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UNCONDITIONAL BIORTHOGONAL WAVELET BASES IN $L^p(\mathbb{R}^d)$

BY

WALDEMAR POMPE (Darmstadt)

Abstract. We prove that a biorthogonal wavelet basis yields an unconditional basis in all spaces $L^p(\mathbb{R}^d)$ with $1 < p < \infty$, provided the biorthogonal wavelet set functions satisfy weak decay conditions. The biorthogonal wavelet set is associated with an arbitrary dilation matrix in any dimension.

0. Introduction. The most common and known wavelet sets are those constructed from the dilation matrix $A = 2 \cdot \mathrm{Id}$. It seems, however, that the existence of wavelet sets for an arbitrary dilation matrix A is also of practical interest. In recent years many such constructions have been found. For instance, in [CD] the multiresolution analysis is used to build bidimensional compactly supported wavelets, provided $|\det A| = 2$ and A has integer entries. In [Ma] there is a construction of a C^r multiresolution analysis for an arbitrary dilation matrix A with integer entries. This gives a $(|\det A| - 1)$-element wavelet set associated to A. Another construction of a wavelet set for any dilation matrix A is shown in [DLS], which actually gives an example of a single wavelet (i.e. the wavelet set has one element). If the dilation matrix A with integer entries has a self-affine tiling, it is possible (see [GM]) to find a wavelet set whose elements take only a finite number of values. This generalizes the classical Haar wavelet in one dimension. With the same assumption on A, the paper [Str] contains a construction of a wavelet set which is r-regular. Although there exist dilation matrices having integer entries and no self-affine tilings (see [LW1] and [LW2]), it is nonetheless possible to construct r-regular wavelets associated to those matrices (see [Bo]).

For practical reasons, biorthogonal wavelets are sometimes considered instead of orthogonal ones. The paper [LG] explains in detail why this generalization is of interest. Some examples of biorthogonal compactly supported wavelets in two dimensions (with $A \neq 2 \cdot \mathrm{Id}$) are given in [CD] and [CS].

It has been proved that, under some (decay) assumptions on the wavelet set functions, (bi)orthogonal wavelet sets yield unconditional bases in all L^p spaces ($1 < p < \infty$) (see [Gr], [Me], [W1], [W2], [FW]). However, all those

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papers consider only the case $A = 2 \cdot \mathrm{Id}$, and most of them are in dimension $d = 1$. A theorem which applies to the widest class of one-dimensional examples is given in [W2]. Its biorthogonal version has been proved in [FW]. The aim of this paper is to extend this theorem to any dimension d and any dilation matrix A, so that the resulting theorem applies to the recently constructed examples (quoted above) and proves their unconditionality in all $L^p(\mathbb{R}^d)$ spaces with $1 < p < \infty$. We will only require a weak decay condition on the wavelet set functions (condition (2.2)), similar to one used in [W2]. Although the general scheme of our proof follows in fact [W2] and [FW], we had to overcome several (technical) difficulties arising from the geometry of A as well as from the dimension d being >1. As a result, we obtain a theorem which generalizes that of [W2]. Moreover it easily applies to the examples constructed in [Bo], [GM], [Str], [CD] and [CS] and proves their unconditionality in all $L^p(\mathbb{R}^d)$ spaces, which has been an open question.

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1. Basic definitions and theorems. In order to make this paper self-contained, we recall/introduce some definitions and a theorem which we will use.

DEFINITION 1.1. A real matrix A is called a *dilation matrix* if every eigenvalue λ of A satisfies the inequality $|\lambda| > 1$.

In the theory of wavelets an additional assumption on A is usually made: The entries of A are integers. This assumption allows one to use the method of multiresolution analysis to construct examples of wavelet sets. Since we will not be using this additional condition in what follows, nor does it simplify the proof, we have not included it in the definition.

DEFINITION 1.2. Let A be a $d \times d$ dilation matrix. An n-element set of pairs of complex-valued functions $(\Psi^1(x), \Phi^1(x)), \ldots, (\Psi^n(x), \Phi^n(x))$ from $L^2(\mathbb{R}^d)$ is called a *biorthogonal wavelet set* if:

(i) the families $\{\Psi^m_{jk}\}$, $\{\Phi^m_{jk}\}$, with $j \in \mathbb{Z}$, $k \in \mathbb{Z}^d$, $m \in \{1, \ldots, n\}$, form Riesz bases in $L^2(\mathbb{R}^d)$, where

$$\Psi^m_{jk}(x) = |\det A|^{j/2} \Psi^m(A^j x - k), \qquad \Phi^m_{jk}(x) = |\det A|^{j/2} \Phi^m(A^j x - k);$$

(ii) the system $(\Psi_{jk}^m, \Phi_{jk}^m)$ is biorthogonal, that is,

$$\langle \Psi_{jk}^m, \Phi_{j'k'}^{m'} \rangle = \int_{\mathbb{R}^d} \Psi_{jk}^m(x) \Phi_{j'k'}^{m'}(x)\, dx = \begin{cases} 1 & \text{if } (j,k,m) = (j',k',m'), \\ 0 & \text{otherwise.} \end{cases}$$

If $\Psi^m = \Phi^m$ then the set $\Psi^1, \ldots, \Psi^n$ is called a *wavelet set*. In case $d = 1$, $n = 1$, $A = [2]$ the function $\psi(x) = \Psi^1(x)$ is called a *wavelet*.

DEFINITION 1.3. A biorthogonal system (x_n, x_n^*) in a Banach space X is called an *unconditional basis* if for every $x \in X$ the series $\sum_{n=1}^{\infty} \langle x, x_n^* \rangle x_n$ converges to x unconditionally in norm.

The following theorem gives a very useful characterization of an unconditional basis.

THEOREM 1.4. *Let (x_n, x_n^*) be a biorthogonal system in a Banach space X such that the set $\{x_n\}$ is linearly dense in X. Assume that there exists a constant c such that for all finite subsets $S \subset \mathbb{N}$ and $x \in X$,*

$$\Big\| \sum_{n \in S} \langle x, x_n^* \rangle x_n \Big\| \le c \|x\|.$$

Then (x_n, x_n^*) is an unconditional basis in X.

The proof can be found in [W1, p. 174, Theorem 7.7(i)].

2. The main theorem. Before we state our main theorem, we need a few definitions.

DEFINITION 2.1. Let A be a $d \times d$ dilation matrix. We will say that the set $B \subset \mathbb{R}^d$ is *A-balanced* if it is bounded, open, convex, centrally symmetric about 0 and there exists $0 < q < 1$ such that $B \subseteq qAB$.

The question arises whether for every dilation matrix A there exists an A-balanced set. The answer turns out to be affirmative and the proof follows immediately from the following theorem:

THEOREM 2.2. *Let L be a $d \times d$ matrix with real entries. Then*

$$\varrho(L) = \inf \|L\|,$$

where $\|\cdot\|$ denotes the norm of L as a linear mapping $L : \mathbb{R}^d \to \mathbb{R}^d$; the infimum is taken over all norms on $\mathbb{R}^d$, and $\varrho(L)$ denotes the spectral radius of the matrix L.

We omit the proof of this theorem; it can be found in [BL, Theorem 2.1.2, p. 33].

Applying Theorem 2.2 to $L = A^{-1}$, we infer that there exist $0 < q < 1$ and a norm $|\cdot|$ on $\mathbb{R}^d$ such that $|u| \le q|Au|$ for $u \in \mathbb{R}^d$. Then the set $B = \{x \in \mathbb{R}^d : |x| < 1\}$ is A-balanced.

We will use the following notation: If B is an A-balanced set, then we put

$$B_j = A^{j+1}B \setminus A^j B \quad \text{for } j \in \mathbb{N}_0 := \{0, 1, 2, \ldots\}.$$

Since $B \subset AB$, the sets B_j are disjoint. From the assumption $B \subseteq qAB$ it follows that $q^{-j}B \subseteq A^j B$ for $j \geq 0$, which gives $\bigcup_{j\in\mathbb{N}_0} B_j = \mathbb{R}^d \setminus B$.

Definition 2.3. Let A be a $d \times d$ dilation matrix. We will say that the function $\varphi : \mathbb{R}^d \to \mathbb{R}$ is A-*radial* if it is integrable, non-negative and for some A-balanced set B, the function φ is constant on the sets $B, B_0, B_1, B_2, \ldots$ and non-increasing in the following sense:

$$\text{if} \quad x \in \mathbb{R}^d \text{ and } i \geq j, \quad \text{then} \quad \varphi(A^i x) \leq \varphi(A^j x). \tag{2.1}$$

Now we are ready to formulate the main theorem of this paper.

Theorem 2.4. *Let* $(\Psi^1(x), \Phi^1(x)), \ldots, (\Psi^n(x), \Phi^n(x))$ *be a biorthogonal wavelet set in* $L^2(\mathbb{R}^d)$ *associated with a dilation matrix* A. *Let* φ *be an* A-*radial function satisfying*

$$\int_{\mathbb{R}^d} \varphi(u) \ln(|u|_e + 1)\, du < \infty, \tag{2.2}$$

where $|u|_e^2 = u_1^2 + \ldots + u_d^2$ *for* $u = (u_1, \ldots, u_d)$. *If* $|\Psi^m(x)| \leq \varphi(x)$ *and* $|\Phi^m(x)| \leq \varphi(x)$ *for* $x \in \mathbb{R}^d$ *and every* $m \in \{1, \ldots, n\}$, *then the biorthogonal system* $(\Psi^m_{jk}, \Phi^m_{jk})$ *form an unconditional basis in all* $L^p(\mathbb{R}^d)$ *spaces with* $1 < p < \infty$.

Remarks. Since all norms in $\mathbb{R}^d$ are equivalent, the condition (2.2) is equivalent to

$$\int_{\mathbb{R}^d} \varphi(u) \ln(|u| + 1)\, du < \infty, \tag{2.3}$$

where $|\cdot|$ is an arbitrary norm of $\mathbb{R}^d$. The function φ is bounded and integrable on $\mathbb{R}^d$, so the condition (2.3) can also be restated in the form

$$\int_{\mathbb{R}^d \setminus B'} \varphi(u)(1 + \ln|u|)\, du < \infty, \tag{2.4}$$

where $B' = \{u \in \mathbb{R}^d : |u| < 1\}$.

The size of the A-balanced set B associated with φ depends on the given A-radial function φ. However, redefining $\varphi(x)$ to be $\varphi(A^s x)$ and taking s sufficiently small, we may assume that the A-balanced set B associated with φ is arbitrarily large. In particular, we will assume that

$$\bigcup_{k\in\mathbb{Z}^d} (B + k) = \mathbb{R}^d. \tag{2.5}$$

3. Proof of Theorem 2.4. In what follows, we use the following notation. For an A-balanced set B, we put

$$B(y,r) = A^r B + y \quad (y \in \mathbb{R}^d,\ r \in \mathbb{Z}).$$

If $x \in \mathbb{R}^d$, we set $|x| = |x|_B = \inf\{t > 0 : t^{-1}x \in B\}$. Then $|\cdot|$ is a norm in $\mathbb{R}^d$. For a measurable set $X \subset \mathbb{R}^d$, $|X|$ denotes the Lebesgue measure of X.

LEMMA 3.1. *Let B be an A-balanced set with $B \subseteq qAB$ $(0 < q < 1)$. Then there exists a positive integer ℓ such that for all $y, \overline{y} \in \mathbb{R}^d$ and $r \in \mathbb{Z}$ the following condition holds*:

$$\text{(3.1)} \qquad \text{if} \quad B(y,r) \cap B(\overline{y},r) \neq \emptyset, \quad \text{then} \quad B(y,r) \subset B(\overline{y}, r+\ell).$$

Moreover, the number ℓ depends only on q.

Proof. Define ℓ to be a positive integer such that $q^\ell \le 1/3$. Since $B(y,r) \cap B(\overline{y},r) \neq \emptyset$, we obtain $(B + A^{-r}(y-\overline{y})) \cap B \neq \emptyset$. Therefore $B + A^{-r}(y-\overline{y}) \subset 3B \subseteq q^{-\ell}B \subseteq A^\ell B$. This gives the inclusion $B(y,r) \subset B(\overline{y}, r+\ell)$.

In this paper, ℓ always denotes the positive integer satisfying (3.1).

THEOREM 3.2 (Calderón–Zygmund decomposition). *Let B be a fixed A-balanced set. Then for every $f \in L^1(\mathbb{R}^d)$ and $\lambda > 0$, there exists an at most countable set $\mathbb{I} \subset \mathbb{R}^d \times \mathbb{Z}$ and a constant $c > 0$, independent of f and λ, such that*:

(i) $|B(y,r)|^{-1} \int_{B(y,r)} |f(t)|\,dt \le c\lambda$ *for* $(y,r) \in \mathbb{I}$,

(ii) $|f(x)| \le \lambda$ *almost everywhere on* $\mathbb{R}^d \setminus \bigcup_{(y,r)\in\mathbb{I}} B(y,r)$,

(iii) $\sum_{(y,r)\in\mathbb{I}} |B(y,r)| \le (c/\lambda)\|f\|_1$.

Moreover, there exists a positive integer κ, independent of f and λ, such that for $(y,r) \in \mathbb{I}$ the sets $B(y, r-\kappa)$ are disjoint.

Proof. For $y \in \mathbb{R}^d$ and $\delta > 0$ set $\mathcal{B}(y,\delta) = B(y, [\log_2 \delta])$. Then the following conditions hold:

(a) $\mathcal{B}(y,\delta_1) \subseteq \mathcal{B}(y,\delta_2)$ for $\delta_1 < \delta_2$;

(b) $\bigcup_{\delta>0} \mathcal{B}(y,\delta) = \mathbb{R}^d$, $\bigcap_{\delta>0} \mathcal{B}(y,\delta) = \{y\}$;

(c) $\mathcal{B}(y,\delta) \cap \mathcal{B}(\overline{y},\delta) \neq \emptyset \ \Rightarrow\ \mathcal{B}(y,\delta) \subset \mathcal{B}(\overline{y}, 2^\ell\delta)$;

(d) $|\mathcal{B}(y,2\delta)| = |\det A| \cdot |\mathcal{B}(y,\delta)|$;

(e) for a fixed measurable set U and $\delta > 0$ the function $y \mapsto |\mathcal{B}(y,\delta) \cap U|$ is continuous.

Indeed, (a) follows immediately from $B \subset AB$; the equalities in (b) are simple consequences of

$$q^{-r}B + y \subseteq A^r B + y = B(y,r), \quad B(y,-r) = A^{-r}B + y \subseteq q^r B + y \quad (r > 0),$$

respectively. Part (c) is just a rewording of Lemma 3.1; (d) follows at once from the definitions of the sets $B(y,r)$ and $\mathcal{B}(y,\delta)$. To see (e) note that

$$|\mathcal{B}(y,\delta)\cap U| = \mathbf{1}_{\mathcal{B}(0,\delta)} * \mathbf{1}_U(y),$$

which is clearly continuous.

Therefore for the sets $\mathcal{B}(y,\delta)$ with $y\in\mathbb{R}^d$ and $\delta>0$ the conditions (i)–(iv) of [St, pp. 8–9] are satisfied. Thus the assertion of the theorem follows immediately from [St, Lemma 2 (p. 15), Theorem 2 (p. 17) and the construction in the proof of Theorem 2 (pp. 17–18)]. (We remark that taking a closer look at the proof of [St, Lemma 2 (p. 15)] one may put $\kappa = 2\ell$.)

Lemma 3.3. *For every A-radial function β and every $j\ge 0$, there exists a constant $c=c(\beta,j)$ such that for every $x\in\mathbb{R}^d$,*

$$\sum_{k\in\mathbb{Z}^d}\beta(x-A^{-j}k)\le c.$$

Proof. Let B be the A-balanced set associated with β. Fix $x\in\mathbb{R}^d$ and choose $0<t<1$ such that the sets $C_k = tB-(A^jx-k)$ are pairwise disjoint. By Lemma 3.1, each $A^{-j}C_k$ may intersect at most $\ell+1$ different sets among $B, B_0, B_1, B_2, \ldots$, whence taking into account the form of β, we have the estimate

$$\beta(x-A^{-j}k)\le \inf_{u\in A^{-(j+\ell)}C_k}\beta(u)\quad \text{for } k\in\mathbb{Z}^d.$$

The sets $A^{-(j+\ell)}C_k$ are disjoint, so

$$\begin{aligned}|tA^{-(j+\ell)}B|\sum_{k\in\mathbb{Z}^d}\beta(x-A^{-j}k) &\le \sum_{k\in\mathbb{Z}^d}|A^{-(j+\ell)}C_k|\inf_{u\in A^{-(j+\ell)}C_k}\beta(u)\\ &\le \int_{\mathbb{R}^d}\beta(u)\,du,\end{aligned}$$

which completes the proof.

Lemma 3.4. *Let $\beta(x)$ be an A-radial function and put $\widetilde{\beta}(x)=\beta(x/2)$. Then for all $t,u\in\mathbb{R}^d$ the following inequality holds*:

$$\sum_{k\in\mathbb{Z}^d}\beta(t-k)\beta(u-k)\le c\widetilde{\beta}(t-u),$$

where c does not depend on t and u.

Proof. Let B be the A-balanced set associated with β. Let p be the largest integer such that the set $B(t,p)$ does not contain the vector $\frac{1}{2}(t+u)$. Since the sets $B(y,r)$ have centers of symmetry, p is the largest integer such that $B(u,p)$ does not contain $\frac{1}{2}(t+u)$. The sets $B(y,r)$ are also convex, so there exists a hyperplane π_1 of dimension $d-1$ separating $B(t,p)$ and $B(u,p)$. Let π_2 be the hyperplane symmetrical to π_1 with respect to the point

$\frac{1}{2}(t+u)$; then π_2 is parallel to π_1 and also separates $B(t,p)$ and $B(u,p)$. Therefore the hyperplane parallel to π_1 and π_2, passing through $\frac{1}{2}(t+u)$, separates $B(t,p)$, $B(u,p)$ as well and divides $\mathbb{R}^d$ into two disjoint half-spaces H_1 and H_2 with $B(t,p) \subset H_1$, $B(u,p) \subset H_2$. Then

$$\begin{aligned} \beta(u-k) &\le \beta\big(u - \tfrac{1}{2}(t+u)\big) = \beta\big(\tfrac{1}{2}(t-u)\big) \quad \text{for } k \in H_1, \\ \beta(t-k) &\le \beta\big(t - \tfrac{1}{2}(t+u)\big) = \beta\big(\tfrac{1}{2}(t-u)\big) \quad \text{for } k \in H_2. \end{aligned}$$

Thus we obtain

$$\begin{aligned} \sum_{k\in\mathbb{Z}^d} \beta(t-k)\beta(u-k) &\le \sum_{k\in H_1} \beta(t-k)\beta\big(\tfrac{1}{2}(t-u)\big) + \sum_{k\in H_2} \beta\big(\tfrac{1}{2}(t-u)\big)\beta(u-k) \\ &\le 2c\widetilde{\beta}(t-u), \end{aligned}$$

where c is the constant from Lemma 3.3.

From now on, let B denote the A-balanced set associated with φ. We set $\widetilde{B}_j = 2B_j$ for $j \in \mathbb{N}_0$, $\widetilde{B} = 2B$, $\widetilde{\varphi}(u) = \varphi(u/2)$ for $u \in \mathbb{R}^d$ and

$$\widetilde{\eta}(u) = \mathbf{1}_{\mathbb{R}^d \setminus \widetilde{B}}(u) \sum_{j\ge 0} |\det A|^j \widetilde{\varphi}(A^j u).$$

We show below (Lemma 3.5) that the above series is convergent for $u \notin \widetilde{B}$, so the above definition makes sense.

Since the function $\widetilde{\varphi}(u)$ is constant on the sets $\widetilde{B}$, $\widetilde{B}_j$ $(j \in \mathbb{N}_0)$, the function $\widetilde{\eta}(u)$ is also constant on these sets and satisfies the condition analogous to (2.1):

$$\text{if} \quad x \in \mathbb{R}^d \setminus \widetilde{B} \text{ and } i \ge j \ge 0, \quad \text{then} \quad \widetilde{\eta}(A^i x) \le \widetilde{\eta}(A^j x).$$

Lemma 3.5. $\widetilde{\eta} \in L^1(\mathbb{R}^d)$.

Proof. Let $\eta(u) = \widetilde{\eta}(2u)$ for $u \in \mathbb{R}^d$. We show that $\eta \in L^1(\mathbb{R}^d)$. We have

$$\eta(u) = \mathbf{1}_{\mathbb{R}^d \setminus B}(u) \sum_{j\ge 0} |\det A|^j \varphi(A^j u).$$

Fix $j \ge 0$ and assume that $u \in B_j$. Then $A^j t = u$ for some $t \in B_0$. Since $B \subseteq qAB$,

$$\|A^{-1}\| = \sup_{|x|=1} |A^{-1}x| \le q < 1,$$

whence we get $1 \le |t| = |A^{-j}A^j t| \le \|A^{-1}\|^j \cdot |A^j t| \le q^j |u|$.

Thus we have shown that there exists a constant $c > 0$ such that if $u \in B_j$ (for some $j \ge 0$), then $e^{cj} \le |u|$. Therefore

$$\int_{\mathbb{R}^d} \eta(u)\,du = \sum_{j\ge 0} \int_{\mathbb{R}^d\setminus B} |\det A|^j \varphi(A^j u)\,du = \sum_{j\ge 0} \int_{\mathbb{R}^d\setminus A^j B} \varphi(u)\,du$$
$$= \sum_{j\ge 0} \int_{B_j} (j+1)\varphi(u)\,du \le c^{-1} \sum_{j\ge 0} \int_{B_j} \varphi(u)(1+\ln|u|)\,du$$
$$= c^{-1} \int_{\mathbb{R}^d\setminus B} \varphi(u)(1+\ln|u|)\,du,$$

which is finite according to (2.4).

Let S be an arbitrary subset of $\{1,\dots,n\}\times\mathbb{Z}\times\mathbb{Z}^d$. Define

$$U_S f = \sum_{(m,j,k)\in S} \langle f, \Phi^m_{jk}\rangle \Psi^m_{jk}. \tag{3.2}$$

Then for every finite set S the operator U_S is bounded on $L^p(\mathbb{R}^d)$ where $1\le p\le\infty$. If S is an arbitrary set, not necessarily finite, then the above series, understood as a limit in $L^2(\mathbb{R}^d)$, is well defined and it follows from Definition 1.2 that $\|U_S f\|_2 \le c\|f\|_2$ for $f\in L^2(\mathbb{R}^d)$. Here c is a constant independent of S.

As a special case, fix $r\in\mathbb{Z}$ and consider

$$S(r) = \{(m,j,k)\in\{1,\dots,n\}\times\mathbb{Z}\times\mathbb{Z}^d : j<-r\},$$
$$T(r) = \{(m,j,k)\in\{1,\dots,n\}\times\mathbb{Z}\times\mathbb{Z}^d : j\ge -r\}.$$

Set $P_r f = U_{S(r)} f$, $Q_r f = U_{T(r)} f$. Then $P_r f + Q_r f = f$ for $f\in L^2(\mathbb{R}^d)$.

If $y\in\mathbb{R}^d$, $r\in\mathbb{Z}$, then we define

$$B^*(y,r) = B(y,r) + 2B(0,r) = \{x+2t : x\in B(y,r),\ t\in B(0,r)\}$$

and analogously $B^* = B + 2B$ $(= B + \widetilde{B})$.

LEMMA 3.6. *There exists an A-radial function α such that for all functions $f\in L^1(\mathbb{R}^d)\cap L^2(\mathbb{R}^d)$ satisfying $f(u)=0$ for $u\notin B(y,r)$, and every $S\subseteq\{1,\dots,n\}\times\mathbb{Z}\times\mathbb{Z}^d$,*

$$|U_S Q_r f(x)| \le \|f\|_1\cdot|\det A|^{-r}\alpha(A^{-r}x - A^{-r}y) \quad \textit{for } x\in\mathbb{R}^d\setminus B^*(y,r), \tag{3.3}$$

$$|P_r f(x)| \le \|f\|_1\cdot|\det A|^{-r}\alpha(A^{-r}x - A^{-r}y) \quad \textit{for } x\in\mathbb{R}^d. \tag{3.4}$$

Moreover, α is constant on the sets $A^\ell\widetilde{B}, \widetilde{B}_\ell, \widetilde{B}_{\ell+1}, \widetilde{B}_{\ell+2}, \dots$

Proof. Using the substitution formula for integration, it suffices to prove (3.3) for $r=0$, $y=0$. Since $f\in L^1(\mathbb{R}^d)\cap L^2(\mathbb{R}^d)$,

$$|U_S Q_0 f(x)| \le \int_{\mathbb{R}^d} |f(t)|\cdot K(x,t)\,dt \quad (x\in\mathbb{R}^d),$$

where

$$K(x,t) = \sum_{m=1}^{n} \sum_{j\geq 0} |\det A|^j \sum_{k\in\mathbb{Z}^d} |\Phi^m(A^j x - k)\Psi^m(A^j t - k)|.$$

Assume that $x \notin B^*$ and $t \in B$. Then $x - t \notin \widetilde{B}$, which by Lemma 3.4 gives

$$\begin{aligned} K(x,t) &\leq n \sum_{j\geq 0} |\det A|^j \sum_{k\in\mathbb{Z}^d} \varphi(A^j x - k)\varphi(A^j t - k) \\ &\leq 2cn \sum_{j\geq 0} |\det A|^j \widetilde{\varphi}(A^j(x-t)) = 2cn\widetilde{\eta}(x-t), \end{aligned}$$

where c is the constant from Lemma 3.3.

Define

$$\alpha(x) = \begin{cases} 2cn\beta & \text{for } x \in A^\ell \widetilde{B}, \\ 2cn\widetilde{\eta}(A^{-\ell}x) & \text{otherwise}, \end{cases}$$

where

$$\beta = \max\left(\text{the value of } \widetilde{\eta} \text{ on } \widetilde{B}_0; \frac{\widetilde{\varphi}(0)}{|\det A| - 1}\right).$$

Then α is an A-radial function. By Lemma 3.1, each of the sets

$$x + B = \{x + u : u \in B\}, \quad \text{where } x \in \mathbb{R}^d,$$

may intersect at most $\ell+1$ of the sets $\widetilde{B}, \widetilde{B}_0, \widetilde{B}_1, \widetilde{B}_2, \ldots$ Therefore we obtain

$$2cn \sup_{t\in x+B} \widetilde{\eta}(t) \leq \alpha(x) \quad \text{for } x \in \mathbb{R}^d.$$

Moreover $B^* \subset A^\ell B \subset A^\ell \widetilde{B}$, whence $\alpha(x) = 2cn\beta$ for $x \in B^*$.

We know that $f(u) = 0$ for $u \notin B$, so if $x \notin B^*$ we get

$$\begin{aligned} |U_S Q_0 f(x)| &\leq \int_B |f(t)| \cdot K(x,t)\,dt \leq 2cn \int_{\mathbb{R}^d} |f(t)|\widetilde{\eta}(x-t)\,dt \\ &\leq \|f\|_1 \cdot 2cn \sup_{t\in x+B} \widetilde{\eta}(t) \leq \|f\|_1 \cdot \alpha(x), \end{aligned}$$

which completes the proof of (3.3).

To prove (3.4), it suffices to take $r = 0$, $y = 0$. Using (3.3) for $S = \{1,\ldots,n\} \times \mathbb{Z} \times \mathbb{Z}^d$, $r = 0$, $y = 0$ and recalling that $f(u) = 0$ for $u \notin B$ we obtain

$$|P_0 f(x)| = |Q_0 f(x)| \leq \|f\|_1 \cdot \alpha(x) \quad \text{for } x \notin B^*.$$

For $x \in \mathbb{R}^d$ we have

$$|P_0 f(x)| \leq \int_{\mathbb{R}^d} |f(t)| \cdot K(x,t)\,dt,$$

where

$$K(x,t)=\sum_{m=1}^{n}\sum_{j<0}|\det A|^{j}\sum_{k\in\mathbb{Z}^d}|\Phi^m(A^jx-k)\Psi^m(A^jt-k)|.$$

Thus

$$\begin{aligned}K(x,t)&\le n\sum_{j<0}|\det A|^{j}\sum_{k\in\mathbb{Z}^d}\varphi(A^jx-k)\varphi(A^jt-k)\\&\le 2cn\sum_{j<0}|\det A|^{j}\widetilde{\varphi}(A^j(x-t))\le 2cn\frac{\widetilde{\varphi}(0)}{|\det A|-1}\le 2cn\beta,\end{aligned}$$

whence $|P_0f(x)|\le\|f\|_1\cdot\alpha(x)$ for $x\in B^*$, which completes the proof of (3.4).

THEOREM 3.7. *If S is a finite subset of $\{1,\ldots,n\}\times\mathbb{Z}\times\mathbb{Z}^d$, then the operators U_S defined by* (3.2) *are of weak type* $(1,1)$ *with the weak-type constant independent of S; i.e. there exists a constant c such that for every finite set S and every $f\in L^1(\mathbb{R}^d)$,*

$$|\{x\in\mathbb{R}^d:|U_Sf(x)|>\lambda\}|\le\frac{c}{\lambda}\|f\|_1.$$

Proof. Fix $\lambda>0$ and assume first that $f\in L^1(\mathbb{R}^d)\cap L^2(\mathbb{R}^d)$. We find an at most countable subset $\mathbb{I}\subset\mathbb{R}^d\times\mathbb{Z}$, sets $B(y,r)$ ($(y,r)\in\mathbb{I}$) and a positive integer κ such that the conclusion of Theorem 3.2 is satisfied. Then there exist disjoint measurable sets $R(y,r)$ such that $B(y,r-\kappa)\subseteq R(y,r)\subseteq B(y,r)$ for $(y,r)\in\mathbb{I}$ and

$$\bigcup_{(y,r)\in\mathbb{I}}R(y,r)=\bigcup_{(y,r)\in\mathbb{I}}B(y,r).$$

Put

$$f_{yr}=f\cdot\mathbf{1}_{R(y,r)},\qquad F=\mathbb{R}^d\setminus\bigcup_{(y,r)\in\mathbb{I}}R(y,r).$$

Let moreover $(\mathbb{I}_N)_{N\in\mathbb{N}}$ be a non-decreasing family of finite subsets of $\mathbb{I}$ such that $\bigcup_{N\in\mathbb{N}}\mathbb{I}_N=\mathbb{I}$. Set

$$f_N=f\cdot\mathbf{1}_F+\sum_{(y,r)\in\mathbb{I}_N}f_{yr}=f\cdot\mathbf{1}_F+\sum_{(y,r)\in\mathbb{I}_N}Q_rf_{yr}+\sum_{(y,r)\in\mathbb{I}_N}P_rf_{yr}=:g_1+g_2+g_3.$$

Since U_S is linear,

$$\{x\in\mathbb{R}^d:|U_Sf_N(x)|>\lambda/4\}\subseteq\bigcup_{j=1}^{3}\{x\in\mathbb{R}^d:|U_Sg_j(x)|>\lambda/12\},$$

whence

$$|\{x\in\mathbb{R}^d:|U_Sf_N(x)|>\lambda/4\}|\le\sum_{j=1}^{3}|\{x\in\mathbb{R}^d:|U_Sg_j(x)|>\lambda/12\}|.\tag{3.5}$$

Using Chebyshev's inequality, the inequality $\|U_Sg_1\|_2\le c\|g_1\|_2$ and Theorem 3.2(ii) we obtain

$$(3.6)\qquad |\{x\in\mathbb{R}^d : |U_S g_1(x)| > \lambda/12\}| \le \frac{144}{\lambda^2}\int_{\mathbb{R}^d} |U_S g_1(x)|^2\,dx \le \frac{c}{\lambda^2}\int_F |f(x)|^2\,dx \le \frac{c}{\lambda}\int_F |f(x)|\,dx \le \frac{c}{\lambda}\|f\|_1.$$

Set $F^* = \mathbb{R}^d \setminus \bigcup_{(y,r)\in\mathbb{I}} B^*(y,r)$. From Lemma 3.6 we have

$$\int_{F^*}\Big|U_S \sum_{(y,r)\in\mathbb{I}_N} Q_r f_{yr}(x)\Big|\,dx \le \sum_{(y,r)\in\mathbb{I}_N}\ \int_{\mathbb{R}^d\setminus B^*(y,r)} |U_S Q_r f_{yr}(x)|\,dx \le \sum_{(y,r)\in\mathbb{I}_N}\|f_{yr}\|_1 \int_{\mathbb{R}^d\setminus B^*(y,r)} |\det A|^{-r}\alpha(A^{-r}x - A^{-r}y)\,dx \le \sum_{(y,r)\in\mathbb{I}_N}\|f_{yr}\|_1 \int_{\mathbb{R}^d}\alpha(x)\,dx \le c\|f\|_1,$$

whence from Chebyshev's inequality we get

$$(3.7)\qquad |\{x\in F^* : |U_S g_2(x)| > \lambda/12\}| \le \frac{12}{\lambda}\int_{F^*}|U_S g_2(x)|\,dx \le \frac{c}{\lambda}\|f\|_1.$$

Since $|B^*(y,r)| = 3^d|B(y,r)|$, using Theorem 3.2(iii) we obtain

$$(3.8)\qquad |\mathbb{R}^d\setminus F^*| \le \sum_{(y,r)\in\mathbb{I}}|B^*(y,r)| = 3^d\sum_{(y,r)\in\mathbb{I}}|B(y,r)| \le \frac{c}{\lambda}\|f\|_1.$$

Inequalities (3.7) and (3.8) give

$$(3.9)\qquad |\{x\in\mathbb{R}^d : |U_S g_2(x)| > \lambda/12\}| \le |\mathbb{R}^d\setminus F^*| + |\{x\in F^* : |U_S g_2(x)| > \lambda/12\}| \le \frac{c}{\lambda}\|f\|_1.$$

We now prove that

$$(3.10)\qquad \Big\|\sum_{(y,r)\in\mathbb{I}_N} P_r f_{yr}\Big\|_2^2 \le c\lambda\|f\|_1,$$

where c is a constant independent of f, λ and N. Using (3.4) and Theorem 3.2(i), we obtain

$$\Big\|\sum_{(y,r)\in\mathbb{I}_N} P_r f_{yr}\Big\|_2^2 \le \sum_{(y',r')\in\mathbb{I}_N}\ \sum_{(y,r)\in\mathbb{I}_N} |\langle P_{r'} f_{y'r'}, P_r f_{yr}\rangle| \le 2\sum_{(y',r')\in\mathbb{I}_N}\ \sum_{\substack{(y,r)\in\mathbb{I}_N\\ r\le r'}} |\langle P_{r'} f_{y'r'}, P_r f_{yr}\rangle|$$

$$\leq 2 \sum_{(y',r')\in\mathbb{I}_N} \sum_{\substack{(y,r)\in\mathbb{I}_N\\ r\leq r'}} \int_{\mathbb{R}^d} |P_{r'} f_{y'r'}(x)| \cdot |P_r f_{yr}(x)|\, dx$$

$$\leq 2 \sum_{(y',r')\in\mathbb{I}_N} \|f_{y'r'}\|_1 \cdot |\det A|^{-r'} \sum_{\substack{(y,r)\in\mathbb{I}_N\\ r\leq r'}} \|f_{yr}\|_1 \cdot |\det A|^{-r}$$

$$\times \int_{\mathbb{R}^d} \alpha(A^{-r'}x - A^{-r'}y')\alpha(A^{-r}x - A^{-r}y)\, dx$$

$$\leq c\lambda \sum_{(y',r')\in\mathbb{I}_N} \|f_{y'r'}\|_1 \cdot |\det A|^{-r'}$$

$$\times \sum_{\substack{(y,r)\in\mathbb{I}_N\\ r\leq r'}} \int_{\mathbb{R}^d} \alpha(A^{-r'}x - A^{-r'}y')\alpha(A^{-r}x - A^{-r}y)\, dx = I_1$$

Substituting $u = A^{-r'}x - A^{-r'}y'$ we get

$$I_1 \leq c\lambda \sum_{(y',r')\in\mathbb{I}_N} \|f_{y'r'}\|_1 \sum_{\substack{(y,r)\in\mathbb{I}_N\\ r\leq r'}} \int_{\mathbb{R}^d} \alpha(u)\alpha(A^{r'-r}u - A^{-r}(y-y'))\, du.$$

To complete the proof of (3.10) it suffices to show that there exists a constant c, independent of y', r' and N, such that

$$\sum_{\substack{(y,r)\in\mathbb{I}_N\\ r\leq r'}} \int_{\mathbb{R}^d} \alpha(u)\alpha(A^{r'-r}u - A^{-r}(y-y'))\, du \leq c. \tag{3.11}$$

Put $z = A^{-r'}(y-y')$, $s = r - r'$. With this notation, we say that $(z,s) \in \mathbb{J}_N$ if and only if $(y,r) \in \mathbb{I}_N$. In other words, the sets $B(y,r)$ with $(y,r) \in \mathbb{I}_N$ are the images of the sets $B(z,s)$ with $(z,s) \in \mathbb{J}_N$ under the mapping $x \mapsto A^{r'}x + y'$. Now (3.11) may be rewritten as

$$\sum_{\substack{(z,s)\in\mathbb{J}_N\\ s\leq 0}} \int_{\mathbb{R}^d} \alpha(u)\alpha(A^{-s}u - A^{-s}z)\, du \leq c. \tag{3.12}$$

For $k \in \mathbb{Z}^d$, let L_k be the set of those $(z,s) \in \mathbb{J}_N$ such that $s \leq 0$ and $\widetilde{B}(z,s)$ intersects $\widetilde{B} + k$. Then

$$\begin{aligned}\sum_{(z,s)\in L_k} |\widetilde{B}(z,s)| &= |\det A|^{\kappa} \sum_{(z,s)\in L_k} |\widetilde{B}(z,s-\kappa)| \leq |\det A|^{\kappa} \cdot |A^{\ell}\widetilde{B} + k| \\ &= |\det A|^{\kappa+\ell} \cdot |\widetilde{B}| \leq c,\end{aligned} \tag{3.13}$$

where c does not depend on k. To prove (3.12), it is enough, in view of (2.5), to show that

$$I_2 + I_3 := \sum_{k\in\mathbb{Z}^d} \sum_{(z,s)\in L_k} \int_{A^{2\ell}\widetilde{B}+k} \alpha(u)\alpha(A^{-s}u - A^{-s}z)\,du$$
$$+ \sum_{k\in\mathbb{Z}^d} \sum_{(z,s)\in L_k} \int_{\mathbb{R}^d\setminus(A^{2\ell}\widetilde{B}+k)} \alpha(u)\alpha(A^{-s}u - A^{-s}z)\,du \le c.$$

We first estimate I_2. By Lemma 3.1, each $A^{2\ell}\widetilde{B}+k$ may intersect at most $2\ell+1$ of the sets $A^\ell\widetilde{B}$, $\widetilde{B}_{\ell+1}$, $\widetilde{B}_{\ell+2}$, ... Therefore, taking into account the form of the function α developed in Lemma 3.6, we get

$$\alpha(u) \le \alpha(A^{-2\ell}k) \quad \text{for } u \in A^{2\ell}\widetilde{B}+k.$$

From this, (3.13) and Lemma 3.3 we obtain

$$I_2 \le \sum_{k\in\mathbb{Z}^d} \alpha(A^{-2\ell}k) \sum_{(z,s)\in L_k} \int_{\mathbb{R}^d} \alpha(A^{-s}u - A^{-s}z)\,du$$
$$= \sum_{k\in\mathbb{Z}^d} \alpha(A^{-2\ell}k) \sum_{(z,s)\in L_k} |\det A|^s \int_{\mathbb{R}^d} \alpha(u)\,du$$
$$\le \frac{c}{|\widetilde{B}|} \sum_{k\in\mathbb{Z}^d} \alpha(A^{-2\ell}k) \sum_{(z,s)\in L_k} |\widetilde{B}(z,s)| \le c.$$

Now we estimate I_3. Define $\gamma(x) = \alpha(A^{-\ell}x)$. If $(z,s)\in L_k$ then $\widetilde{B}(k,r)$ intersects $\widetilde{B}(z,r)$. Therefore by Lemma 3.1,

$$A^r\widetilde{B} + z \subset A^{r+\ell}\widetilde{B} + k \quad \text{for } r \ge 0,$$

which according to the definition of $\gamma(x)$ gives

$$\alpha(A^{-s}u - A^{-s}z) \le \gamma(A^{-s}u - A^{-s}k) \quad \text{for } (z,s)\in L_k,\ u\in\mathbb{R}^d. \tag{3.14}$$

By the formula for $\alpha(x)$, given in the proof of Lemma 3.6, we obtain

$$\gamma(x) = \sum_{j\ge 0} |\det A|^j \widetilde{\varphi}(A^{j-2\ell}x) \quad \text{for } x \notin A^{2\ell}\widetilde{B},$$

whence

$$|\det A|^{-s}\gamma(A^{-s}x) = \sum_{j\ge -s} |\det A|^j \widetilde{\varphi}(A^{j-2\ell}x) \tag{3.15}$$
$$\le \gamma(x) \quad \text{for } x \notin A^{2\ell}\widetilde{B},\ s\le 0.$$

Now using (3.14) and (3.15) we have

$$I_3 \le \sum_{k\in\mathbb{Z}^d} \sum_{(z,s)\in L_k} \int_{\mathbb{R}^d\setminus(A^{2\ell}\widetilde{B}+k)} \alpha(u)\gamma(A^{-s}u - A^{-s}k)\,du$$
$$= \sum_{k\in\mathbb{Z}^d} \sum_{(z,s)\in L_k} \int_{\mathbb{R}^d\setminus A^{2\ell}\widetilde{B}} \alpha(u+k)\gamma(A^{-s}u)\,du$$

$$\leq \sum_{k\in\mathbb{Z}^d} \sum_{(z,s)\in L_k} |\det A|^s \int_{\mathbb{R}^d} \alpha(u+k)\gamma(u)\,du$$
$$= \sum_{k\in\mathbb{Z}^d} m(k) \int_{\mathbb{R}^d} \alpha(u+k)\gamma(u)\,du = I_4,$$

where $m(k)$ denotes the sum of the measures of the sets $\widetilde{B}(z,s)$ with $(z,s) \in L_k$. Therefore by (3.13) and Lemma 3.3 we get

$$I_4 \leq c \int_{\mathbb{R}^d} \sum_{k\in\mathbb{Z}^d} \alpha(u+k)\gamma(u)\,du \leq c,$$

which completes the proof of the inequality (3.10).

Thus we have

$$\begin{aligned}(3.16)\quad |\{x\in\mathbb{R}^d : |U_S g_3(x)| > \lambda/12\}| &\leq \frac{144}{\lambda^2} \int_{\mathbb{R}^d} |U_S g_3(x)|^2\,dx \\ &\leq \frac{144}{\lambda^2} \Big\| \sum_{(y,r)\in\mathbb{I}_N} P_r f_{yr} \Big\|_2^2 \leq \frac{c}{\lambda}\|f\|_1.\end{aligned}$$

From (3.5), (3.6), (3.9) and (3.16) we get

$$|\{x\in\mathbb{R}^d : |U_S f_N(x)| > \lambda/4\}| \leq \frac{c}{\lambda}\|f\|_1$$

for some constant c. Therefore

$$\begin{aligned}&|\{x\in\mathbb{R}^d : |U_S f(x)| > \lambda/2\}| \\ &\quad\leq |\{x\in\mathbb{R}^d : |U_S(f-f_N)(x)| > \lambda/4\}| + |\{x\in\mathbb{R}^d : |U_S f_N(x)| > \lambda/4\}| \\ &\quad\leq \frac{16}{\lambda^2}\|f-f_N\|_2^2 + \frac{c}{\lambda}\|f\|_1.\end{aligned}$$

Letting $N\to\infty$, we infer that there exists a constant c such that for every finite set S,

$$|\{x\in\mathbb{R}^d : |U_S f(x)| > \lambda/2\}| \leq \frac{c}{\lambda}\|f\|_1 \quad \text{for } f\in L^1(\mathbb{R}^d)\cap L^2(\mathbb{R}^d).$$

Let now f be an arbitrary function from $L^1(\mathbb{R}^d)$ and let f_N be this time a sequence of functions from $L^1(\mathbb{R}^d)\cap L^2(\mathbb{R}^d)$ convergent to f in the norm of $L^1(\mathbb{R}^d)$. For a fixed finite set S, there exists a constant c_S such that $\|U_S f\|_1 \leq c_S\|f\|_1$ for $f\in L^1(\mathbb{R}^d)$. Hence

$$\begin{aligned}&|\{x\in\mathbb{R}^d : |U_S f(x)| > \lambda\}| \\ &\quad\leq |\{x\in\mathbb{R}^d : |U_S(f-f_N)(x)| > \lambda/2\}| + |\{x\in\mathbb{R}^d : |U_S f_N(x)| > \lambda/2\}| \\ &\quad\leq \frac{2}{\lambda}\|U_S(f-f_N)\|_1 + \frac{c}{\lambda}\|f_N\|_1 \leq \frac{2c_S}{\lambda}\|f-f_N\|_1 + \frac{c}{\lambda}\|f_N\|_1.\end{aligned}$$

Letting $N \to \infty$ we obtain

$$|\{x \in \mathbb{R}^d : |U_S f(x)| > \lambda\}| \le \frac{c}{\lambda}\|f\|_1 \quad \text{for } f \in L^1(\mathbb{R}^d),$$

which completes the proof.

Proof of Theorem 2.4. For an arbitrary subset S of $\{1,\dots,n\} \times \mathbb{Z} \times \mathbb{Z}^d$ the operator U_S is of type $(2,2)$. Thus Theorem 3.7 and the Marcinkiewicz interpolation theorem imply that there exists a constant $c = c_p$ $(1 < p < 2)$ such that for every finite set S,

$$\|U_S f\|_p \le c\|f\|_p \quad \text{for } f \in L^p(\mathbb{R}^d). \tag{3.17}$$

Fix $f \in L^1(\mathbb{R}^d) \cap L^2(\mathbb{R}^d)$. Let (S_N) be an increasing family of subsets of $\{1,\dots,n\} \times \mathbb{Z} \times \mathbb{Z}^d$ such that S_N has N elements and $\bigcup_{N\in\mathbb{N}} S_N = \{1,\dots,n\} \times \mathbb{Z} \times \mathbb{Z}^d$. Then $\|U_{S_N} f - f\|_2 \to 0$ as $N \to \infty$. Fix $1 < p < 2$ and choose an arbitrary $1 < p' < p$. Let

$$r = \frac{2-p'}{2-p}, \qquad s = \frac{2-p'}{p-p'}, \qquad \alpha = \frac{p'}{r}, \qquad \beta = \frac{2}{s}.$$

Then

$$\frac{1}{r} + \frac{1}{s} = 1 \quad \text{and} \quad \alpha + \beta = p.$$

From Hölder's inequality and from (3.17) we obtain

$$\begin{aligned}\int_{\mathbb{R}^d} |U_{S_N} f - f|^p &\le \Big(\int_{\mathbb{R}^d} |U_{S_N} f - f|^{p'}\Big)^{1/r} \cdot \Big(\int_{\mathbb{R}^d} |U_{S_N} f - f|^2\Big)^{1/s} \\ &\le c\Big(\int_{\mathbb{R}^d} |U_{S_N} f - f|^2\Big)^{1/s},\end{aligned}$$

where $c = c(p, p', f)$ is a constant independent of N. Letting $N \to \infty$ in the above inequalities we get $\|U_{S_N} f - f\|_p \to 0$. This means that the set of functions

$$\Psi^m_{jk}(x) = |\det A|^{j/2}\Psi^m(A^j x - k), \qquad \text{where } j \in \mathbb{Z},\ k \in \mathbb{Z}^d,\ m = 1,\dots,n,$$

is linearly dense in $L^p(\mathbb{R}^d)$. This, together with (3.17) and Theorem 1.4, shows that the biorthogonal system $(\Psi^m_{jk}, \Phi^m_{jk})$ is an unconditional basis in all $L^p(\mathbb{R}^d)$ spaces with $1 < p < 2$.

The spaces $L^p(\mathbb{R}^d)$ with $1 < p < 2$ are reflexive, so by duality (see for example Proposition 7.12 in [W1]) the system $(\Psi^m_{jk}, \Phi^m_{jk})$ is an unconditional basis in $L^p(\mathbb{R}^d)$ with $2 < p < \infty$.

This completes the proof of Theorem 2.4.

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FB Mathematik, AG6
Schloßgartenstr. 7
64289 Darmstadt, Germany
E-mail: pompe@mathematik.tu-darmstadt.de

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