

Uncertainty shocks, asset supply and pricing over the business cycle*

Francesco Bianchi
Duke

Cosmin Ilut
Duke & NBER

Martin Schneider
Stanford & NBER

September 2013

Abstract

This paper studies a DSGE model with endogenous financial asset supply and ambiguity averse investors. An increase in uncertainty about financial conditions leads firms to substitute away from debt and reduce shareholder payout in bad times when measured risk premia are high. Regime shifts in volatility generate large low frequency movements in asset prices due to uncertainty premia that are disconnected from the business cycle.

1 Introduction

This paper studies a DSGE model with endogenous financial asset supply and ambiguity averse investors. Firms face frictions in debt and equity markets and decide on capital structure and net payout. Investors perceive time varying uncertainty about real and financial technology. Uncertainty shocks lead firms to reoptimize capital structure as relative asset prices such as risk premia change. In an estimated model that allows for both smooth changes in ambiguity and regime shifts in volatility, concerns about financial conditions generates low frequency movements in asset prices that are disconnected from the business cycle.

We model ambiguity aversion by recursive multiple priors utility. When agents evaluate an uncertain consumption plan, they use a worst case conditional probability drawn from a set of beliefs. A larger set indicates higher uncertainty. In our DSGE context, beliefs are parameterized by the conditional means of innovations to real or financial technology. Conditional means are drawn from intervals centered around zero. The width of the interval

*PRELIMINARY & INCOMPLETE. Comments welcome! Email addresses: Bianchi fb36@duke.edu, Ilut cosmin.ilut@duke.edu, Schneider schneidr@stanford.edu. We would like to thank Lars Hansen, Nir Jaimovich, Konstantinos Theodoridis and Nic Vincent as well as workshop and conference participants at Chicago Fed, Cornell, CREI, Duke, LBS, LSE, Mannheim, Minneapolis Fed, NBER Summer Institute, Queen's, SITE, UCSD and University of Chicago for helpful discussions and comments.

measures the amount of ambiguity. It can change either smoothly with the arrival of intangible information or it can jump discretely across regimes with different stochastic volatility. Both types of change in uncertainty work like a drop in the conditional mean and hence have first order effects on decisions.

Time variation in ambiguity leads econometricians to measure time varying premia in asset markets. Indeed, when investors evaluate an asset *as if* the mean payoff is low, then they are willing to pay only a low price for it. To an econometrician, the return on the asset – actual payoff minus price – will then look unusually high. The more ambiguity investors perceive, the lower is the price and the higher is the subsequent return. An econometrician who runs a regression of return on price (normalized by dividends) will thus find a positive coefficient. If interest rates are stable – say because bonds are less ambiguous than stocks – then the price-dividend ratio helps forecast *excess* returns on stocks, that is, there are time varying risk premia on stocks. A convenient feature of our model is that asset premia are due to perceptions of low mean, and therefore appear in a standard loglinear approximation to the equilibrium.

Our model determines investment, production and financing choices of the US nonfinancial corporate sector. It is driven by shocks to real production technology as well as shocks to financial technology. Equity and corporate debt are priced by a representative agent. The supply of equity and debt, and hence leverage, is endogenously determined. Firms face an upward sloping marginal cost curve for debt: debt is cheaper than equity at low levels of debt, but becomes eventually more expensive as debt increases. Firms also have a preference for dividend smoothing. To maximize shareholder value, they find interior optima for leverage and net shareholder payout. Firm decisions are sensitive to ambiguity since shareholder value incorporates uncertainty premia. In particular, an increase in ambiguity about real or financial technology leads firms to substitute away from debt and reduce leverage.

We estimate the model with postwar US data on five observables. We include three key quantities chosen by the nonfinancial corporate sector: investment growth, net payout to shareholders relative to GDP and market leverage. We also include the value of nonfinancial corporate equity relative to GDP. We thus also consider the corporate price/payout ratio, which behaves similarly to the price-dividend ratio. Finally, we include the real short term interest rate. In sum, we ask our model to account for the price and quantity dynamics of equity and debt, along with real investment.

Estimation delivers two main results. First, regime shifts in volatility help understand jointly the heteroskedasticity of quantities and the low frequency movements in asset prices. When we allow for two regimes for stochastic volatilities with symmetric priors, we identify a low and a high volatility regime. The latter dominates a prolonged period of time from

the early '70s to the beginning of the '90s, when financial variables were volatile and the price/payout ratio was low. A switch from the low volatility regime to the high volatility regime determines a drop in stock prices of around 60% on impact that is followed by a further drawn out decline that can last for decades. This is because higher volatility increases ambiguity and generates a substantial price discount.¹

The second result is that financial quantities depend relatively more on uncertainty shocks than real variables. In particular, changes in uncertainty about future financial conditions are important for understanding the positive comovement of debt and net payout to shareholders. Those changes also help our model account for the excess volatility of stock prices. Indeed, since financing costs affect corporate cash flow relatively more than consumption, uncertainty about financing costs moves stock prices more than bond prices. Moreover, the model can generate movements in stock prices that are somewhat decoupled from the business cycles. Importantly, both dividends and prices are endogenously determined in our model as optimal responses to uncertainty shocks.

Relative to the literature, the paper makes three contributions. First it introduces a class of linear DSGE models that accommodate both endogenous asset supply and time varying uncertainty premia. Second, it shows how to extend that class of models to allow for first order effects of stochastic volatility. Finally, the results suggest a prominent role for uncertainty shocks in driving jointly asset prices and firm financing decisions.

There are a number of papers that study asset pricing in production economies with aggregate uncertainty shocks. Several authors have studied rational expectations models that allow for time variation in higher moments of the shock distributions. The latter can take the form of time varying disaster risk (Gourio (2012)) or stochastic volatility (Basu and Bundick (2011), Caldara et al. (2012), Malkhozov and Shamloo (2012)). Another line of work investigates uncertainty shocks when agents have a preference for robustness (Cagetti et al. (2002), Bidder and Smith (2012), Jahan-Parvar and Liu (2012)). Most of these papers identify equity with the value of firm capital or introduce leverage exogenously. In contrast, our interest is in how uncertainty shocks drive valuation when leverage responds optimally to those shocks.

Recent work has explored whether the interaction of uncertainty shocks and financial frictions can jointly account for credit spreads and investment. Most of this work considers changes in firm-level volatility (Arellano et al. (2010), Gilchrist et al. (2010), Christiano et al. (2013)). Gourio (2013) incorporates time varying aggregate risk and thus allows risk premia to contribute to spreads. In contrast to our paper, this line of work does not focus on the

¹If the economy happens to revert to the low volatility regime, a symmetric pattern occurs, with a stock market boom followed by a slow return to the low volatility conditional steady state.

determination of equity prices.

We also build on a recent literature that tries to jointly understand financial flows and macro quantities. Jermann and Quadrini (2012) and Covas and den Haan (2011, 2012) develop evidence on the cyclical behavior of debt and equity flows. Their modeling exercises point out the importance of shocks to financial technology, a finding that is confirmed by our results. We also emphasize, however, the role of uncertainty about financial conditions. The latter is essential in order for our estimated model to account for time variation in risk premia on equity.

Glover et al. (2011) and Croce et al. (2012) study the effects of taxation in the presence of uncertainty shocks. Their setups are similar to ours in that they combine a representative household, a tradeoff theory of capital structure and aggregate uncertainty shocks (in their case, changes in stochastic volatility under rational expectations). While their interest is in quantifying policy effects, our goal is to assess the overall importance of different uncertainty shocks.

The paper is structured as follows. Section 2 presents the model. Section 3 uses first order conditions for households and firms to explain the effect of uncertainty shocks on firm asset supply and asset prices. Section 4 describes our solution and estimation strategy, and then discusses the estimation results.

2 Model

Our model determines investment, production and financing choices of the US nonfinancial corporate sector as well as the pricing of claims on that sector by a representative household. Firms are owned by the infinitely-lived representative household who maximizes shareholder value.

We consider time varying uncertainty about two sources of shocks. The first is “real” technology. Ilut and Schneider (2011) showed that uncertainty about TFP can lead to large fluctuations in labor input over the business cycle in a model with nominal frictions. Building on this result, we model the “real” shock here as a *joint* change in marginal product of capital today and uncertainty perceived about the marginal product of capital in the future. This approach allows us to accommodate business cycle implications of uncertainty shocks without explicitly modeling nominal frictions.

The second source of shocks is financial – it reflects the cost of restructuring in the corporate sector. The key feature of this “financial technology” shock is that it affects firm cash flow without affecting production. To assess the relative importance and possible interaction of real and financial uncertainty, we then allow uncertainty perceived about both

real and financial technology to move over time.

2.1 Technology and accounting

There is a single perishable good that serves as numeraire.

Profits and production

The corporate sector produces numeraire from physical capital K_t according to the production function

$$Y_t = Z_t K_t^\alpha \xi^{(1-\alpha)t} \quad (1)$$

where ξ is the trend growth rate of the economy and α is the capital share. The shock Z_t to the marginal product of capital accounts for fluctuations in variable factors. In particular, it can reflect the effects of uncertainty shocks on labor input.

Capital is produced from numeraire and depreciates at rate δ

$$K_{t+1} = (1 - \delta)K_t + \left[1 - \frac{S''}{2} (I_t/I_{t-1} - \xi)^2\right] I_t, \quad (2)$$

Capital accumulation is thus subject to adjustment costs that are convex in the growth rate of investment I_t , as in Christiano et al. (2005). This functional form captures the idea that the scale of investment affects the organization of the firm. For example, investing at some scale I_t requires allocating the right share of managerial effort to guiding expansion rather than overseeing production. Moving to a different scale entails reallocating managerial effort accordingly.

Financing

In addition to investment, shareholders choose firms' net payout and their level of debt. Two types of frictions are relevant here. First, there are costs of restructuring the corporate sector. Every period, shareholders pay an adjustment cost

$$\phi(D_t, D_{t-1}) = f_t \xi^t + \frac{\phi'' \xi^t}{2} (D_t/D_{t-1} - \xi)^2 \quad (3)$$

where f_t is random. The idea here is that the corporate sector consists of many firms that are managed independently, for example because of limited managerial span of control. Shareholders determine the ownership structure of those firms through mergers and spinoffs. In addition, they choose net aggregate payout.

The fixed component of the restructuring cost reflects changes in financial market conditions that affect the cost of restructuring measures such as mergers and spinoffs. The variable

component is motivated by costs that occur as the scale of payout is changed, analogously to (2). For example, paying out at a large positive scale (continually repurchasing many shares or paying dividends at a high rate) requires shareholders to pressure managers to relinquish cash flow. In contrast, paying out at a large negative scale (continually raising a lot of new capital) requires the firm to focus more on maintaining relationships with primary investors. In both cases, refocusing the firm quickly is difficult.

The second friction arises in in the credit market. Firms issue one period noncontingent debt. Let Q_t^b denote the price of a riskless short bond. Suppose the corporate sector issues $Q_{t-1}^b B_{t-1}^f$ worth of bonds at date $t - 1$. At date t , it not only repays B_t^f to lenders, but also incurs the financing cost

$$\kappa \left(B_{t-1}^f \right) = \frac{\Psi}{2} \frac{1}{\xi^t} \left(B_{t-1}^f \right)^2$$

The marginal cost of issuing debt is thus upward sloping. This feature naturally arises if there is a idiosyncratic risk at the firm level and costly default. When firms choose capital structure, they trade off this cost against the tax advantage of debt.

Consider the firm's cash flow statement at date t . Denoting the corporate income tax rate by τ_k , we can write net payout as

$$\begin{aligned} D_t = & \alpha Y_t - I_t - \kappa(B_{t-1}^f) - \phi(D_t, D_{t-1}) - (B_{t-1}^f - Q_t^b B_t^f) \\ & - \tau_k \left[\alpha Y_t - B_{t-1}^f (1 - Q_{t-1}^b) - \delta Q_{t-1}^k K_{t-1} - I_t \right] \end{aligned} \quad (4)$$

The first line records cash flow in the absence of taxation: payout equals revenue less investment, restructuring and financing costs as well as net debt repayment. The second line subtracts the corporate income tax bill: the tax rate τ_k is applied to profits, that is, income less interest, depreciation and investment.

Household wealth

We denote the price of aggregate corporate sector equity by P_t . In addition to owning the firm, the household receives an endowment of goods $\pi \xi^t$ and government transfers $t_r \xi^t$. We assume a proportional capital income tax. Moreover, capital gains on equity are taxed immediately at the same rate. The household budget constraint is then

$$\begin{aligned} C_t + P_t \theta_t + Q_t^b B_t^h = & (1 - \alpha) Y_t + \pi \xi^t + t_r \xi^t + B_{t-1}^h + (P_t + D_t) \theta_{t-1} \\ & - \tau_l \left[(1 - \alpha) Y_t + (1 - Q_{t-1}^b) B_{t-1}^h + D_t \theta_{t-1} + (P_t - P_{t-1}) \theta_{t-1} + \pi \xi^t \right] - \tau_c C_t \end{aligned} \quad (5)$$

The first line is the budget in the absence of taxation: consumption plus holdings of equity and bonds – equals labor and endowment income plus the (cum dividend) value of assets.

The second line subtracts the tax bill. The income tax rate applies to labor income, interest, dividends as well as capital gains. The consumption tax rate is denoted τ_c .

We do not explicitly model the government, since we do not include observables that identify its behavior in our estimation. To close the model, one may think of a government that collects taxes based on the rates τ_l, τ_c, τ_k and uses lump sum transfers to follow a Ricardian policy of stabilizing its debt. The market clearing condition then states that $B_t^f = B_t^h$. The model is thus consistent with households owning not only corporate debt but also government debt.

2.2 Uncertainty and preferences

We denote information that becomes available to agents at date t by a vector of random variables ε_t and write $\varepsilon^t = (\varepsilon_t, \varepsilon_{t-1}, \dots)$ for the entire information set as of date t . Agents perceive ambiguity about real and financial technology shocks. The dynamics of these shocks can be written as

$$\begin{aligned} \log Z_{t+1} &= \tilde{z}(\varepsilon^t) + \mu_{t,z}^* + \tilde{\sigma}_{t,z} \varepsilon_{t+1}^z + v_{t+1}^z \\ \log f_{t+1} &= \tilde{f}(\varepsilon^t) + \mu_{t,f}^* + \tilde{\sigma}_{t,f} \varepsilon_{t+1}^f \end{aligned} \quad (6)$$

where $\varepsilon_{t+1}^z, \varepsilon_{t+1}^f$ and v_{t+1}^z are iid with $\varepsilon_{t+1}^i \sim \mathcal{N}(0, 1)$, $i = z, f$ and where $\mu_{t,z}^*$ and $\mu_{t,f}^*$ are deterministic sequences.² The decomposition of the innovations into deterministic and a random components serves to distinguish between ambiguity and risk, respectively. In particular, changes in risk are modeled in the usual way as changes in realized volatility. We assume throughout that the volatilities are bounded away from zero.

Consider the ambiguous components $\mu_{t,i}^*$. We assume that agents know the long run empirical moments of the sequences $\mu_{t,i}^*$; in particular, they know that the long empirical distribution of $\mu_{t,i}^*$ is iid normal with mean zero and variance $\sigma_{i,\mu}^2$ that is independent of the shocks ε_t^i and v_t^z . However, when making decisions at date t , agents do not know the current $\mu_{t,i}^*$. In fact, it is impossible for the agent (or an econometrician) to learn the sequences $\mu_{t,i}^*$ in (6), even with a large amount of data: the sequence $\mu_{t,i}^*$ cannot be distinguished from the realization ε_t^i .

In our econometric work below, we resolve this uncertainty probabilistically: we work below with volatility processes $\sigma_{t,i}^2 = \tilde{\sigma}_{t,i}^2 + \sigma_{i,\mu}^2$ and an iid innovation process, that is, $\mu_{t,i}^* = 0$. However, the probability we use is not the only one that is consistent with the data – there are

²Allowing for a nonnormal innovation v_{t+1}^z in addition to ε_{t+1}^z is helpful for a more flexible specification of business cycle risk below.

many others corresponding to different sequences $\mu_{t,i}^*$. Agents in the model treat uncertainty as ambiguity: they do not resolve uncertainty about $\mu_{t,i}^*$ by thinking in terms of a single probability.

Uncertainty shocks and changes in confidence

Based on date t information, agents contemplate an interval of conditional means $\mu_{t,i} \in [-a_{t,i}, a_{t,i}]$ for each component i . They are not confident enough to further integrate over alternative forecasts (and so in particular they do not use a single forecast). The vector $a_t = (a_{t,z}, a_{t,f})'$ summarizes ambiguity perceived about Z and f given date t information. It can be thought of as an (inverse) measure of confidence. If $a_{t,i}$ is low, then agents find it relatively easy to forecast the fundamental shock i and their behavior is relatively close to that of expected utility maximizers (who use a single probability when making decisions). In contrast, when $a_{t,i}$ is high, then agents do not feel confident about forecasting.

We allow for two sources of changes in confidence (and thus perceived ambiguity). On the one hand, confidence can depend on observed volatility. It is plausible that in more turbulent times agents find it harder to settle on a forecast of the future. On the other hand, confidence can move with intangible information that is not reflected in current fundamentals or volatility. To accommodate both cases, we let

$$a_{t,i} = \eta_{t,i} \sigma_{t,i}; \quad i = f, z \quad (7)$$

Here the $\eta_{t,i}$ s are stochastic processes that describes change in confidence due to the arrival of intangible information. Their laws of motion, like those of the volatilities $\sigma_{t,i}$, are known to agents. The information ε_t received at date t thus includes not only ε_t^z and ε_t^f , but also innovations to $\sigma_{i,t}$ and $\eta_{i,t}$.

We can interpret the linear relationship in (7) as arising when allowing $\mu_{t,i} \in [-a_{t,i}, a_{t,i}]$ if and only if

$$\frac{\mu_{t,i}^2}{2\sigma_{t,i}^2} \leq \frac{1}{2}\eta_{t,i}^2$$

The left hand side is the relative entropy between two normal distributions that share the same standard deviation $\sigma_{t,i}$ but have different means $\mu_{t,i}$ and zero, respectively. The agent thus contemplates only those conditional means that are sufficiently close to the long run average of zero in the sense of conditional relative entropy. The relative entropy distance captures that intuition through the fact that when $\sigma_{t,i}$ increases it is harder to distinguish different models.

Preferences

The representative household has recursive multiple priors utility. A consumption plan is a family of functions $c_t(\varepsilon^t)$. Conditional utilities derived from a given consumption plan c are defined by the recursion

$$U(c; \varepsilon^t) = \log c_t(\varepsilon^t) + \beta \min_{\mu_{t,i} \in \times_i [-a_{t,i}, a_{t,i}]} E^\mu [U(c; \varepsilon^t, \varepsilon_{t+1})], \quad (8)$$

where the conditional distribution over ε_{t+1} uses the means $\mu_{t,i}$ that minimize expected continuation utility. If $a_t = 0$, we obtain standard separable log utility with those conditional beliefs. If $a_t > 0$, then lack of information prevents agents from narrowing down their belief set to a singleton. In response, households take a cautious approach to decision making – they act as if the worst case mean is relevant.³

Given the specification of the ambiguous shocks, it is easy to solve the minimization step in (8) at the equilibrium consumption plan: the worst case expected cash flow is low and the worst case expected restructuring cost is high. Indeed, consumption depends positively on cash flow and negatively on the restructuring cost. It follows that agents act throughout as if forecasting under the worst case mean $\mu_{f,t} = a_{f,t}$ and $\mu_{z,t} = -a_{z,t}$. This property pins down the representative household's worst case belief after every history and thereby a worst case belief over entire sequences of data. We can thus also compute worst case expectations many periods ahead, which we denote by stars. For example $E^* D_{t+k}$ is the worst case expected dividend k periods in the future.

3 Uncertainty shocks, firm financing and asset prices

In this section we describe the main tradeoffs faced by investors and firms when pricing assets and deciding asset supply, respectively. To ease notation, we set the trend growth rate $\log \xi$ equal to zero. The solution of the model with positive growth is provided in the appendix.

Contingent claims prices and shareholder value

To describe t -period ahead contingent claims prices, we define random variables M_0^t that represent prices normalized by conditional *worst case* probabilities. This normalization is

³In the expected utility case, time t conditional utility can be represented as $E_t [\sum_{\tau=0}^{\infty} \log c_{t+\tau}]$ where the expectation is taken under a conditional probability measure over sequences that is updated by Bayes' rule from a measure that describes time zero beliefs. An analogous representation exists under ambiguity: time t utility can be written as $\min_{\pi \in \mathcal{P}} E_t^\pi [\sum_{\tau=0}^{\infty} \log c_{t+\tau}]$. The time zero set of beliefs \mathcal{P} can be derived from the one step ahead conditionals \mathcal{P}_t as in the Bayesian case, see Epstein and Schneider (2003) for details.

convenient for summarizing the properties of prices, which are derived from households' and firms' first order conditions. We also define a one-period-ahead pricing kernel as $M_{t+1} = M_0^{t+1}/M_0^t$. From household utility maximization, we obtain

$$M_{t+1} = \beta \frac{C_t}{C_{t+1}} \frac{1 - \tau_l}{1 - \tau_l \beta E_t^* [C_t/C_{t+1}]}$$

The pricing kernel is the marginal rate of substitution, multiplied by a factor that corrects for taxes. Since the qualitative effects we emphasize here do not depend on the level of personal income taxation, we set $\tau_l = 0$ for the remainder of this section.

The formulas for the bond and stock price are then standard, except that expectations are taken under the worst case belief:

$$\begin{aligned} Q_t &= E_t^* [M_{t+1}] \\ P_t &= E_t^* [M_{t+1} (P_{t+1} + D_{t+1})] \end{aligned} \tag{9}$$

An increase in ambiguity makes the worst case belief worse and thereby changes asset prices. For example, if agents perceive more ambiguity about future consumption, then the bond price rises and the interest rate falls. Similarly, more ambiguity about dividends tends to lower the stock price.

The firm maximizes shareholder value

$$E_0^* \sum_{t=1}^{\infty} M_0^t D_t$$

Shareholder value also depends on worst case expectations. Indeed, state prices determined in financial markets reflect households' attitudes to uncertainty, as illustrated by the household Euler equations. For example, when there is more ambiguity about future consumption then – other things equal – cash flows are discounted less. When there is more ambiguity about future cash flow, the firm tends to be worth less.

3.1 Payout and capital structure choice

Let λ_t denote the multiplier on the firm's date t budget constraint (4), normalized by the contingent claims price M_0^t . In the presence of restructuring and financing costs, the shadow value of funds inside the firm can be different from one. The firm's first order equations for debt is

$$Q_t^b \lambda_t = E_t^* [M_{t+1} \lambda_{t+1}] (1 - \tau_k (1 - Q_t^b) + \kappa'(B_t^f)) \tag{10}$$

The marginal benefit of issuing an additional dollar of debt is the bond price multiplied by the firm's shadow value of funds. The marginal cost includes not only the present value of a dollar to the firm, but also the tax advantage of debt and the marginal financing cost. The tax advantage implies that marginal cost is typically below marginal benefit at low levels of debt. At the optimal capital structure, the tax advantage is traded off against the financing cost.

The firm's first order condition for payout is

$$D_t(1 - \lambda_t) = \lambda_t \tilde{\phi} \left(\frac{D_t}{D_{t-1}} \right) - E_t^* \left[M_{t+1} \lambda_{t+1} \tilde{\phi} \left(\frac{D_{t+1}}{D_t} \right) \right] \quad (11)$$

where the function $\tilde{\phi}(D_t/D_{t-1}) := D_t D_{t-1} \phi_1(D_t, D_{t-1})$ is increasing with $\tilde{\phi}(1) = 0$. The optimal payout choice thus stabilizes the growth rate of payout in uncertainty adjusted terms. Indeed, at the steady state we have $\lambda_t = 1$. Near a steady state, payout will thus be set to equate the uncertainty adjusted growth rates.

Consider now the firm's response to an increase in uncertainty. In particular, suppose that, under the worst case belief, future dividends are low and funds are scarce, that is, the relative shadow value of funds $E_t^*[M_{t+1}\lambda_{t+1}]/\lambda_t$ increases. From (10), holding fixed the riskless rate, the marginal cost of debt has increased and the firm responds by cutting current debt B_t^f . At the same time, (11) suggests that the firm will decrease payout already at date t in order to smooth the drop in the growth rate of payout. As a result, uncertainty shocks make payout and debt move together.

In contrast, suppose that there is a shock to cash flow, say that temporarily lowers dividends and makes current funds more scarce relative to funds in the future. In this case, (10) suggests that the firm should borrow temporarily so as to cover the shortfall in funds. Cash flow shocks thus tend to move payout and debt in opposite directions.

3.2 Asset pricing

To see how asset pricing works in our model, we consider an approximate solution that is also used in our estimation approach. The approximation proceeds in three steps. First, we find the "worst case steady state", that is, the state to which the model would converge if there were no shocks and the data were generated by the worst case probability belief. Second, we linearize the model around the worst case steady state. Finally, we derive the true dynamics of the system, taking into account that the exogenous variables follow the

data generating process (6).⁴

Linearization around the worst case steady state

At the worst case steady state, the Euler equations (9) imply that the bond price is β and the price dividend ratio is $\beta / (1 - \beta)$. These values are the same as in the deterministic perfect foresight steady state. However, the level of consumption and dividends as well as other variables will be lower than in a perfect foresight steady state. This is because they are computed using the worst case mean productivity $\bar{Z}e^{-\bar{a}z}$ and restructuring cost $\bar{f}e^{\bar{a}f}$.

We mark log deviations from the worst case steady state by both a hat (for log deviation) and a star (to indicate that the perturbation is around the worst case steady state). The loglinearized pricing kernel and the household Euler equation for bonds and equity are

$$\begin{aligned}\hat{m}_{t+1}^* &= \hat{c}_t^* - \hat{c}_{t+1}^* \\ \hat{q}_t^* &= E_t^*[\hat{m}_{t+1}^*] \\ \hat{p}_t^* &= E_t^*[\hat{m}_{t+1}^* + \beta\hat{p}_{t+1}^* + (1 - \beta)\hat{d}_{t+1}^*]\end{aligned}\tag{12}$$

The short term interest rate is $\hat{r}_t^* = -\hat{q}_t^* = -E_t^*\hat{m}_{t+1}^*$. Linearization implies that asset prices do not reflect risk compensation. However, they still reflect uncertainty premia since expectations are computed under the worst case mean.

Stock price and interest rate volatilities

We can use the loglinearized Euler equations to understand the relative volatility of stock prices and interest rates in a model with ambiguity shocks. Substituting into the Euler equation for stocks, the price dividend ratio, or more precisely the price payout ratio, can be written as

$$\hat{p}_t^* - \hat{d}_t^* = -\hat{r}_t^* + E_t^*[\beta(\hat{p}_{t+1}^* - \hat{d}_{t+1}^*) + \hat{d}_{t+1}^* - \hat{d}_t^*]\tag{13}$$

The first line expresses the price dividend ratio as the worst case expected payoff relative to dividends, discounted at the riskless interest rate. In general equilibrium, an increase in uncertainty can move both the payoff term (if cash flow becomes more uncertain), and the interest rate (if consumption becomes more uncertain). In the data, interest rates are relatively stable whereas the price dividend ratio moves around a lot. As a result, the first effect must dominate the second if uncertainty shocks are to play an important role.

We can solve forward to express the price dividend ratio as the present value of future

⁴The worst case steady used in steps 1 and 2 should be viewed a computational tool that helps describe agents' optimal choices. Agents choose conservative policies in the face of uncertainty, and this looks *as if* the economy were converging to the worst case steady state.

growth rates in the dividend-consumption ratio

$$\begin{aligned}\hat{p}_t^* - \hat{d}_t^* &= E_t^* \left[\beta(\hat{p}_{t+1}^* - \hat{d}_{t+1}^*) + (\hat{d}_{t+1}^* - \hat{c}_{t+1}^*) - (\hat{d}_t^* - \hat{c}_t^*) \right] \\ &= E_t^* \sum_{\tau=1}^{\infty} \beta^{\tau-1} \left((\hat{d}_{t+\tau}^* - \hat{c}_{t+\tau}^*) - (\hat{d}_t^* - \hat{c}_t^*) \right)\end{aligned}\quad (14)$$

If dividends are proportional to consumption, then the price dividend ratio is constant – with log utility, income and substitution effects cancel. In contrast, if dividends are a small share of consumption (as in the data), then uncertainty about dividends will tend to dominate and an increase in uncertainty can decrease the price dividend ratio. The formula also shows that the price dividend ratio reflects expected worst case growth rates. If firms smooth these growth rates in response to uncertainty shocks, this tends to contribute to price volatility.

Zero risk steady state and unconditional premia

Unconditional premia predicted by the model depend on the average amount of ambiguity reflected in decisions. Suppose all shocks are equal to zero, but agents still use decision rules that reflect their aversion to ambiguity. In particular, agents perceive constant ambiguity, as in the worst case steady state. We can study this “zero risk” steady state using decision rules derived by linearization around the worst case steady state. From this perspective, the true steady state productivity and restructuring cost (\bar{Z}, \bar{f}) look like a positive deviation from steady state summarized by the vector $(-\bar{a}_z, \bar{a}_f)$. Mechanically, we are looking at the steady state of a system in which technology is always at $(-\bar{a}_z, \bar{a}_f)$, but in which agents act *as if* the economy is on an impulse response towards the worst case steady state.

Consider the impulse response that moves from the zero risk steady state log consumption and dividend, (\bar{C}, \bar{D}) say, to their worst case counterparts (\bar{C}^*, \bar{D}^*) . We work with loglinearized impulse responses and write $\bar{c} = \log \bar{C} - \log \bar{C}^*$. Along the linearized impulse response, the Euler equations (12) hold deterministically. For example, the steady state log bond price is

$$\bar{Q} = \beta \exp(\bar{q}) = \beta \exp(\bar{c} - \hat{c}_1),$$

where \hat{c}_1 is the first value along the impulse response. If there is ambiguity about consumption we would expect the impulse response to decline towards the worst case. In this case, the bond price is higher than β , the worst case (as well as rational expectations) steady state bond price. In other words, ambiguity about consumption lowers the interest rate – a precautionary savings effect.

Consider now the steady state price dividend ratio. The log deviation $\bar{p}^* - \bar{d}^*$ of from the worst case value $\beta/(1-\beta)$ is given by (13), where the sum is over the consumption

and dividend path along the linearized impulse response. For example, if the dividend-consumption ratio declines along the impulse response – say because there is a lot of average ambiguity about dividends and dividends are a small share of consumption – then $\bar{p}^* - \bar{d}^*$ is negative, that is, the steady state price dividend ratio \bar{P}/\bar{D} is below $\beta/(1 - \beta)$. The presence of ambiguity thus induces a price discount.

Combining the bond and stock price calculations, the equity premium at the zero risk steady state is⁵

$$\log(\bar{P} + \bar{D}) - \log \bar{P} + \log \bar{Q} = (1 - \beta)(\bar{d}^* - \bar{p}^*) - (\bar{c} - \hat{c}_1)$$

Ambiguity can generate a steady state equity premium positive for two reasons. First, the average stock return can be higher than under rational expectation because the price dividend ratio is lower. This is the first term. Second, the interest rate can be lower. The second effect is small if dividends are a small share of consumption and ambiguity is largely about dividends. We emphasize the role of the first effect: it says that average equity returns themselves are higher than in the rational expectations steady state. Ambiguity need not simply work through low real interest rates.

Predictability of excess returns

A standard measure of uncertainty premia in asset markets is the expected excess return on an asset computed from a regression on a set of predictor variables. The log excess stock return implied by our model can be approximated as

$$\begin{aligned} x_{t+1}^e &= \log(p_{t+1} + d_{t+1}) - \log p_t - \log(i_t) \\ &\approx \beta \hat{p}_{t+1}^* + (1 - \beta) \hat{d}_{t+1}^* - \hat{p}_t^* + \hat{q}_t^* \\ &= \beta \left(\hat{p}_{t+1}^* - \hat{d}_{t+1}^* - E_t^*[\hat{p}_{t+1}^* - \hat{d}_{t+1}^*] \right) + \hat{d}_{t+1}^* - E_t^* \hat{d}_{t+1}^* \end{aligned}$$

Here the second line is due to loglinearization of the return around the worst case steady state. The third line follows from the household Euler equation for stocks.

Consider now an econometrician who attempts to predict excess stock returns in the model economy. Suppose for concreteness that he has enough predictor variables to actually recover theoretical conditional expectation of payoff next period given the state variables

⁵The log stock return at the zero risk steady state is

$$\log(\bar{P} + \bar{D}) - \log \bar{P} \approx (1 - \beta)(\bar{d} - \bar{p}) - \log \beta$$

where we are using the fact that all asset returns are equal to $-\log \beta$ at the worst case steady state.

of the model. With a large enough sample, he will measure the expected excess return $E_t x_{t+1}^e$, where the expectation is taken with the conditional mean $\mu_t^* = 0$.⁶ Using the above expression, we can write the measured risk premium as

$$E_t x_{t+1}^e = \beta(E_t - E_t^*)[\hat{p}_{t+1}^* - \hat{d}_{t+1}^*] + (E_t - E_t^*)[\hat{d}_{t+1}^* - E_t^* \hat{d}_{t+1}^*],$$

where $(E_t - E_t^*)$ represents the difference between the expectation under $\mu_t^* = 0$ and the worst case expectation. This is a term that is proportional to ambiguity a_t . This expression suggests an interesting approach to quantify ambiguity in a linear model. Since risk premia must be due to ambiguity, it is possible to learn about ambiguity parameters up front from simple linear regressions without solving the DSGE model fully.

4 Estimation

4.1 Dynamics of shocks, volatility and ambiguity

We now present functional forms for shocks to real and financial technology as well as uncertainty about those shocks. In other words, we fill in the details of the dynamics in (6) and (7) above.

Volatility and ambiguity regimes

To parsimoniously model correlated changes in uncertainty, and to account for nonlinear dynamics in uncertainty, we use finite state Markov chains. The standard deviations $\sigma_{t,z}$ and $\sigma_{t,f}$ in (6) are driven by a two-state Markov chain s_t^σ with transition matrix H^σ . Each state in s_t^σ represents a "volatility regime"; a regime switch simultaneously moves both standard deviations. The regime is known at date t so that volatility is known one period in advance. Movements in confidence due to intangible information, denoted by $\eta_{t,i}$ also depend on the realization of a two-state Markov chain. This chain, denoted s_t^η , is independent from s_t^σ and is governed by a transition matrix H^η . As for the volatility regimes, each state in s_t^η simultaneously changes both levels $\eta_{t,z}$ and $\eta_{t,f}$.

To derive a loglinear approximation to equilibrium in the presence of stochastic volatility, it is helpful to write the chains as VARs. For example, s_t^σ can be written as

$$\begin{bmatrix} e_{1,t}^\sigma \\ e_{2,t}^\sigma \end{bmatrix} = H^\sigma \begin{bmatrix} e_{1,t-1}^\sigma \\ e_{2,t-1}^\sigma \end{bmatrix} + \begin{bmatrix} v_{1,t}^\sigma \\ v_{2,t}^\sigma \end{bmatrix} \quad (15)$$

⁶Indeed, since all unconditional empirical moments converge to those of a process with $\mu_t^* = 0$ by construction, the same is true for conditional moments.

where $e_{j,t}^\sigma = 1_{s_t^\sigma=j}$ is an indicator operator if the volatility regime s_t^σ is in place, and the shock v_t^σ is defined such that $E_{t-1}[v_t^\sigma] = 0$. A similar VAR representation is available for s_t^η .

Financial technology

Financial technology is modeled as a persistent AR(1) process in logs, with ambiguity driven by the regimes:

$$\begin{aligned}\tilde{f}(\varepsilon^t) &= \log \bar{f} + \rho_f (\log f_t - \log \bar{f}), \\ a_{t,f} &= \eta_f(s_t^\eta) \sigma_f(s_t^\sigma),\end{aligned}\tag{16}$$

In the first line, \tilde{f} is the mean function used in (6) and \bar{f} is the constant steady state restructuring cost. The second line shows that ambiguity about financial technology follows a four-state Markov chain.

Real technology

Real technology is handled differently because we view Z_t as incorporating the response of variable inputs to an underlying uncertainty shock. We model shocks to Z_t as a *joint* change in marginal product of capital today and ambiguity perceived about the marginal product of capital in the future.⁷ This requires two extensions to the functional form (16). On the one hand, the current shock should depend (negatively) on the current uncertainty regime. Here is where we use the nonnormal shock v_{t+1}^z introduced in (6). We define

$$\begin{aligned}\tilde{z}(\varepsilon^t) &= \log \bar{Z} + \rho_z (\log Z_t - \log \bar{Z}) - \varkappa E_t [\eta_z(s_{t+1}^\eta) \sigma_z(s_{t+1}^\sigma) | s_t^\eta, s_t^\sigma] \\ v_{t+1}^z &= -\varkappa (\eta_z(s_{t+1}^\eta) \sigma_z(s_{t+1}^\sigma) - E_t [\eta_z(s_{t+1}^\eta) \sigma_z(s_{t+1}^\sigma) | s_t^\eta, s_t^\sigma])\end{aligned}\tag{17}$$

On the other hand, ambiguity should move continuously (negatively) with the shock. To this effect, we introduce an AR(1) component of a_t with innovations that are perfectly negatively correlated with those to $\log Z_t$:

$$\begin{aligned}a_{t,z} &= \eta_z(s_t^\eta) \sigma_z(s_t^\sigma) + \tilde{a}_{t,z}^c \\ \tilde{a}_{t,z}^c &= \rho_a \tilde{a}_{t-1,z}^c - \sigma_a \sigma_z(s_{t-1}^\sigma) \varepsilon_t^z\end{aligned}\tag{18}$$

⁷In particular, Ilut and Schneider (2012) show that an increase in ambiguity about total factor productivity makes firms and households act cautiously so that hours worked and economic activity can contract even if current labor productivity did not change. We capture similar effects here by making the innovations to real technology negatively correlated with the current innovation to ambiguity about it.

4.2 Markov-switching VAR representation of the equilibrium

We can write the equilibrium representation of our DSGE model as a Markov-switching VAR. The interval of one-step ahead conditional means given by $[-a_{t,i}, a_{t,i}]$ for each shock i is affected by the product $a_{t,i} = \eta_{t,i}\sigma_{t,i}$ of the two sources of ambiguity. Both uncertainty chains s_t^η and s_t^σ linearly affect $a_{t,i}$ and hence the worst-case conditional expectation. Moreover, the chains are stationary and ergodic and their dynamics are the same under the true and worst case dynamics. The worst case steady state depends on their long run averages $\bar{\eta}_i$ and $\bar{\sigma}_i$. As intangible ambiguity $\eta_{t,i}$ or volatility $\sigma_{t,i}$ fluctuate around their respective longrun means, there are "shocks" to ambiguity $a_{t,i}$ and therefore shifts in the constants of the VAR representation.

Formally, we define the vector of linear deviations of the product $\eta_i(s_t^\eta)\sigma_i(s_t^\sigma)$ for $i = z, f$ from its ergodic values of $\bar{\eta}_i\bar{\sigma}_i$ as following a four-state Markov chain, which is obtained by mixing the two independent chains s_t^η and s_t^σ . We can then write the VAR representation of this composite Markov chain as

$$\begin{bmatrix} e_{1,1,t}^{\eta\sigma} \\ e_{1,2,t}^{\eta\sigma} \\ e_{2,1,t}^{\eta\sigma} \\ e_{2,2,t}^{\eta\sigma} \end{bmatrix} = H^{\eta\sigma} \begin{bmatrix} e_{1,1,t-1}^{\eta\sigma} \\ e_{1,2,t-1}^{\eta\sigma} \\ e_{2,1,t-1}^{\eta\sigma} \\ e_{2,2,t-1}^{\eta\sigma} \end{bmatrix} + \begin{bmatrix} v_{1,t}^{\eta\sigma} \\ v_{2,t}^{\eta\sigma} \\ v_{3,t}^{\eta\sigma} \\ v_{4,t}^{\eta\sigma} \end{bmatrix} \quad (19)$$

where $e_{m,n,t}^{\eta\sigma} = 1_{s_t^\eta=m, s_t^\sigma=n}$ is an indicator operator if at time t the intangible ambiguity regime m and the volatility regime n are in place, where $m, n \in \{1, 2\}$. The realizations of the shock $v_t^{\eta\sigma}$ are such that $E_{t-1}[v_t^{\eta\sigma}] = 0$. The transition matrix is $H^{\eta\sigma} = H^\eta \otimes H^\sigma$.

For each shock i and the four m, n combinations we can define $a_i(m, n) \equiv \eta_i(s_t^\eta = m)\sigma_i(s_t^\sigma = n) - \bar{\eta}_i\bar{\sigma}_i$. For example, when the intangible information regime 1 and volatility regime 1 are in place, $e_{1,1,t}^{\eta\sigma} = 1$ and the rest of the three $e_{m,m,t}^{\eta\sigma} = 0$. This means that our system of equations will load $a_i(1, 1)e_{1,1,t}^{\eta\sigma} = a_i(1, 1)$ and put zero weight on the other three realizations $a_i(1, 2), a_i(2, 1)$ and $a_i(2, 2)$. In this case, the realization of the $v_t^{\eta\sigma}$ shock such that $e_{1,1,t}^{\eta\sigma} = 1$ makes the linear deviation $\eta_i(s_t^\eta = 1)\sigma_i(s_t^\sigma = 1) - \bar{\eta}_i\bar{\sigma}_i$ hit the economy as a discrete shock. By augmenting our DSGE state vector with the vector $e_t^{\eta\sigma}$ we control for the first order effects of the shifts in intangible ambiguity and volatility.

Given these first-order shifts, we can then proceed to linearize the rest of equilibrium conditions of the model. We then use an observational equivalence result according to which our economy behaves *as if* the agent maximizes expected utility under the worst-case belief. Given this equivalence, we use standard perturbation techniques that are a good approximation of the nonlinear decision rules under expected utility. We then obtain a VAR

representation of the linearized DSGE model. The uncertainty regimes produce Markov-switching constants in that VAR. Further details on the representation are presented in Appendix 5.1.

4.3 Estimates

For our estimation we include five observables based on US data: investment growth, dividend to GDP ratio, equity price to GDP ratio, the short term real interest rate, and firm debt to equity value ratio. The time period is 1959Q1 to 2011Q3. We estimate the model using Bayesian methods.⁸ Because we only have two continuously distributed shocks but we have five observables, to avoid stochastic singularity we need to introduce three observation errors. We set these errors on the dividend to GDP ratio, real interest rate, and the firm debt to equity value ratio. real interest rate, and firm debt to equity value ratio.

There are two key differences from a standard Bayesian estimation of a homoskedastic linear DSGE model. First, we have to account for heteroskedasticity in the shocks of our model, as in the literature on regime-switching volatilities. Second, differently from that literature, the volatility regimes, as well as the intangible ambiguity ones, have first order effects on the endogenous variables of our linear model.

Our approach achieves identification of the volatility and intangible ambiguity regimes through two channels: on the one hand, since they enter as a product in the linearized model, both these uncertainty regimes shift the constant of the Markov-switching VAR. On the other hand, the two types of regimes can be differentiated through the properties of the fundamental shocks. While the intangible information regime s_t^η leaves unaffected the moments of the shocks, the volatility regime s_t^σ shows up as changes in the second moment of the next period innovations. Through the use of the Kalman smoother, the estimation can then identify how likely it is that a shift in the constant is due to the high volatility or the high intangible ambiguity regime.

Choosing ambiguity parameters

The regime-switching dynamics of risk and ambiguity are governed by the Markov chains s_t^η and s_t^σ . There we estimate directly the corresponding two values of $\sigma_{t,i}$ and $\eta_{t,i}$ together with the transition matrices H^σ and H^η . We are then left with choosing parameters σ_a, ρ_a and \varkappa . In order for the set $[-a_{t,z}, a_{t,z}]$ to be well-behaved we need the process for $a_{t,z}$ to remain nonnegative. Similarly to the parametrization used in Ilut and Schneider (2011), we

⁸Details about the estimation strategy are contained in Bianchi (2012).

then set

$$3\sigma_a = \eta_{z,L} \sqrt{1 - \rho_a^2} \quad (20)$$

where $\eta_{z,L}$ is the value for η_z in the low ambiguity regime. This ensures that even in the low ambiguity regime, the probability that $\eta_{t,z}$ becomes negative is .13%, and any negative $\eta_{t,z}$ will be small.

The second consideration is that we want to bound the lack of confidence by the measured variance of the shock that agents perceive as ambiguous. Ilut and Schneider (2011) argue that a reasonable upper bound for $a_{t,i}$ is given by $2\sigma_{t,i}$. We can impose this bound directly on the two values for η_f by making them lower than 2.⁹ However, for the ambiguity on the real technology, we cannot enforce the bound exactly. Here we assume that it is violated with probability .13% even in the high ambiguity regime

$$\eta_{z,H} + 3 \frac{\sigma_a}{\sqrt{1 - \rho_a^2}} \leq 2. \quad (21)$$

We satisfy the constraints in (20) and (21) by imposing that both value $\eta_{z,L}$ and $\eta_{z,H}$ are lower than 1. Finally, we set the proportionality factor \varkappa in (17) equal to σ_a^{-1} . This means that the negative effect on the current \hat{z}_t of one unit increase in $\tilde{a}_{t,z}^c$ is the same as that of $\eta_z(s_t^\eta)\sigma_z(s_t^\sigma)$. We then are left with estimating the values $\eta_{z,L}, \eta_{z,H}$ and ρ_a as we can then infer σ_a from (20).

Parameter estimates

We estimate a subset of the parameters, with values reported Table 1. The other parameters are fixed, as reported in Table 2, to values relatively standard in the literature, or calibrated to match some key ratios from the NIPA accounts. Further details are in Appendix 5.2.1.

4.4 Results

In this subsection we describe the economic forces identified by our estimation through a series of figures.

The role of measurement error

Figure 1 shows our five observables together with their smoothed model implied counterparts. The model exactly matches investment and the equity-gdp-ratio. For the three series where we include measurement error, the latter is small especially for leverage and the

⁹As shown in Table 1, we impose a Beta prior on $0.5\eta_f$.

dividend-gdp ratio. The model generates movements in those variable both at low frequency and business cycle frequencies. It misses some of the low frequency movements in the real interest rate in the 1970s and 80s. However, the model implied real interest rate is not very volatile, consistent with the data.

Regime shifts

There are four sources of exogenous variation: the two shocks Z_t and f_t as well as the regimes for volatility and intangible ambiguity. The top and bottom panels of Figure 2 display, for each sample date, the conditional probabilities that a high volatility or a high ambiguity regime was in place. The high volatility regime was most likely in place from the mid 1970s to the late 1980s. Table 1 shows that its main effect is an increase in the volatility of restructuring costs. The high ambiguity regime is characterized by a lower confidence in both the real technology Z_t and the restructuring costs f_t , with a stronger effect in the latter. In terms of time-variation, our model suggests that there were pronounced increases in confidence, especially towards the end of recent booms such as in the late 1980s and again in the late 1990s. Our estimation interprets the last 5 years as a period of higher econometric uncertainty over which ambiguity regime dominated. There is some evidence that the drop in prices associated with the last recession is associated with a loss of confidence and that the bounceback following that is a return to a high confidence regime. However, inference at the end of the sample is weak. We interpret this as an effect of the lack of significant evidence to separate the end-of-sample movements in stock prices into uncertainty premia or measured cash-flow changes.

Real uncertainty and the business cycle

Movements in business cycle quantities are mostly accounted for by the joint shock to real technology and uncertainty Z_t . Figure 3 shows the contributions of different sources of variation to year-on-year investment growth. Each panel focuses on a different source of exogenous variation: the top panel looks at Z_t , the middle panel at the restructuring cost f_t , and the bottom panel at the regime shifts. For each source of variation, its panel shows the data series as a red dash-dotted line. The solid blue line is what the model would predict if all variation came from the source considered in the panel. The bulk of the variation in investment is clearly due to movements in Z_t , although shifts in regimes also play a role.

Intuitively, movements in Z_t have two effects. On the one hand, they move the marginal product of capital. It is natural that a decrease in Z_t lowers output and investment, as it would in a standard RBC model. On the other hand, the change in the *current* marginal product of capital comoves negatively with ambiguity about future capital. This further lowers the return on investment. In addition, it induces ambiguity about future consumption

and lowers real interest rate.

Figure 4 shows the contribution of Z_t to the other observables. In addition to its cyclical effect on investment, it also plays an important role for dividends and the real interest rate. In particular it helps account for sharp drops in the real interest rate in both recent recessions. This effect would not occur if Z_t were purely a TFP shock: mean reversion in TFP would then tend to raise interest rates in recessions. While the real uncertainty shock matters for real quantities, its effect on the stock price is relatively small.

Uncertainty about financial conditions

Figure 5 displays the impulse response to a boost in confidence due to intangible information, that is, a regime shift in the $\eta_{t,i}$ s. This type of shift affects mostly confidence about financial conditions; the estimation suggests it took place for example in the late 1980s and 1990s. The figure plots not only our five observables, but adds also the ratio of debt to GDP so as to make the effects on capital structure more transparent. In each panel, the model is initially in the steady state for the high ambiguity regime. It then experiences a regime shift to the low ambiguity regime in period 20.

A boost in confidence leads to a joint increase in payout and debt (see the middle panels in the top and bottom rows). At the same time, the stock market rises. The price effect is strong enough that leverage of the corporate sector falls even as debt expands. An initial drop in investment – accompanied by a short term upward spike in the interest rate – quickly turns into an investment boom.

The intuition for the comovement of financial quantities follows from the firm’s optimal policies discussed in subsection 3.1. Shareholders would like to issue debt to exploit the tax advantage, but they worry that an increase in restructuring costs might make internal funds scarce. When they become more confident that funds will be cheap, they effectively substitute away from equity financing by issuing debt and paying dividends to themselves.

The increase in the equity-gdp ratio is stronger than that of the dividend-gdp ratio (see the first two panels in the top row). In other words, the boost in confidence increases the price dividend ratio. As discussed in subsection 3.2, this is due in part because a decline in ambiguity reduces the uncertainty premium on stocks. The stock market boom is not due to a decline in interest rates – in fact the real rate rises as confidence goes up.

Figure 6 displays the impulse response to an increase in volatility. The effect on financial quantities and the stock price are essentially the opposite as in Figure 5. This is to be expected since the increase in volatility affects decision rules by increasing ambiguity. Another similarity is that the effects are quite drawn out over time. This raises the question of what a sequence of regime changes contributes to our interpretation of the data.

Figure 7 starts the model at the ergodic steady state and then shocks it with a sequence

of regime changes. In particular, we construct a deterministic sequence of regimes from the estimates reported in Figure 2 by assuming that a regime occurs if its probability is larger than one half. The constructed sequence is shown in the bottom right panel of Figure 7. The other panels report the contribution of the regimes to financial quantities, investment and prices. The resulting movements are sizable and account for a large chunk of the comovement in stock prices, debt and dividends.

References

- ARELLANO, C., Y. BAI, AND P. KEHOE (2010): “Financial Markets and Fluctuations in Uncertainty,” FRB of Minneapolis, Research Department Staff Report.
- BASU, S. AND B. BUNDICK (2011): “Uncertainty Shocks in a Model of Effective Demand,” Tech. Rep. 774, Boston College Working Papers.
- BIDDER, R. AND M. SMITH (2012): “Robust Animal Spirits,” *Journal of Monetary Economics*, 59, 738–750.
- CAGETTI, M., L. P. HANSEN, T. SARGENT, AND N. WILLIAMS (2002): “Robustness and Pricing with Uncertain Growth,” *Review of Financial Studies*, 15, 363–404.
- CALDARA, D., J. FERNANDEZ-VILLAYERDE, J. F. RUBIO-RAMIREZ, AND W. YAO (2012): “Computing DSGE models with recursive preferences and stochastic volatility,” *Review of Economic Dynamics*, 15, 188–206.
- CHRISTIANO, L., M. EICHENBAUM, AND C. EVANS (2005): “Nominal Rigidities and the Dynamic Effects of a Shock to Monetary Policy,” *Journal of Political Economy*, 113.
- CHRISTIANO, L., R. MOTTO, AND M. ROSTAGNO (2013): “Risk Shocks,” *The American Economic Review*, forthcoming.
- CROCE, M. M., H. KUNG, T. T. NGUYEN, AND L. SCHMID (2012): “Fiscal policies and asset prices,” *Review of Financial Studies*, 25, 2635–2672.
- EPSTEIN, L. G. AND M. SCHNEIDER (2003): “Recursive Multiple-Priors,” *Journal of Economic Theory*, 113, 1–31.
- GILCHRIST, S., J. W. SIM, AND E. ZAKRAJSEK (2010): “Uncertainty, financial frictions, and investment dynamics,” Unpublished working Paper.

- GLOVER, B., J. F. GOMES, AND A. YARON (2011): “Corporate taxes, leverage, and business cycles,” Unpublished working Paper.
- GOURIO, F. (2012): “Disaster Risk and Business Cycles,” *The American Economic Review*, 102, 2734–2766.
- (2013): “Credit risk and disaster risk,” *American Economic Journal: Macroeconomics*, 5, 1–34.
- ILUT, C. AND M. SCHNEIDER (2011): “Ambiguous Business Cycles,” Unpublished manuscript, Duke University.
- JAHAN-PARVAR, M. AND H. LIU (2012): “Ambiguity Aversion and Asset Prices in Production Economies,” Unpublished working Paper.
- JERMANN, U. AND V. QUADRINI (2012): “Macroeconomic Effects of Financial Shocks,” *American Economic Review*, 102, 238–271.
- MALKHOZOV, A. AND M. SHAMLOO (2012): “News Shocks, Volatility Shocks, and Asset Prices,” Unpublished working paper.

5 Appendix

5.1 Solution method for a model with ambiguity and MS volatility

Here we describe our approach to solve the model with regime switching ambiguity and volatility. The steps of the solution are the following:

1. Describe the law of motion for the shocks

(a) The perceived law of motion for the continuous shocks:

$$\begin{aligned}\widehat{z}_{t+1} &= \rho_z \widehat{z}_t - \varkappa \eta_z(s_{t+1}^\eta) \sigma_z(s_{t+1}^\sigma) + \mu_{t,z}^* + \sigma_z(s_t^\sigma) \varepsilon_{t+1}^z \\ \widehat{f}_{t+1} &= \rho_f \widehat{f}_t + \mu_{t,f}^* + \sigma_f(s_t^\sigma) \varepsilon_{t+1}^f\end{aligned}$$

where each element i in the vector μ_t belongs to a set

$$\mu_{t,f} \in [-\eta_f(s_t^\eta) \sigma_f(s_t^\sigma), \eta_f(s_t^\eta) \sigma_f(s_t^\sigma)] \quad (22)$$

$$\mu_{t,z} \in [-\eta_{t,z} \sigma_z(s_t^\sigma), \eta_{t,z} \sigma_z(s_t^\sigma)] \quad (23)$$

(b) Volatility follows a two-state Markov chain s_t^σ with transition matrix H^σ .

(c) There is an independent two-state Markov chain s_t^η that governs intangible ambiguity for financial technology. For the real technology, ambiguity follows the process

$$\begin{aligned}a_{t,z} &= \eta_z(s_t^\eta) \sigma_z(s_t^\sigma) + \widetilde{a}_{t,z}^c \\ \widetilde{a}_{t,z}^c &= \rho_a \widetilde{a}_{t-1,z}^c - \sigma_a \sigma_z(s_{t-1}^\sigma) \varepsilon_t^z\end{aligned}$$

2. Guess and verify the worst-case scenario. As discussed in detail in Ilut and Schneider (2011), the solution to the equilibrium dynamics of the model can be found through a guess-and-verify approach. To solve for the worst-case belief that minimizes expected continuation utility over the i sets in (8), we propose the following procedure:

- (a) guess the worst case belief μ^0
- (b) solve the model assuming that agents have expected utility and beliefs μ^0 .
- (c) compute the agent's value function V
- (d) verify that the guess μ^0 indeed achieves the minima.

The following steps detail the point 2.b) above. Here we use an observational equivalence result saying that our economy can be solved *as if* the agent maximizes expected utility under the belief μ^0 . Given this equivalence, we can use standard perturbation techniques that are a good approximation of the nonlinear decision rules under expected utility. In particular, we will use linearization. When we refer to the guess below, we use $\mu_z^* = -a_z$ and $\mu_f^* = a_f$.

3. Compute worst-case steady states

- (a) Compute the ergodic values $\bar{\eta}_i$ and $\bar{\sigma}_i$.
- (b) Based on the guess above compute the worst-case steady states for the shocks, denoted by $\bar{\tau} = (\bar{z}, \bar{f})$.
- (c) Compute the worst-case steady state \bar{Y} of the endogenous variables. For this, use the FOCs of the economy based on their deterministic version in which the one step ahead expectations are computed under the guessed worst-case belief.

4. Dynamics:

- (a) Linearize around \bar{Y} and $\bar{\tau}$ by finding the coefficient matrices from linearizing the FOCs. Here use that

$$a_{t,i} - \bar{\eta}_i \sigma_{t,i} = \eta_{t,i} \sigma_{t,i} - \bar{\eta}_i \bar{\sigma}_i$$

and define a composite Markov-chain for the product $\eta_{t,i} \sigma_{t,i}$ as in equation (19). The linearized FOCs can be written in the canonical form of solving rational expectations models as:

$$\tilde{\Gamma}_0 \tilde{S}_t = \tilde{\Gamma}_1 \tilde{S}_{t-1} + \tilde{\Psi} [\varepsilon'_t, v_t^{\eta\sigma}]' + \Pi \omega_t$$

where S_t is the DSGE state.

- (b) Given that the shock v_t is defined such that $E_{t-1}[v_t^{\eta\sigma}] = 0$, a standard solution method to solve rational expectations general equilibrium models can be employed. The solution can then be rewritten as a MS-VAR with stochastic volatility in which the constant is also time-varying:

$$\tilde{S}_t = C_t + T \tilde{S}_{t-1} + R \sigma(s_{t-1}^\sigma) \varepsilon_t \tag{24}$$

The changes in the constant arise from the first order effects of the composite regimes of stochastic volatility and ambiguity. Notice that the solution in (24) is

expressed in terms of the original DSGE state variables as the regime variables $e_{1,1,t}^{\eta\sigma}$, $e_{1,2,t}^{\eta\sigma}$, $e_{2,1,t}^{\eta\sigma}$, and $e_{2,2,t}^{\eta\sigma}$ have been replaced with the MS constant C_t .

- (c) Verify that the guess μ^0 indeed achieves the minima of the time t expected continuation utility over the sets in (8).

5. Equilibrium dynamics under the true DGP. The above equilibrium was derived under the worst-case beliefs. We need to characterize the economy under the econometrician's law of motion. There are two objects of interest: the zero-risk steady state of our economy and the dynamics around that steady state.

- (a) The zero-risk steady state, denoted by Y^* . This is characterized by shocks, including the volatility regimes, being set to their ergodic values under the true DGP. Y^* can then be found by looking directly at the linearized solution, adding $R_z\bar{\eta}_z\bar{\sigma}_z$ and subtracting $R_f\bar{\eta}_f\bar{\sigma}_f$:

$$Y^* - \bar{Y} = T(Y^* - \bar{Y}) + R_z\bar{\eta}_z\bar{\sigma}_z - R_f\bar{\eta}_f\bar{\sigma}_f \quad (25)$$

where R_z and R_f are the equilibrium response to positive innovations to \hat{z}_t and \hat{f}_t respectively.

- (b) Dynamics. The law of motion in (24) needs to take into account that expectations are under the worst-case beliefs which differ from the true DGP. Then, defining $\hat{S}_t \equiv S_t - S^*$ and using (24) together with (25) we have:

$$\begin{aligned} \hat{S}_t = & C_t + T\hat{S}_{t-1} + R\sigma(s_{t-1}^\sigma)\varepsilon_t + R_z(\eta_z(s_{t-1}^\eta)\sigma_z(s_{t-1}^\sigma) - \bar{\eta}_z\bar{\sigma}_z + \tilde{a}_{t-1,z}^c) - \\ & - R_f(\eta_f(s_{t-1}^\eta)\sigma_f(s_{t-1}^\sigma) - \bar{\eta}_f\bar{\sigma}_f) \end{aligned}$$

5.2 Equilibrium conditions for the estimated model

Here we describe the equations that characterize the equilibrium of the estimated model in Section 4. To solve the model, we first scale the variables in order to induce stationarity. The variables are scaled as follows:

$$c_t = \frac{C_t}{\xi^t}, y_t = \frac{Y_t}{\xi^t}, k_t = \frac{K_t}{\xi^t}, i_t = \frac{I_t}{\xi^t}$$

Financial variables:

$$p_t = \frac{P_t}{\xi^t}, d_t = \frac{D_t}{\xi^t}, b_t^i = \frac{B_t^i}{\xi^t}; i = f, h$$

The borrowing costs:

$$\frac{\kappa(B_{t-1}^f)}{\xi^t} = f_t + \frac{\Psi}{2} \frac{1}{\xi^2} (b_{t-1}^f)^2; \quad \frac{\phi(D_t, D_{t-1})}{\xi^t} = \frac{\phi'' \xi^2}{2} \left(\frac{d_t}{d_{t-1}} - 1 \right)^2$$

We now present the nonlinear equilibrium conditions characterizing the model, in scaled form. The expectation operator in these equations, denoted by E_t^* , is the one-step ahead conditional expectation under the worst case belief μ^0 . According to our model, the worst case is that future z_{t+1} is low, and that the financing cost f_{t+1} is high.

The firm problem is

$$\max E_0^* \sum M_{0,t}^f D_t$$

subject to the budget constraint

$$\begin{aligned} d_t = (1 - \tau_k) & \left[\alpha y_t - \frac{b_{t-1}^f}{\xi} (1 - Q_{t-1}^b) - \frac{\phi'' \xi^2}{2} \left(\frac{d_t}{d_{t-1}} - 1 \right)^2 - i_t \right] - \\ & - f_t - \frac{\Psi}{2} \frac{1}{\xi^2} (b_{t-1}^f)^2 + \delta \tau_k q_{t-1}^k \frac{k_{t-1}}{\xi} - \frac{b_{t-1}^f}{\xi} Q_{t-1}^b + b_t^f Q_t^b \end{aligned} \quad (26)$$

and the capital accumulation equation

$$k_t = \frac{(1 - \delta)k_{t-1}}{\xi} + \left[1 - \left(\frac{S''}{2} \frac{i_t \xi}{i_{t-1}} - \xi \right)^2 \right] i_t \quad (27)$$

Let the LM on the budget constraint be $\lambda_t M_{0,t}^f \epsilon_t$ and on the capital accumulation be $\mu_t M_{0,t}^f \epsilon_t$. Then the scaled pricing kernel is

$$m_{t+1}^f \equiv M_{t+1} \frac{\xi^{t+1}}{\xi^t} = \beta \frac{c_t}{c_{t+1}} \frac{1 - \tau_l}{1 - \tau_l \beta E_t^* [c_t / (c_{t+1} \xi)]}. \quad (28)$$

The FOCs associated with the firm problem are then:

1. Dividends:

$$1 = \lambda_t \left[1 + (1 - \tau_k) \phi'' \xi^2 \frac{1}{d_{t-1}} \left(\frac{d_t}{d_{t-1}} - 1 \right) \right] - E_t m_{t+1}^f \lambda_{t+1} \left[(1 - \tau_k) \phi'' \xi^2 \frac{d_{t+1}}{d_t^2} \left(\frac{d_{t+1}}{d_t} - 1 \right) \right] \quad (29)$$

2. Bonds:

$$Q_t^b \lambda_t = E_t^* m_{t+1}^f \lambda_{t+1} \frac{1}{\xi} \left[1 - \tau_k (1 - Q_t^b) + \frac{\Psi}{\xi} b_t^f \right] \quad (30)$$

3. Investment:

$$1 = \frac{q_t^k}{(1 - \tau_k)} \left[1 - \frac{S''}{2} \xi^2 \left(\frac{i_t}{i_{t-1}} - 1 \right)^2 - S'' \xi^2 \left(\frac{i_t}{i_{t-1}} - 1 \right) \frac{1}{i_{t-1}} \right] + \quad (31)$$

$$+ E_t^* m_{t+1}^f \frac{\lambda_{t+1}}{\lambda_t} q_{t+1}^k S'' \xi^2 \frac{i_{t+1}^2}{i_t^2} \left(\frac{i_{t+1}}{i_t} - 1 \right)$$

where

$$q_t^k \equiv \frac{\mu_t}{\lambda_t}$$

4. Capital:

$$1 = E_t^* m_{t+1}^f \frac{\lambda_{t+1}}{\xi \lambda_t} R_{t+1}^K \quad (32)$$

$$R_{t+1}^k \equiv \frac{(1 - \tau_k) \alpha \left(\frac{k_t}{\xi} \right)^{\alpha-1} L^{1-\alpha} + (1 - \delta) q_{t+1}^k}{q_t^k} + \delta \tau_k$$

The household problem is as follows:

$$\max E_0^* \sum \beta^t \log c_t$$

$$(1 + \tau_c) c_t + p_t \theta_t = (1 - \tau_l) \left[(1 - \alpha) y_t + \pi + d_t \theta_{t-1} + \frac{b_{t-1}^h}{\xi} (1 - Q_{t-1}^b) \right] + \quad (33)$$

$$+ p_t \theta_{t-1} - \tau_l (p_t - \frac{1}{\xi} p_{t-1}) \theta_{t-1} + \frac{b_{t-1}^h}{\xi} Q_{t-1}^b - b_t^h Q_t^b + t_r$$

Thus, the FOCs associated to the household problem are:

1. Bond demand:

$$Q_t^b = \beta E_t^* \frac{c_t}{\xi c_{t+1}} [1 - \tau_l (1 - Q_t^b)] \quad (34)$$

2. Equity holding:

$$p_t = \beta E_t^* \frac{c_t}{c_{t+1}} \left[(1 - \tau_l) (p_{t+1} + d_{t+1}) + \frac{\tau_l}{\xi} p_t \right] \quad (35)$$

The market clearing conditions characterizing this economy are:

$$b_t^h = b_t^f \quad (36)$$

$$c_t + i_t + \frac{\phi'' \xi^2}{2} \left(\frac{d_t}{d_{t-1}} - 1 \right)^2 + f_t + \frac{\Psi}{2} \frac{1}{\xi^2} \left(b_{t-1}^f \right)^2 = y_t + \pi \quad (37)$$

$$\theta_t = 1$$

corresponding to the market for bonds, goods and equity shares, respectively.

Thus, we have the following 11 unknowns:

$$k_t, i_t, b_t^f, b_t^h, Q_t^b, p_t, c_t, d_t, q_t^k, \lambda_t, m_t^f$$

The equations (27), (28), (29), (30), (31), (32), (34), (35), (36),(37) give us 10 equations. By Walras' law, we can then use one out of the two budget constraints in (26) and (33) (using $\theta_t = 1$). This gives us a total of 11 equations.

5.2.1 Parametrization

Rescaling and calibrating parameters

For the steady state calculation of the model it is helpful to rescale some parameters. Specifically, denote by \bar{y}^{gdp} the worst-case steady state measured GDP, i.e. total goods $y_t + \pi$ minus financing costs. Then, define the following ratios:

$$f_y = \frac{f}{\bar{y}^{gdp}}; \Psi_y = \frac{\Psi}{\bar{y}^{gdp}}; \pi_y = \frac{\pi}{\bar{y}^{gdp}}, t_y = \frac{t_r}{\bar{y}^{gdp}}$$

where t is the steady state level of transfers.

We estimate a subset of the parameters, with values reported in Table 1. The other parameters are fixed, as reported in Table 2, to values relatively standard in the literature. The constant L of hours worked only scales the economy. Parameters are calibrated to match some key ratios from the NIPA accounts. First, total measured GDP in our model, denoted here by y^{gdp} , corresponds to the non-financial corporate sector (NFB) output plus goods produced by the other productive sectors- financial, non-corporate and household. We associate the firm in our model with the NFB sector and thus π_y equals goods produced by other productive sectors divided by y^{gdp} . The tax parameters are computed as follows: τ_l equals total personal taxes and social security contributions divided by total income, where the latter is defined as total wages plus dividends. τ_k equals NFB taxes divided by NFB profits and τ_c equals NFB sales taxes divided by NFB output. The government spending ratio g equals government net purchases from other sectors plus net exports divided by y^{gdp} . The ratio t_y equals government transfers (including social security and medicare) plus after-tax government wages divided by y^{gdp} .

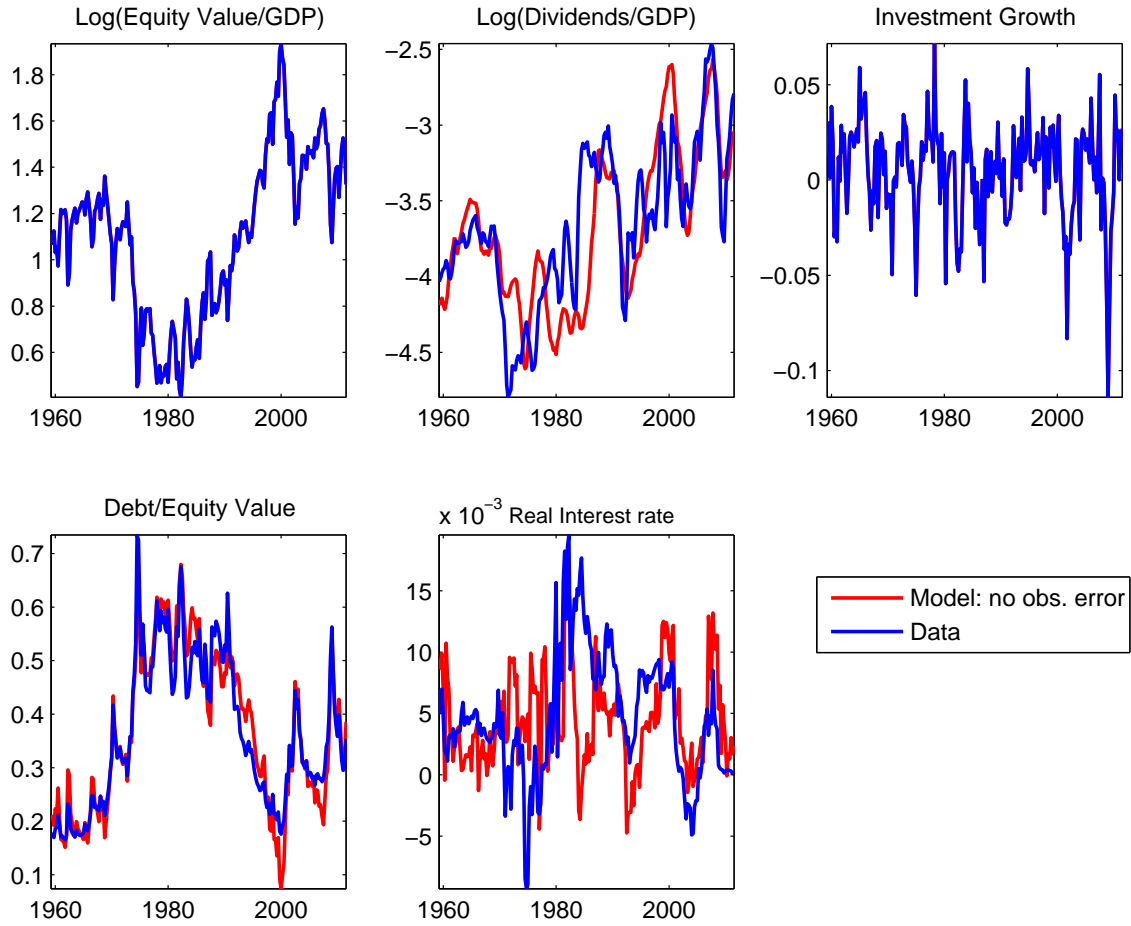


Figure 1: Variables used for estimation. Red line is the smoothed model-implied path subtracting the estimated observation error. The blue line represents the data.

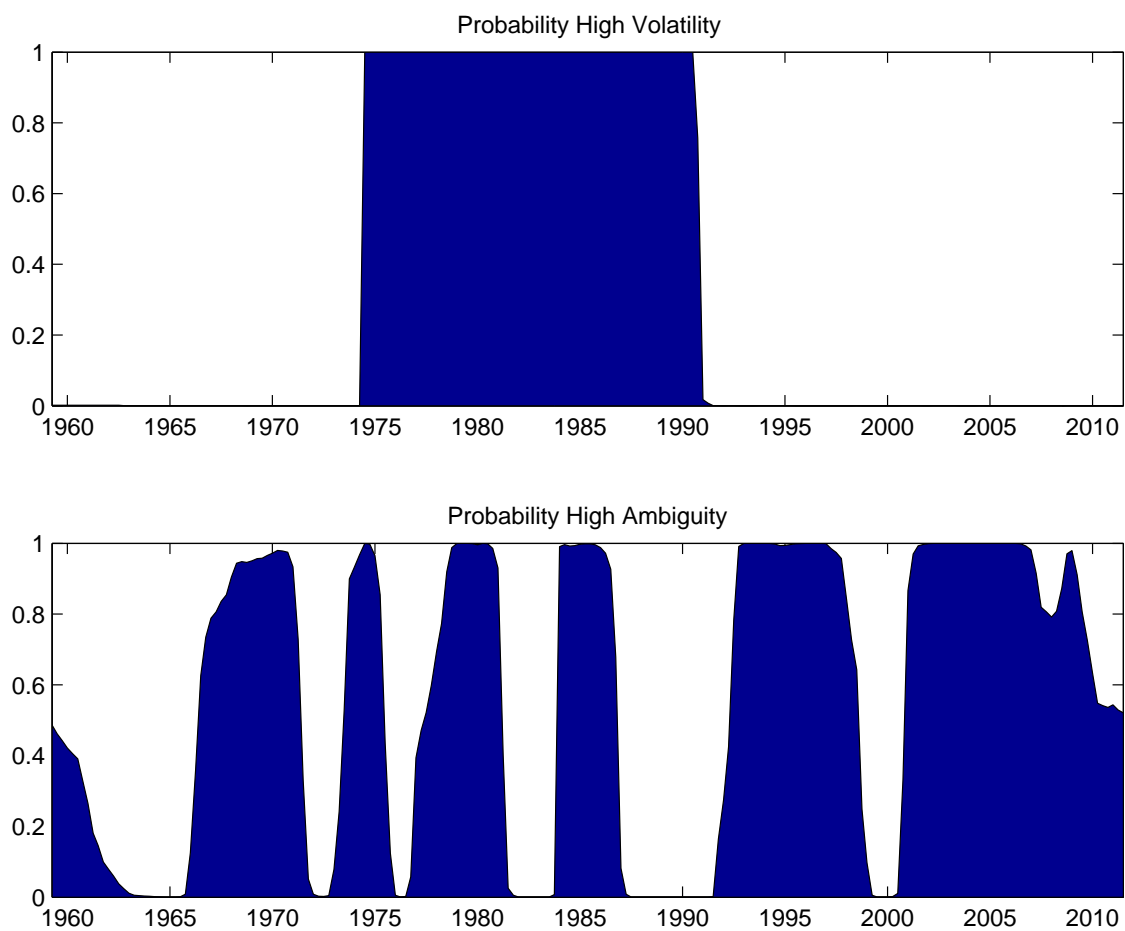


Figure 2: Smoothed regime probabilities. Top panel refers to volatility regime number 2, which we refer to as the High volatility regime. Bottom panel refers to intangible ambiguity regime number 2, which we refer to as the High ambiguity regime. Estimates for the values across regimes are shown in Table 1.

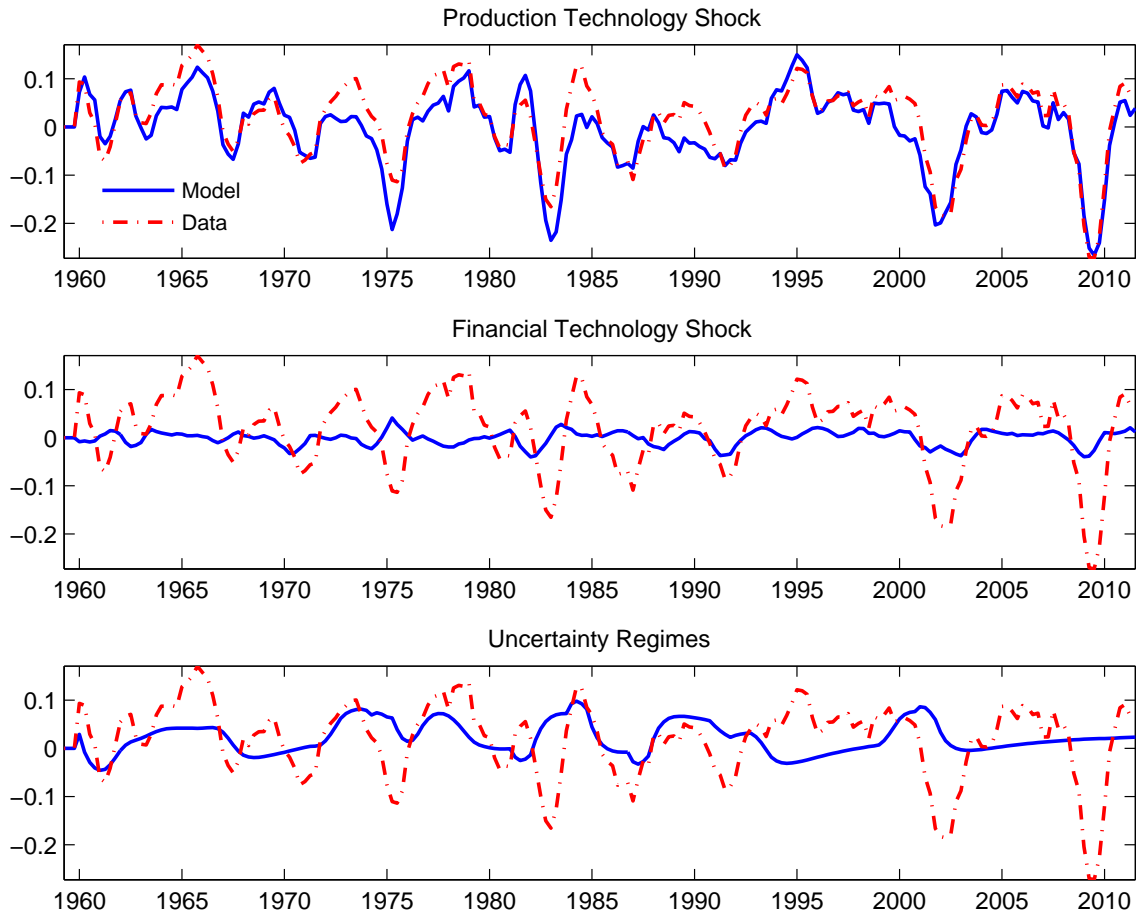


Figure 3: Contribution of different sources of variation on the year-on-year investment growth. Blue line is the counterfactual model-implied evolution based only on that source.

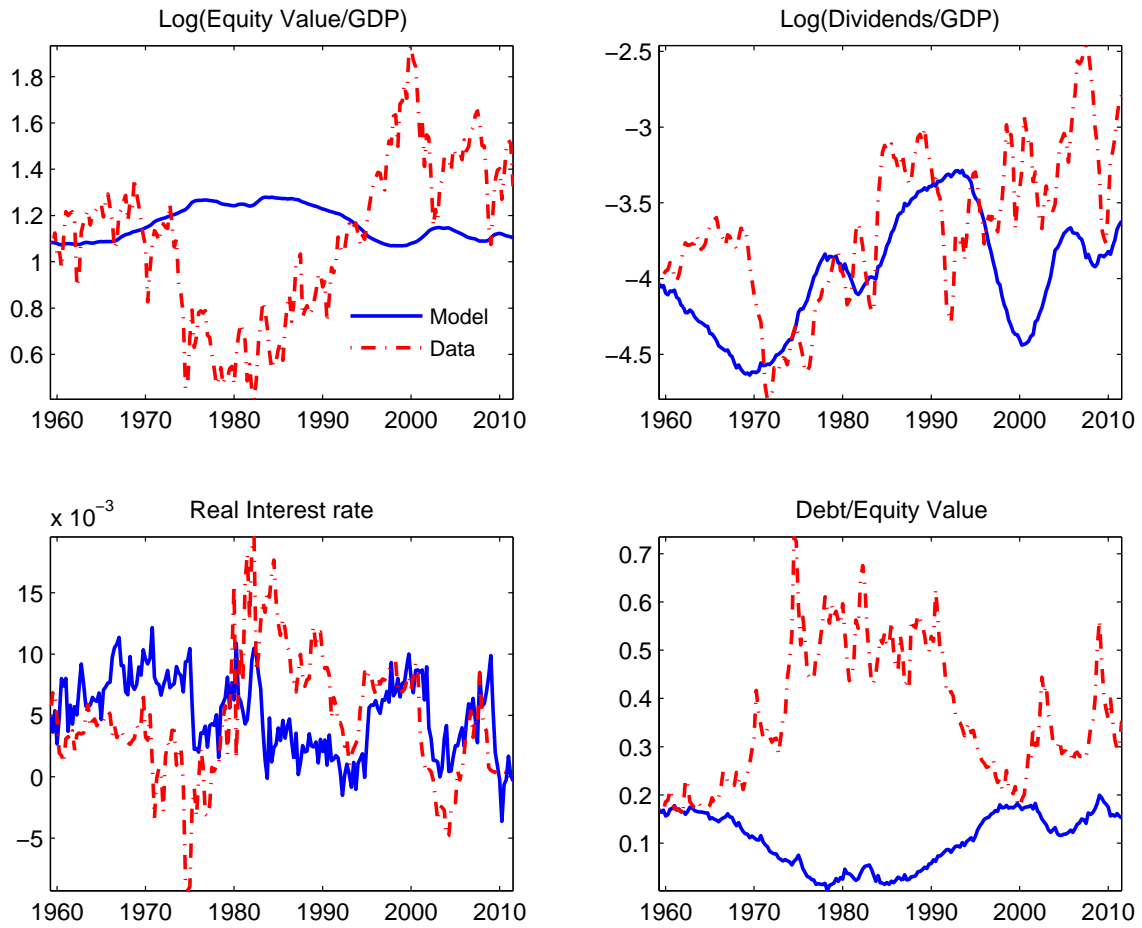


Figure 4: Contribution of the real technology shock to different financial variables. The blue line is the model-implied counterfactual evolution based only on that shock.

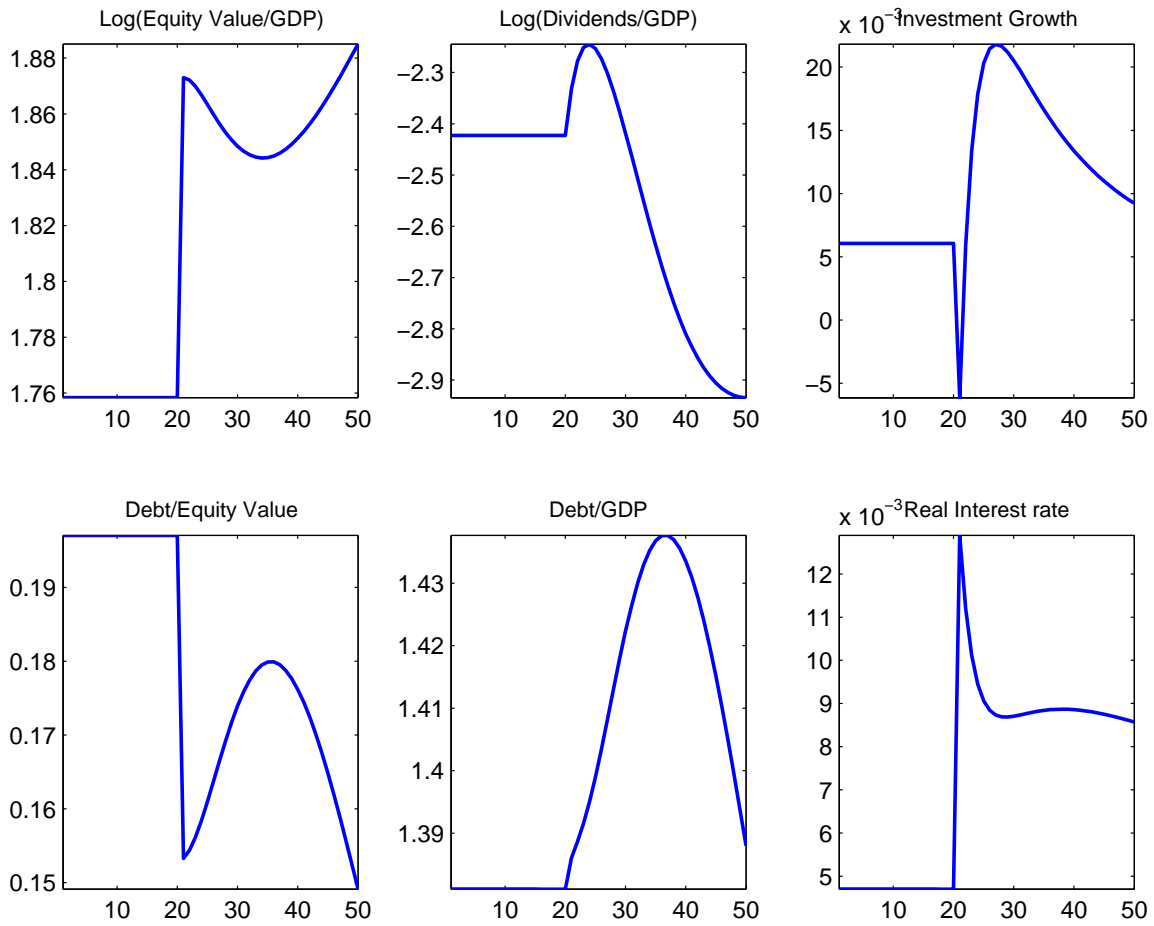


Figure 5: Impulse response for a switch from a high ambiguity-low volatility regime to a low ambiguity-low volatility regime. The switch occurs in period 20.

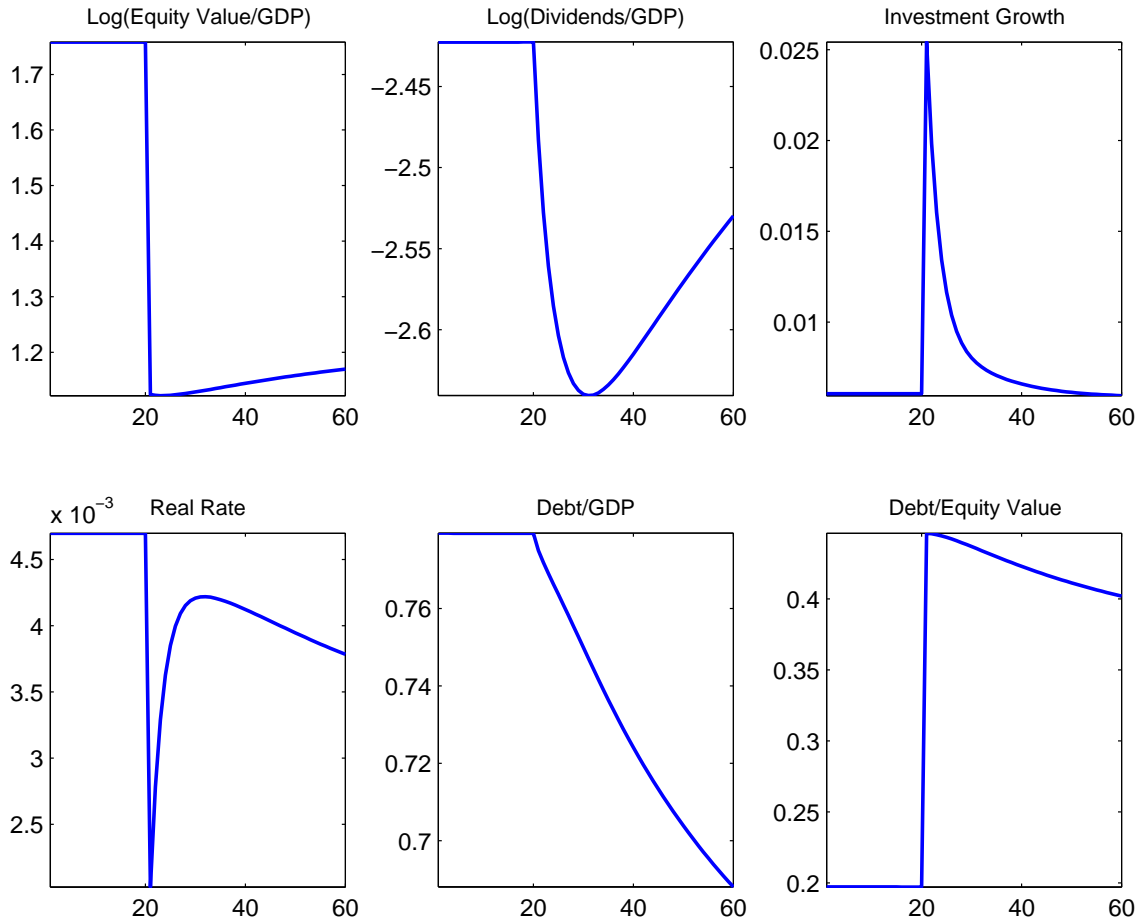


Figure 6: Impulse response for a switch from a high ambiguity-low volatility regime to a high ambiguity-high volatility regime. The switch occurs in period 20.

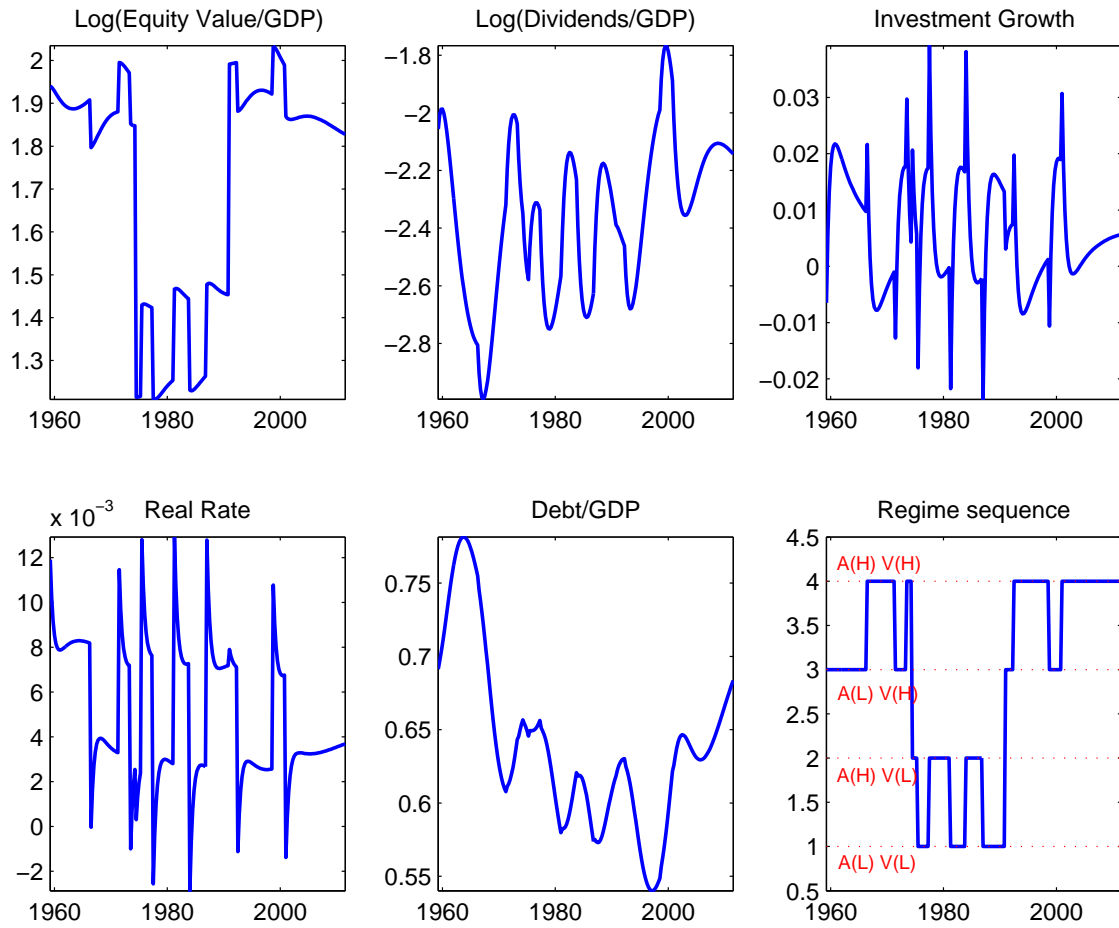


Figure 7: Evolution induced by the typical regime sequence based on the smooth probabilities of Figure 2.

| Parameter | Mode | Mean | 5% | 95% | Type | Mean | Std |
|-------------------------|--------|--------|--------|--------|-----------|--------|--------|
| $\eta_z(1)$ | 0.8176 | 0.8138 | 0.8031 | 0.8212 | <i>B</i> | 0.5 | 0.2 |
| $0.5\eta_f(1)$ | 0.0334 | 0.0294 | 0.0260 | 0.0352 | <i>B</i> | 0.5 | 0.2 |
| $\eta_z(2)$ | 0.8667 | 0.8579 | 0.8463 | 0.8677 | <i>B</i> | 0.5 | 0.2 |
| $0.5\eta_f(2)$ | 0.0995 | 0.0836 | 0.0768 | 0.0999 | <i>B</i> | 0.5 | 0.2 |
| $\sigma_z(1)$ | 0.0171 | 0.0175 | 0.0171 | 0.0178 | <i>IG</i> | 0.05 | 0.05 |
| $\sigma_f(1)$ | 0.6172 | 0.6129 | 0.6056 | 0.6198 | <i>IG</i> | 0.05 | 0.05 |
| $\sigma_z(2)$ | 0.0170 | 0.0174 | 0.0168 | 0.0178 | <i>IG</i> | 0.05 | 0.05 |
| $\sigma_f(2)$ | 1.0788 | 1.0806 | 1.0685 | 1.0962 | <i>IG</i> | 0.05 | 0.05 |
| $100f_y$ | 0.1687 | 0.1737 | 0.1638 | 0.1861 | <i>B</i> | 0.3 | 0.2 |
| ϕ'' | 0.0058 | 0.0053 | 0.0045 | 0.0061 | <i>G</i> | 0.1 | 0.08 |
| Ψ_y | 0.0027 | 0.0027 | 0.0026 | 0.0028 | <i>G</i> | 0.005 | 0.004 |
| $100(\xi - 1)$ | 0.6070 | 0.6056 | 0.5820 | 0.6265 | <i>G</i> | 0.3 | 0.05 |
| δ | 0.0193 | 0.0179 | 0.0169 | 0.0199 | <i>B</i> | 0.025 | 0.003 |
| S'' | 0.9260 | 0.9020 | 0.7420 | 0.9394 | <i>G</i> | 2 | 1 |
| α | 0.2934 | 0.2828 | 0.2770 | 0.2989 | <i>B</i> | 0.35 | 0.05 |
| $100(\beta^{-1} - 1)$ | 0.8880 | 0.9004 | 0.8636 | 0.9369 | <i>G</i> | 0.3 | 0.2 |
| ρ_z | 0.9542 | 0.9570 | 0.9534 | 0.9610 | <i>B</i> | 0.5 | 0.15 |
| ρ_f | 0.9881 | 0.9882 | 0.9875 | 0.9887 | <i>B</i> | 0.5 | 0.15 |
| ρ_a | 0.8968 | 0.9093 | 0.8901 | 0.9199 | <i>B</i> | 0.5 | 0.15 |
| H_{11}^σ | 0.9993 | 0.9995 | 0.9993 | 0.9997 | <i>D</i> | 0.9048 | 0.0626 |
| H_{22}^σ | 0.9985 | 0.9986 | 0.9983 | 0.9989 | <i>D</i> | 0.9048 | 0.0626 |
| H_{11}^η | 0.9853 | 0.9796 | 0.9636 | 0.9895 | <i>D</i> | 0.9048 | 0.0626 |
| H_{22}^η | 0.9360 | 0.9515 | 0.9308 | 0.9638 | <i>D</i> | 0.9048 | 0.0626 |
| $100\sigma_{oe,D/Y}$ | 18.07 | 17.26 | 17.10 | 18.47 | <i>IG</i> | 5.05 | 1.01 |
| $100\sigma_{oe,Debt/P}$ | 3.27 | 3.27 | 3.09 | 3.47 | <i>IG</i> | 1.39 | 0.28 |
| $100\sigma_{oe,RIR}$ | 0.45 | 0.45 | 0.42 | 0.47 | <i>IG</i> | 0.05 | 0.01 |

Table 1: Modes, means, 90% error bands, and priors of the DSGE parameters.

| τ_l | τ_k | τ_c | π_y | t_y | g |
|----------|----------|----------|---------|-------|------|
| 0.189 | 0.193 | 0.09 | 0.3 | 0.21 | 0.05 |

Table 2: Calibrated parameters