UNIVERSAL ALGEBRA AND APPLICATIONS
BANACH CENTER PUBLICATIONS, VOLUME 9
PWN—POLISH SCIENTIFIC PUBLISHERS
WARSAW 1982

UNDECIDABILITY OF THE COMPLETENESS PROBLEM OF MODAL LOGIC*

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A normal modal logic is said to be *complete* if for each non-thesis α of the logic there is a Kripke frame for the logic on which α is not valid. This paper addresses the question whether there is an effective procedure for determining whether or not a given finitely axiomatized normal modal logic is complete.

We shall consider logics with several necessity operators: if $S \subseteq N = \{1, 2, \ldots\}$ let K_S be the logic whose language has $\neg$, $\vee$, and distinct unary operators $\Box_i$ for $i \in S$, and which has the axioms and rules of K for each $\Box_i$. That is, the axioms of K_S are the classical tautologies and $\Box_i(p \to q) \to (\Box_i p \to \Box_i q)$ $(i \in S)$, and the rules of inference are Substitution, Detachment, and Necessitation (from α, infer $\Box_i \alpha$ $(i \in S)$).

Let Fla_S be the set of all formulas of K_S. If $\beta \in \mathrm{Fla}_S$, let $K_S(\beta)$ be the logic obtained from K_S by adding β as a new axiom; $K_S(\beta) \vdash \gamma$ means that γ is a thesis of $K_S(\beta)$.

A *Kripke frame* $\boldsymbol{W}$ for K_S consists of a non-empty set W together with binary relations $\prec_i$ $(i \in S)$ on W. The relation $\boldsymbol{W} \models \beta$ (β is *valid on* $\boldsymbol{W}$) is defined as follows: a *valuation* θ assigns to each $\alpha \in \mathrm{Fla}_S$ a subset $\theta(\alpha)$ of W subject to the conditions

$$\theta(\neg\alpha) = W - \theta(\alpha),$$
$$\theta(\alpha \vee \beta) = \theta(\alpha) \cup \theta(\beta),$$
$$\theta(\Box_i \alpha) = \{w \mid (\forall v)(w \prec_i v \Rightarrow v \in \theta(\alpha))\},$$

and $\boldsymbol{W} \models \beta$ if $\theta(\beta) = W$ for every valuation θ.

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* This paper is a revised and extended version of a lecture presented at the Stefan Banach International Mathematical Center, Warsaw, in April, 1978. The author is grateful to the National Research Council of Canada, the University of Warsaw, and the Polish Academy of Sciences for their support of this research.

A *general frame* $(\boldsymbol{W}, P)$ for K_S consists of a Kripke frame $\boldsymbol{W}$ together with a non-empty family $P \subseteq P(W)$ such that

$$X \in P \Rightarrow W - X \in P,$$

$$X, Y \in P \Rightarrow X \cup Y \in P,$$

$$X \in P \Rightarrow \{w \mid (\forall v)(w \prec_i v \Rightarrow v \in X)\} \in P \quad (i \in S).$$

A valuation for $(\boldsymbol{W}, P)$ is a valuation θ for $\boldsymbol{W}$ such that $\theta(\alpha) \in P$ for each $\alpha \in \mathrm{Fla}_S$, and $(\boldsymbol{W}, P) \models \beta$ if $\theta(\beta) = W$ for every such θ.

It is easily seen (cf. [4]) that α is a thesis of $K_S(\beta)$ if and only if α is valid on every general frame on which β is valid; $K_S(\beta)$ is *complete* if this remains true (for all α) when "general frame" is replaced by "Kripke frame".

A general frame $(\boldsymbol{W}, P)$ is *connected* if the smallest equivalence relation containing all the $\prec_i$'s is $W \times W$, and *refined* if P is the basis of a Hausdorff topology on W such that each $\prec_i$ is closed in $W \times W$ (cf. [4] for a more explicit definition). It is not difficult to show (cf. [7]) that for each general frame $(\boldsymbol{W}, P)$ and formula α not valid on $(\boldsymbol{W}, P)$ there is a connected, refined, general frame $(\boldsymbol{W}', P')$ such that every formula valid on $(\boldsymbol{W}, P)$ is valid on $(\boldsymbol{W}', P')$ but α is not valid on $(\boldsymbol{W}', P')$. Hence $K_S(\beta) \vdash \alpha$ if and only if α is valid on every connected, refined, general frame on which β is valid.

If $S \subseteq T$, then $K_T(\beta)$ is a *conservative extension* of $K_S(\gamma)$ if the theses of $K_S(\gamma)$ are exactly the formulas of K_S which are theses of $K_T(\beta)$.

PROPOSITION. *If $S \subseteq T$ and $\beta \in \mathrm{Fla}_S$, then $K_T(\beta)$ is a conservative extension of $K_S(\beta)$. If $S \cap T = \emptyset$, $\beta \in \mathrm{Fla}_S$, $\gamma \in \mathrm{Fla}_T$, and $K_S(\beta)$ is consistent, then $K_{S \cup T}(\beta \wedge \gamma)$ is a conservative extension of $K_T(\gamma)$.*

To prove the first statement, it suffices to note that if $(\boldsymbol{W}, P)$ is a general frame for K_S, $(\boldsymbol{W}, P) \models \beta$, and $\neg(\boldsymbol{W}, P) \models \gamma$, then we can find a general frame $(\boldsymbol{W}', P)$ for K_T such that $(\boldsymbol{W}', P) \models \beta$ and $\neg(\boldsymbol{W}', P) \models \gamma$; namely, $\boldsymbol{W}'$ is $\boldsymbol{W}$ together with $\prec_i = W \times W$ for $i \in T - S$. The second statement is proved in [8].

An intuitive understanding of certain recursion-theoretic notions will be necessary (and sufficent). A set $C \subseteq N$ is *recursively enumerable* if there is an effective procedure for generating the members of C (not necessarily in any particular order) and *recursive* if there is an effective procedure for determining of an arbitrary number whether or not it is a member of C. A set C is *reducible* (or *many-one reducible*, in the usual terminology) to another set D if there is a recursive (that is, effectively calculable) function f such that $(\forall n)(n \in C \Leftrightarrow f(n) \in D)$, and D is *universal* (respectively, *co-universal*) if every recursively enumerable set is reducible to D (respectively, to the complement of D). There exist recursively enumerable, non-recursive sets; such a set is not reducible to any recursive set, so no universal or co-universal set is recursive. Reducibility is transitive, and one set is reducible to another if and only if its complement is reducible to the complement of the other, so any set to which a universal or co-universal set is reducible is itself universal or co-universal respectively. We identify formulas with natural numbers via a suitable Gödel numbering of Fla_N.

If $S = \{1, \ldots, m\}$ we write Fla_m and K_m for Fla_S and K_S respectively. Let $A_m = \{\beta \in \mathrm{Fla}_m \mid K_m(\beta)$ is consistent$\}$ and $B_m = \{\beta \in \mathrm{Fla}_m \mid K_m(\beta)$ is complete$\}$. Our main result is the following.

THEOREM. (a) *A_1 is recursive, but A_m is co-universal for all $m \geqslant 2$.* (b) *B_m is universal for all $m \geqslant 3$.*

It is easy to see that A_1 is recursive, since $K_1(\beta)$ is consistent if and only if β is valid on some one-element Kripke frame [2]. Since K_m is a conservative extension of K_n when $n \leqslant m$, A_n is reducible to A_m ($\beta \in A_n \Leftrightarrow f(\beta) \in A_m$, where $f(\beta)$ is β or $p \wedge \neg p$ according as $\beta \in \mathrm{Fla}_n$ or not). So to complete the proof of (a) it suffices to show that A_2 is co-universal.

We shall make use of some constructions appearing in [7]. Let C be any recursively enumerable set. By [7], Section 4, there is a number m, a formula β of K_m, and a recursive sequence $\alpha_1, \alpha_2, \ldots$ of formulas of K_m, such that

$$(*) \qquad (\forall n)\big(n \in C \Leftrightarrow K_m(\beta) \vdash \alpha_n\big).$$

Moreover, examination of this construction reveals that if $(\boldsymbol{W}, P)$ is any connected, refined, general frame for K_m and $(\boldsymbol{W}, P) \models \beta$ then, for each n,

$$(**) \qquad (\boldsymbol{W}, P) \models \alpha_n \quad \text{or} \quad (\boldsymbol{W}, P) \models \neg \alpha_n.$$

By the reduction procedure of [7], Section 2, one obtains a formula β' of K_1, and a recursive sequence $\alpha_1', \alpha_2', \ldots$ of formulas of K_1 such that $(*)$ is true of $\beta', \alpha_1', \alpha_2', \ldots$ But examination of *this* construction shows that $(**)$ no longer holds. All that can be said is that $\theta(\alpha_n')$ is independent of θ, that is, if $(\boldsymbol{W}, P)$ is a connected, refined, general frame for K_1, $(\boldsymbol{W}, P) \models \beta'$, and θ_1 and θ_2 are valuations for $(\boldsymbol{W}, P)$ then, for each n, $\theta_1(\alpha_n') = \theta_2(\alpha_n')$. In order to restore $(**)$ we must retreat to K_2.

LEMMA. *If C is a recursively enumerable set, then there is a formula β of K_2 and a recursive sequence $\alpha_1, \alpha_2, \ldots$ of formulas of K_2 such that $(*)$ and $(**)$ hold.*

To prove the lemma, let $\beta', \alpha_1', \alpha_2', \ldots$ be formulas of K_1 as in the preceding paragraph. Let β be the conjunction of β', $\Box_2 p \to \Box_1 p$, and the S5 axioms for $\Box_2$, and for each n let α_n be $\Box_2 \alpha_n'$. If $(\boldsymbol{W}, P)$ is a general frame for K_1 and $(\boldsymbol{W}, P) \models \beta'$, then $(\boldsymbol{W}', P) \models \beta$, where $\boldsymbol{W}'$ is $\boldsymbol{W}$ together with $\prec_2 = W \times W$; moreover, $(\boldsymbol{W}, P) \models \gamma \Leftrightarrow (\boldsymbol{W}', P) \models \gamma$ for all $\gamma \in \mathrm{Fla}_1$. Thus

$K_2(\beta)$ is a conservative extension of $K_1(\beta')$, so $n \in C \Leftrightarrow K_1(\beta') \vdash \alpha'_n \Leftrightarrow K_2(\beta) \vdash \alpha'_n \Leftrightarrow K_2(\beta) \vdash \alpha_n$, and (*) holds. Moreover, if $(\boldsymbol{W}, P) = (W, \prec_1, \prec_2, P)$ is connected and refined and $(\boldsymbol{W}, P) \models \beta$, then $\prec_2 = W \times W$ so $\theta(\alpha_n) \in \{\emptyset, W\}$ for any valuation θ for $(\boldsymbol{W}, P)$; but $\theta(\alpha_n)$ is independent of θ (any valuation for $(\boldsymbol{W}, P)$ is a valuation for $(W, \prec_1, P)$, and $\theta(\alpha'_n)$ is independent of θ), so $(\boldsymbol{W}, P) \models \alpha_n$ or $(\boldsymbol{W}, P) \models \neg\alpha_n$; that is, (**) holds as well. This completes the proof of the lemma.

Let C be any recursively enumerable set and let $\beta, \alpha_1, \alpha_2, \ldots$ be as in the lemma. If $n \in C$, then $K_2(\beta) \vdash \alpha_n$ so $K_2(\beta \wedge \neg\alpha_n)$ is inconsistent and $\beta \wedge \neg\alpha_n \notin A_2$. Conversely, if $n \notin C$, then $\neg K_2(\beta) \vdash \alpha_n$, so there is a connected, refined, general frame $(\boldsymbol{W}, P)$ such that $(\boldsymbol{W}, P) \models \beta$ but $\neg(\boldsymbol{W}, P) \models \alpha_n$. Because of (**), $(\boldsymbol{W}, P) \models \beta \wedge \neg\alpha_n$, so $K_2(\beta \wedge \neg\alpha_n)$ is consistent and $\beta \wedge \neg\alpha_n \in A_2$. Thus $(\forall n)(n \in C \Leftrightarrow \beta \wedge \neg\alpha_n \notin A_2)$. Since C was arbitrary, A_2 is co-universal.

We turn now to the proof of (b). If $m \geqslant 3$, let $\gamma \in \mathrm{Fla}_{\{m\}}$ be such that $K_{\{m\}}(\gamma)$ is incomplete ([1], [5]). If C is any recursively enumerable set, let $\beta, \alpha_1, \alpha_2, \ldots$ be as in the lemma, so $n \in C \Leftrightarrow K_2(\beta) \vdash \alpha_n \Leftrightarrow (\beta \wedge \neg\alpha_n) \notin A_2$. If $n \in C$, then $K_2(\beta \wedge \neg\alpha_n)$ is inconsistent, so $K_m(\beta \wedge \neg\alpha_n \wedge \gamma)$ is inconsistent and thus complete, and $(\beta \wedge \neg\alpha_n \wedge \gamma) \in B_m$. Conversely, if $n \notin C$, then $K_2(\beta \wedge \neg\alpha_n)$ is consistent. By the proposition $K_{m-1}(\beta \wedge \neg\alpha_n)$ is consistent and $K_m(\beta \wedge \neg\alpha_n \wedge \gamma)$ is a conservative extension of $K_{\{m\}}(\gamma)$. Then since $K_{\{m\}}(\gamma)$ is incomplete, so is $K_m(\beta \wedge \neg\alpha_n \wedge \gamma)$ and $(\beta \wedge \neg\alpha_n \wedge \gamma) \notin B_m$. Thus we have $n \in C \Leftrightarrow (\beta \wedge \neg\alpha_n \wedge \gamma) \in B_m$, and C is reducible to B_m. This completes the proof of the theorem.

It is clear that each A_m has recursively enumerable complement. Thus part (a) of the theorem specifies each A_m up to recursive isomorphism type ([3], Chapter 7) and so determines exactly the complexity of each A_m. Part (b) is much less definitive. In the first place, the condition $m \geqslant 3$ should not, it seems to me, be required. The reduction procedure of ([7], Sections 2, 3) provides, when $m \geqslant n \geqslant 1$, a formula δ of Fla_n and a recursive function φ: $\mathrm{Fla}_m \to \mathrm{Fla}_n$ such that if $K_m(\beta)$ is incomplete, then so is $K_n(\delta \wedge \varphi(\beta))$, but (contrary to what I said in April) it is not clear whether the converse holds as well. If so, then B_n is reducible to B_m when $n \geqslant m \geqslant 1$ and so B_m is universal for all $m \geqslant 1$. In the second place, part (b) leaves open the possibility that there may be an (infinite) recursive set Γ of formulas of K_m such that, for every $\beta \in \mathrm{Fla}_m$, $K_m(\beta)$ is complete if and only if β is equivalent to some formula in Γ. I do not believe that this is the case.

An interesting problem, related to the ones discussed here, was mentioned by a member of the audience at the Banach Center. (I hope he has had more success with it than I have.) Is $\{\beta \in \mathrm{Fla}_1 |\ K_1(\beta)$ is decidable$\}$ decidable?

Addendum (added August 25, 1978). Robert E. Woodrow showed me a way to prove that B_1 and B_2 are universal, by modifying the reduction procedure of [7]. Another way, which does not require such a modification, is as follows.

Let a formula $\beta \in K_m$ be called *r-persistent* if whenever $(\boldsymbol{W}, P)$ is refined and $(\boldsymbol{W}, P) \models \beta$, then $\boldsymbol{W} \models \beta$. It is clear that if $K_m(\beta)$ is inconsistent, then β is r-persistent, and if β is r-persistent, then $K_m(\beta)$ is complete. It is not difficult to show that the reduction procedure of [7] preserves r-persistence, that is, if β is r-persistent, then so is $\delta \wedge \varphi(\beta)$. So we may apply the reduction procedure to the formulas $\beta \wedge -\alpha_n \wedge \gamma$ of K_3 to obtain formulas $\eta_n = \delta \wedge \varphi(\beta \wedge \neg\alpha_n \wedge \gamma)$ of K_2 (or of K_1) such that $n \in C \Leftrightarrow \eta_n \in B_2$ (or B_1).

References

[1] Kit Fine, *An incomplete logic containing S4*, Theoria 40 (1974), 23–29.

[2] David Makinson, *Some embedding theorems for modal logic*, Notre Dame J. Formal Logic 12 (1971), 251–254.

[3] Hartley Rogers, Jr., *Theory of recursive functions and effective computability*, McGraw-Hill, New York 1967.

[4] S. K. Thomason, *Semantic analysis of tense logics*, J. Symbolic Logic 37 (1972), 150–158.

[5] —, *An incompleteness theorem in modal logic*, Theoria 40 (1974), 30–34.

[6] —, *Reduction of second-order logic to modal logic*, Z. Math. Logik Grundl. Math. 21 (1975), 107–114.

[7] —, *Reduction of tense logic to modal logic, II*, Theoria 41 (1975), 154–169.

[8] —, *Independent propositional modal logics*, to appear.

Presented to the Semester
Universal Algebra and Applications
(February 15 – June 9, 1978)