control systems. For example, variation of physical system parameters is typically real-valued.

Definition 3.1: A scalar real-valued norm-bounded perturbation Δ_r is the set of real numbers Δ which are bounded in magnitude to some real number $\bar{\delta} \in \mathscr{R}^+$:

$$\Delta_r = \{\Delta \mid \Delta \in [-\bar{\delta}, +\bar{\delta}]\} \quad (4)$$

The normalised version of Δ_r is $B\Delta_r = \{\Delta \mid \Delta \in [-1, +1]\}$.

The sets Δ_r and Δ_c share the property of being bounded in maximum singular value. However, there are three important differences. First, the elements of Δ_r are scalars, while Δ_c may have matrices as its (block) elements. Secondly, the set Δ_r contains only real numbers. Thirdly, the maximum singular value of the elements in Δ_c can vary with frequency while the maximum singular value of a real perturbation, which is equal to the maximum absolute value, is fixed.

One special structure, which is important in the application in Section 5, is the real repeated uncertainty for one parameter.

Definition 3.2: A $p \times p$ real-valued repeated perturbation Δ_{rr} is defined as

$$\Delta_{rr} = \{\Delta \mid \Delta = \delta I, \delta \in [-\bar{\delta}, +\bar{\delta}]\} \quad (5)$$

with $\bar{\delta} \in \mathscr{R}^+$ some real number. The normalised version is $B\Delta_{rr} = \{\Delta \mid \Delta = \delta I, \delta \in [-1, +1]\}$. In both equations I is the $p \times p$ identity matrix.

The starting point for parametric uncertainty modelling is a state-space description of an uncertain system. A procedure is described which can be used to derive an LFT form of a model with parametric uncertainties in the entries of its state-space matrices. This procedure involves three steps: (i) scaling the parameter variations such that they belong to $B\Delta_r$ (or $B\Delta_{rr}$), (ii) uncertainty extraction resulting in a separation between the nominal (constant) part of a system and a varying (uncertain) part and (iii) obtaining an LFT description.

3.2 General case

Consider a vector $p = (p_1, \ldots, p_t) \in \mathscr{R}^t$ containing t scalar parameters, for example spring stiffness, resistance etc. Let the model of the perturbed system be given as a state-space realisation in which the entries of the matrices depend on the parameter vector p:

$$\begin{aligned} \dot{x} &= A(p)x + B(p)u \quad x \in \mathscr{R}^n, u \in \mathscr{R}^m \\ y &= C(p)x + D(p)u \quad y \in \mathscr{R}^l \end{aligned} \quad (6)$$

Restrict attention to the case of 'smooth' perturbations in the form of parametric uncertainties. More specifically, assume that each entry of the matrices in eqn. 6 is described as a rational multidimensional (ND) polynomial function of the parameters p. For example, the (i, j)th entry of the A-matrix can have the form $A_{ij}(p) = \{p_1 + p_2 a_0 + p_2^2 p_3\}/\{p_1^4 p_3 + a_1 p_4\}$ in which a_0 and a_1 are constants.

For this general class of systems the following procedure provides a way to derive an LFT uncertainty description.

Step 1: Scaling: Let the parameter vector p be given with lower and upper bound vectors p_{min} and p_{max} respectively: $p_{min_i} \leqslant p_i \leqslant p_{max_i}$ for $i = 1, \ldots, t$. Define $p_{nom} = (p_{min} + p_{max})/2$, $s = (p_{max} - p_{min})/2$, $\boldsymbol{\delta} = (\delta_1, \ldots, \delta_t)$, $\delta_i \in [-1, +1]$ then $p_i = p_{nom_i} + s_i \delta_i$. In this way the varying parameter vector p is decomposed into a nominal part p_{nom}, the constant scaling factors s_i and the normalised real-valued perturbations δ_i collected in the vector $\boldsymbol{\delta}$.

Step 2: Uncertainty extraction: Let the state-space model eqn. 6 be given and assume the parameter vector p has been scaled. Define the $(n + l) \times (n + m)$ matrix

$$S(p) = \begin{pmatrix} A(p) & B(p) \\ C(p) & D(p) \end{pmatrix} \tag{7}$$

The nominal part of the state-space model is given by $S(p_{nom})$. The uncertain part of the state-space model is defined as an $(n + l) \times (n + m)$ matrix $S_\Delta(\delta)$ with entries

$$[S_\Delta]_{ij}(\delta) = S_{ij}(p) - S_{ij}(p_{nom}) \tag{8}$$

Hence, $[S_\Delta]_{ij}(\delta) = 0$ if no uncertain parameter enters the (i, j)th entry of S, for $i = 1, \ldots, n + l$ and $j = 1, \ldots, n + m$.

Using this definition the perturbed state-space model eqn. 6 can be written as

$$\begin{pmatrix} \dot{x} \\ y \end{pmatrix} = S(p_{nom})\begin{pmatrix} x \\ y \end{pmatrix} + S_\Delta(\delta)\begin{pmatrix} x \\ u \end{pmatrix} \tag{9}$$

from which it is clear that the uncertain part is now separated from the nominal part.

Step 3. Obtaining a linear fractional transformation: The third step is to rewrite eqn. 9 into a linear fractional form. We construct this by defining a new input vector u_Δ and a new output vector y_Δ. The output y_Δ is fed back to the input u_Δ through a diagonal perturbation $\Delta(\delta) = \operatorname{diag}(\delta_1 I_1, \ldots, \delta_t I_t)$. Furthermore, constant matrices B_Δ, C_Δ and D_Δ are defined which contain information on how the uncertainties affect the nominal model:

$$\begin{aligned} \begin{pmatrix} \dot{x} \\ y \end{pmatrix} &= S(p_{nom})\begin{pmatrix} x \\ u \end{pmatrix} + B_\Delta u_\Delta \\ y_\Delta &= C_\Delta\begin{pmatrix} x \\ u \end{pmatrix} + D_\Delta u_\Delta \\ u_\Delta &= \Delta(\delta) y_\Delta \end{aligned} \tag{10}$$

where $\Delta(\delta) = \operatorname{diag}(\delta_1 I_1, \ldots, \delta_t I_t)$, in which I_i denotes an identity matrix with dimensions related to the repeatedness of perturbation δ_i (see also Definition 3.2).

Rewriting eqn. 9 as an LFT involves finding the constant matrices B_Δ, C_Δ and D_Δ such that eqn. 9 is equivalent to eqn. 10. Eliminating u_Δ and y_Δ in eqn. 10 yields

$$\begin{pmatrix} \dot{x} \\ y \end{pmatrix} = S(p_{nom})\begin{pmatrix} x \\ u \end{pmatrix} + B_\Delta(I - \Delta(\delta)D_\Delta)^{-1}\Delta(\delta)C_\Delta\begin{pmatrix} x \\ u \end{pmatrix} \tag{11}$$

which must be equivalent to eqn. 9. This implies that the following realisation problem has to be solved.

General problem definition: Find constant matrices B_Δ, C_Δ and D_Δ and $\Delta(\delta) = \operatorname{diag}(\delta_1 I_1, \ldots, \delta_t I_t)$ with dimensions as small as possible such that

$$B_\Delta(I - \Delta(\delta)D_\Delta)^{-1}\Delta(\delta)C_\Delta = S_\Delta(\delta) \tag{12}$$

where $S_\Delta(\delta)$ is the matrix from eqn. 8.

Eqn. 12 can be interpreted as follows. Consider only the nontrivial case that $\delta_i \neq 0$, $i = 1, \ldots, t$. In that case, $\Delta(\delta) = \operatorname{diag}(\delta_1 I_1, \ldots, \delta_t I_t)$ is invertible and eqn. 12 can be rewritten as $B_\Delta(\Delta^{-1}(\delta) - D_\Delta)^{-1}C_\Delta = S_\Delta(\delta)$. Defining $\rho_i = 1/\delta_i$ yields

$$B_\Delta\left(\begin{bmatrix} \rho_1 I_1 & & 0 \\ & \ddots & \\ 0 & & \rho_t I_t \end{bmatrix} - D_\Delta\right)^{-1} C_\Delta = S_\Delta(\delta) \tag{13}$$

which can be considered as a multidimensional (minimal) realisation problem. Note that in general there may be freedom in choosing $(D_\Delta, C_\Delta, B_\Delta)$ as a minimal realisation. This means that an LFT for a given analysis problem need not be unique.

General problem solution: Eqn. 12 is solvable for the general case. In this paper we will make this statement tractable, without giving a rigorous proof.

Recall that the uncertain model has rational ND polynomial parameter-dependent entries of the state-space matrices. Hence, every entry in $S_\Delta(\delta)$ can be written as a scalar function of the parameter vector $\boldsymbol{\delta}$: $[S_\Delta]_{ij}(\delta) = k(\delta)/(1 + l(\delta))$, with $k(0) = 0$ and $l(0) = 0$ and with the specific structure of the denominator because $S_\Delta(0) = 0$. The denominator can be represented as an LFT being a negative feedback $l(\delta)$ over a gain 1. The numerator $k(\delta)$ and the function $l(\delta)$ consist both of several terms with products and powers of the parameters δ_i. Each of these terms can be represented by an LFT and hence also the sum of them. This gives an LFT for $k(\delta)$ and one for $1/(1 + l(\delta))$. The product of two LFTs is another LFT, so that we have found an LFT for $[S_\Delta]_{ij}(\delta)$. After the combination of all entries of $S_\Delta(\delta)$ into one large LFT structure, a minimal realisation step is necessary for each individual element of δ. For more details see Reference 7.

3.3 Special cases

3.3.1 One varying parameter: Important examples of uncertainty models for the one parameter case are those where entries of the model depend as rational functions on one varying parameter, for instance the operating condition for linearised systems. Consider the system of eqn. 9 with δ a scalar (i.e. $t = 1$).

Lemma 3.3: Define $\rho = 1/\delta$. Then $S_\Delta(\rho^{-1})$ is strictly proper in ρ.

Proof: According to eqn. 8, for $\delta = 0$, $[S_\Delta]_{ij}(0) = 0$ if no uncertain parameter enters entry (i, j) in eqn. 6 and $\lim_{\delta \to 0} [S_\Delta]_{ij}(\delta) = S_{ij}(p_{nom}) - S_{ij}(p_{nom}) = 0$ otherwise, implying that $\lim_{\delta \to 0} S_\Delta(\delta) = \lim_{\rho \to \infty} S_\Delta(\rho^{-1}) = 0$.

Theorem 3.4: Assuming that $S_\Delta(\rho^{-1})$ is rational and strictly proper, the uncertainty modelling problem is to find constant matrices B_Δ, C_Δ and D_Δ and $\Delta(\delta) = \delta I$, δ scalar, with dimensions as small as possible such that eqn. 12 holds for δ being scalar. This is equivalent to the realisation problem: $B_\Delta(\rho I - D_\Delta)^{-1}C_\Delta = S_\Delta(\rho^{-1})$.

Proof: Follows immediately from eqns. 12 and 13 for $t = 1$.

Lemma 3.3 shows that the uncertainty modelling for the one parameter case can always be carried out such that $S_\Delta(\rho^{-1})$ is strictly proper. Therefore a solution always exists, since the problem is equivalent to a standard state-space realisation problem [8].

Corollary 3.5: If $(D_\Delta, C_\Delta, B_\Delta)$ is a minimal realisation, the solution to Theorem 3.4 yield $\Delta(\delta) = \delta I$ with the smallest possible dimensions for which an LFT can be found.

Remark 3.6: The connection between state-space realisation and parametric uncertainty modelling can also be reversed: a state-space model as an uncertainty. In Reference 9 this has been worked out by defining in discrete time the z-variable as a repeated block perturbation ('state-space μ').

Example 3.7: Suppose a first-order system has a state-space A-matrix which can be written as $A = A_{nom} + \delta^2$, then

$$\dot{x} = A_{nom}x + \delta^2 x \tag{14}$$

and constructing an LFT is fairly simple in this case:

$$\dot{x} = A_{nom}x + B_\Delta u_\Delta$$
$$y_\Delta = C_\Delta x + D_\Delta u_\Delta$$
$$u_\Delta = \Delta(\delta) y_\Delta$$

This set equals eqn. 14 if

$$\Delta(\delta) = \begin{bmatrix} \delta & 0 \\ 0 & \delta \end{bmatrix} \quad D_\Delta = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \quad C_\Delta = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

and

$$B_\Delta = [0 \quad 1]$$

Owing to the structure of D_Δ a polynomial in δ is created.

In this example $\phi = \delta^2$ could have been modelled and ϕ treated as a simple linear perturbation. However, when this concept is applied more generally for example if δ appears somewhere else in the state equation as another polynomial, a procedure as in the example is necessary.

3.3.2 Linear parametric uncertainties: If the parameters $\delta = (\delta_1, \ldots, \delta_t)$ enter the state-space matrices in a linear way [10, 11], D_Δ can be taken as identically zero, as is clear from eqn. 12. For this case it is obvious that $S_\Delta(\delta) = \sum_{i=1}^{t} \delta_i S_{\Delta_i}$.

Theorem 3.8: Let $S_\Delta(\delta) = \sum_{i=1}^{t} \delta_i S_{\Delta_i}$. The problem to find constant matrices B_Δ, C_Δ and $\Delta(\delta) = \text{diag}(\delta_1 I_1, \ldots, \delta_t I_t)$ with dimensions as small as possible such that eqn. 12 holds with $D_\Delta = 0$, is always solvable. The solution is given by the solution to

$$B_\Delta \Delta(\delta) C_\Delta = \sum_{i=1}^{t} \delta_i S_{\Delta_i} \tag{15}$$

with $\Delta(\delta) = \text{diag}(\delta_1, I_1, \ldots, \delta_t I_t)$.

Proof: From the general problem definition (eqn. 12) eqn. 15 results for the linear parameter case. That a solution to this problem always exists can be seen as follows. Suppose that S_{Δ_i} has rank r_i, then there exist matrices P_i and Q_i where P_i is $(n + l) \times (r_i)$ and Q_i is $(r_i) \times (n + m)$ such that $\delta_i S_{\Delta_i} = \delta_i P_i Q_i = P_i[\delta_i I_{r_i}]Q_i$. Choosing $\Delta = \text{diag}(\delta_1 I_{r_1}, \ldots, \delta_t I_{r_t})$, $B_\Delta = (P_1, \ldots, P_t)$, and $C_\Delta = (Q_1^T, \ldots, Q_t^T)^T$ yields eqn. 15.

From Theorem 3.8 it follows that generically the uncertainty $\Delta(\delta) = \text{diag}(\delta_1 I_{r_1}, \ldots, \delta_t I_{r_t})$ for which a solution exists has at least dimension $\sum_{i=1}^{t} r_i$ where r_i is the rank of S_{Δ_i}. However, in some cases perturbations can be taken together which is formulated in the following result.

Corollary 3.9: The dimension of an uncertainty $\Delta(\delta) = \text{diag}(\delta_1 I_{r_1}, \ldots, \delta_t I_{r_t})$ can be made smaller than $\sum_{i=1}^{t} r_i$ if rank $\sum_{i=1}^{t} \alpha_i S_{\Delta_i} < \sum_{i=1}^{t} r_i$, with α_i any nonzero real number. In such a case, some δ_i are perturbing the system in a similar way and can be taken together. This is called a reducible uncertainty model. An example has been worked out in Reference 6; see also Reference 7.

3.3.3 Other special problems: The two special cases described previously are formalised with Theorem 3.4 and Theorem 3.8. Two examples are presented to show solutions for the case where products and quotients of parameters appear. In both, the following relations for the varying parameters are assumed $a = a_{nom} + s_a \delta_a$, $b = b_{nom} + s_b \delta_b$.

Example 3.10: Consider the state equation

$$\dot{x} = abx = a_{nom} b_{nom} x + (\gamma_1 \delta_a + \gamma_2 \delta_a \delta_b + \gamma_3 \delta_b)x$$
$$= A_{nom} x + S_\Delta(\delta_a, \delta_b)x$$

where $\gamma_1 = s_a b_{nom}$, $\gamma_2 = s_a s_b$, $\gamma_3 = s_b a_{nom}$ (uncertainty extraction). The problem is to find matrices $(B_\Delta, C_\Delta, D_\Delta)$ such that $\dot{x} = A_{nom}x + S_\Delta(\delta_a, \delta_b)x$ is equivalent to the linear fractional form:

$$\dot{x} = A_{nom}x + B_\Delta u_\Delta$$
$$y_\Delta = C_\Delta x + D_\Delta u_\Delta \tag{16}$$
$$u_\Delta = \begin{pmatrix} \delta_a & 0 \\ 0 & \delta_b \end{pmatrix} y_\Delta$$

The equivalence is satisfied for

$$B_\Delta = (\gamma_1 \quad \gamma_2) \quad C_\Delta = \begin{pmatrix} 1 \\ \gamma_3/\gamma_2 \end{pmatrix} \quad D_\Delta = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

Example 3.11: Consider

$$\dot{x} = \frac{a}{b}x = \frac{a_{nom}}{b_{nom}}x + \frac{(\gamma_1 \delta_a + \gamma_2 \delta_b)}{(1 + \gamma_3 \delta_b)}x$$
$$= A_{nom}x + S_\Delta(\delta_a, \delta_b)x$$

where $\gamma_1 = s_a b_{nom}$, $\gamma_2 = -a_{nom} s_b / b_{nom}^2$, $\gamma_3 = s_b / b_{nom}$. Again we are looking for matrices $(B_\Delta, C_\Delta, D_\Delta)$ such that $\dot{x} = A_{nom}x + S_\Delta(\delta_a, \delta_b)x$ is equivalent to eqn. 16. This is satisfied for

$$B_\Delta = (1 \quad 1) \quad C_\Delta = \begin{pmatrix} \gamma_1 \\ \gamma_2 \end{pmatrix} \quad D_\Delta = \begin{pmatrix} 0 & 0 \\ -\gamma_3 & -\gamma_3 \end{pmatrix}$$

3.4 General μ-interconnection structure

For practical problems in general both complex and parametric uncertainties have to be taken into account. This can be done by deriving LFTs for each of the perturbations, and collecting these models into one μ-interconnection structure. The uncertainty matrix Δ then consists of complex and real-valued entries, as defined in the following general block structure.

Given two non-negative integers m_r and m_c define a vector $\boldsymbol{\kappa}$ with length $m_r + m_c$ and with positive integer entries:

$$\boldsymbol{\kappa} = (k_1, \ldots, k_{m_r}, k_{m_r+1}, \ldots, k_{m_r+m_c}) \tag{17}$$

Definition 3.12: Given the vector $\boldsymbol{\kappa}$ (eqn. 17), the associated block-diagonal perturbation Δ_b is defined by the set

$$\Delta_b = \{\Delta \mid \Delta = \text{diag}(\Delta_1^r I_{k_1}, \ldots, \Delta_{m_r}^r I_{k_{m_r}}, \Delta_1^c I_{k_{m_r+1}}, \ldots, \Delta_{m_c}^c I_{k_{m_r+m_c}}\} \tag{18}$$

where $\Delta_i^r \in \Delta_r$ (Definition 3.1), $i = 1, \ldots, m_r$, $\Delta_i^c \in \Delta_c$ (Definition 2.1) but with the additional constraint that Δ_i^c is a scalar if $k_i > 1$, $i = m_r + 1, \ldots, m_r + m_c$ and where I_k is a $k \times k$ identity matrix. The normalised block-diagonal perturbation set is denoted as $B\Delta_b$ with $\Delta_i^r \in B\Delta_r$, $\Delta_i^c \in B\Delta_c$.

Notice that $\Delta_i^r I_{k_i}$ is a real repeated block and $\Delta_i^c I_{k_{m_r+i}}$ is a complex repeated block. If $k_i = 1$ the uncertainty is nonrepeated and the complex uncertainties are allowed to be matrices in that case. The vector $\boldsymbol{\kappa}$ thus comprises the structure information (real/complex, repeatedness). The following section shows how robustness analysis can be done for general block-structures given by Definition 3.12.

4 Structured singular value analysis

This section briefly describes the structured singular value analysis for both the complex and the real case; for more details see References 3, 4, 12. In the sequel, we assume that $\Delta(s)$ as well as the nominal system $M(s)$ are stable. First, well-known results for the unstructured complex (normalised) case are reviewed.

Consider the system of Fig. 1 in which the uncertainty feedback Δ is assumed to be a full complex uncertainty ($\Delta \in B\Delta_c$, Definition 2.1). Denote the partition of $M(s)$ which is coupled to Δ by $M_{11}(s)$. For this unstructured case the well known small-gain theorem provides necessary and sufficient conditions for internal stability of the perturbed system $F_u(M, \Delta)$: the system in Fig. 1 is internally stable if and only if $\sup_\omega\{\bar{\sigma}(M_{11}(j\omega))\} < 1$, $\omega \in (-\infty, \infty)$ [13]. Now consider the case that $B\Delta_c$ is replaced by $B\Delta_b$ with $B\Delta_b$ some block-diagonal structure. Then the small gain theorem does not necessarily hold. For that reason Doyle [3] introduced the structured singular value μ.

Theorem 4.1: Let $M(s)$ be stable and let $\Delta \in B\Delta_b$, then the system $F_u(M, \Delta)$ in Fig. 1 is internally stable if and only if

$$\det(I - \Delta(j\omega)M_{11}(j\omega)) \neq 0 \qquad \forall \Delta \in B\Delta_b \quad \omega \in (-\infty, \infty) \tag{19}$$

which holds if and only if

$$\sup_\omega \{\mu(M_{11}(j\omega))\} < 1 \quad \omega \in (-\infty, \infty) \tag{20}$$

Proof: see Reference 3.

The difference between the structured singular-value theorem (Theorem 4.1) and the small-gain theorem is that the maximum singular value of a matrix can be computed easily and exactly, which is not the case for the structured singular value. Computing μ requires the optimisation of an expression in several independent variables. It is known that this optimisation problem leads to an upper or lower bound for μ and that the exact value can only be determined in special cases [3, 12].

Define a block-diagonal set of invertible matrices D_b with a structure related to the set Δ_b :

$$D_b = \{D \mid D = \operatorname{diag}(D_1, \ldots, D_{m_r+m_c})\} \tag{21}$$

where for all $i = 1, \ldots, m_r + m_c$ for which $k_i > 1$: $D_i = D_i^H > 0$, $D_i \in \mathbf{C}^{k_i \times k_i}$, and for all $i = 1, \ldots, m_r + m_c$ for which $k_i = 1$: $D_i = d_i I_p$, $d_i \in \mathscr{R}^+$, $p = \dim(\Delta_i)$, and with $(\cdot)^H$ the complex conjugate transpose. Notice that for all real uncertainties ($i = 1, \ldots, m_r$) $p = 1$. Then $D^{-1}\Delta D = \Delta$ for $D \in D_b$, $\Delta \in \Delta_b$. For such matrices D it can be proven that $\mu(M_{11}) = \mu(DM_{11}D^{-1})$ and because $\mu(M_{11}) \leqslant \bar{\sigma}(M_{11})$ we construct an upper bound [3]:

$$\mu(M_{11}) \leqslant \inf_{D \in D_b} \bar{\sigma}(DM_{11}D^{-1}) \tag{22}$$

This property can be used to compute an upper bound for μ, by optimisation of the entries of D. A lower bound can also be constructed, see References 3, 12. For the purely complex case ($m_r = 0$) this minimisation problem is convex [14] which implies that every local minimum of the $\bar{\sigma}$ expression is global. Unfortunately, when there are real-valued uncertainties the bound may be arbitrary far off and as such the minimisation of eqn. 22 may yield conservative results. A solution to this problem has been proposed by Fan and others [4, 5] in the form of a new upper bound for μ:

$$\mu(M_{11}) \leqslant \sqrt{\left(\max\left\{0, \inf_{D,G} \bar{\lambda}(M_D^H M_D + j[GM_D - M_D^H G])\right\}\right)} \tag{23}$$

with $M_D = DM_{11}D^{-1}$, $D \in D_b$, and with $G \in G_b$ defined as

$$G_b = \{G \mid G = \operatorname{diag}(G_1, \ldots, G_{m_r}, O_{m_r+1}, \ldots, O_{m_r+m_c})\} \tag{24}$$

where $G_i = G_i^H \in \mathbf{C}^{k_i \times k_i}$, $i = 1, \ldots, m_r$, and with O_i the null-matrix with dimension $k_i \times k_i$ if $k_i > 1$ and $p \times p$, $p = \dim(\Delta_i)$ if $k_i = 1$. Notice that $G_i = g_i \in \mathscr{R}$ if $k_i = 1$.

If there are no real blocks ($m_r = 0$), then $G = 0$ and eqn. 23 simplifies to 22. This shows that eqn. 23 is based on the same principle as the earlier upper bound for purely complex structures, namely the minimisation of a maximum singular value. Note also that the inequality in eqn. 23 still holds if $G = 0$ is chosen. Hence, the complex structured singular value bound (eqn. 22) is a sufficient condition (an upper bound) for the real case. However, less conservative results (a smaller upper bound for μ) may be obtained for $G \neq 0$. The computation involves a minimisation over the free parameters in D and G. From the definitions of D_b and G_b it follows that the number of parameters involved is given by

$$D \text{ scaling:} \left[\sum_{i=1}^{m_r+m_c} k_i + 2(k_i - 1)^2\right] - 1 \tag{25}$$

$$G \text{ scaling:} \sum_{i=1}^{m_r} k_i + 2(k_i - 1)^2 \tag{26}$$

with k_i the entries of $\boldsymbol{\kappa}$. For example, a complex nonrepeated problem with three uncertainties, $m_r = 0$, $m_c = 3$, $\boldsymbol{\kappa} = (k_1, k_2, k_3) = (1, 1, 1)$, has only two parameters for D-scaling and none for G-scaling. For a 3×3 real-repeated one parameter problem, $m_r + m_c = m_r = 1$, $\boldsymbol{\kappa} = k_1 = 3$, has 17 parameters for D-scaling and 18 for G-scaling.

An algorithm has been written to compute the upper bound (eqn. 23). In fact, all possible combinations of real, real repeated, complex and (scalar) complex repeated can be handled with it. The algorithm is used in the following section.

To give some insight into the effect of the G-scaling on the value of the upper bound, this section concludes with a simple example.

Example 4.2: Suppose we have one real scalar uncertainty $\Delta = \delta \in [-1, 1]$. Denote the related μ-interconnection structure $M_{11}(s)$ as $m(s)$. Let $m(s)$ be evaluated at some frequency ω_0 : $m(j\omega_0) = r + qj$ where $r, q \in \mathscr{R}$. In this case, the upper bound (eqn. 23) can be written as ($D = 1$)

$$\mu(m(j\omega_0)) \leqslant \sqrt{\left(\max\left\{0, \inf_g (r^2 + q^2 - 2qg)\right\}\right)} \tag{27}$$

with $G = g$. First suppose that $q \neq 0$, then for any given (r, q) a g can be found such that $(r^2 + q^2 - 2qg) < 0$ and hence $\mu(m(j\omega_0)) = 0$. Now assume that $q = 0$, i.e. $m(j\omega_0)$ crosses the real axis, then eqn. 27 gives $\mu(m(j\omega_0)) \leqslant |r|$ for any choice of g. This is equivalent to the well known amplitude margin of a scalar system.

The example shows that the G-scaling in fact pushes the upper bound down in those cases where the interconnection matrix has only complex values, if calculated for a real uncertainty. This result can be generalised for multivariable systems, using the eigenvalues of $DM_{11}D^{-1}$, see Reference 15. It also shows that the optimisation problem is not continuous on G, see also Reference 16. Generically, complex perturbations always prevent this situation.

5 Robustness analysis of a 5D actuator

In optical recording (Compact Disc), a very high information density is applied. To detect this information, high precision mechanisms are needed to position the laser spot on the disc with an accuracy $\leqslant 0.1$ μm. Using servoactuators with a high bandwidth (500–1000 Hz) it is possible to keep on the track despite disturbances from outside the mechanism such as mechanical shocks and disc eccentricity. An actuator which makes it possible to achieve a very high bandwidth is the 5D-actuator [17]. This consists of a magnetic ring with a lens in it, which is magnetically positioned by an active system of nine coils, Fig. 3. Using a mirror underneath the magnetic ring, the

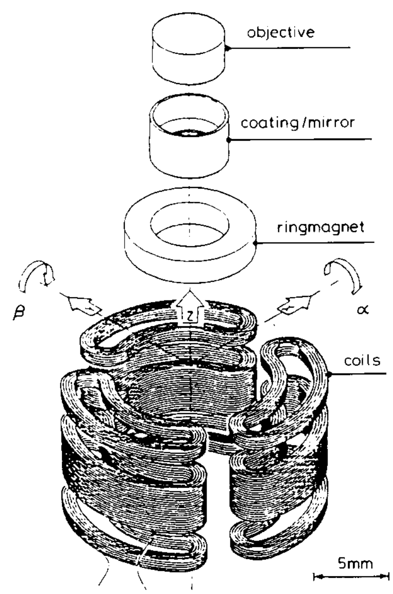

Fig. 3 *Schematic view of the 5D-actuator*

positions (z, α, β) can be detected, while the x (tracking) and z (focusing) positions are measured relative to the disc above. The position of the lens is controllable in 5 degrees of freedom by means of the electromagnetic forces.

A major problem with this system is that it has severe couplings between the magnetic forces as a function of position z. This gives interaction problems between the degrees of freedom to be controlled. From a nonlinear model it follows that with the aid of a decoupling matrix the (x, β) and (y, α) degrees of freedom can be decoupled from one another. We restrict attention to the 2D problem in the (y, α)-direction only. A state-space model has been derived, linearised with respect to vertical position z

$$\begin{pmatrix} \dot{y} \\ \ddot{y} \\ \dot{\alpha} \\ \ddot{\alpha} \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} y \\ \dot{y} \\ \alpha \\ \dot{\alpha} \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 1.77 - 0.24z^2 & -1.27z \\ 0 & 0 \\ 5.34z & 1.97 - 1.77z^2 \end{pmatrix} \begin{pmatrix} u_y \\ u_\alpha \end{pmatrix} \tag{28}$$

and with the outputs y and α. The z-dependency of the model appears nonlinearly in the input matrix B and is caused by a nonlinear distribution of the magnetic field lines as a function of z. The entries of B are polynomial fits on data obtained from a finite element calculation of the magnetic field distribution. The interaction terms are for $z > 0$ of opposite sign compared to those for $z < 0$. Using a multimodel design method [18] a diagonal controller has been designed for three operating points: $z = -1$, 0 and $+1$ mm, resulting in a compensator for y: $0.51 * 10^7\ (8.23 * 10^{-4}s + 1)/(3.26 * 10^{-5}s + 1)$ and for α: $0.30 * 10^7(9.54 * 10^{-4}s + 1)/(2.85 * 10^{-5}s + 1)$. We are interested if this system is stable in all operating points. To be more precise, whether it is stable for every position z, where z can vary between -1.8 mm and $+1.8$ mm. We restrict attention to the variations in z only, hence we have a one-parameter problem. Using the uncertainty modelling procedure described, a μ-interconnection structure can be derived with a real repeated uncertainty $\Delta = zI$ (scaled to $-1, \ldots, +1$, and with I the 4×4 identity matrix), with the matrices in eqn. 6 model as follows:

$$B_{nom} = \begin{pmatrix} 0 & 0 \\ 1.77 & 0 \\ 0 & 0 \\ 0 & 1.97 \end{pmatrix}$$

$$B_\Delta = \left(\begin{array}{cccc} 0 & 0 & 0 & 0 \\ 0 & 0.24 & -1.27 & 0 \\ 0 & 0 & 0 & 0 \\ 5.34 & 0 & 0 & -1.77 \\ \hline 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

$$C_\Delta = \left(\begin{array}{cccc|cc} 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

$$D_\Delta = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

For this case, the following calculations have been done: (i) small-gain theorem, (ii) structured singular-value computation (eqn. 22) assuming that Δ is a 4×4 diagonal complex uncertainty: $\kappa = (1, 1, 1, 1)$, $m_r = 0$, $m_c = 4$, $p = 1$ (four times), i.e. three D-scalings, (iii) Δ assumed to be complex repeated (eqn. 22), $\kappa = 4$, $m_r = 0$, $m_c = 1$, i.e. 21 D-scalings, and (iv) real-repeated (eqn. 23): $\kappa = 4$, $m_r = 1$, $m_c = 0$, i.e. 21 D-scalings and 22 G-scalings. Results of the computations are given in Fig. 4.

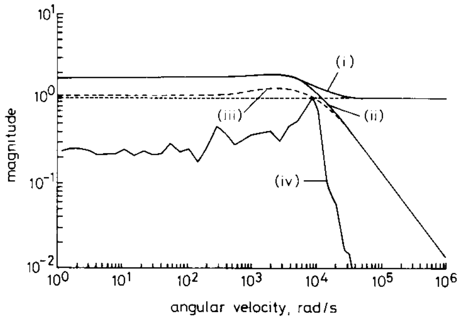

Fig. 4 *Real and complex structured singular value for 5D actuator*
(i) $\bar{\sigma}$
(ii) μ complex
(iii) μ complex repeated
(iv) μ real repeated

The singular-value test holds for unstructured complex uncertainties and since the perturbations in this problem are structured, repeated and real the test is expected to be very conservative. This can be seen in Fig. 4 where $\bar{\sigma}(M_{11}(j\omega))$ has a peak value of two, implying that only an uncertainty two times smaller (i.e. $|z| < 0.9$ mm) then the actual uncertainty would satisfy the test. The test for case (ii) takes the structure of the perturbations into account (but nonrepeated: D is diagonal), and therefore is less conservative. The third line is again the complex structured singular value but now for repeated uncertainties, case (iii), which shows to be less conservative. Finally, the real structured singular value test (iv) computes an upper bound for structured and real (repeated) perturbations. Fig. 4 shows that for this case the computed upper bound equals 1 and hence is on the edge of stability. The results are not smooth because of the nonzero stopping criterion of the algorithm.

In this case, it is possible to calculate stability for all operating conditions in another way. The stability criterion is $\det(I - zM_{11}) \neq 0$ for all real-valued z (scaled to $-1, \ldots, +1$). This is the same as evaluating the characteristic values $\lambda(M_{11}(j\omega))$ along the real axis, Fig. 5.

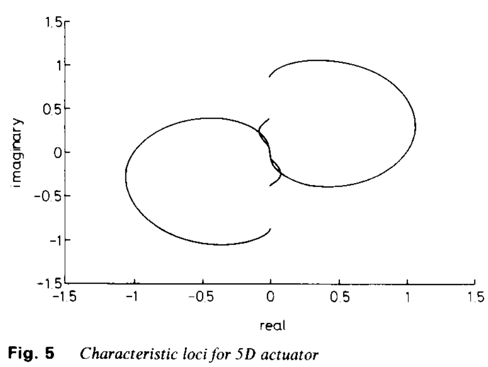

Fig. 5 *Characteristic loci for 5D actuator*

From the figure it follows that the system is indeed on the edge of stability for these operating points.

6 Conclusions

Robustness analysis for systems with complex and real-valued uncertainties consists of uncertainty modelling and computing stability bounds. A procedure has been described to model complex and real perturbations, comprising scaling of the individual perturbations, extracting the varying part from the constant part of a system and creating a linear fractional form. For those types of models, recent developments of structured singular value computation for complex and real, possibly repeated, uncertainties are applicable. An electromechanical positioning device, to be used in optical recording, has been analysed for stability over a range of operating conditions.

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