Journal for Research in Mathematics Education
2004, Vol. 35, No. 3, 164–186

Understanding Primes: The Role of Representation

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In this article we investigate how preservice elementary school (K–7) teachers understand the concept of prime numbers. We describe participants' understanding of primes and attempt to detect factors that influence their understanding. Representation of number properties serves as a lens for the analysis of participants' responses. We suggest that an obstacle to the conceptual understanding of primality of numbers is the lack of a *transparent* representation for a prime number.

Key words: Conceptual knowledge; Content knowledge; Number sense; Preservice teacher education; Representations; Teacher education; Teacher knowledge; Whole numbers

Prime numbers are often described as building blocks of natural numbers. The term *building blocks* can be viewed as a metaphorical interpretation of the Fundamental Theorem of Arithmetic, which claims that the prime decomposition of a composite number to prime factors exists and is unique. Although the uniqueness of prime decomposition presents a challenge for many learners, its existence is the property that is usually taken for granted (Zazkis & Campbell, 1996b). However, it is the existence property that is behind the building-blocks metaphor, creating an image of composite numbers being built up multiplicatively from primes. What are the structure and the properties of these building blocks?

There are two properties in particular that seem to present a mystery to the learner. One is the existence of infinitely many prime numbers, which entails very large primes. Another is the property that prime numbers are not generated by a simple polynomial function. In fact, mathematicians of different origins have struggled for centuries to discover a prime number generator. A few successes in this area have been recorded in the early 1970s (see for example Gandhi's formula in Ribenboim, 1996), but these developments present considerable mathematical complexity and are beyond the scope of our investigation.

Although the understanding of elementary number theory has been the topic of a few recent studies (Campbell & Zazkis, 2002), there has not been any study focusing specifically on prime numbers. On a related matter, Zazkis and Campbell (1996b) investigated preservice teachers' understanding of prime decomposition and concluded that "if the concepts of prime and composite numbers have not been

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The study reported in this article was supported in part by grant #410-2002-1120 from the Social Sciences and Humanities Research Council of Canada.

adequately constructed, this will likely inhibit any meaningful conceptualization of prime decomposition" (p. 217).

Understanding numbers, ways of representing numbers, and relationships among numbers have been identified by the National Council of Teachers of Mathematics as important aspects in the Number and Operations Standard for all grade levels (NCTM, 2000). We believe that the significance of understanding primes is in each of these three aspects. As building blocks, primes are crucial in understanding numbers and multiplicative relationships among numbers. Further, understanding primes relies heavily on the representation of numbers, as we illustrate in the next section.

In this article we describe and analyze preservice elementary school (K–7) teachers' understanding of prime numbers and attempt to detect factors that influence their understanding. We use representation of number properties, and what can be learned by considering representations of a number, as a lens for the analysis of participants' responses. We argue that the lack of a *transparent* representation of prime numbers presents an obstacle for students' understanding of prime numbers.

REPRESENTATION OF NUMBER PROPERTIES

A number of research studies in mathematics education have addressed the issue of representation (e.g., Cuoco, 2001; Goldin & Janvier, 1998; Janvier, 1987a). Various meanings have been attributed to the notion of representation and various distinctions made. However, only a small part of the work on representation has addressed representation of numbers, focusing primarily on fractions and rational numbers (e.g., Lamon, 2001; Lesh, Behr, & Post, 1987), or on negative numbers (e.g., Goldin & Shteingold, 2001). In the study reported here we focused on the representations of natural numbers. Prior studies dealing with the representations of natural numbers in the decimal number system focused on the canonical place-value representation and students' understanding of the place-value concept (e.g., Boulton-Lewis, 1998) as well as the connection between number-names and their representations (e.g., Fuson, 1990; Miura, 2001). Properties of natural numbers that can be expressed by various representations remain largely unexplored.

Lesh, Behr, and Post (1987) introduced the distinction between transparent and opaque representational systems. According to these researchers, a *transparent* representation has no more and no less meaning than the represented idea(s) or structure(s). An *opaque* representation emphasizes some aspects of the ideas or structures and de-emphasizes others. Meira (1998) referred to this distinction in consideration of instructional devices and suggested that transparency is not a feature of an artifact per se but of its use in a specific instructional activity. His research suggested that acknowledging transparency in use supports learning.

Borrowing the terminology used by Lesh and his colleagues (1987) in drawing the distinction between transparent and opaque representations, Zazkis and Gadowsky (2001) focused on representations of numbers and suggested that all such representations are opaque; however, they may have transparent features. For example, representing the number 784 as 28^2 emphasizes that this is a perfect square but de-emphasizes the divisibility of this number by 98. That is, in representing 784 as 28^2, the property of 784 being a perfect square is *transparent* and the property of 784 being divisible by 98 is *opaque*. Representing the same number as $13 \times 60 + 4$ makes it transparent that the remainder of 784 in division by 13 is 4, but makes opaque its property of being a perfect square. Zazkis and Gadowsky addressed the importance of students' awareness to properties of numbers embedded in their different representations and suggested several instructional activities to promote this awareness.

In this article we extend the representation of specific numbers to the representation of sets of numbers possessing the same property through the use of algebraic notation. We say that, for a whole number k, $17k$ is a *transparent* representation for a multiple of 17, as this property is embedded or "can be seen" in this form of the representation. However, it is impossible to determine whether $17k$ is a multiple of 3 by considering the representation alone. In this case, we say that the representation is *opaque* with respect to divisibility by 3.

It often happens that the definition for a set of numbers relies on the existence of certain representation. For example, a rational number is a number that can be represented as a/b, where a is an integer and b is a nonzero integer. Furthermore, the existence of a certain transparent representation can serve as a discriminating property. For example, a number is even if and only if it can be represented as $2k$ for some whole number k. A number is a perfect square if it can be represented as k^2 for some whole number k. Similarly, we consider $5k$ as a transparent representation of multiples of 5 and $17k + 3$ as a transparent representation of numbers that leave a remainder of 3 when divided by 17. Finding an appropriate representation for a given property is crucial for manipulating representatives of this set.

In learning elementary number theory, consideration of the properties of oddness/evenness or divisibility of numbers is naturally accompanied by consideration of the property of primality. However, there is no transparent representation for prime numbers. We often denote a prime number by p, but this representation is opaque in every regard. Primality cannot be derived from this representation in a way that, for example, oddness of a number can be derived from the representation $2k + 1$. How does this lack of transparent representation influence students' understanding of primes? In order to address this issue we first need to examine the significance of the roles that representation plays in acquisition of mathematical knowledge.

ROLES OF REPRESENTATION IN MATHEMATICS

The issue of representation is not new to mathematics education. However, it has recently attracted fresh attention and examination in mathematics education research and practice (Cuoco, 2001; Goldin & Janvier, 1998). In the study reported here we are interested in representations that unravel properties of numbers. We consider the *standard* representational systems of symbols and operations in arithmetic and algebra, such as decimal representation of numbers and letter representation of variables. Historically, the achievement of representation we denote today as standard was a lengthy process guided by need and human ingenuity. However, for today's purposes we consider these representations as *tools* for learning and communicating mathematics. The following summary draws on a variety of ideas presented in prior research (e.g., Cuoco, 2001; Goldin, 1998; Goldin & Janvier, 1998; Goldin & Kaput, 1996; Janvier, 1987a; Skemp, 1986).

Tools for Manipulation and Communication

Representation is often presented as a tool for manipulating objects. Having a representation in hand allows an individual to detach himself or herself from the meaning of this representation and operate on the symbols alone, making the manipulations automatic, and returning later to interpreting the result of the symbolic manipulation (Skemp, 1986). Further, the nature of the manipulation to be performed may influence the choice of representation. For example, multiplication of large numbers is better manipulated when these numbers are represented in Hindu-Arabic, rather than in the Roman, numeration system. Likewise, for the purpose of multiplying complex numbers, the polar representation is preferable to the ordered-pair representation of such numbers.

Communication is often mentioned as an important role of representation (e.g., Kaput, 1991; Skemp, 1986). As a tool for communicating, representations serve a dual purpose: they help in the communication of ideas, and they help in the communication between individuals. However, a representation itself is just a string of symbols. Instead, representation comes to life when learners map the symbols to the mathematical notions. This mapping is a two-way street in that it enables the learner to communicate ideas more efficiently and it enables the learner to recognize and interpret what ideas are being communicated by the symbols. Moreover, availability and awareness of shared representations—be it among classmates or among research mathematicians—creates a social milieu for mathematical discourse.

Tools for Conceptual Understanding

An important role of representation in mathematics is as a tool for thinking and gaining insights (Diezmann & English, 2001; Kaput, 1987). Researchers have drawn strong connections between the representations that students use and their understanding (Friedlander & Tabach, 2001; Lamon, 2001), with understanding connected to the ability to apply various representations and to choose one that is appropriate to the problem situation. Janvier (1987b) describes understanding as a "cumulative process mainly based upon the capacity of dealing with an 'ever-enriching' set of representations" (p. 67). Furthermore, representations are often considered as a means to form conceptual understanding. The ability to move smoothly between various representations of the same concept is seen as an indication of conceptual understanding and also as a goal for instruction (Lesh, Behr, & Post, 1987). Likewise, research by Even (1998) suggests that knowledge of different representations is intertwined with knowledge of underlying notions and the context.

Since acting on mathematical objects promotes the construction of corresponding mental objects (Dubinsky, 1991; Sfard, 1991), representation aids learners in their mental constructions. Representations are also described as tools for generalization and abstraction in that expressing generality can be achieved by an appropriate choice of representation. Moreover, according to Kaput (1991), possessing an abstract mathematical concept "is better regarded as a notationally rich web of representations and applications" (p. 61).

METHODOLOGY

Participants and Setting

The study reported in this article is part of ongoing research on the learning of elementary topics in number theory by preservice elementary school teachers (e.g., Zazkis, 1998b; Zazkis, 2000; Zazkis & Campbell, 1996a, 1996b; Zazkis & Liljedahl, 2002). In this ongoing research we shadow the learners in their mathematics courses, examine their written work, and invite volunteers to participate in clinical interviews. In our analyses of extensive amounts of data, various themes emerge that are worthy of separate focused attention. In this article we present fragments of this work that attend to one such theme: prime numbers and representations. These fragments draw on data collected from university students enrolled in a mathematics course "Principles of Mathematics for Teachers," which is a core course for teacher certification at the elementary (K–7) level.

The course involved 4 hours per week of classroom instruction that were followed by an open lab tutorial, where teaching assistants were available to address students' questions and to engage them in additional problems or activities. Despite the limitations of a large class of 116 students, the instruction was interactive, and group work was implemented during class time as well as in the preparation of students' assignments. The usual practice in the course was for the instructor to engage students in a mathematical activity or a problem and then draw conclusions or discuss concepts with references to the activity. For example, students were asked to list all the possible arrangements of n objects ($1 < n < 40$) in a rectangular array. Figure 1 demonstrates four possible arrangements for 6 objects and two possible arrangements for 5 objects.

The notions of prime and composite number, factor, multiple, and divisibility were discussed and formally defined with a reference to this rectangular array activity. Other topics in the chapter on number theory, which were studied for the period of 3–4 weeks, included divisibility rules, prime decomposition and the Fundamental Theorem of Arithmetic, and least common multiple and greatest common divisor. The data were collected from students after the completion of this chapter on elementary number theory.

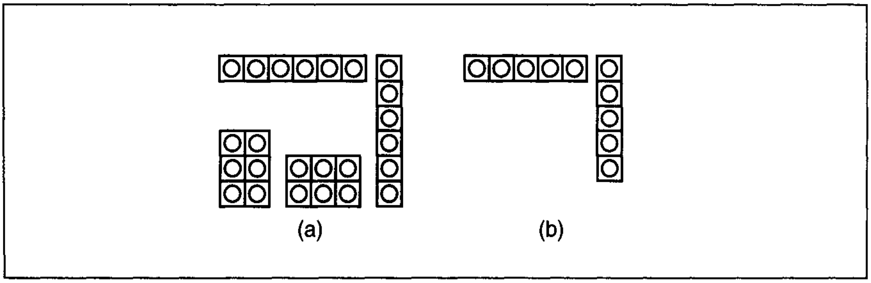

Figure 1. Rectangular array arrangements for 6 objects (a) and 5 objects (b).

Questions

Subject matter knowledge is essential in learning to teach for understanding (e.g., Ball, 1996). As such, in designing questions we considered what preservice teaches' knowledge of prime numbers could and should entail. Moreover, we build on the properties of *connectedness* and *basic ideas*, as discussed by Ma (1999) as being among her indicators of Profound Understanding of Fundamental Mathematics (PUFM).

The fact that prime numbers are a basic idea, or building block, of number theory was a starting point for this research. Further, this basic idea cannot be approached in isolation because understanding of a mathematical concept presents a complex web of relations and connections with other concepts. As such, a student's understanding of a prime number is connected to the understanding of multiplicative relationships among natural numbers, that is, of factors, multiples, composite numbers, and divisibility. Therefore, we believe that an understanding of a concept of a prime number by an elementary school teacher should include at least the following:

a) Awareness that any natural number greater than 1 is either prime or composite and the ability to cite and explain the definition of a prime number;
b) Understanding that if a number is represented as a product it is composite unless the factors are 1 and a prime; and
c) Awareness that composite numbers have a unique prime decomposition and that the number of primes is infinite (though not necessarily the ability to provide a formal mathematical proof for these claims).

For the purpose of this study, we designed the following questions and analyzed participants' responses to them.

(1) How do you describe a prime number? A composite number? What is the relationship between prime and composite numbers?
(2) Consider $F = 151 \times 157$. Is F a prime number? Circle YES/NO, and explain your decision.

(3) Consider $m(2k + 1)$, where m and k are whole numbers. Is this number prime? Can it ever be prime?

Our choice of questions related primarily to the understanding listed above in that Question 1 related to (a) and Questions 2 and 3 related to (b), although some aspects related to (c) surfaced in the students' responses. Question 1 was initially included in the interview as an easy "warm-up" question, intended to create a conversation and an atmosphere of cooperation. However, the way in which students describe a concept provides a window into their understanding. Therefore, Question 1 was analyzed for the ways in which students described prime numbers. Although students learned the formal definition, we expected to find in their responses their ways of thinking about prime and composite numbers rather than, or in addition to, the citation of the definition. Furthermore, we were interested in the relationships that participants identify between prime and composite numbers.

Question 2 requires no work because the number F is represented as a product of natural numbers. We were interested in seeing to what degree this representation would play a role in the responses of participants.

Question 3 is also concerned with a number represented as a product. However, since the product is represented in algebraic notation, whether or not this product is trivial is not specified by constraining the values of k and m; that is, we consider a product of two numbers to be trivial if one of the factors is 1. In this case, the product can be trivial only if m is 1 or if $2k + 1$ is 1. Similar to Question 2, the focus of the analysis of responses to this question was on the participants' attention to representation.

Data Collection and Analysis

Question 2 sought a written response from 116 students. Questions 1 and 3 were presented in an interview setting to a subset of 18 volunteers. Because group work was an essential component of the course, several students requested to be interviewed together with their team members. The researchers responded positively to these students' request, which resulted in the 18 volunteers being distributed into two groups of 3, one group of 2 and 10 individual interviews. However, the differences between individual responses and group interaction are not analyzed here.

Calculators were available to students during the data collection for this study. This was consistent with the availability of calculators during their coursework. Despite the unrestricted availability of calculators, we urged participants to use them only where appropriate.

Students' responses were compiled according to the common themes that emerged within each question. Furthermore, building on the analysis of the students' responses across the three questions, we suggested possible avenues for constructing primality as a mental object.

RESULTS AND ANALYSIS

Question 1

In Question 1, participants in the interview were asked to describe the meaning of prime and composite numbers and to describe further how they saw the relationship between them. Prior to this interview, the formal definition of prime numbers presented to students in the textbook and by the instructor was this: Prime numbers are numbers that have exactly 2 factors. A class discussion took place to clarify whether this definition was equivalent to what students recalled from their elementary school experience; that is, prime numbers are those that are divisible only by 1 and itself. However, the phrase "divisible only by 1 and itself" created ambiguity in the consideration of the primality of number 1. It was pointed out to the students that by mathematical convention the number 1 is not considered to be prime, and therefore "having exactly 2 factors" was a more accurate indicator of the property of primality.

Meaning of prime and composite numbers. All the interviewed participants provided a reasonable description of prime and composite numbers. The wording of the question—how do you *describe* a prime number—did not imply that the formal definition was expected. Eight students used a variation of a formal definition (referring to exactly 2 factors or 2 distinct factors), whereas the other 10 used a variation of a definition familiar from their prior schooling (divisible by 1 and itself). Two interrelated themes emerged in participants' responses— providing a *negative* description and supplementing the learned definition with personal understanding. We exemplify these tendencies with these excerpts from the interviews; all names that follow are pseudonyms.

Sally: Prime numbers cannot be divided by anything other than 1 and itself. Composite numbers are not prime numbers. You would have 1 and itself and you would have other um, what do you call them . . . multiples?

Tom: Primes are those that cannot be factored, yea, like cannot be factored any further. Composite numbers are like, you can always factor out primes out of them.

Karen: Prime number would be a number that would be divisible only by 1 and itself, it means it wouldn't have any other factor, besides these two factors, 1 and itself are the only factors. Composite numbers would have other factors besides these two, these two you would always have.

Jenny: Prime numbers have only two factors. They are not having other factors to be broken down into. Composite numbers have more than two factors, it means just one more and it makes the number composite.

Sally and Tom appeared to think of prime numbers in terms of properties these numbers do not hold in that that the numbers "cannot be divided" or "cannot be factored." We refer to these as *negative* descriptions. Karen's response contains the formal definition, but it is accompanied by an additional negative explanation, "wouldn't have any other factor." Likewise, in Jenny's description the formal definition is cited, but also supplemented with further explanation of what prime numbers "are not having." For Karen and Jenny, the definition itself is not sufficient for creating and communicating meaning, and as such, it is supplemented with an additional explanation that attempts to interpret the definition. Negative descriptions appeared in the responses of 15 out of 18 participants either in their initial description of prime numbers (e.g., Sally and Tom) or in their additional explanations (e.g., Karen and Jenny).

Relationship between prime and composite numbers. The following excerpts from two individual interviews with Jenny and Sally are examples of students' descriptions of this relationship.

Interviewer: And how would you describe the relationship between prime numbers and composite numbers?

Jenny: A composite number is made up of prime numbers.

Interviewer: Made up? In what way?

Jenny: It can be decomposed into prime numbers, like broken down into.

Interviewer: When you say decomposed, what do you mean?

Jenny: Breaking it down into smaller numbers that when put together they come up to the bigger number.

Interviewer: Put together?

Jenny: Multiplied together.

[...]

Interviewer: How would you describe the relationship between prime and composite numbers?

Sally: Well, not really, there is no relationship, they are one or the other. If you have a number and it's prime you know that it's not composite.

In Jenny's response we recognize her awareness of the existence of prime decomposition. In fact, the majority of the participants (12 out of 18) made some reference to prime decomposition. Sally sees prime and composite primarily as disjoint sets: "They are one or the other." This theme appeared in the responses of 10 participants, either exclusively or in conjunction with the allusion to prime decomposition. Although for this particular question Sally's view appears incomplete, it is exactly the understanding that is necessary to deal with Question 2 in an elegant manner.

Question 2

Question 2 asked students to determine whether the number F, given as $F = 151 \times 157$, was prime and to explain their decision. Table 1 provides a summary of students' responses to Question 2. Since students were asked to circle their decision (YES/NO), it was unambiguous how to classify their choice as correct or incorrect. However, it was the students' explanations of their choice, rather than the choice itself, that revealed their understanding of primes. We identified and clustered the justifications students provided for their decision about the primality of number F. In Table 1, *correct* (C) or *incorrect* (I) refers to the claim itself, rather than the justification provided to support the claim. In the sections that follow, we discuss the categories of responses that students gave to Question 2.

Table 1
Summary of Responses to Question 2

Consider $F = 151 \times 157$. Is F a prime number? Circle YES/NO and explain your decision.			
Correct claims (NO)			74
Justification:	C(1)	Definition of primes	33
	C(2)	Definition of composite number	19
	C(3)	Application of algorithm	14
	C(4)	Lack of closure	2
	C(5)	Reasoning by example(s)	3
	C(6)	Other	3
Incorrect claims (YES)			42
Justification:	I(1)	Product of primes is prime	24
	I(2)	Misapplication of an algorithm	8
	I(3)	Misapplication of divisibility rules	6
	I(4)	Other	4

Definition of prime and composite numbers. As shown in Table 1, a total of 74 out of 116 students (64%) have claimed correctly that F was a composite number. However, not all the correct answers were accompanied by correct justifications. The majority of the arguments ($n = 52$) used either the definition of prime numbers ($n = 33$; C(1) in the table) or the complimentary definition of composite numbers ($n = 19$; C(2) in the table). The following excerpts exemplify students' responses in these two categories.

C(1): A prime number has exactly 2 factors, 1 and itself. $F = 151 \times 157 = 23707$, thus it is not prime because 151 and 157 are its factors, as well as 1 and 23707.

C(1): Prime numbers are only divisible by 1 and themselves. F will be divisible not only by 1 and F but also by 151 and 157. Therefore F is not prime.

C(2): 151 and 157 are both prime. Therefore they are prime factors of their product, making their product a composite number.

C(2): Because it is composed of at least 2 factors, 151 and 157, other than one and itself. Therefore it is composite.

It is interesting to note here that 20 out of 52 arguments classified as either C(1) or C(2) involved considerations that are not essential for answering this question; it is unnecessary to calculate the number F to address the question. Furthermore, the claim that 151 and 157 are both prime, although correct, is irrelevant to the case. In some responses this claim was not only mentioned in passing, but also the property of primality of 151 and 157 was derived after students engaged in a considerable amount of calculations, following the learned algorithm for determining primality or some variation of it. This time and energy investment suggests that primality of 151 and 157 was important to these students in making their decision about F.

Application of algorithm. In this category, which we call C(3), the arguments that accompanied correct answers were based on an application of an algorithm for determining prime numbers, carried out for the number 23707. As we reported earlier in this article, students had access to calculators while working on this question. An example of a participant's response follows.

C(3): $\sqrt{23707} = 153.9$. Now we check if any of the prime numbers lower than 153 divide 23707. 23707 is divisible by 151 and 157 so it is not prime.

In this case, the algorithm was applied correctly and lead to a correct conclusion. However, the need for the algorithm is somewhat troublesome because it shows that these students could not conclude that *F* was a composite number from considering its representation as a product. Furthermore, the ability to determine divisibility of *F* by 151 only after checking by division implies that these students did not have a strong connection between divisibility and multiplication, a finding that is consistent with findings of prior research (Zazkis & Campbell, 1996a).

Lack of closure. Two students based their argument on the lack of closure of prime numbers under multiplication (labeled C(4) in the table). An example of this argument follows:

C(4): *F* is not prime because 151 and 157 are primes and the set of prime numbers is not closed under multiplication.

This response may appear as a rather impressive application of recently acquired terminology. However, a closer look reveals an incorrect logical implication. Although the claim that primes are not closed under multiplication is correct, the use of this claim to determine that *F* is not prime is inappropriate. Lack of closure means that there exists at least one pair of elements in the set such that the result of a binary operation applied on this pair is not an element in the set. This does not imply that the result of a binary operation applied to *any* two elements does not belong to the set.

Reasoning by examples. There were 2 students who based their arguments on considering examples, which we classified as C(5) responses. One example appears below:

C(5): $2 \times 3 = 6$, $5 \times 7 = 35$, $2 \times 5 = 10$, $5 \times 3 = 15$

All of these examples illustrate that when you multiply a prime number by a prime number your result is not a prime number. Prime numbers are 2, 3, 5, 7, 13, 17, 19, ... None of the products were prime numbers.

This is a clear illustration of an empirical inductive proof scheme (Harel & Sowder, 1998). That is to say, what convinces the student is a consideration of a finite number of examples, rather than *F*'s representation as a product. We note here that all the examples in this student's illustration are examples of familiar *small* primes, and we will return to this observation later.

We turn now to describing students' arguments justifying the incorrect claim that *F* is prime. As shown in Table 1, this claim was made by 42 out of 116 participants (36%).

Product of primes. Out of these 42 participants, 24 claimed that *F* was a prime number based on the primality of 151 and 157. An example of responses that we called I(1) follows:

I(1): Both 151 and 157 are prime numbers, and 2 prime numbers multiplied together are going to give another prime number

Of these 24 participants, 8 simply claimed that 151 and 157 were prime, whereas the other 16 explained in detail how they confirmed the primality of these two numbers. The absurdity of this argument could be an indication of a profound psychological inclination toward closure, that two of a kind produce a third of the same kind.

Misapplication of an algorithm. In responses classified as I(2), 8 students calculated the product of 151 and 157 to be 23707 and then tested for primality of this product. The approach of 7 of these students was similar to the algorithmic approach in C(3). The students followed the algorithmic part of what they learned in that they determined the square root of 23707 and then attempted to perform division of 23707 by all primes smaller that the square root. However, unlike students applying the correct approach in C(3), they failed to apply the algorithm to its full extend, considering only a partial list of prime numbers. In some cases, the list of primes resulted in using familiar primes up to 19 or 29. In other cases, it is difficult to determine which primes were considered (because the list was not explicitly mentioned) to reach the conclusion. An example of an I(2) response is this:

I(2): $151 \times 157 = 23707$, $\sqrt{23707} = 154$

Check all prime numbers lower than 154 to see if the number is prime and if none of them can divide 23707 then the number is prime.

One student used her knowledge of divisibility rules to check primality of 23707. She checked all the familiar divisibility rules, that is, rules for divisibility by 2, 3, 4, 5, 6, 8, 9, and 10 and concluded that *F* was prime.

In the responses included in this category we identify an implicit belief that if a number is composite, it must be divisible by a small prime. Similar observations were reported by Zazkis and Campbell (1996b). This belief seems to co-exist with the explicitly stated awareness that in order to determine primality of a number, all the primes smaller than its square root must be checked. This belief also co-exists with the awareness of existence of "very large" prime numbers and infinitely many prime numbers.

Misapplication of divisibility rules. Another justification of the primality of *F* was based on the misapplication of divisibility rules—I(3) in the table. For example:

I(3): It is prime because the last digit in the number is 7 and the sum of the digits is the number 19. 19 is a prime number and is not divisible by anything but itself and 1. So F is prime.

Consideration of the sum of the digits is a common test for divisibility by 3 and by 9. In the aforementioned response, this divisibility test was overgeneralized to claim that the number was prime based on the primality of the sum of its digits. The last digit is also mentioned, but not actually used in the argument. In several other cases, students justified the primality of F by the primality of its last digit. Although the number of students making these types of claims is relatively small (5%), their arguments confirm the observations of prior research, which reported several similar cases of incorrect generalization of divisibility rules (Zazkis & Campbell, 1996a).

In category I(4), we clustered other students' arguments that were either unique in this group or cases where the argument was incomplete or a student' strategy was inconclusive based on the written response. One of the arguments in this category requires further attention:

I(4): The product of two odd numbers is always odd. Therefore making F odd too.

Although only one student demonstrated confusion between *prime* and *odd* in response to Question 2, a similar confusion surfaced in other encounters as well, and we discuss these next.

Question 3

In Question 3 students in a clinical interview setting were asked to comment on whether or not a number represented by $m(2k + 1)$ could be prime. This question is similar to Question 2 in that it asks the students to consider the primality of a number presented as a product. However, the explicit use of algebraic notation to represent the number limited the possibility of applying any of the algorithmic methods familiar to the participants. Without the option to regress to algorithms, two main themes surfaced in participants' responses: (a) attention to definition (or a perceived definition) of prime and composite numbers and (b) the convincing power of examples.

We note here that the participants tended to change their minds—at times more than once—during the interview. Those interviewed in a group setting were at times persuaded by their group members to change their responses. Others did so as a result of prompting by the interviewer. Thus, rather than summarize frequencies of occurrences, in the next section we illustrate the two main themes.

Definition of prime and composite numbers. In the following excerpt from a group interview, Dina and Sally attempt to convince Dan that the number $m(2k + 1)$ cannot be prime.

Dina: Primes are only divisible by 1 and itself, so. . . , yeah, yeah, so if you were just to write it out, like that could be 1, m, something times m and then the number itself right? It would have 4, at least 4 yeah.

[...]

Sally: Because you're multiplying it in, [pause], because this is just saying *m* times this right, do you know what I'm saying? It would be *m* times this to be the final product right, so *m* would be one, and then this would be another, and then 1 and whatever the answer of this would be right, so it couldn't be prime because if it were prime it would just be this times this, right. You've got these two things in the middle, wouldn't it? Do you know what I mean, do you know what I'm doing here?

Dan: No.

Sally: Oh, I don't know like, uh, I don't know what it's called, but you know when you like, let's say you're going to say 6 is 2 times 3...

Dan: Factors. You mean factors.

Sally: Factors, yeah it's going to look like that...

Dina: Because if it were prime, it would only be multiplied by 1 and it, its factors would be, prime factors would be 1 and itself.

Dan: You're saying it has factors...

Sally: It has other factors, so it can't be prime...

Interviewer: And those are?

Sally: m and $2k + 1$...

Dina: Yeah, because that's the whole point of prime numbers.

Dina claims in the very beginning of the excerpt that "it would have 4"—implying 4 factors. Sally communicates the same idea by explicitly listing the factors as 1, m, $2k + 1$, and $m(2k + 1)$. She identifies "These two things in the middle"—pointing to m and $2k + 1$—as additional factors, leading Dina and Sally to the conclusion that the number cannot be prime. This view of primes is incomplete, rather than incorrect. In Question 2, representation of F as a product 151×157 assisted participants in determining that F was a composite number. In Question 3, for Dina and Sally, representation of the number as a product presented a distraction, rather than an asset. This is consistent with Tom's perception in Question 1 that "prime numbers cannot be factored," in ignoring the possibility of trivial factorization.

Considering a prime as a number that is only divisible by 1 and itself, it was a popular choice in this group of participants to assign 1 to m. For Karen, however, this choice appeared sufficient, and her partner Debra accepted the argument.

Karen: I was trying to make the tree, and I thought that I couldn't... but if *m*, if any number besides 1, you can start making a tree.

[...]

Debra: Ok, just start doing it because I'm not quite following you.

Karen: If you have this number and *m* is 3, you can immediately factor out a 3 right?

Debra: Okay yeah.

Karen: But I'm, that means if it's, *m* is 2 you can immediately factor out a 2, if *m* is 4 you can immediately factor out a 4. However, if *m* is 1 you can't, there's a possibility that you can't, actually there's maybe a certainty because this is odd, the $2k + 1$ is odd that you can't start factoring out anything if m is 1. It freezes this as an odd number. So my proposal is, yes it's prime if *m* is 1.

Debra: You're right.

Karen's first inclination is related to a familiar procedure of making a factor tree. She noticed that in order to "factor out" m, the value of m should be "any number besides 1." This leads her to the conclusion that m should be 1 in order for $m(2k + 1)$ to be prime. Further, Karen recognizes $2k + 1$ as a representation of an odd number. However, she could be, at least temporarily, confusing the notions of prime and odd—a theme that emerged several times in our investigation.

Following the same view of primes as numbers divisible only by 1 and itself, Nora concluded that the number is prime if m is 1 and $2k + 1$ is "itself."

Nora: Primes you cannot break, you can just write it as 1 and itself, like the number itself, like 7 is 1 times 7, but nothing else. This will be prime if you can't break it any further. So if here you have here $m(2k + 1)$, to be prime m should be 1 and this [$2k + 1$] should be itself.

She further developed several examples by solving for k several equations of the kind $2k + 1 =$ <prime>. Paul, on the other hand, identified two options:

Paul: It's possible because then it would only be, like $2k + 1$ as one factor, if $2k + 1$ is a prime number and m is 1, or of m is a prime number and $2k + 1$ equals 1, then it would also work, it would also be prime.

However, this complete argument was provided only after some prompting from the interviewer. Paul's initial attempt was to represent the number $m(2k + 1)$ as $2km + m$, which led him astray. Only after refocusing on the original representation as a product was Paul able to provide a comprehensive solution.

Convincing power of examples. Different perspectives on the role of examples in learning mathematics have been addressed by researchers (Mason, 2002; Mason & Pimm, 1984; Rissland, 1991; Wilson, 1990; Zaslavsky & Peled, 1996). In particular, the convincing power of examples was acknowledged by Harel and Sowder (1998) as an empirical inductive proof scheme. Although it is sufficient to show an example to support the claim that the number $m(2k + 1)$ can be prime, the way in which students found such an example reveals their understanding of prime numbers. The following two excerpts from individual interviews with Sharon and Lisa illustrate the search for examples.

Interviewer: Our next number is $m(2k + 1)$, m is a whole number, k is a whole number, can a number written like this be prime?

Sharon: [Pause.] Um, okay, I'm just looking at the 2 here and I just thought if it can be prime, say if k was a 2 and $2 \times 2 = 4$, plus 1 equals 5, and if m was 1 then yeah, it could work.

Interviewer: So you showed that when $m = 1$ and $k = 2$ we get a prime number. Good. Are there other examples?

[Sharon substitutes different values for k.]

Sharon: [Pause.] So it works for 3, doesn't work for 4, works for 5 and 6, not for 7, I don't know, doesn't seem to be a rule here.

[...]

Interviewer: Let's ask a different question. Our number is $m(2k + 1)$.

Lisa: Okay.

Interviewer: Can it be a prime number?

Lisa: I guess it would depend on what you put in for k and for m.

Interviewer: Okay.

Lisa: If you put 2 for m, I figure because you're multiplying something by 2 then it automatically means that it's divisible by 2, which would in my brain make it not prime and then it would be divisible by more than . . .

Interviewer: Okay.

[Lisa tries several examples at random.]

Interviewer: So now you're trying 5 for k and 6 for m, any insights?

Lisa: Well, it's looking to me like it's not going to get prime.

Both Lisa and Sharon turned to examples as their first choice of strategy. However, Sharon's first choice was a lucky one. Lisa, on the other hand, attempts several substitutions at random, concluding that "it's not going to get prime." Despite the fact that Sharon answers correctly and Lisa does not, their understanding of primes appears to be similar. What makes the difference in their responses is Sharon's systematic approach in choosing numbers to substitute. However, her system seems to be guided by neatness and discipline, rather than understanding of primes. Such understanding is demonstrated, for example, by Nora's response in acknowledging the fact that $2k + 1$ should be prime and calculating k from there. It is also interesting to note that Lisa immediately knew that substituting 2 for m will result in an even number that would definitely not be a prime number. However, this realization did not prevent her from substituting 6 for m minutes later.

In the next excerpt with Wanda, the power of examples is even more explicit. Having observed Wanda's quick substitution of numbers to produce a prime result, the interviewer asks her to consider a different argument, provided by another student.

Wanda: Yes, it is possible. I substituted 5 for k and ended up with a prime number. But k doesn't have to be prime, just a whole number, so I'm just saying 2×4 is 8, +1 is 9 which is not prime. So that just means it can be prime, but it isn't always.

Interviewer: Okay, you found an example where it is possible. A student was here earlier and she claimed that this number cannot be prime because it is divisible by m and by $2k + 1$ and it is always divisible by 1 and by itself, so it has all these factors so it cannot be prime, what would you tell her?

Wanda: I would try and show her examples where it didn't work that way.

Interviewer: Examples of what?

Wanda: I would use the example that I just found where it does end up being prime. It's not such abstract thinking, it's concrete numbers. But then in terms of explaining why her reasoning didn't work, I don't know what I would do.

Wanda claims she would convince her classmate by showing her examples that contradict her claim. However, she does not attend to the flaw in the reasoning, explicitly claiming "I don't know what I would do."

In summary, considering primes as numbers that cannot be represented as a product ignores the possibility of trivial factorization. Acknowledging trivial factorization served for some students as a guide in generating examples. Others were led to believe that the number represented by $m(2k + 1)$ cannot be prime or turned to a guess and check by substitution strategy.

UNDERSTANDING PRIMES: WHAT HAPPENS WITHOUT REPRESENTATION

The previous sections presented participants' responses to each of the three questions separately. In this section, we comment on their responses to the three questions in an integrated manner. We provide a short summary of the results and then consider possible approaches to understanding primality.

As mentioned previously, the subset of participants who responded to Question 1 provided a reasonable description for a prime number. It is not surprising that all knew what prime numbers were, and it is reasonable to expect that the results would have been similar had Question 1 been posed to each of the 116 participants. However, the responses to Question 2 and 3 indicate that this knowledge is often not implemented in practice. In fact, only 52 out of 116 students (45%, combining C(1) and C(2)) implemented their knowledge in responding to Question 2. Thus, understanding of primes appeared incomplete, inconsistent, fragile, algorithm-driven, and significantly influenced by particular examples.

Treating mathematical concepts as objects supports construction of corresponding mental objects in the mind of students (Dubinsky, 1991; Sfard, 1991). One possible way to treat concepts as objects is to involve them as inputs in mathematical processes, that is, to act on them or to perform operations on them. In order to act on representatives of certain sets of numbers, representation is an asset. For example, in order to consider the sum of two odd numbers, we represent them as $2k + 1$ and $2m + 1$ and perform the operation of addition acting on this representation. Of course one can use symbols such as x and y for the two odd numbers, but in this case, no conclusion about the parity of the sum can be drawn from considering $x + y$.

Researchers agree that achieving an object conception of mathematical concepts is challenging (e.g., Sfard, 1991). Supported by the data and informed by the theories of object construction and the role that representation may play in constructing mental objects, we suggest that the lack of transparent representation for prime numbers creates an obstacle for acting on them, and, thus, creates an additional difficulty for constructing a mental object. In the context of our study, an object conception for a prime number is consistent with responses categorized as C(1) and C(2) to Question 2 (see Table 1) and with Paul's and Nora's responses to Question 3. The majority of students, however, appear not to have reached this stage. We next suggest three interrelated approaches in which students construct their understanding of prime numbers.

Primality as an Outcome of Factoring

One way to construct understanding of primality is as an outcome of the process of factoring natural numbers. If a number has a nontrivial factor, where trivial factors are 1 and the number itself, then it is not a prime. If a number has only trivial factors, it is prime. Factoring is closely related to the notion of representation, because factoring a number is synonymous with representing it as a product of natural numbers. In this approach, the property of primality is added as a yes/no property assigned to the object of a natural number that has been constructed by learners at earlier stages as an outcome of generalization of equivalence and seriation (Piaget, 1965).

In an attempt to factor a number, how does one look for possible factors? Sophisticated and fast algorithms for factoring very large numbers used in ciphering are beyond the scope of our interests here. The students in our study initially recalled their school experiences and associated the search for factors with the building of factor trees. Further into the course they were introduced to a standard algorithm, where in search for nontrivial factors we check for divisibility by primes smaller than the square root of the number to be factored. This suggests that the property of primality is assigned to small numbers before it can be assigned to large numbers. In our study, both correct and incorrect decisions on the primality of F (Question 2) were accompanied by the algorithm. Out of 116 students, 22 (19%, combining categories C(3) and I(2); see Table 1) referred to the algorithm in their response. However, as the data demonstrate, correct application of the algorithm could be quite meaningless, as it is not needed for determining primality of a number represented in a factored form.

A few observations deserve attention in discussing the standard algorithm for determining primality. One is that students often have difficulty in explaining, and even in reproducing the explanation, as to why it is sufficient to consider only divisibility by primes smaller than the square root. This inability to explain leads to a distrust of the algorithm. We witnessed students who, being familiar with the algorithm, would check for divisibility by all the primes up to the number. We observed students who attempt to check for divisibility by *all* the numbers, not only primes, up to the square root or even up to the number itself. "To be on the safe side" was the usual reason students provided to justify these unnecessary calculations.

It is clear from the data that the majority of students' ideas of factors, multiples, and divisibility are not well connected. Earlier research (Zazkis, 2000) demonstrated that some preservice teachers do not recognize the claim "B is a factor of A" as equivalent to the claim of "A is divisible by B." This research also showed instances where, given a number in its prime factorization, $117 = 33 \times 13^2$, and a request to list all its factors, students attempted to factor 117 by building a factor tree, rather than to build factors from its prime factors. In the study reported here a similar idea is illustrated with a more striking example: students attempted to perform factoring, and in some cases failed to do so correctly, when all the factors were explicitly presented to them.

Primality by Observing Examples

The existence of a transparent representation for a specific number property can help in abstracting and generalizing that property. However, a lack of transparent representation for prime numbers may lead students to generalize from examples. Of course, generalizing from examples takes place regardless of the available representations; however, with the lack of transparent representation, this approach may be preferable for constructing understanding of primes.

Based on examples from their experience, the following conclusions are drawn, either explicitly or implicitly, by some learners:

- Prime numbers are small;
- Every large number, if composite, is divisible by a small prime number.

This understanding of primes is illustrated explicitly in the excerpt from an interview with Tanya, reported in Zazkis and Campbell (1996b). This student, similarly to some participants in our study, claimed that 391 ($391 = 17 \times 23$) was prime after dividing it by a few "small" primes.

Tanya: I guess it's probably more experience than anything, but it just seems to me that when you factor a number into its primes, I mean when you're doing this, you're trying to find the smallest, I mean numbers that can no longer be broken into anything smaller aside from 1 and itself, so that. I guess, it's just the whole idea of factoring things down into their smallest parts gives me the idea that those parts are themselves going to be small (p. 216).

In the study reported here the belief in small primes was a *belief in action.* That is, in their attempt to determine primality of F (Question 2), eight participants checked the divisibility of F only by using a small number or small primes in order to draw their conclusion. Although the frequency of occurrence of this intuitive belief among the participants in this study was small, we have included it here because of the connection to prior research.

Another property of primes derived by considering examples is that prime numbers are, for the most part, odd. This observation leads at times to a mathematically incorrect implication that *all* prime numbers are odd (Zazkis, 1995). Furthermore, by confusing the relationship between prime and odd, some students believe that odd numbers are prime. This appears to be a logical misinterpretation in considering a necessary condition for primality (for numbers greater than 2) as a sufficient condition.

Primality by Exclusion: What Primes Are Not

Another approach for constructing primality is by exclusion—by considering *not* what prime numbers *are* but what prime numbers *are not.* This is the idea underlying the famous Sieve of Eratosthenes—the process of "sieving out" composite numbers from a list of natural numbers in order to be left with primes only.

As indicated by the responses to Question 1, considering *what primes are not* is a salient feature in students' descriptions of prime numbers. In the words of participants, "Prime numbers are those that *cannot* be divided by anything, other than 1 and itself" or "Primes *cannot* be factored." Rather than, or in addition to, repeating a positive definition such as that prime numbers have exactly 2 factors, the majority of students phrased their description of primes by attending to features that prime numbers do not possess. In other words, they are not composite numbers. When the participants were asked specifically in Question 1 about the relationship between prime and composite numbers, there was an expectation that the theme of prime factorization would emerge more frequently in participants' responses. However, for some students, the relationship between prime and composite numbers consisted entirely of their membership in disjoint sets. In students' words, "They are one or the other, cannot be both" or "They are sort of opposites of each other."

In considering *what primes are not*, a number's representation can be a major indicator. If primes are numbers that cannot be nontrivially factored, then representing a number in a nontrivially factored form is definitely evidence that the numbers are *not* prime. However, less than half (52 out of 116, or 45%) of the participants responding to Question 2 based their decision on such a representation. We believe that the reason for this finding could be the lack of a strong connection (at least in the minds of the majority of participants) between the factored form representation of F as 151×157 and the ability to identify 151 and 157 as factors or divisors of F. Polysemy[1] of the word *factor* may be a confusing factor here, as numbers that appear as factors (multiplier and multiplicand) *in* a number sentence are not necessarily factors *of* the product. Another possible explanation for ignoring the given representation could be related to participants' school experience, in which the decimal representation of numbers prevails, and all the alternative representations are treated as exercises aimed at uncovering the standard decimal representation of a number.

Although arguing for the importance of attending to representation, we also acknowledge that for some students attention to representation was a hindrance. In responding to Question 3, some students claimed that a number given as a product cannot be prime, ignoring the possibility of a trivial factorization as $1 \times p$.

CONCLUSION

In the study reported here, we investigated preservice elementary school teachers' understanding of the concept of primality of numbers. We have shown that *possession* of the knowledge of what prime numbers are often does not result in the *implementation* of this knowledge in a problem situation. Furthermore, we have presented three intertwined ways in which participants understand the concept of prime number. We have described number representations as transparent or opaque with respect to a certain property and argued that the lack of transparent representation for primality is an obstacle in constructing this concept. As Skemp (1986) proposes, "Making an idea conscious seems to be closely connected with associating it with a symbol" (p. 83). However, which symbol can we associate with the idea of primality? As mentioned above, there is no transparent representation for prime

[1] The term *polysemy* refers to the property of words having different but related meanings. The issue of polysemy is discussed in detail in Zazkis (1998a).

numbers. However, there is a transparent representation for a number that is not prime as a product of two natural numbers, neither of which is equal to 1. While a lack of representation was judged to be an obstacle, the availability of representation indeed served, for some, as an asset and as a basis for decision making. Representation is representing *something* only if there is connections in students' mind between the notions that *are* being represented and the notions they are being represented *by*.

We agree with Lamon (2001) in her observation that mathematics education research is faster in identifying students' deficiencies than in suggesting alternatives. One pedagogical suggestion that has yet to be tested empirically is to involve students in the consideration of large numbers. By *large* we mean numbers that are beyond the manipulation abilities of a hand-held calculator. A possible variation of Question 2 would ask students to determine whether, for example, the number 151^{157} is prime. Our assumption is that the inability to rely on calculations will force more students to attend to the representation that makes the conclusion transparent.

Over a decade and a half ago, Kaput (1987) claimed that "We give virtually no explicit attention at any level in mathematics education to the relation between the transparency of certain mathematical properties or operations and the representation in which they are encoded" (p. 21). He further suggested that "naturally, there is a tight connection between the omissions of the curriculum and the omissions of the research community" (p. 21). Our research complements the recent work on representations in an attempt to make up for this omission.

Research on whole numbers in mathematics education has traditionally focused on number operations and decimal representations. This implicit tradition is reflected in a recent National Research Council report *Adding It Up* (Kilpatrick, Swafford, & Findell, 2001) that defined its explicit focus on the notion of a number and attempted to synthesize the rich and diverse research on pre-K–8 mathematical learning. It is not surprising that this report does not mention concepts related to the multiplicative structure of whole numbers, such as primality or divisibility. We consider this to be yet another serious omission of research on the K–8 level as well as on the teacher education level.

Acknowledging these omissions, our study makes a contribution in two arenas. We enrich research on representations by focusing on representation of numbers, distinguishing between transparent and opaque representations, and provide a theoretical lens for the consideration of the effects that representation has on mathematical learning. Further, we extend the research on preservice teachers' understanding of concepts in elementary number theory by focusing on prime numbers. We hope that this study raises significant concerns that are worthy of further research on the understanding of primes in particular and on the understanding of multiplicative structures and relations of whole numbers in general, as well as on the role that representations (or lack thereof) play in the learning of specific mathematical concepts.

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