with $u(w,s) = |w|_\infty u(\hat{w},s)$ where $\hat{w} = w/|w|_\infty$. For any $w \in \mathbf{R}^2$ and $s \in \{0,1\}$, $|z(w,s)|_\infty \le 6|w|_\infty$.

We now turn our attention to signals $\mathbf{w}(k) \in l_\infty^n$ and $\mathbf{w}(t) \in L_\infty^n$, and to distinguish them from vectors in $\mathbf{R}^n$ we will use boldface type. We introduce a control law $\hat{K}_h : l_\infty^2 \to l_\infty^1$, with

$$(\hat{K}_h \mathbf{w})(k) = \begin{cases} u(w(k),0), & \text{if } k \text{ is even} \\ u(w(k),1), & \text{if } k \text{ is odd.} \end{cases} \tag{12}$$

Then, $\hat{K}_h$ is a nonlinear and two-periodic controller. As $P_{22} = 0$, (4) in Theorem 4.1 yields $\hat{K}_h = \hat{Q}$ and, hence, we may define an NTI controller $\hat{K}_{TI}$ as in Theorem 4.2 by $\hat{K}_{TI} = (1/2)(\hat{K}_h + q\hat{K}_h q^{-1})$. Then, S_{zw} and $qS_{zw}q^{-1}$ are given by

$$(S_{zw}\mathbf{w})(t) = \begin{cases} z(w([t/h]),0), & \text{if } [t/h] \text{ is even} \\ z(w([t/h]),1), & \text{if } [t/h] \text{ is odd} \end{cases} \tag{13}$$

$$(qS_{zw}q^{-1}\mathbf{w})(t) = \begin{cases} z(w([t/h]),1), & \text{if } [t/h] \text{ is even} \\ z(w([t/h]),0), & \text{if } [t/h] \text{ is odd.} \end{cases} \tag{14}$$

By Theorem 4.2, $\hat{K}_{TI}$ gives closed-loop disturbance response system

$$\begin{aligned}(S_{TI}\mathbf{w})(t) &= \frac{1}{2}(S_{zw} + qS_{zw}q^{-1})(t) \\ &= \frac{1}{2}(z(w([t/h]),0) + z(w([t/h]),1)).\end{aligned} \tag{15}$$

If we consider the specific unit disturbance signal $\tilde{\mathbf{w}} \in L_\infty^2$ with $\tilde{\mathbf{w}}(t) = w_3$ for all $t \in \mathbf{R}$, we obtain $\|S_{zw}\tilde{\mathbf{w}}\|_\infty = \|qS_{zw}q^{-1}\tilde{\mathbf{w}}\|_\infty = 6$. So S_{zw} is L_∞ norm h-periodic to $\tilde{\mathbf{w}}$. Also $\|S_{TI}\tilde{\mathbf{w}}\|_\infty = 4.5$, so S_{zw} is not L_∞ h-periodic to $\tilde{\mathbf{w}}$. As $\tilde{\mathbf{w}}$ is a constant signal, $\sup_{0\le\tau<Nh}\|S_{zw}\tilde{\mathbf{w}}_\tau\|_\infty = 6$ and $\sup_{0\le\tau<h}\|S_{TI}\tilde{\mathbf{w}}_\tau\|_\infty = 4.5$. So Theorem 5.1 is verified for the signal $\tilde{\mathbf{w}}$.

We can show that for any $\mathbf{w} \in L_\infty^2$, $\|S_{zw}\mathbf{w}\|_\infty \le 6\|\mathbf{w}\|_\infty$, and $\|S_{TI}\mathbf{w}\|_\infty \le 4.5\|\mathbf{w}\|_\infty$. Hence $\|S_{zw}\|_\infty = 6$ and $\|S_{TI}\|_\infty = 4.5$. Thus S_{zw} attains its system norm on the constant sequence $\{\mathbf{w}^{[k]} = \tilde{\mathbf{w}}\}$, but is not $L_\infty h$-periodic to it, verifying Theorem 5.2.

VII. Conclusion

In this note, the use of NPTV sampled-data control of continuous time LTI plants for disturbance rejection performance is analyzed. For a given strictly NPTV discrete controller yielding a closed-loop disturbance response system that is not $L_p h$-periodic to the disturbance inputs, an NTI discrete controller is constructed and shown to provide strictly better rejection of L_p specific and uniform disturbances, for all $p \in [1,\infty]$. The authors consider that, in general, the closed-loop system with a strictly NPTV discrete controller will be h-periodic to very few inputs. Indeed, the likelihood of h-periodicity is reduced for larger values of the controller period N. The results obtained include linear periodic sampled-data control systems, and similar results can be straightforwardly obtained for nonlinear periodic discrete systems.

As the system L_p norm can be related to the system robustness under norm bounded model perturbations, the results imply that periodic controllers amplify the system L_p norm which can lead to deterioration of the system robustness against norm bounded model perturbations. Hence, the use of periodic control will in general be at the cost of inferior system robustness performance, relative to that achievable by time invariant control.

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Undershoot and Settling Time Tradeoffs for Nonminimum Phase Systems

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***Abstract*—It has been known for some time that real nonminimum phase zeros imply undershoot in the step response of linear systems. Bounds on such undershoot depend on the settling time demanded and the zero locations. In this note, we review such constraints for linear time invariant systems and provide new stronger bounds that consider simultaneously the effect of two real nonminimum phase zeros. Using the concept of zero dynamics, we extend these results to a class of nonlinear systems.**

***Index Terms*—Nonminimum phase (NMP), settling time, undershoot, zero dynamics.**

I. Introduction

The analysis of performance tradeoffs in feedback control systems aims to expound and quantify fundamental compromises in the design of a feedback controller. This area of research has been the focus of many studies through the years (see, e.g., the research monographs [1] and [2]). Such studies have used both frequency domain (sensitivity functions, interpolation constraints, achievable H_∞ performance) and time domain (cheap control, undershoot–overshoot, settling time and rise time, L_∞, and L_2 performance) approaches. However, the analysis

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of performance tradeoffs is most clearly developed for linear systems from a frequency domain perspective.

In attempting to extend these results to nonlinear systems, one natural avenue to explore is the extension of time domain constraints. Such an approach has been taken in [3] to extend the cheap control results of [4] to a class of nonlinear systems.

An additional advantage of the time domain analysis of performance tradeoffs, is the characterization of design constraints directly related to the transient response specifications of the system, e.g., undershoot and settling time, which are of immediate practical value.

In this note, we consider the extension of the results on undershoot and settling time constraints for nonminimum phase linear systems obtained in [5] using frequency domain tools. As in [3], we use the concept of zero dynamics [6], [7] for the characterization of nonminimum phase nonlinear systems. In the context of the zero dynamics formulation, we first rederive the results of [5] for linear systems, and provide new, stronger undershoot bounds that consider simultaneously the effect of two real nonminimum phase zeros.

We then present an extension of these results for a class of nonlinear systems with unstable zero dynamics. Using concepts of constrained reachability ([8, Sec. 3.6], [9, Ch. 6]) we show that the step response of systems with scalar separable unstable zero dynamics must necessarily undershoot. Furthermore, as is the case for linear systems, the undershoot cannot be small if a short settling time is required when the zero dynamics are comparatively slow.

A. Background and Definitions

Consider the general problem of trying to control a single-input–single-output nonlinear time invariant plant

$$y = \mathcal{P} * u \tag{1}$$

where $\mathcal{P}$ denotes, in the most general setting, a dynamic nonlinear operator mapping the input signal $u(t)$ to the output signal $y(t)$. In this note, we are primarily interested in step response tracking examples, where we wish to move from an equilibrium[1] $(y = 0)$ to a new equilibrium $(y = \bar{y} \neq 0)$. We are particularly interested in the *undershoot* that may occur during such a transition. Given an output signal $y(t)$ we define the *relative undershoot* r_{us} as

$$r_{\mathrm{us}}(y(\cdot)) = -\inf_{t \in (0,\infty)} \left\{ \frac{y(t)}{\bar{y}} \right\}.$$

Undershoot is an important time domain characteristic of the response of a control system. There are several reasons why it may be undesirable to permit excessive undershoot in a plant response. First, large undershoot may cause state constraints to be violated. Second, large undershoot may be, and in the linear case definitely is, an indication of poor sensitivity robustness properties. Third, large undershoot may deceive a supervisor or operator into believing the control system is faulty, and therefore manually intervening in a control system.

We are also interested in the *settling time* of a feedback control system. The settling time has a variety of definitions by different authors. In this note, for simplicity, we use the following definition of an exact settling time T_{es}:

$$T_{\mathrm{es}}(y(\cdot)) = \inf_{\tau \in (0,\infty)} \{ \tau : t > \tau \Rightarrow y(t) = \bar{y} \}.$$

Note that for many real control systems [for example, linear time invariant (LTI) systems], the output y may not be able to settle exactly in finite time. However, the output may be able to approximate, to an arbitrary degree of accuracy, an "ideal" output signal which does have finite exact settling time. This approximation significantly simplifies the analysis in this note.

[1]We assume, without loss of generality, that the initial equilibrium is at the origin.

II. Linear Time Invariant Plants

In the case where the plant can be described by a linear, causal, finite-dimensional operator, we replace the description of (1) by the following rational transfer function description:

$$y = P(s) * u$$

where the notation $P(s) * u$ represents convolution of the impulse response of $P(s)$ with u.

In this case, we say that the plant transfer function $P(s)$ is *minimum phase* if all of its zeros have nonpositive real parts. If any of the zeros of $P(s)$ have positive real part, then we say that $P(s)$ is *nonminimum phase* (NMP). The study of NMP zeros, and their effects on time domain control properties have been studied previously by several authors [2], [5], [10, Ch. 4]–[12]. We first review the situation for a single real NMP zero.

A. Single Real NMP Zero

Consider the case where we have a single real NMP zero at $s = \lambda$. We then have the following result.

Proposition II.1: [5] Suppose that $P(\lambda) = 0$ where λ is a positive real number. Consider any input–output signals such that $u(t)$ is bounded and the output $y(t)$ settles exactly to $\bar{y}$ in time T. Then, the relative undershoot must satisfy

$$r_{\mathrm{us}}(y) \geq \frac{1}{e^{\lambda T} - 1}. \tag{2}$$

Proof[2]*:* Since $u(t)$ is bounded, λ is in the region of convergence of $U(s) = \mathsf{L}\{u(t)\}$ (the Laplace transform of $u(t)$). Therefore, since $Y(\lambda) = P(\lambda)U(\lambda) = 0$, we get the interpolation constraint $\int_0^\infty y(t) e^{-\lambda t} dt = 0$. By splitting the interval of integration, dividing by $\bar{y}$, and using the definition of exact settling, we obtain

$$\int_T^\infty e^{-\lambda t} dt = \int_0^T -\frac{y(t)}{\bar{y}} e^{-\lambda t}\, dt. \tag{3}$$

The result then follows from the fact that $-y(t)/\bar{y} \leq r_{\mathrm{us}}(y)$. ■

We now wish to expand on this result to consider the case where we have two real NMP zeros.

B. Two Real NMP Zeros

Suppose that we have an LTI plant with two real NMP zeros at $s = \lambda_1$ and $s = \lambda_2$. Then using the same arguments as in the previous section, we obtain one interpolation constraint for each NMP zero. Of course, Proposition II.1 applies (at least as a lower bound) individually to each of these two NMP zeros. However, the interaction of the two interpolation constraints gives stronger results as we now show.

Proposition II.2: Suppose that $P(\lambda_1) = P(\lambda_2) = 0$ where $\lambda_1 > \lambda_2 > 0$ are real numbers. Consider any input–output signals such that $u(t)$ is bounded and the output $y(t)$ settles exactly to $\bar{y}$ in time T. Then, the relative undershoot must satisfy

$$r_{\mathrm{us}}(y) \geq \frac{\lambda_1 e^{-\lambda_2 T} - \lambda_2 e^{-\lambda_1 T}}{\lambda_1 (1 - e^{-\lambda_2 T}) - \lambda_2 (1 - e^{-\lambda_1 T})}. \tag{4}$$

Proof: Since λ_1 and λ_2 can be considered as two single NMP zeros, the proof of Proposition II.1 can be used to show that (3) holds for λ_1 and λ_2. Thus, for $j = 1, 2$ we have

$$\frac{e^{-\lambda_j T}}{\lambda_j} = \int_0^T -\frac{y(t)}{\bar{y}} e^{-\lambda_j t} dt. \tag{5}$$

Subtracting (5) with $j = 1$ from the same equation with $j = 2$ yields

$$\frac{e^{-\lambda_2 T}}{\lambda_2} - \frac{e^{-\lambda_1 T}}{\lambda_1} = \int_0^T -\frac{y(t)}{\bar{y}} (e^{-\lambda_2 t} - e^{-\lambda_1 t})\, dt.$$

[2]We include the proof here since it is brief, and is instructive for the remaining development of the note.

Since $e^{-\lambda_2 t} - e^{-\lambda_1 t} \geq 0$ for $t \geq 0$, this implies that

$$\frac{e^{-\lambda_2 T}}{\lambda_2} - \frac{e^{-\lambda_1 T}}{\lambda_1} \leq r_{\rm us}(y) \int_0^T (e^{-\lambda_2 t} - e^{-\lambda_1 t})\, dt$$

from which the desired result follows. ∎

This result may be illustrated as shown in Fig. 1. Note that by taking each constraint individually, we would only get the results for $\lambda_1/\lambda_2 \rightarrow \infty$. Clearly, from the figure, if the zeros are not widely spaced, the results when the two constraints are considered together may be many times worse than for the individual constraints. For example, consider $\lambda_2 T = 1$. As $\lambda_1 \rightarrow \infty$ the lower bound on the relative undershoot is 0.6 but when λ_1 is close to λ_2 the bound on $r_{\rm us}(y)$ is approximately 2.78.

The results have been derived based on Laplace transform analysis of an LTI plant. However, when considering nonlinear plants, such analysis may be inappropriate and difficult to generalize. To facilitate analysis of nonlinear plants, we now consider the same problems, using a zero dynamics formulation.

C. Linear Zero Dynamics Formulation

The zero dynamics formulation (see, for example, [7, p. 538] and [6]) for a linear system performs a state transformation from the generic state-space form

$$\dot{x} = Ax + Bu \quad y = Cx$$

to the zero dynamics form

$$\begin{aligned} \dot{\xi} &= A_c \xi + B_c C_0 z + B_c u \\ \dot{z} &= A_0 z + B_0 y \\ y &= C_c \xi. \end{aligned} \tag{6}$$

In this case, the zero dynamics refers to the dynamics of the equation $\dot{z} = A_0 z$ and in particular, the eigenvalues of A_0 are the zeros of the transfer function, $P(s)$. Hence, an NMP system has unstable zero dynamics.

With the system in zero dynamics form, one way of considering the relationship of undershoot to settling time and NMP zeros is as follows: Take the target output $\bar{y}$ and form the target final state $\bar{z} = -A_0^{-1} B_0 \bar{y}$. Without loss of generality, let $\bar{y} > 0$. For each $\alpha \geq 0$, let $\mathcal{Y}_\alpha$ denote the set of functions y which satisfy

$$y(t) \geq -\alpha \qquad \forall t \geq 0.$$

Bounds on α, and hence the relative undershoot $\alpha/\bar{y}$, can then be found by assuming that $\exists\, y \in \mathcal{Y}_\alpha$ which takes z from 0 to $\bar{z}$ in time T. We now use this approach to rederive the results for one and two NMP zeros.

1) First-Order Linear Zero Dynamics: Without loss of generality, we take $A_0 = \lambda > 0$ and $B_0 = 1, \bar{y} > 0$. In this case, clearly $\bar{z} = -\bar{y}/\lambda$ and the general solution to the linear zero dynamics equation is given by

$$z(t) = \int_0^t e^{\lambda(t-\tau)} y(\tau)\, d\tau. \tag{7}$$

From (7), and for $y \in \mathcal{Y}_\alpha, \alpha \geq 0$, we obtain

$$-z(T) \leq \int_0^T e^{\lambda(T-\tau)} \alpha\, d\tau = \alpha \frac{e^{\lambda T} - 1}{\lambda}. \tag{8}$$

Clearly, therefore, unless $\alpha(e^{\lambda T} - 1) \geq \bar{y}$, (8) contradicts $z(T) = \bar{z}$. It follows that $r_{\rm us}(y)(e^{\lambda T} - 1) \geq 1$ which is equivalent to (2). Thus, we have rederived the result of Section II.A without using Laplace transforms.

2) Second-Order Linear Zero Dynamics: Suppose that $\bar{y} > 0$ and we have two distinct real NMP zeros $\lambda_1 > \lambda_2 > 0$. Without loss of generality, we can take A_0 to be diagonal,

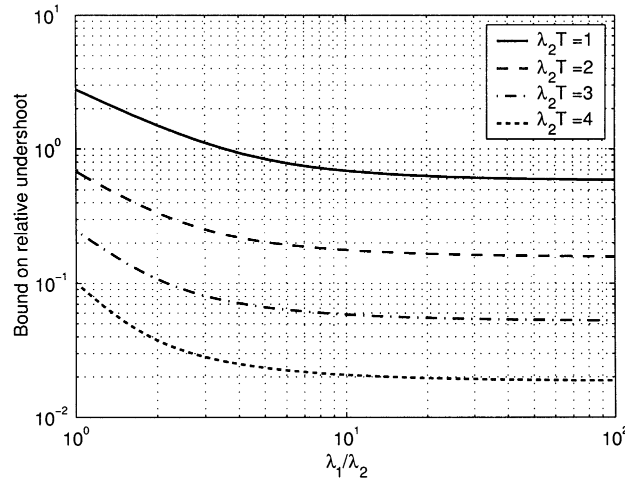

Fig. 1. Bound on $r_{\rm us}(y)$ given in (4) versus λ_1/λ_2 for $\lambda_1 > \lambda_2$ and $\lambda_2 T = 1, 2, 3$, and 4.

that is, $A_0 = \operatorname{diag}\{\lambda_1, \lambda_2\}$ and $B_0 = [1, 1]^T$. In this case, $\bar{z} = -A_0^{-1} B_0 \bar{y} = -\bar{y}[(1/\lambda_1), (1/\lambda_2)]^T$. We also use the notation $B_0^\perp = [1, -1]$ and the definition

$$z_\alpha(T) \triangleq -\alpha \int_0^T e^{A_0(T-\tau)} B_0\, d\tau \tag{9}$$

$$= -\alpha A_0^{-1}(e^{A_0 T} - I) B_0. \tag{10}$$

With this background, we state our first proposition.

Proposition II.3: For the second order zero dynamics system previously defined, any state $z(T)$ reachable in time $T > 0$ with $y \in \mathcal{Y}_\alpha, \alpha \geq 0$ must satisfy both

$$B_0^\perp (z(T) - z_\alpha(T)) \geq 0 \tag{11}$$

and

$$B_0^\perp e^{-A_0 T}(z(T) - z_\alpha(T)) \leq 0. \tag{12}$$

Proof: First, we note that using the Cayley Hamilton theorem (see, for example, [13, p. 167]), $e^{A_0 t} = I\phi_0(t) + A_0 \phi_1(t)$, where $\phi_0(t) = (\lambda_1 e^{\lambda_2 t} - \lambda_2 e^{\lambda_1 t}/((\lambda_1 - \lambda_2)$ and $\phi_1(t) = (e^{\lambda_1 t} - e^{\lambda_2 t})/(\lambda_1 - \lambda_2)$.

We prove (12). The proof of (11) is similar. From the definition of z_α (9) and the linear zero dynamics (6), we obtain

$$\begin{aligned} &B_0^\perp e^{-A_0 T}(z(T) - z_\alpha(T)) \\ &\quad = B_0^\perp e^{-A_0 T} \int_0^T e^{A_0(T-\tau)} B_0 (y(\tau) + \alpha)\, d\tau \\ &\quad = \int_0^T B_0^\perp (I\phi_0(-\tau) + A_0\phi_1(-\tau)) B_0 (y(\tau) + \alpha)\, d\tau \\ &\quad = B_0^\perp A_0 B_0 \int_0^T \phi_1(-\tau)(y(\tau) + \alpha)\, d\tau \leq 0 \end{aligned}$$

because both $B_0^\perp A_0 B_0$ and $(y(\tau) + \alpha)$ are nonnegative, and $\phi_1(-\tau)$ is nonpositive for $\tau \geq 0$. ∎

At first sight, there might not appear to be any correspondence between this result and the earlier result (4) based on Laplace transforms. However, as we now show, the two results give identical bounds on the undershoot.

Corollary II.4: If the conditions of Proposition II.2 are satisfied, then (12) implies (4).

Proof: Let $\rho = r_{\rm us}(y)$ and $\alpha = \rho\bar{y}$. Inequality (12), together with $z(T) = \bar{z} = -\bar{y} A_0^{-1} B_0$ and (10), gives

$$\begin{aligned} 0 &\geq B_0^\perp e^{-A_0 T}(z(T) - z_\alpha(T)) \\ &= B_0^\perp e^{-A_0 T}\left(-\bar{y} A_0^{-1} B_0 + \rho\bar{y} A_0^{-1}(e^{A_0 T} - I) B_0\right). \end{aligned}$$

Since $\bar{y} > 0$, and A_0^{-1} and $e^{-A_0 T}$ commute, this is equivalent to

$$0 \geq -B_0^{\perp} A_0^{-1} e^{-A_0 T} B_0 + \rho B_0^{\perp} A_0^{-1}(I - e^{-A_0 T}) B_0.$$

This implies, therefore, that since $B_0^{\perp} A_0^{-1}(I - e^{-A_0 T}) B_0 < 0$

$$r_{\text{us}}(y) \geq \frac{B_0^{\perp} A_0^{-1} e^{-A_0 T} B_0}{B_0^{\perp} A_0^{-1}(I - e^{-A_0 T}) B_0} = \frac{\frac{e^{-\lambda_1 T}}{\lambda_1} - \frac{e^{-\lambda_2 T}}{\lambda_2}}{\frac{1-e^{-\lambda_1 T}}{\lambda_1} - \frac{1-e^{-\lambda_2 T}}{\lambda_2}}$$

which is equivalent to (4). ■

We therefore see that in the case of a linear time invariant plant, the zero dynamics form allows the same results as those obtained using Laplace transforms for relating zeros, undershoot and settling time.

III. Performance Limitations for Nonlinear Systems With Unstable Zero Dynamics

In the previous section, the zero dynamics form of a linear system was used to derive a bound on the undershoot for a given settling time. This was achieved by finding a relationship between the relative undershoot at the output and the reachable states of the zero dynamics. In this section, we will show that these ideas can be extended to nonlinear systems with unstable zero dynamics.

Suppose that a nonlinear system has the form

$$\begin{aligned} \dot{\xi} &= F(\xi, z, u), \qquad \xi \in \mathbf{R}^l \\ \dot{z} &= F_0(z, y), \qquad z \in \mathbf{R}^m \\ y &= H(\xi) \end{aligned} \tag{13}$$

where u is the input and y is the output. We focus on the zero dynamics (13). Note that although we assume that (13) represents the full zero dynamics, for the purpose of this note, it is sufficient for this equation to describe part of the zero dynamics. The solution to the zero dynamics equation with the initial condition $z(0) = z_0$ shall be denoted $\phi(t, z_0, y)$.

The following assumptions will be made.

Assumptions:

A1) $\forall \bar{y} \in \mathbf{R}$, $\dot{z} = F_0(z, \bar{y})$ has a unique equilibrium point $\bar{z}$ which implies $0 = F_0(\bar{z}, \bar{y})$.

A2) $F_0(0, 0) = 0$.

We shall be concerned with the problem of taking the system from rest to the equilibrium at $y(t) = \bar{y} > 0$. This is equivalent to finding y, which satisfies the following constraints:

$$\lim_{t \to \infty} y(t) = \bar{y} > 0 \tag{14}$$

$$\lim_{t \to \infty} \phi(t, 0, y) = \bar{z}. \tag{15}$$

As stated previously, y has a *finite (exact) settling time* if $y(t) = \bar{y}\ \forall t > T$.

Definition 1 (Stability Definitions): The equilibrium point $\bar{z}$ previously defined is *unstable* if it is not (locally) asymptotically stable. It is *anti-stable* if $\dot{z} = -F_0(z, \bar{y})$ is (locally) asymptotically stable. The zero dynamics are unstable (antistable) if $\forall \bar{y}$, the corresponding equilibrium point is unstable (antistable). If $\bar{z}$ is unstable then the *stable manifold* $\mathcal{M}_{\bar{z}}$, corresponding to $\bar{z}$, is given by

$$\mathcal{M}_{\bar{z}} = \left\{ z_0 \in \mathbf{R}^m : \lim_{t \to \infty} \phi(t, z_0, \bar{y}) = \bar{z} \right\}. \qquad \square$$

Note that in the case where $\bar{z}$ is antistable, $\mathcal{M}_{\bar{z}} = \{\bar{z}\}$. Also, if $\bar{z}$ is globally asymptotically stable, then $\mathcal{M}_{\bar{z}} = \mathbf{R}^m$.

Recall that, for each $\alpha \geq 0, \mathcal{Y}_\alpha$ is the set of functions y which satisfy $y(t) \geq -\alpha\ \forall t \geq 0$.

Definition 2 (Reachability Definitions): Consider the system described by (13). For each triple (z_0, α, T) the *reachable set*, $\mathcal{R}_{z_0, \alpha, T}$ is the set given by

$$\mathcal{R}_{z_0, \alpha, T} = \{z^* \in \mathbf{R}^m : \exists\, y \in \mathcal{Y}_\alpha \text{ s.t. } \phi(T, z_0, y) = z^*\}$$

and a set $\mathcal{S} \subseteq \mathbf{R}^m$ is *reachable* if $\mathcal{S} \subseteq \mathcal{R}_{z_0, \alpha, T}$. A set $\mathcal{S}_u \subseteq \mathbf{R}^m$ is *unreachable* if $\mathcal{R}_{z_0, \alpha, T} \subseteq \mathcal{S}_u^c$, where $\mathcal{S}_u^c = \mathbf{R}^m \setminus \mathcal{S}_u$. $\square$

We observe that if y satisfies constraints (14) and (15), and $\bar{z}$ is unstable, then y must stabilise the zero dynamics by driving z to $\mathcal{M}_{\bar{z}}$. This leads to the following lemma.

Lemma III.1: Consider the system described by (13). Suppose that Assumptions A1) and A2) are satisfied, $\bar{y} > 0$ and y_1 satisfies constraint (15). Then, the following statements hold

a) If the open set $\mathcal{S}_u$ is unreachable $\forall y \in \mathcal{Y}_0$ (and $\forall t > 0$), and $\bar{z} \in \mathcal{S}_u$, then y_1 must undershoot.
b) If $T_{\text{es}}(y_1) = T$, and $\mathcal{M}_{\bar{z}}$ is unreachable at $t = T\ \forall y \in \mathcal{Y}_\alpha$, then $r_{\text{us}}(y_1) \geq \alpha/\bar{y}$.

Proof: The proof is immediate, in both cases, by contradiction. ■

We now apply the aforementioned result to the case of scalar antistable zero dynamics.

A. Scalar Zero Dynamics

1) General Case: Suppose that the zero dynamics satisfy

$$\dot{z} = F_0(z, y) = f_0(z) + g_0(z) y, \qquad z(0) = 0 \tag{16}$$

where $z \in \mathbf{R}$, $f_0(z)$ is continuous and increasing $(df_0/dz > 0$ almost everywhere), $f_0(0) = 0$, and $g_0(z)$ has constant sign $\forall z$. Without loss of generality, we take $g_0(z) > 0$. Note that the conditions on f_0 ensure that the system satisfies Assumptions A1) and A2).

Suppose that y is required to track a step of height $\bar{y} > 0$. Let the corresponding equilibrium point be $\bar{z}$. $\bar{z} < 0$ because $f(\bar{z}) = -g(\bar{z})\bar{y} < 0$. $\bar{z}$ is also anti-stable because f_0 is an increasing function. It follows that y must drive z to $\bar{z}$.

For this system, the following proposition holds [14].

Proposition III.2: Consider the previous system. Suppose that $y \in \mathcal{Y}_\alpha$ and let $z_\alpha(t)$ be the solution to initial value problem (16) with $y(t) = -\alpha$

$$z(t) \geq z_\alpha(t).$$

Proof: The proof [14, Prop. 2] is a direct application of the comparison principle. ■

Suppose that $y \in \mathcal{Y}_0$. When $\alpha = 0, z_\alpha(t) = 0$. Thus, from the proposition, $z(t) \geq 0\ \forall t$. But then $\bar{z}$ is unreachable, and so y must undershoot.

We can also quantify the required undershoot for a given exact settling time. Clearly, from the previous proposition, for a given permissible level of undershoot $\rho, y(t) = \rho\bar{y}$ takes z from 0 to $\bar{z}$ in minimum time. It was also shown in [14] that, as a consequence of this, $\bar{z}$ is unreachable for $t < T_{\text{es}}^*(\rho, \bar{y})$, where

$$T_{\text{es}}^*(\rho, \bar{y}) = \int_0^{\bar{z}} \frac{1}{f_0(z) - \rho\bar{y} g_0(z)}\, dz.$$

We note that $T_{\text{es}}^*(\rho, \bar{y})$ is the infimal settling time for a given ρ and $\bar{y}$. Let $r_{\text{us}}^*(T, \bar{y})$ be the infimal undershoot for a given $\bar{y}$ and settling time T. Since $T_{\text{es}}^*(\rho, \bar{y})$ is a decreasing (and, hence, one-to-one) function of ρ

$$r_{\text{us}}^*(T_{\text{es}}^*(\rho, \bar{y}), \bar{y}) = \rho \tag{17}$$

and so, $r_{\text{us}}^*(T, \bar{y}) = \hat{\rho}$, where $\hat{\rho}$ is the solution of $T_{\text{es}}^*(\hat{\rho}, \bar{y}) = T$.

2) Particular Example: Consider the particular example in which $f_0(z) = z^3$ and $g_0(z) = 1$, i.e., $\dot{z} = z^3 + y$. Then, $\bar{z} = -\sqrt[3]{\bar{y}}$

$$\text{and for } \rho > 0, \qquad T_{\text{es}}^*(\rho, \bar{y}) = \int_0^{\bar{z}} \frac{1}{z^3 - \rho\bar{y}}\, dz.$$

Let $b = \rho^{1/3}$. An expression for T_{es}^* can be derived as follows:

$$\begin{aligned} T_{\text{es}}^*(\rho, \bar{y}) &= \int_0^{\bar{z}} \frac{1}{z^3 + (b\bar{z})^3}\, dz \\ &= \frac{1}{3(b\bar{z})^2} \left[\ln|z + b\bar{z}| - \frac{1}{2} \ln|z^2 - b\bar{z}z + (b\bar{z})^2| \right]_0^{\bar{z}} \end{aligned}$$

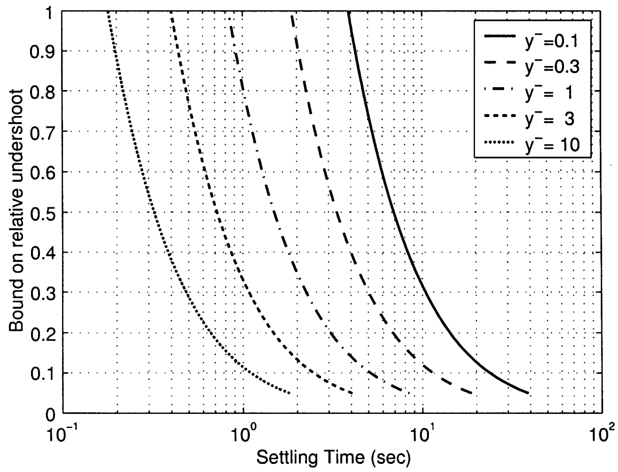

Fig. 2. Scalar example. Bound on relative undershoot versus settling time for several values of $\bar{\boldsymbol{y}}$.

$$
\begin{aligned}
&+ \frac{1}{2b\bar{z}} \int_0^{\bar{z}} \frac{1}{\left(z - \frac{b\bar{z}}{2}\right)^2 + \frac{3(b\bar{z})^2}{4}}\, dz \\
&= \frac{1}{6(b\bar{z})^2} \ln\left(\frac{(1+b)^2}{1-b+b^2}\right) + \frac{1}{\sqrt{3}(b\bar{z})^2} \\
&\quad \times \left[\tan^{-1}\left(\frac{2z - b\bar{z}}{\sqrt{3}b\bar{z}}\right)\right]_0^{\bar{z}} \\
&= \frac{1}{6b^2(\bar{y})^{2/3}} \left[\ln\left(\frac{(1+b)^2}{1-b+b^2}\right) + 2\sqrt{3}\right. \\
&\quad \left.\times \left[\tan^{-1}\left(\frac{2-b}{\sqrt{3}b}\right) + \frac{\pi}{6}\right]\right].
\end{aligned}
$$

It follows from (17) that a plot of $r_{\rm us}^*(T, \bar{y})$ as a function of T may be obtained by plotting ρ against $T_{\rm es}^*$. Several of these plots are shown in Fig. 2. Note that the bound on the relative undershoot increases for fast settling times and smaller $\bar{y}$ (slower zero dynamics). This is qualitatively similar to the linear case where the bound $1/(e^{\lambda T} - 1)$ is worse for fast settling and slow zero dynamics.

IV. Conclusion

NMP behavior can be understood in the linear and nonlinear case using the zero-dynamics formulation. In this formulation, the "constraints" imposed by plant NMP behavior can be examined. In particular, the permissible output behavior must drive the state of the zero dynamics onto the stable manifold. Furthermore, in cases where we wish to achieve this in a finite time, a lower bound on the required output deviation is imposed. For the case of scalar nonlinear NMP zero dynamics, we show fast settling and small undershoot are incompatible requirements. This is consistent with linear system conclusions for real NMP zeros.

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Selection of Variables for Stabilizing Control Using Pole Vectors

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***Abstract*—For a linear multivariable plant, it is known from earlier work that the easy computable pole vectors provide useful information about in which input channel (actuator) a given mode is controllable and in which output channel (sensor) it is observable. In this note, we provide a rigorous theoretical basis for the use of pole vectors, by providing a link to previous results on performance limitations for unstable plants.**

***Index Terms*—Actuator selection, control structure design, $\boldsymbol{H}$-infinity control, $\boldsymbol{H_2}$-control, input usage, linear systems, performance limitations, sensor selection.**

I. Introduction

Most available control theories consider the problem of designing an optimal multivariable controller for a well-defined case with given inputs, outputs, measurements, performance specifications, and so on. The following important *structural decisions* [14] that come before the actual controller design are therefore not considered.

1) Selection of inputs u (manipulated variables, actuators).
2) Selection of primary outputs y_1: controlled variables with specified reference values.
3) Selection of secondary outputs (measurements, sensors) y_2: Extra variables that we select to measure and control in order to stabilize the plant and achieve local disturbance rejection.

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