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UNCONDITIONAL CONVERGENT SERIES ON LOCALLY CONVEX SPACES

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Abstract. A characterization of unconditional convergent series is given for the case of sequentially complete locally convex spaces. From it we show that if E is a barrelled space with continuous dual E', then $(E', \beta(E', E))$ contains no copy of $(c_0, \|\cdot\|_\infty)$ if and only if every continuous linear operator $T: E \to l_1$ is both compact and sequentially compact.

Bessaga and Pelczynski [2] proved that a Banach space X contains no copy of $(c_0, \|\cdot\|_\infty)$ if and only if every weakly unconditional Cauchy series in X is unconditional convergent. Li Ronglu [5] proved that this is equivalent to every continuous linear operator $T: c_0 \to X$ being both compact and sequentially compact. Li Ronglu and Bu Qingying showed in [6] that these properties are valid for sequentially complete locally convex spaces.

In this paper, by using the Basic Matrix Theorem due to Antosik and Mikusinski [7], we present a characterization of unconditional convergent series on a sequentially complete locally convex space. From it we show that if E is a barrelled space with continuous dual E' , then $(E', \beta(E', E))$ contains no copy of $(c_0, \|\cdot\|_\infty)$ if and only if every continuous linear operator $T: E \to l_1$ is both compact and sequentially compact, i.e., for every bounded subset B of E, $\overline{T(B)}$ is both compact and sequentially compact in l_1.

Let E be a sequentially complete locally convex space. A series $\sum_n x_n$ in E is said to be unconditional convergent if for every permutation π of N, the series $\sum_{n=1}^{\infty} x_{\pi(n)}$ is convergent. It is easy to see that the following conditions are equivalent [8]:

(1) The series $\sum_n x_n$ is unconditional convergent.

(2) The series $\sum_n x_n$ is subseries convergent.

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(3) For every neighbourhood V of 0 in E there exists an integer n such that for every finite subset σ of N which satisfies min $\{i \in \sigma\} > n$ we have $\sum_{i\in\sigma} x_i \in V$.

(4) The series $\sum_n t_n x_n$ is convergent for every $\{t_n\} \in l_\infty$.

A series $\sum_n x_n$ in E is said to be weakly unconditional Cauchy if for every $f \in \mathrm{E}'$ we have $\sum_n |f(x_n)| < \infty$.

Now, we present a characterization of unconditional convergent series which is the analogue of condition (H) in [9].

Theorem 1. *Let E be a sequentially complete locally convex space with continuous dual E'. Then the series $\sum_n x_n$ is unconditional convergent in E if and only if for every equicontinuous subset A of E' and $\varepsilon > 0$, there exists $n_\varepsilon \in N$ such that*

$$\sup_{f\in A} \sum_{n=n_\varepsilon+1}^{\infty} |f(x_n)| < \varepsilon. \tag{5}$$

Proof. "$\Leftarrow$": For any neighbourhood V of 0 in E, there exist a continuous seminorm p of E and $\varepsilon > 0$ such that $\{x|p(x) \leq \varepsilon\} \subseteq V$. Let $A = \{f|f \in E'$, $\sup_{p(x)\leq 1} |f(x)| \leq 1\}$. Then A is an equicontinuous subset of E' and from (5) it follows that there exists an $n_\varepsilon \in N$ such that

$$\sup_{f\in A} \sum_{n=n_\varepsilon+1}^{\infty} |f(x_n)| \leq \varepsilon.$$

Let σ be a finite subset of N which satisfies $\min\{n \in \sigma\} > n_\varepsilon$. By the Hahn-Banach Theorem we can find an $f \in A$ such that

$$p\left(\sum_{n\in\sigma} x_n\right) = f\left(\sum_{n\in\sigma} x_n\right) \leq \sum_{n=n_\varepsilon+1} |f(x_n)| \leq \sup_{f\in A} \sum_{n=n_\varepsilon+1}^{\infty} |f(x_n)| \leq \varepsilon.$$

That is, $\sum_{n\in\sigma} x_n \in V$. This proves (5) $\Rightarrow$ (3). So the series $\sum_n x_n$ is unconditional convergent.

"$\Rightarrow$": Assume that the series $\sum_n x_n$ is unconditional convergent but (5) is not valid. Since $\sum_n x_n$ must be weakly unconditional Cauchy, it is easily seen that there exist $\varepsilon_0 > 0$, an equicontinuous sequence $\{f_k\}$ and an integer sequence $n_1 < m_1 < n_2 < m_2 < \ldots$ such that

$$\sum_{n=n_k}^{m_k} |f_k(x_n)| \geq \varepsilon_0,\ k \in N.$$

Let $\sigma_k = \{n | n \in N,\ n_k \le n \le m_k\},\ \sigma = \bigcup_{k=1}^{\infty} \sigma_k$ and

$$t_n^0 = \begin{cases} \dfrac{\overline{f_k(x_n)}}{|f_k(x_n)|} & \text{if } n \in \sigma_k \text{ and } f_k(x_n) \ne 0, \\ 0 & \text{otherwise.} \end{cases}$$

Then $\{t_n^0\} \in l_\infty$ and $\sum_{n=n_k}^{m_k} |f_k(x_n)| = \sum_{n \in \sigma_k} |f_k(x_n)| = \sum_{n \in \sigma_k} f_k(t_n^0 x_n) \ge \varepsilon_0$. This shows that for every $k \in N$ we have

$$\text{(a)} \qquad \sum_{n \in \sigma_k} f_k(t_n^0 x_n) = f_k\left(\sum_{n \in \sigma_k} t_n^0 x_n\right) \ge \varepsilon_0.$$

Let $S = \overline{\text{Span}\{x_n\}}$. Since the series $\sum_n x_n$ is unconditional convergent, for every $\{t_n\} \in l_\infty$ the series $\sum_n t_n x_n$ is convergent and $\sum_n t_n x_n \in S$.

Let A_1 be the $\sigma(E', E)$ closure of $\{f_k\}$. Then A_1 is a $\sigma(E', E)$ compact subset of E' by the Banach-Alaoglu Theorem [4, 20.9 (4)]. Note that $S \subseteq E$ and, therefore, A_1 is also a $\sigma(E', S)$ compact subset of E', and is metrizable since S is separable [4, 21.3(4)]. Therefore, $\{f_k\}$ has a subsequence $\{f_{k_i}\}$ which is $\sigma(E', S)$-convergent to an $f_0 \in A_1$. That is, for every $x \in S$ we have

$$\text{(b)} \qquad \lim_i f_{k_i}\ (x) = f_0(x).$$

Now consider the infinite matrix $\left[f_{k_i}\left(\sum_{n \in \sigma_{k_j}}\ t_n^0 x_n\right)\right]_{ij}$. From (b) it follows that for every $j \in$ N,

$$\lim_i f_{k_i}\left(\sum_{n \in \sigma_{k_j}} t_n^0 x_n\right) = f_0\left(\sum_{n \in \sigma_{k_j}} t_n^0 x_n\right).$$

Moreover, if $\{j_k\}$ is an increasing sequence of N and $\sigma_0 = \bigcup_{l=1}^{\infty} \sigma_{k_{j_l}}$, then $\sum_{n \in \sigma_0} t_n^0 x_n \in$ S. Thus, we have

$$\lim_i f_{k_i}\left(\sum_{n \in \sigma_0} t_n^0 x_n\right) = f_0\left(\sum_{n \in \sigma_0} t_n^0 x_n\right).$$

From the Basic Matrix Theorem of Antosik and Mikusinski [7] it follows that $\lim_i f_{k_i}\left(\sum_{n \in \sigma_{k_i}} t_n^0 x_n\right) = 0$. This contradicts (a) and the theorem is obtained. ■

Let E be a barrelled space with continuous dual E', $\beta(E', E)$ be the strong topology on E' and E'' be the continuous dual of $(E', \beta(E', E))$. It is clear that $E \subseteq E''$ and every bounded subset B of E is an equicontinuous set on

$(E',\ \beta(E',E))$. Since E is a barrelled space, it follows that $(E',\ \beta(E',E))$ is sequentially complete [10].

Theorem 2. *Let E be a barrelled space. Then the following conditions are equivalent*:

(1°) $(E',\ \beta(E',\ E))$ *contains no copy of* $(c_0,\ \|\cdot\|_\infty)$.

(2°) *Every weakly unconditional Cauchy series* $\sum_n f_n$ *in* $(E',\ \beta(E',\ E))$ *is unconditional convergent.*

(3°) *Let* $\sum_n f_n$ *be a series in* $(E',\ \beta(E',\ E))$. *If for every* $x \in E$ *we have* $\sum_n |f_n(x)| < \infty$, *then for every bounded subset* B *of* E *and* $\varepsilon > 0$, *there exists an* $n_\varepsilon \in N$ *such that*

$$\sup_{x\in B} \sum_{n=n_\varepsilon+1}^{\infty} |f_n(x)| < \varepsilon.$$

(4°) *Every continuous linear operator* $T : E \to l_1$ *is both compact and sequentially compact.*

(5°) $(E',\ \beta(E',\ E))$ *contains no copy of* $(l_\infty,\ \|\cdot\|_\infty)$.

Proof. Since $(E', \beta(E',E))$ is sequentially complete, from [6, Th.4] it follows that $(1°) \Leftrightarrow (2°)$.

If for every $x \in E, \sum_{n=1}^{\infty} |f_n(x)| < \infty$, then the series $\sum_n f_n$ must be weakly unconditional Cauchy. Indeed, for any $F \in E''$, denote

$$a_n = \begin{cases} \dfrac{\overline{F(f_n)}}{|F(f_n)|} & \text{if} \quad F(f_n) \neq 0, \\ \qquad 0 & \text{if} \quad F(f_n) = 0. \end{cases}$$

Then it is easily seen that $\{\sum_{n=1}^{m} a_n f_n\}_{m=1}^{\infty}$ is pointwise bounded on E. Since E is a barrelled space, $\{\sum_{n=1}^{m} a_n f_n\}_{m=1}^{\infty}$ is equicontinuous and therefore is a bounded subset of $(E',\ \beta(E',\ E))$. That is, there exists $M > 0$ such that for every $m \in N$ we have

$$F\left(\sum_{n=1}^{m} a_n f_n\right) = \sum_{n=1}^{m} |F(f_n)| \leq M.$$

This shows that $\sum_n f_n$ is a weakly unconditional Cauchy series in $(E',\ \beta(E',E))$. From (2°) it follows that $\sum_n f_n$ is unconditional convergent and by Theorem 1 we obtain (3°). That is, $(2°) \Rightarrow (3°)$.

If (3°) holds, let $T : E \to l_1$ be a continuous linear operator and $T(x) = (f_1(x),\ f_2(x), \ldots, f_n(x), \ldots)$. Since $|f_n(x)| \leq \sum_n |f_n(x)| = \|T(x)\|$, therefore, for every $n \in N$, we have $f_n \in E'$ and $\sum_n |f_n(x)| = \|Tx\| < \infty$. This shows that the series $\sum_n f_n$ satisfies condition (3°) and therefore, for every bounded subset B of E, $T(B)$ is a bounded subset of l_1 and for any $\varepsilon > 0$, there exists an $n_\varepsilon \in N$ such that

$$\sup_{x\in B} \sum_{n=n_\varepsilon+1}^{\infty} |f_n(x)| < \varepsilon.$$

Thus, $\overline{T(B)}$ is both a compact and sequentially compact subset of l_1 [3]. That is, (3°) $\Rightarrow$ (4°).

We now show (4°) $\Rightarrow$ (2°). If (2°) is not valid, there exists a series $\sum_n f_n$ in $(E', \beta(E', E))$ which is weakly unconditional Cauchy but is not unconditional convergent. Now we define an operator $T : E \to l_1$ by $T(x) = \{f_n(x)\}$ for every $x \in E$. It is easily seen that T is a linear operator. We show that T is also continuous.

At first, for any $\{t_n\} \in B_{l_\infty}$, the unit ball of l_∞, and $m \in N$, denote $\sum_{n=1}^m t_n f_n = \mathrm{F}_{m,t}$. Then for every $x \in \mathrm{E}$, $\{t_n\} \in B_{l_\infty}$ and $m \in N$, we have

$$|\mathrm{F}_{m,t}(x)| = |\sum_{n=1}^{m} t_n f_n(x)| \leq \sum_{n=1}^{m} |t_n f_n(x)| \leq \sum_{n=1}^{\infty} |f_n(x)|.$$

This shows that $\{\mathrm{F}_{m,t}\}$ is pointwise bounded on E and, therefore, is equicontinuous.

For every neighbourhood V of 0 in l_1, there exists $\varepsilon > 0$ such that $\{y|y \in l_1,\ \|y\| \leq \varepsilon\} \subseteq V$. Since $\{F_{m,t}\}$ is equicontinuous, there exists a neighbourhood U of 0 in E such that for every $F_{m,t}$ we have $\sup_{x\in U} |F_{m,t}(x)| \leq \varepsilon$. From the definition of $F_{m,t}$ it follows that $\sup_{x\in U} \|T(x)\| \leq \varepsilon$. Thus, T is continuous.

Finally, we show that T is not compact. Indeed, since $\sum_n f_n$ is not unconditional convergent, there exists a neighbourhood W of 0 in $(E', \beta(E', E))$ and a sequence of finite subsets $\{\sigma_n\}$ of N such that for every $n \in N$, we have $\max \sigma_n < \min \sigma_{n+1}$ and $\sum_{k\in\sigma_n} f_k \notin W$. So there exist a bounded subset B of E and $\varepsilon > 0$ such that $\{f|f \in E',\ |f(x)| \leq \varepsilon,\ x \in B\} \subseteq W$. Thus, we can obtain a sequence $\{x_n\}$ of B such that

$$|\sum_{k\in\sigma_n} f_k(x_n)| \geq \frac{\varepsilon}{2},\ n \in N.$$

Moreover,

$$\sum_{k\in\sigma_n} |f_k(x_n)| \geq \frac{\varepsilon}{2},\ n \in N.$$

From [3, Prop. 6.11], it follows that $\overline{\{T(x_n)\}}$ is not a compact subset of l_1 and, therefore, is not a sequentially compact subset of l_1. That is, (4°) $\Rightarrow$ (2°).

Since $(c_0,\ \|\cdot\|_\infty) \subseteq (l_\infty,\ \|\cdot\|_\infty)$, it follows that (1°) $\Rightarrow$ (5°). If (1°) is not valid, from (1°) $\Leftrightarrow$ (2°) it follows that there exists a weakly unconditional

Cauchy series $\sum_n f_n$ in $(E', \beta(E', E))$ which is not unconditional convergent. Since E is a barrelled space, $(E',\ \sigma(E',\ E))$ and $(E', \beta(E', E))$ are sequentially complete. Therefore, it is easily seen that the series $\sum_n f_n$ is subseries $\sigma(E', E)$ convergent. Define $\mu : 2^N \to E'$ by $\mu(\sigma) = \sum_{n\in\sigma} f_n$. Note that $\sum_n f_n$ is not $\beta(E', E)$ unconditional convergent and, therefore, is not subseries $\beta(E', E)$ convergent. From the sequential completeness of $\beta(E', E)$, it follows that there exists a disjoint sequence $\{\sigma_j\} \subseteq 2^N$ such that $\{\mu(\sigma_j)\}$ does not converge to 0 in $\beta(E', E)$. As proved in [1, Th. 10.7 and Cor. 10.8] that $(E', \beta(E', E))$ contains a copy of $(l_\infty,\ \|\cdot\|_\infty)$. This contradicts (5°). The proof is completed. ∎

Corollary 3 [11]. *Let X be a Banach space with continuous dual X'. Then $(X',\ \|\cdot\|)$ contains no copy of $(c_0,\ \|\cdot\|)$ if and only if every continuous linear operator $T : E \to l_1$ is compact.*

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