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Understanding Ratio and Proportion as an Example of the Apprehending Zone and Conceptual-Phase Problem-Solving Models

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Learning mathematics requires learning to use culturally specific mathematical language, formats, and methods (math tools). To use these math tools effectively in a problem situation, one must learn to identify the mathematical elements of that problem situation; i.e., one must learn to *mathematize*. In traditional approaches to mathematics learning, these aspects are often separated, with problem solving following learning about mathematical tools. We present in this chapter a model for learning mathematics with understanding that highlights the kinds of connections that can facilitate sense-making by the learner. We exemplify this model with a new approach to the learning of ratio and proportion. This approach addresses two major learning difficulties in this domain (e.g., Behr, Harel, Post, & Lesh, 1993; Harel & Confrey, 1994; Kaput & West, 1994; Lamon, 1999). First, students typically use additive rather than multiplicative solution methods (e.g., to solve 6:14 = ?:35, they find the difference between 6 and 14 and subtract it from 35 to find 27:35 rather than seek multiplicative relationships). Second, they have difficulty moving from easy problems that use the basic ratio (e.g., 3:7 = ?:14) to middle-difficulty nondivisible problems in which neither ratio is a multiple of the other (e.g., 6:14 = ?:35).

The two models of learning and teaching mathematics introduced here seek to enlarge our view of mathematical cognition by examining such cognition in process during teaching and learning. It is conceptually and methodologically difficult to capture such rich thick data, but our models are intended to serve as lenses to focus on certain central elements of such teaching and learning. We also introduce a new methodological tool for data organization and presentation, our *transcliptions*. These give the reader brief views of interactions in our classrooms via tables focused on the learning issues we identified in the domain of ratio and proportion.

The short length of this chapter prevents us from discussing the models, the ratio and proportion design, or the empirical results in any detail, but we provide references for more details in other papers.

THE APPREHENDING ZONE MODEL

Our *Apprehending Zone* (AZ) *Model* focuses attention on several aspects of teaching and learning. The first is relationships between the mathematical tools educational designers plan for students to use, how these tools are used in the classroom, and the conceptual schemes students develop (these are shown rising vertically in Figure 13.1). The design researcher's understandings of a conceptual domain become manifested in the *Design Space* as domain tools and activities that are designed (Figure 13.1, bottom) and then become part of students' understanding through their participation in the classroom interactions using the *Classroom Action Tools* (Figure 13.1, center), so that students come to share close-enough taken-as-shared mathematical interpretations of the domain tools in their own individual *Internalized Space* (Figure 13.1, top). A second aspect of the model is how students learn classroom-shared domain-specific body-based dynamic mathematical images (see Figure 13.1) through interacting with and communicating about the math tools. We combine Piagetian and Vygotskiian theoretical constructs in this model of how design research–based tools and materials are taught and learned in a constructivist social—cultural classroom. We intend both meanings of *apprehending* (holding and understanding) to highlight the crucial links between body-based perceptual actions (seeing, hearing, speaking, body sensing, and gesturing) and conceptual actions (a Piagetian perspective). The *Apprehending Zone* includes each student's *peri-personal space* (the space around a student within which the student can reach) and the *classroom communicative social space* within which students operate by externalizing–internalizing actions, words, inscriptions, and visual structures (a Vygotskiian perspective).

A third aspect of the model is how a teacher creates a classroom learning zone of focus, engagement, and participation by leading students' attention to critical mathematical elements and by continually helping students to make three kinds of crucial links. The first is conceptual links among the domain math formats (shown on the right center in Figure 13.1). The second is conceptual links among the real-world situations that can be recorded in and solved by the domain math formats (shown on the left center in Figure 13.1). The third is links across problem situations and math tools by *mathematizing* the problem situations and by *storyizing* the mathematical tools (shown in the arrows across the center of Figure 13.1). In making these links, and in understanding the situations and the math formats, the teacher and all student participants use body focusing, indicating acts, and gestures to lead their own attention and that of others in the class who may be watching or listening to them. Thus, the Apprehending Zone Model focuses on the time–space of problem-solving activity and cultural communicating.

The Apprehending Zone Model foregrounds the agency of student body and sensory perception in assimilating and linking mathematical formats, situational attributes, and relations among all of these. In our design research classrooms, teachers taught and students learned ratio and proportion by tacitly internalizing—externalizing dynamic visuo/body-sensed schematic images that systematically linked the word-problem situations with the spatial—numerical mathematical formats and solution methods (see Abrahamson, 2004a, for an analysis of the roles of gesturing in our classrooms). These visual and body-based images constructed and mediated the classroom *semiotic network* (Greeno, 1998) of ratio and proportion by linking the various row and column formats, by linking problem situation texts and the math tools through mathematizing and storyizing, and by linking students' developing interpretations of the domain tools with those of their peers and their teacher. The body-based structures of proportion

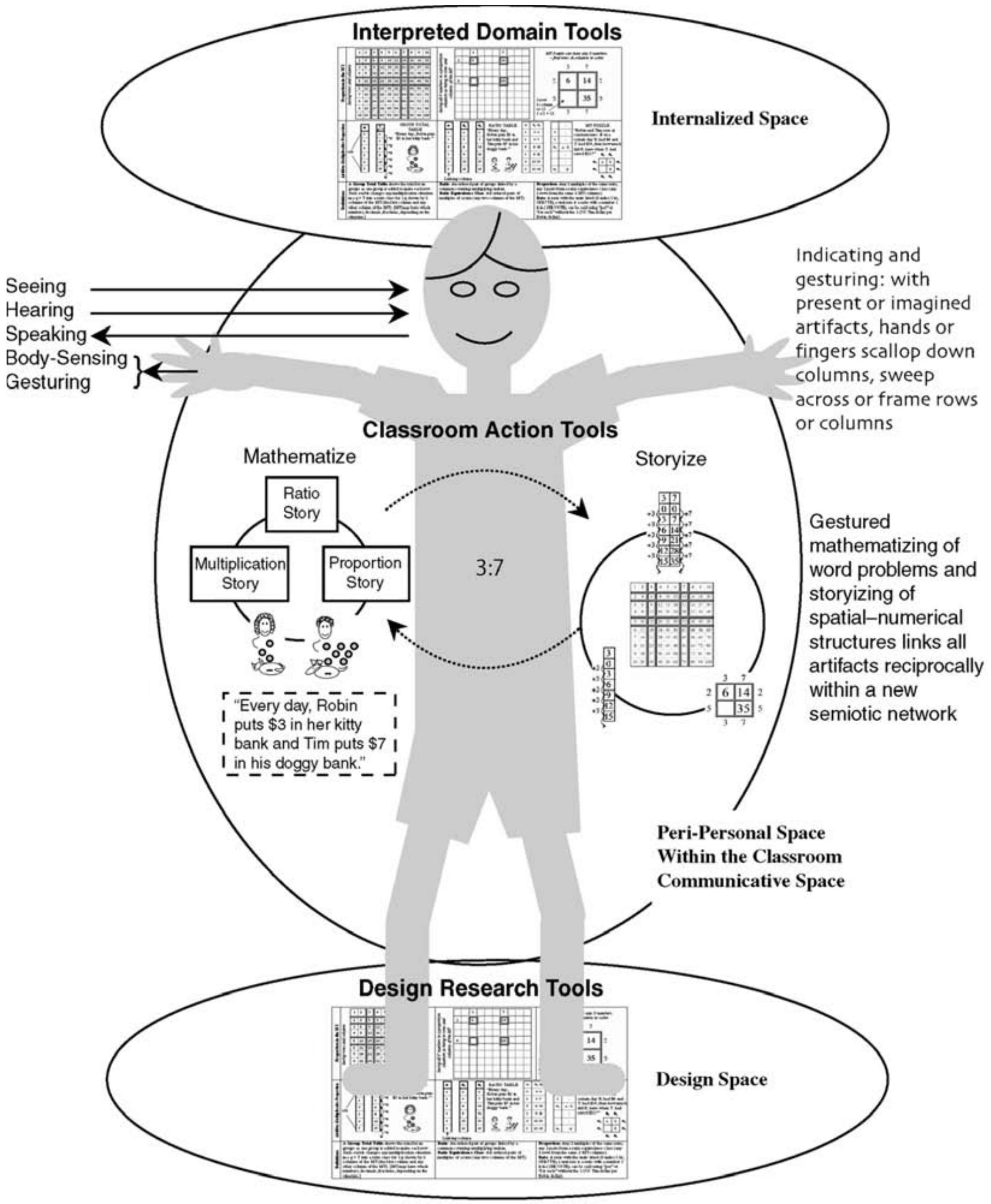


Figure 13.1. The body-based teaching–learning *Apprehending Zone Model*: The time–space of problem and cultural communicating.

became a classroom taken-as-shared artifact or *grammar of space–time* (Urton, 1997) linked to real-world proportional situations and to the various column forms of the *a:b* format. These attentional and gestural links became generative mathematical activity tools, or what Stetsenko (2002) calls "crystallized templates of action" (p. 129; see also Barsalou, 1999, p. 599, on *shared embodiment*). Examples of all three of these crucial kinds of linking during classroom discourse will be given after the ratio and proportion design is described.

THE CONCEPTUAL PHASE PROBLEM-SOLVING MODEL

Our *Conceptual Phase Problem-Solving Model* (see Figure 13.2) describes phases in the reciprocal meaning-making processes of *mathematizing* the problem situation (foregrounding the key mathematical aspects of the situation) and *storyizing* the math notations and methods (telling stories for each math tool). Mathematizing is shown as moving up in the model, and storyizing is shown as moving down. The vertical boxes show the different conceptual models solvers must form as they move from a real-world conception to a solution action sequence. Some of our designed classroom activities moved through all phases for a given problem, and others concentrated at particular spots (e.g., sharing different solution methods for the same problem). This model was developed for describing addition–subtraction word problem solving by students in kindergarten through grade 3 (Fuson, Hudson, & Ron, 1997; Ron, 1999). We modified it here to show in the left column the activities at each phase in which students and teachers need to engage in the classroom and individually when understanding ratio or solving a proportion problem. The fit of the model for these domains that span the elementary school years suggests that it is widely applicable, at least in numerical situations.

THE DESIGN FOR LEARNING ADDITIVE–MULTIPLICATIVE AND MULTIPLICATIVE SOLUTION METHODS FOR PROPORTION PROBLEMS

In our design we sought to introduce fifth graders to ratio and proportion by grounding it in multiplicative contexts that would enable students to avoid the usual additive solution errors. We also used middle-difficulty problem numbers in which the ratio pairs were not multiples of each other but were multiples of the smallest basic ratio (e.g., 3:7, such as 6 to 14 and 15 to 35), so that students would learn more general solution methods than the simple multiply/divide methods used with the easiest problem numbers involving only a basic ratio and a multiple of it (e.g., 3 to 7 and 15 to 35). We discuss at the end of the chapter how this approach can generalize to proportions involving fractions and decimals and to advanced solution methods such as finding unit rates and cross-multiplying. The math formats and situations linked in our design are shown in Figure 13.3 and will be explained in this section.

All approaches to teaching ratio and proportion need to help students use and understand ratio tables and some format for representing and solving proportions. Ratio tables are vertical or horizontal formats that record the results of repeated coordinated additions of a basic ratio pair of numbers. The middle cell of Figure 13.3 shows a ratio table for the ratio 3:7 in which the columns are made by repeatedly adding 3 to the left column and 7 to the right column. Any two rows from a ratio table are proportional because they are each multiples of the basic ratio (e.g., they are $3m_1{:}7m_1$ and $3m_2{:}7m_2$ and thus are multiples of each other). Ratio stories are situations that generate ratio tables. They involve two linked situations that begin at zero and in which the repeated addings are made together. Figure 13.3 gives one of our design ratio stories: "Every day Robin puts $3 in her kitty bank and Tim puts $7 in his doggy bank." Each ratio story involves an explicit or implicit linking column that coordinates the repeated adding actions in the two linked stories; in this story, the linking column numbers the days. The linking column enables one to find a given row by multiplying rather than by repeatedly building up within the ratio table to that row (e.g., on Day 5, Robin has 5 × $3 = $15 in her kitty bank and Tim has 5 × $7 = $35 in his doggy bank). If the rows are made by multiplying instead of by repeatedly adding, ratio tables can have rows omitted or reversed.

Each of the coordinated situations in a ratio story can be seen in isolation as a multiplication story made by repeatedly adding the same number. The Group Total Table shown in the left cell of the middle row in Figure 13.3 shows the multiplication story of Robin putting $3 in her

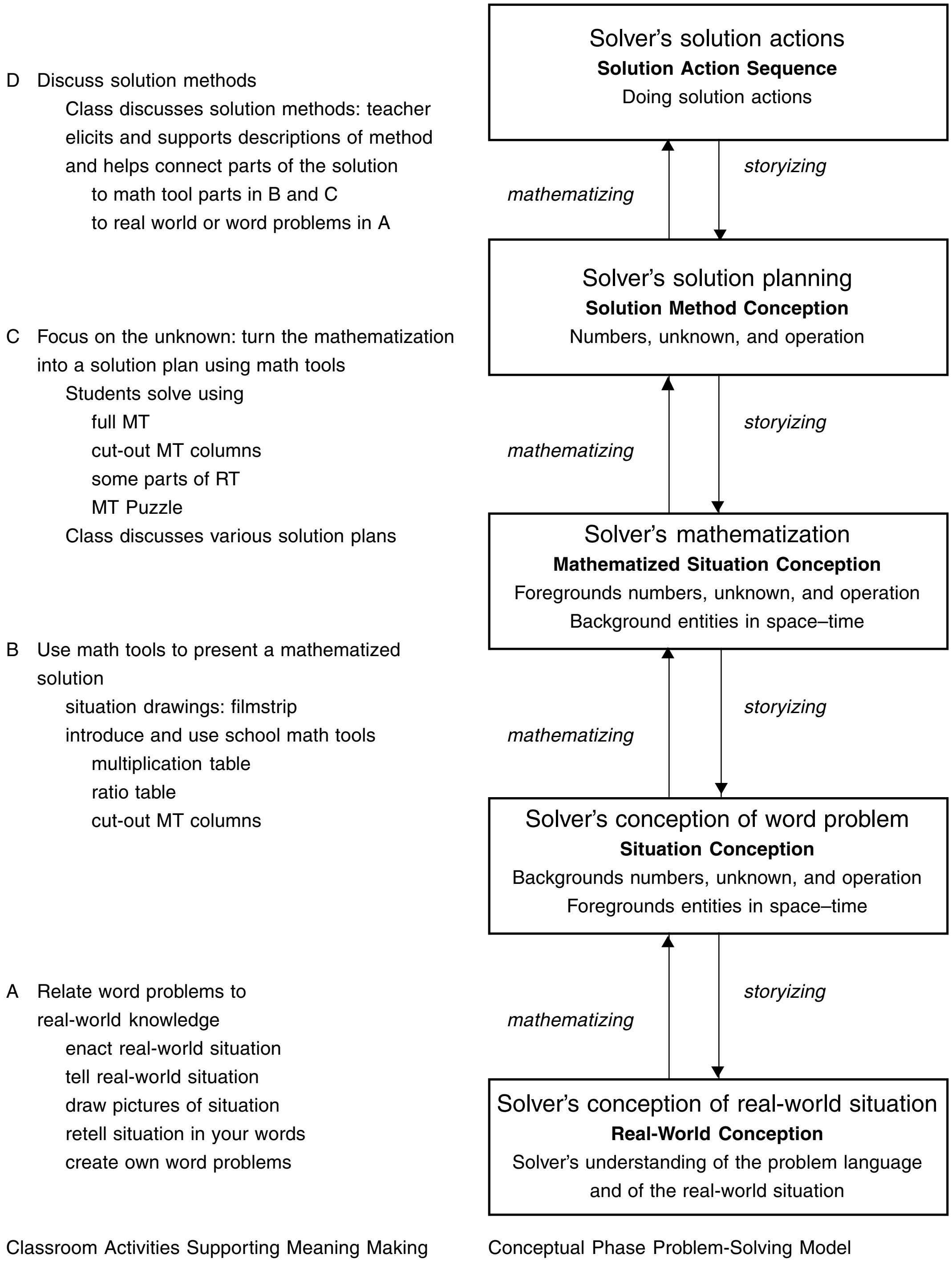


Figure 13.2. The Conceptual Phase Problem-Solving Model in the classroom: mathematizing the situations and storyizing the math tools.

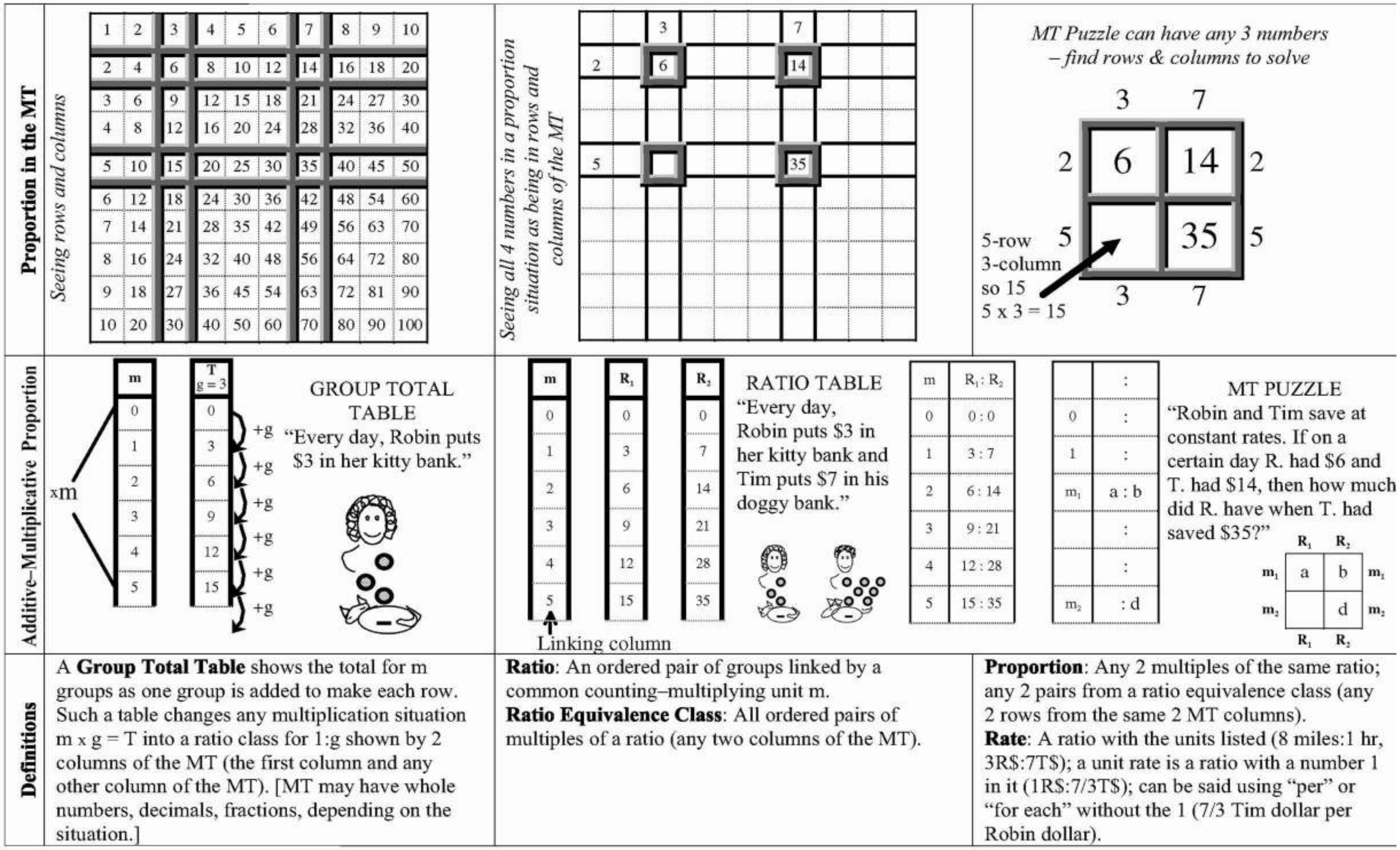


Figure 13.3. Mathematical tools for teaching and learning ratio and proportion.

kitty bank each day. Thus, we can begin students' introduction to ratio with multiplication stories involving repeatedly adding some amount. We call such a single column table (or its related two-column form with the ones column on the left) a Group Total Table to link such tables to students' early experience with multiplication as repeated groups (we would prefer to call these tables the same name as their stories but cannot call them multiplication tables because of confusion with the usual 9 × 9 or 10 × 10 table with that name). Use of these Group Total Tables allows students to explore different kinds of multiplication situations, see the Additive–Multiplicative repeated adding of the same amount in one column of the ratio table format, and relate this to making a particular group total by multiplication (multiplying the row number times the group number, as in "on Day 5, Robin has 5 × \$3 = \$15 in her kitty bank"). Connecting Additive–Multiplication and Multiplication meanings is an important basis for the continued growth of full understanding of all of the aspects of ratio and proportion (see Fuson and Abrahamson, 2004, for a fuller discussion of relationships to more advanced meanings of ratio and proportion and Fuson, Kalchman, Abrahamson, & Izsák, 2002, for discussion of such connections in multidigit multiplication, fractions, and linear functions).

The multiplication table (MT) is a cultural format widely used to display the products of numbers 1 through 10 (see the top left cell in Figure 13.3). Any ratio table using a basic ratio from these numbers can be made by pulling two columns from the multiplication table and putting them together (see the three and seven columns highlighted in the multiplication table). For students who understand the structure of the MT as columns made by repeatedly adding the same number (sometimes called "count-bys" as in "count by 3s"), making a ratio table from two MT columns can facilitate comprehension of the ratio table. For students who do not understand the structure of MT columns, linking multiplication stories to a single MT column and linking ratio stories to a ratio table as columns from an MT can facilitate comprehension of the repeated adding structure of the MT because of the repeated adding actions in the stories.

A proportion is made from any two rows of a ratio table. In the top row of Figure 13.3, one can see that a proportion also arises within an MT as the four corners of a rectangle made by the two columns that form the proportion's ratio table and the two rows containing the proportional pairs. One can solve a proportion with one unknown by thinking about which rows and which columns of the MT form that proportion. This is facilitated by writing the proportion as a mini-MT with one empty cell (see top right of Figure 13.3). Initially, only the three known values appear inside this mini-MT, and there are no values outside it. We call this format an MT puzzle (it was called a proportion quartet in early versions of the design). There are six solution paths for each MT Puzzle. You write the row and column numbers outside the MT Puzzle as you solve it. Working with an MT, you could simply copy these row and column numbers. Working without an MT, you need to determine which MT rows and columns make the MT Puzzle (i.e., find the common factors for both rows and both columns). You start with either the row or column in which you know two numbers. You then can move to the column or row perpendicular to what you just solved or to the other row or column in which you know two numbers. We found that all students from grades 5 through 7 could learn to solve MT Puzzles with, and then without, the support of an MT and that they loved to solve them. Given any proportion problem, one can set up an MT Puzzle (see the example in the right-most middle cell of Figure 13.3) and then solve it to find the unknown number in the proportion problem. It is helpful to label the rows and columns with situation labels to connect the MT Puzzle to the situation.

The usual format for setting up a proportion in the United States is the equivalent fraction format. This does not provide the conceptual links to multiplication given by the MT Puzzle format. It also introduces conceptual confusions between ratios and fractions. And because many students experience difficulties operating with fractions, it does not necessarily suggest helpful solution strategies. The MT Puzzle format used with the MT eliminates these problems.

These tools also can be used to examine and clarify differences and similarities between ratios and equivalent fractions. Fraction equivalents can be seen as two rows from the MT (see the chain of equivalents for 2/5 in the 2 row over the 5 row in the MT in Figure 13.3). Also an MT Puzzle can be used to find any of the four numbers unknown in a fraction equivalence. For example, the MT Puzzle in Figure 13.3 can be taken as the equivalent fractions 6/? = 14/35, and these numbers as vertical fractions can be seen in the MT in the 2s and 5s rows. For a more detailed analysis of these issues and related issues involving rates, see Fuson and Abrahamson (2004, where relationships between our perspective and those of others, especially Confrey, 1994, and Vergnaud, 1983, are discussed).

Our design for teaching ratio and proportion evolved through a series of design research studies (see Cobb, Confrey, diSessa, Lehrer, & Schauble, 2003, for a discussion of design research). Our design work in various domains has been directed by several core design principles. We continually seek to create teaching–learning materials that "start where children are and keep learning meaningful" (Fuson, De La Cruz, Smith, Lo Cicero, Hudson, Ron, & Steeby, 2000; Fuson, 1988). We use innovative approaches that are "intuitively convincing" (Abrahamson, 2000), but we continuously consider pragmatic constraints to ensure widespread usability of the design. We design to help students ground mathematical notations and methods in perception, intuition, and experience (Gelman & Williams, 1998; Wilensky, 1997; Freudenthal, 1981). We find or develop supports for core curricular concepts (Fuson, 1998, 2004) that align with and build on students' implicit models (Fischbein, Deri, Nello, & Marino, 1985) and their developing image-based understandings (Kieren, Pirie, & Gordon Calvert, 1999). Our design studies in ratio and proportion have included (a) two case studies, each with an individual high-achieving third-grade student, that explored an optical stretch–shrink model as grounding for proportional equivalence (Abrahamson, 2002a); (b) a constructionist study with 3 fourth-grade students who were asked to design and build a multiplication table that had no numerals (Abrahamson, 2002b); (c) one summer intervention study with fifth-grade low achievers who initiated creative transition links between the multiplication table, ratio table, and MT Puzzle, including a ratio table with empty rows between the given-values rows that they then collapsed into a MT Puzzle with those values (Abrahamson, 2002c); and (d) six classroom teaching experiments with four different teachers of grades 5 through 7 who taught successive iterations of the curricular unit evolving from our design research (Abrahamson, 2003; Abrahamson & Cigan, 2003; Abrahamson, 2004b; Abrahamson & Fuson, 2004). The classroom studies were a collaborative effort involving the teachers and district math resource staff.

Our most recent design is outlined in Table 13.1. The table details the three kinds of links described by the Apprehending Zone Model within and across the situations and the math formats (amplified in Table 13.1 as the "MT–RT numerical additive–multiplicative and multiplicative stream") and how these links were organized across days into streams of situational and math format numerical activities. Within the numerical stream, links were made between the multiplication table, ratio table, and MT Puzzle formats. Within the real-world situational stream, multiplication stories, ratio stories, and proportion stories were linked to each other. Continuing daily links were made between these two streams by mathematizing the stories (focusing on the mathematical elements to record them in the spatial–numerical math formats) and by storyizing the math formats (telling stories for each kind of format and usually working within a story context when using a math format).

Special materials for the unit included a large whole-class laminated multiplication table, individual multiplication tables from which students cut MT columns to form ratio tables, and filmstrips the students drew to show repeated adding situations (multiplication stories and ratio stories). Students also filled in scrambled multiplication tables in which the rows and columns of an MT were switched around and most of the products were missing. These were like big puzzles, and students used different strategies to solve them. These provided practice

Table 13.1
Mathematizing and Storyizing within the Designed Situational and Numerical Streams

Number of class periods (60 min)	Real-world situation stream	*mathematizing* →	← *storyizing*	MT–RT numerical additive-multiplication and multiplication stream
1				Find patterns in the MT using big class MT and small student MTs
4	Multiplication situations (stories) as a class built up by repeated adding of the same amount: Show actions of adding the same amount on filmstrip drawings.			Make group total tables from filmstrip drawings
				Cut columns from MT and show group total table with 2 columns (1s column and group column)
	Many different multiplication situations (Multiplication Stories)			Match group total tables to multiplication stories
	Continual mixing in of non-multiplication situations			Introduce and practice MT puzzles as coming from 2 rows and 2 columns of the MT
	Writing multiplication and non-multiplication stories (continues into writing ratio and non-ratio stories)			Identify group total and non-group total tables; match to stories
				Introduce and practice scrambled multiplication tables
5 to 8	Linked multiplication stories (group total situations) are Ratio Stories			Ratio tables are made from a common linking column; show with 3 cut-out MT columns (left-most is 1s column); students continue to use MT or RT to solve problems while gaining fluency with MT Puzzle solutions
	Proportion problems come from Ratio Stories			
	Writing proportion and nonproportion stories (continues all unit)			Solve proportion problems with MT puzzles; rows and columns can be scrambled in any order
	Gaining fluency solving a range of proportion problems and differentiating these from nonproportion problems			Gaining fluency with MT Puzzles, Scrambled MTs, and setting up and solving MT Puzzles from proportion problem situations

Note: Relationships were continually established within and between elements in these two streams by *storyizing* the mathematical notations and *mathematizing* the situations through gestured discussions.

with multiplications and divisions and gave models for proportions in MT Puzzles that had smaller numbers in the second column or column.

LEARNING ISSUES IN USING MATH TOOLS IN THE CLASSROOM

Part of our ongoing design research was to identify learning issues that presented difficulties to students and then seek to minimize these in subsequent designs. This recursive process finally resulted in the eight learning issues given in Table 13.2. Of these, six (all but the first and the sixth issues) are learning issues for any teaching design in the domain of ratio and

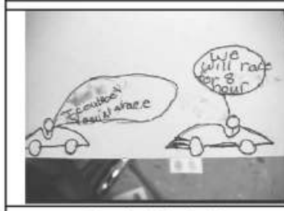
Table 13.2
Learning Issues of the Ratio-and-Proportion Design Grouped by Type of Reasoning

Learning Issue	Definition
Theorems-in-Action	
Solving By Looking Up	Locating unknown values on a Multiplication Table or on a prefilled Ratio Table or MT Puzzle or selecting Multiplication Table cut-out columns. Students may only know how to find relevant parts of the tool to answer a question.
Additive–Multiplicative	
Zero Starting Point	Multiplication and ratio stories have a starting point at zero, from which the repeated addend is iterated (in some versions of the mathematical formats, this zero moment is omitted and the table begins with the addend that will be repeatedly added).
Columns (or Rows) Are Repeated Addition Sequences	Attending to, parsing, constructing, and articulating multiplication stories or ratio stories as MT columns. Columns begin either at 0 or at the column number and then iterate the constant addend (the column number) without repeating or skipping rows.
Repeated Addends Versus Totals	Interpreting the sequence of values running down MT columns as running totals in multiplication stories or ratio stories and specifically distinguishing between these running totals and the constant addend (the column number).
Multiplicative	
Multiplicative Structure and Use of the Table Formats Multiplication Table, Ratio Table, and MT Puzzle	These grid-structured formats have rows and columns with uniform cell sizes, and each number is the product of the left row number and top column number. A 10-by-10 Multiplication Table can be modified by reordering (to make Scrambled Multiplication Tables or MT Puzzles) and by extending rows and columns.
Ratio Table	
Linking Column for the 2 Sequences	The number of iterations so far in the linked Ratio-Story columns can be represented in a separate left-most column, but it may be implicit rather than physically present (as in standard two-column ratio tables).
General	
Vocabulary	Using new and familiar terms for the formats (e.g., row, column), the situations that are grounded in these formats (e.g., miles an hour, growth unit, per), and the "pure" mathematics of the domain (e.g., multiple, common factor, rate, ratio, proportion).
Labeling ("Table Manners")	Labeling columns or rows with the kinds of quantities they represent in the story situation.

proportion. The first—Solving By Looking Up—would occur only in our design for the MT and MT cut-out columns, though it might be used for ratio tables in other designs. However, Solving By Looking Up was crucial in our approach because less-advanced students initially solved problems by finding relevant numbers on the MT (e.g., they found the three known proportion numbers as three corners of a rectangle in the MT and chose the fourth cell of the rectangle as the answer). Such uses of the MT enabled students of all levels to participate in problem solving from the beginning. The sixth learning issue—Linking Column for the 2 Sequences—also is much more explicit in our design than in most approaches. This explicit linking column supports mathematizing and storyizing, and it facilitates moving to multiplication methods because the multiplying number is written.

To exemplify the major learning issues, and to enable readers to see the design math tools in use in discussions in the classroom, we constructed Tables 13.3 through 13.7 to show one

Table 13.3
Understanding of and Difficulty with the Additive–Multiplicative Learning Issue
"Columns/Rows Are Repeated-Addition Sequences"

Tools Used	Example from classroom data			
Understanding Multiplication Story Filmstrip Multiplication Table Ratio Table MT Cutout Columns				
	<Day5_H1_53:12>	<53:17>	<53:21>	<53:43>
	Fernando: Ok, this comic is about the man and his beard. As you can see here, his beard started out as 3 inches right *here*,	and then each day it keeps growing… [scallops his pointer down from top picture along penciled arc with the label "+3"] it keeps growing by 3…	by 3 more inches every *day*. [note beard getting longer] **Ms W**: [to class] What are the columns you need to create a rate table for this situation?	[class responds that they need 2 columns, "1" and "3," that are labeled "Days" and "Inches." Students work with the MT and its column cutouts]
Difficulty Ratio Story Filmstrip Multiplication Table				
	<Day3_T1_6:14>	<6:57>	<7:05>	<7:15>
	[Students are composing ratio-stories and preparing filmstrip columns that illustrate these stories. M'Buto's ratio story: "Bob was a race car driver so was M'Buto. One day Bob said I bet you in a race. M'Buto said we will race for 8 hours. Bob's car was 3 miles an hour and M'Buto's car was 5 miles an hour. Was Bob right?"]	[D. asks if that's the same as saying that "Every hour, Bob's car goes 3 miles, etc.," and M. says it's the same. D. asks how many pictures there will be in the ratio-story filmstrip that M. is preparing, and M. says there will be 3: beginning, middle, and end. D. asks for more details, and M. looks up to the MT poster and says that there will be a picture after 1 hour.]	**D**: Oh, super, so '1, 2, 3, 4,…'— like that? **M**: Yeah. [thinks a moment— that's not what he had said— he had only promised *three* pictures, regardless of the hours or the miles in the story] No— wait a moment—[looks up at the MT, looks back at Dor] then I'll show it after 1 hour again.	**D**: Oh, so you're like a reporter, and you're taking a photograph every hour. **M**: Mm'hmm. **D**: Excellent.

display of understanding and one display of difficulty for each of the second through sixth learning issues. These tables used *transcliptions,* our method of data preparation that enabled the research team to see and communicate about the rich data in the raw videotapes of the classroom. These transcliptions were made by transcribing the videotapes and pasting in clips from the videotape to accompany the transcriptions. Tables 13.3 through 13.7 show typical examples of teaching and learning in action in the classroom. The limited space in this chapter does not permit discussion of these examples (see Abrahamson, 2004b, and Fuson & Abrahamson, 2004, for results of analyzing students' understanding of and difficulty with these learning issues).

These transcliption data made salient to us the centrality of gesturing and body-based communication within the classroom, and they contributed to the development of our Apprehending Zone Model (see also Alibali, Basso, Olseth, Syc, & Goldin-Meadow, 1999, concerning gesturing). In gesturing to mathematical objects within or displaced from their peri-personal space, students had opportunities to: (a) relate various math tools by *folding back* on the MT (i.e., use its familiar structure to make sense of a new format or situation; Kieren et al., 1999); (b) learn and practice verbalized mathematical terminology (e.g., row, column, common factor, multiple, product) supported by touching or gesturing; and (c) develop and practice the *body-based* imaging and action structures (or *haptic experience;* see Nemirovsky, Noble, Ramos-Oliveira, & DiMattia, 2003) necessary for problem solving and communicating in the absence of the math tools (e.g., draw an MT Puzzle in the air and point to cells of it). The results of the analysis of gesturing in student classroom learning are summarized in Abrahamson (2004a).

A BRIEF OVERVIEW OF LEARNING IN OUR CLASSROOMS

We briefly summarize here some of the learning results from the two classes (n = 19 and 20) that were videotaped and studied most intensely (see Abrahamson, 2004b, and Fuson & Abrahamson, 2004, for details). These both were extremely heterogeneous classrooms of fifth grade students, with 20% African American students, 20% Latino students, 38% of students on free lunch, and learning-disabled and English-language learners in both classrooms. Many students had had Everyday Mathematics (Bell et al., 1998) since kindergarten, so they were used to discussing their thinking in class. Both teachers were excellent teachers and led math discussions well. The transcliption data in Tables 13.3 through 13.7 are from one of these classrooms.

On the pretest, most of the students in both classrooms approached ratio-and-proportion word problems using *additive reasoning,* the typical error made even by older students. In independent work-alones given at the beginning of each class to track student learning or on the posttest, all students used *additive–multiplicative* or *multiplicative reasoning* successfully to solve at least some middle-difficulty nondivisible proportion problems in which neither ratio is a multiple of the other. The low-achieving students depended longer into the interventions on the MT as a support for their participation in classroom problem solving and discussion (e.g., they found the three numbers from a problem in the MT and then traced down and across to find the fourth number), but eventually all students successfully made ratio tables and/or MT Puzzles to solve some problems.

Throughout the unit, students showed variability in the ways in which they made ratio tables and MT Puzzles to solve problems. This variability suggested that they were assimilating the formats to their own ways of understanding. Examples of such variability are shown in Figure 13.4.

On the critical three posttest items that involved medium difficulty problems in which the proportions were not multiples of each other (e.g., see Figure 13.5), the middle half of the students (20 of 39) rose from 0% correct on the pretest to 100% correct on the posttest. In the

(*Text continued on page 230*)

Table 13.4

Understanding of and Difficulty with the Additive–Multiplicative Learning Issue "Repeated Addends Versus Totals"

Tools Used	Example from classroom data			
Understanding Multiplication Story Multiplication Table MT Cutout Columns				
	<Day5_H1_01:04 – 9:20>	<09:21>	<10:33>	<10:41>
	Ms W: [reads] "Big Bird collects snails. Every day, he adds 4 snails into his terrarium. How many snails does Big Bird have after 1 day?" [Students use their MT and cutout columns to model the situation then discuss the situation, labels, and vocabulary]	**Ms W**: How many snails does Big Bird collect on the second day? **Class**: 8…. 4…. **Odelia**: 4 more, because you said *on the second day how many did he collect,* and he collects 4 each day …so on the second day he would have 4 more, which would equal 8 in total. **Ms W**: Right.	**Dor:** I'm just confused—Odelia told us that every day he collected 4, but I see here that opposite the 2 there's an 8, so, how does that work out? **Ms W**: Odelia?	**O**: She asked how many snails he collected on the 2nd day, and he collects 4 snails every day, so on the second day he would have collected 4 *more*, and not 8—he would have collected 8 in total.
Difficulty Multiplication Story Multiplication Table				
	<Day4_H1_4:35>	<4:41>	<4:57>	<5:09>
	Violet: Duffy Duck was going one day to Porky Pig's house,	**Violet:** and, uhhm, [turns around to look at the MT poster; turns back] one hour… in one hour he walked 3 miles […] So the first mile… the first hour…hnnn... Duffy Duck walked 3 miles…	**Violet:** In the second hour [turns around to look at the MT poster; turns back] Duffy Duck walked 6 miles, in the third hour Duffy Duck walked 9 miles…	**Ms. W**: That would be a total of 18 miles. [scallops down from 3 to 6, waits for V.'s response, then scallops to 9, 12, etc.] **Violet**: In the first hour he walked 3 miles, in the second hour he walked 3 *more* miles…

Table 13.5
Understanding of and Difficulty with the Additive–Multiplicative Learning Issue
"Multiplicative Structure and Use of the Table Formats"

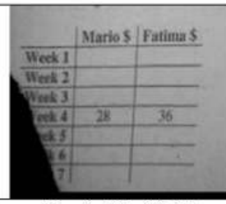
Tools Used	Example from classroom data			
Understanding Ratio Story Ratio Table				
	<Day3_H1_36:18>	<36:52>	<36:57>	<36:58>
	[Ms W projects the next problem on the overhead screen: "Mario and Fatima save their money on a regular basis. Fill in the numbers that are missing in this table." The format is a labeled ratio table with missing values.]	**Saul**: You see what times 4 <u>equ</u>… gives you… [points from '4' to '28' and '36'] This would give you the first week,	**Saul**: …and <u>then</u> you keep on going [drives pencil in the air, down along column while looking away from table and up at Ms. W.]	**Saul**: 'What times 4 <u>equals</u> 36,' 'What times 4 equals 28.'
Difficulty Multiplication Table MT Puzzle	 7 [21 35] 7 3 [27 9] 3	3 7 [21 35] 7 [27] 3	3 5 7 [21 35] 7 [27] 3 5	3 5 7 [21 35] 7 9 [27] 9 3 5
	<Day6_T2_29:19>	<30:20>	<30:26>	<30:33>
	[Margarita has completed an unknown-value MT Puzzle with '9' (should be 45). Also, she has written the factors 3 in the 2nd row rather than above and below the left column. She is copying from the MT that is on her desk directly into the MT Puzzle on her worksheet. Dor asks her to explain her solution.] **Margarita**: 9… 9 times 3 is 27, so you put the 3 here [to left and right of the bottom row] and you put the 7 here [top row] **Dor**: You know what, let's do this one together. Erase the 9 and we'll work together. [Following, D. and M. move between the MT and M's writing of numbers into the MT Puzzle]	**Dor**: Ok, so you found that 7 is a common multiple of 21 and 35. Ok, so what column is 21 in? **M**: The 7… oh, no, in the 3. **D**: So put a 3 here [on top]…and immediately put a 3 across [below] because it's in the same column.	**Dor**: Good. 35—what column is it in? **M**: 5 [makes '5' w/ LH five fingers splayed]. **D**: ok, so put a 5… and immediately put a '5' across. [M. writes in the 5's] Alright!	**D**: 27, what… see, 3 times what? It has to be the… **M**: 9. **D**: Right, and immediately put it across. **D**: You see, so now you have an empty cell here. Kind of '9*5 is 'what'? **M**: 9*9? **D**: Look, <u>you</u> wrote it: 5… **M:** Ahh, 45 [writes 45].

Table 13.6
Understanding of and Difficulty with the Ratio-Table Learning Issue “Zero Starting Point”

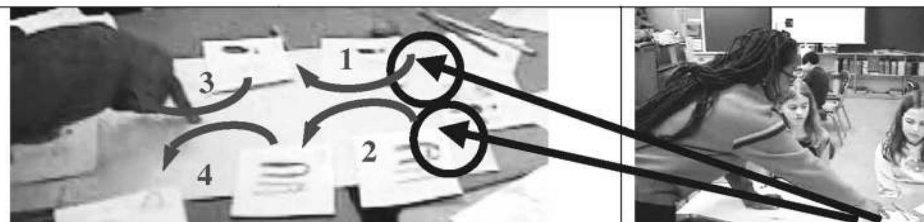
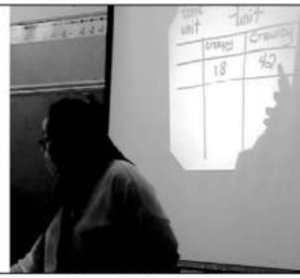
Tools Used	Example from classroom data			
Understanding Ratio Story Filmstrip	 <Day3_H1_3:33> **Moses**: [LD student. Reads his ratio story towards creating a filmstrip] “One day Bart at work for AJ Car Store. He makes $5 an hour. Lisa makes $3 an hour…”	 <later:.23:17> **Moses**: I put zero so it will be at zero dollars. They didn’t make money yet. And the next day,	 <23:20> Bart’s going to have 5, uhhhm 5 [jerks LH forward] dollars <u>and</u> [twists paper clockwise]	 <23:21> Lisa’s going to have <u>3</u> [quick anticlockwise t dollars.
Difficulty Ratio Story Filmstrip Multiplication Table	 <Day 7_H1_25:12zoom_in> [Ms W. discusses with students Kay’s filmstrip ratio-story. The story is about a hair-growing competition between girls and boys. Form day to day, beards and hair grow longer, each by some constant increment. Numerals 1 – 4 show order of Ms W.’s gesturing, as she explains how to insert arcs with the constant addend marked in them. Note, at the top of Kay’s poster is the “zero moment,” before the hair-growing competition had even started.]	<25:26> **Ms W**: [pointing above the boys column] Because in fact, these two points—on our chart—what would these two points be? [Ms W. compares picture columns to MT columns; prompts students to see this.]	 <25:31> [Students look back at tl MT] **Kay**: [returns gaze Ms W] 1? **Ms W**: 1? Do we start at **K**: Zero. **Ms W**: Zero. And in son ways, this space represei zero. [points again above picture columns in poste	

Table 13.7
Understanding of and Difficulty with the Ratio-Table Learning Issue "Linking Column for the 2 Sequences"

Tools Used	Example from classroom data			
Understanding and Difficulty Ratio Story Ratio Table Multiplication Table				
	<Day4_H1_41:08>	<41:37>	<42:15>	<44:28>
	[Students are discussing the following problem: "A pair of lizards—Creepy and Crawley—walk down the side of the Sears Tower. When Creepy has walked 18 floors down, Crawley walks 42 floors down. What could their walking rates be?" That is, students are asked to reduce 18:42 to a smaller ratio] **Ms W**: In some unit of time, Creepy walked 18 and Crawley walked 42. How are we going to find out how many time-units this all happened at?	**Alice**: I know the way: you can look on the chart and you can go to 18 and 42, and since they're across from each other… you could go up and see what column they're in; and 42's in 7, and 18's in the 3 column…[…] Crawley's rate would be 7 and Creepy's rate would be 3.	**Dor**: So, tell me, Alice, how did you know to go to *this* 18 and *this* 42? ['18' and '42' each appear twice on the MT] **Alice:** I just tried it—I used 'guess and check' […] I looked at *this* 18 and it didn't have a 42, so I looked at *this* one and it had a 42. **Ms W**: <u>How did you know they had to be next to each other?</u> **Alice**: I donno. **Ms W**: You're not sure, but you knew that they had to be next to each other? **Alice**: Yeah. **Ms W**: Ok, just like in our table…	**Ms W**: Can anybody support or explain why the 18 and 42 have to be in the same row? **Odelia**: Because it's in the same amount of time, so that this [points to 1-column] will be the 'time,' sort of, so that's how you could tell [gestures across the 6-row, beginning from '6' in the 1-column and towards the right, covering 18 and stopping at 42] **Ms W**: They have to be next to each other because they represent the same period of time. **Odelia**: Yeah.

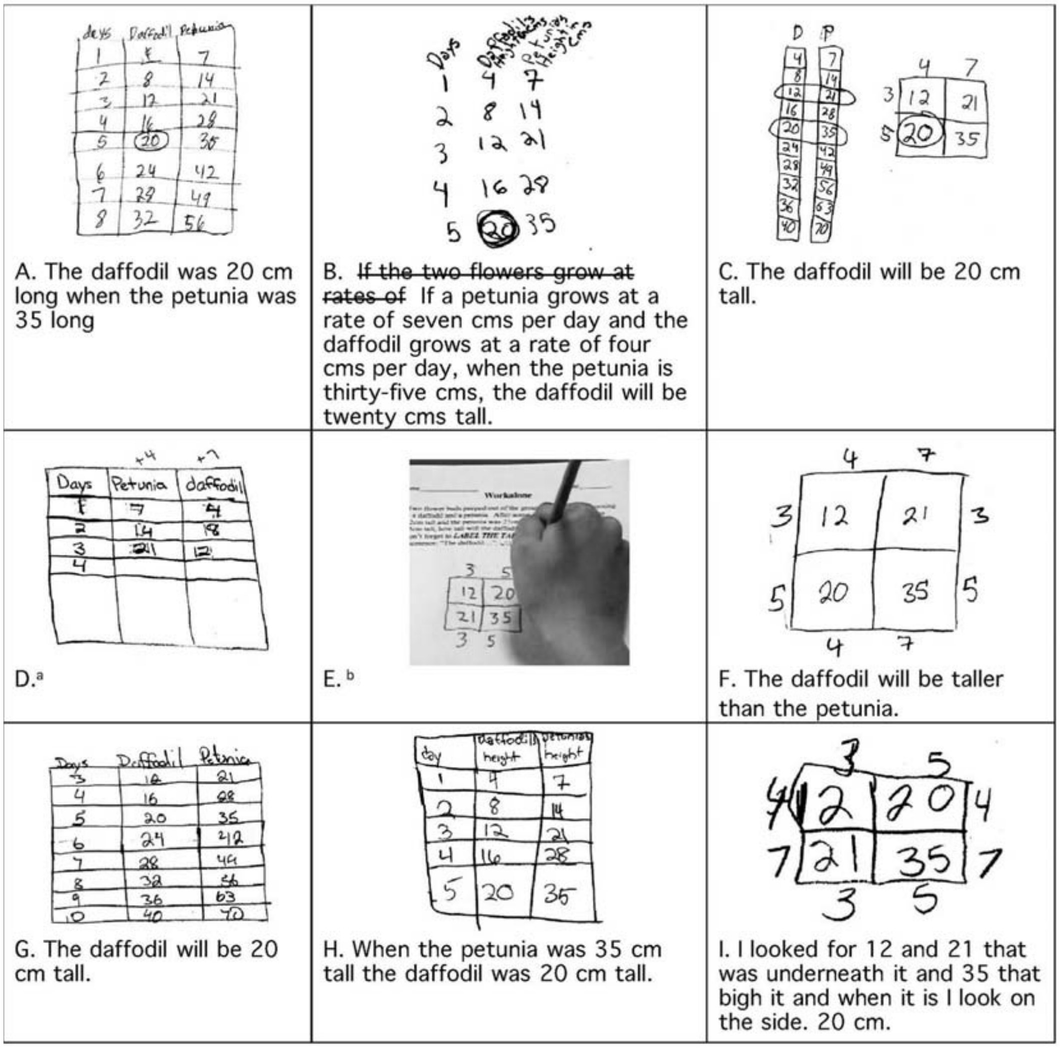


Figure 13.4. Variation in student solution formats and accompanying verbatim written responses in solving individually the Day 5 in-classroom word problem, "Two flower buds peeped out of the ground on the same morning—a daffodil and a petunia. After some days, the daffodil was 12 cm tall and the petunia was 21 cm tall. When the petunia is 35 cm, how tall will the daffodil be?" Students' work suggests a classroom bootstrapping—each student at their personal pace and along their personal path—the familiar structure and function of the multiplication table that was available for their use in developing an understanding of additive–multiplicative properties of situated ratio and proportion. Although the MT-Puzzle format was taught after the ratio table (RT) in the design, students using the MT-Puzzle do not necessarily evidence deeper understanding (e.g., compare B., an RT solution and full explanation, to I., an MT Puzzle copied out of an MT).

[a]This student's incomplete table was included to demonstrate students' flexibility in column order in the RT format (e.g., compare to Item A.) as well as in the PQ format (e.g., compare Items C. and F.). On the posttest (1 week later), she correctly solved all three *critical* items (items with proportional ratios that are related by a non-integer multiple).

[b]Picture E. has been included to demonstrate both our access to work-in-the-making, and specifically to show that such access informed us of students' strategies: this student apparently consulted the MT rather than factoring the PQ.

top fourth of the students, eight rose from 33% to 100% correct, and two were at ceiling on the pretest (but moved from pictorial to numerical solution methods). Of the lowest nine students, five rose from 0% to 53% correct, and four did not solve any of these nondivisible problems completely correctly. Thus, overall, the percent correct on the posttest was 84%.

The errors of the lowest-achieving students showed that they were at various points of mastery ranging from correct setting up of problems in an MT Puzzle with some multiplication or division error to still using a pictorial strategy. Typical pre- and posttest responses for the successful low achievers are shown in Figure 13.5. Each set up a different MT Puzzle, which interviews and observations in class indicated that they could explain. The lowest-achieving students could have benefited from more time on the unit.

Some high-achieving students did use multiplicative strategies on the pretest; these were elaborate concrete pictorial–numerical solutions involving units. All of these students began to use MT Puzzles during the unit and related in interviews or in whole-class discussions their initial solution methods to this more general method.

Much of the class became able to extend the MT Puzzle to three columns to solve complex proportion problems that involved a total, such as: “Monica and Lin are putting together a pony puzzle. Each kid is working at a steady pace. At a certain moment Monica has put in 24 pieces and Lin has put in 32. When they have put in 63 pieces *together* how many pieces has each put in?”

Teachers were very positive about the ratio-and-proportion unit. They said that they felt that their less advanced students had understood multiplication more deeply, that all students had learned a great deal about ratio and proportion, and that their more advanced students had connected multiplication, division, proportions, and fractions. Their students expressed liking the MT-Puzzle solution method because most trusted and were familiar with the MT and

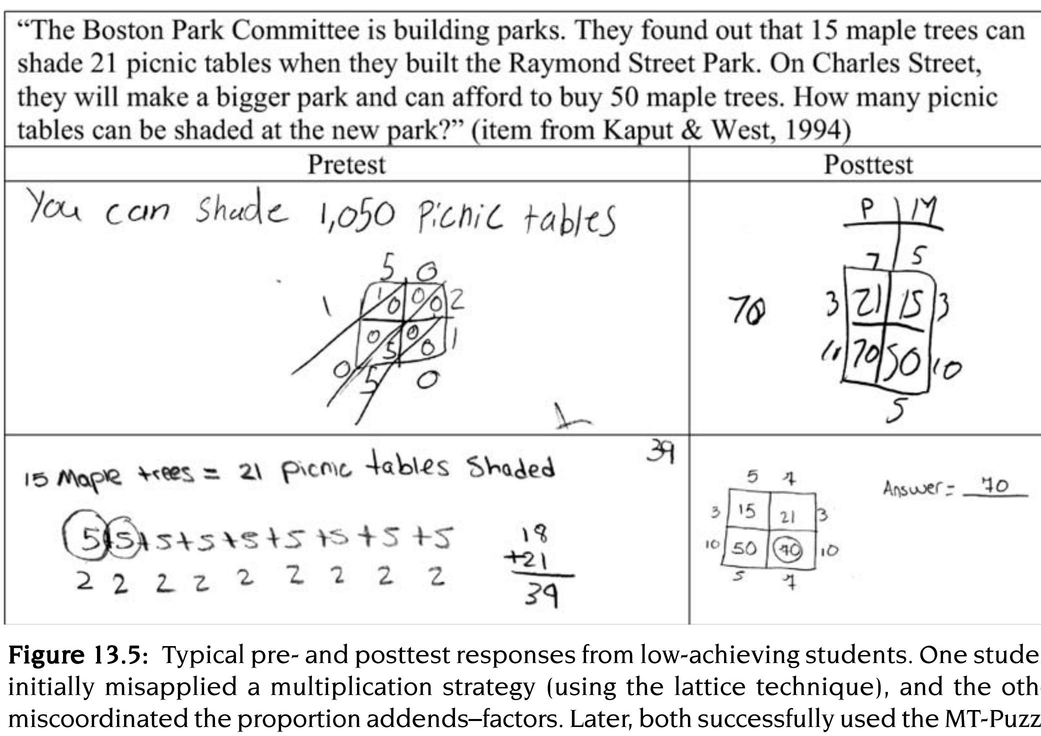


Figure 13.5: Typical pre- and posttest responses from low-achieving students. One student initially misapplied a multiplication strategy (using the lattice technique), and the other miscoordinated the proportion addends–factors. Later, both successfully used the MT-Puzzle format, each setting up a different MT Puzzle. Interviews and observations in class indicated that they could explain their use of the MT Puzzle to solve such problems.

because the format helped them organize the given information during problem solving. The ideas in the unit were presented at a district workshop to thirty fifth-grade teachers. These teachers found the approach accessible and appealing. Thus, the approach seems to fit within the learning zone of a wide range of fifth-grade students, and it appears to move all of them along their own learning paths to using additive–multiplicative and multiplicative solution methods for ratio and proportion.

EXTENDING THE MULTIPLICATION TABLE APPROACH BEYOND WHOLE NUMBER RATIO MULTIPLES

The MT-Puzzle approach transfers to all cases of proportional, fraction, and measure equivalencies and to other topics that use proportions. We did three types of mini-extensions with different classes: these were on percentage, similarity geometry, and coordinate graphing. Following a three-day mini-unit on percentage, one class progressed from 26% to 76% correct on percentage items. Following a two-day mini-unit on finding sides of similar figures, one class progressed from 5% to 78% correct on the nondivisible item. This unit was introduced through the *eye trick* (Abrahamson, 2002a), an optical illusion in which two proportionally equivalent pictures look identical when you hold them up, shutting one eye, with the smaller picture nearer your eye, followed by measuring and tabulating lengths in the pictures. For coordinate graphing, students aligned sets of proportionally equivalent (i.e., similar) rectangles from the eye-trick activity so that their corners were together and used the line made by the diagonally opposite corners to introduce graphs into the coordinate system (Abrahamson, 2002c).

Students can also move from our approach to using unit ratio and cross-multiplication strategies. Unit ratios can be seen as a row in the ratio table in which one number is 1. For the Robin and Tim story in Figure 13.1, these unit ratio rows would be 3/7:1 and 1:7/3. Some of the high-achieving fifth-grade students came to understand fraction unit rates (e.g., 3/7) that they had been using before the intervention as such unit ratio rows above the 3:7 row. A general unit ratio solution method involves moving up the ratio table (by dividing) to find the unit ratio row and then down a mostly empty ratio table (by multiplying) to find the row containing the unknown.

Cross-multiplication can be developed by factoring within the MT Puzzle and observing that the same four factors are in both diagonals. Therefore, the products of the two diagonals are identical, i.e., *top-left x bottom-right = top-right x bottom-left}*. One can then pursue cases in which different cells are the unknown to see that this factor structure allows you to find any unknown by multiplying the two known numbers in a diagonal and dividing by the number in the diagonal with the unknown, e.g., *top-left = (top-right x bottom-left)/bottom-right*. Such cross-multiplication was successfully pursued in a multisession individual tutorial with a low achiever in one of the classrooms. The unit ratio strategy, based within the ratio table, and the cross-multiplication strategy, based within the MT Puzzle, are general enough to be used with any numbers (whole numbers, fractions, or decimals) or with algebraic expressions. If one solves the same problem in all three ways (unit ratio, cross-multiplication, and MT Puzzle), one can see the relationships among the multiplication–division expressions involved in each strategy. Thus, all of these solution methods can come to be understood as different ways of seeing and solving a proportion. However grounding the unit ratio and the cross-multiplication strategies in MT Puzzles may help to avoid the rote learning of these strategies that is often decried in the literature.

CONCLUSION

Our studies indicate that fifth-grade students are ready to learn ratio and proportion. Their comfort with addition and growing facility with multiplication enabled them to understand

and relate the additive–multiplicative multiplication story situations created by repeated addition, the filmstrip drawings of these situations, and vertical ratio tables, all within the supportive context of the multiplication table. Students then moved on to solve proportion problems multiplicatively by thinking of MT Puzzles as two rows of two columns from the multiplication table or as two rows of a ratio table. The continual focus on ratio pairs as rows in vertical multiplication columns almost completely eliminated the typical inappropriate additive reasoning of subtracting within a ratio pair. The MT Puzzle enabled students to set up and solve middle-difficulty nondivisible proportion problems such as 6:14 = ?:35).

The Apprehending Zone Model and the Conceptual Phase Problem-Solving Model identify key learning processes that support sense-making in the classroom. They go beyond static or even dynamic models of mathematical cognition to address desirable features of the teaching–learning setting in action. They also focus on crucial attributes of designed teaching/learning experiences and thus support analyses that can result in recursive improvements in such designs. Our new transcliption method of video data organization and presentation enable a research team and readers to experience classroom teaching and learning data in a richer way than is usual. As these models and methods are applied to other domains, we are hopeful that richer and deeper understandings of mathematical cognition will emerge.

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