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Undirected Distances and the Postman-Structure of Graphs

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We present some properties of the distance function and of shortest paths in ± 1*-weighted undirected graphs. These extend some basic results, e.g., on matchings, on the Chinese postman problem, and on plane multicommodity flows.* Furthermore, distances turn out to be efficient tools to generalize the matching-structure of graphs to a structure related to subgraphs having only *parity constraints on their degrees* (these are called joins or postman sets), a problem posed by Lovász and Plummer. The special cases include *the generalization of the matching-structure of graphs to the weighted case.* The main result of the paper is *a good characterization (linear in the number of edges), conjectured by A. Frank, of the minimum weights of paths from a fixed vertex of an undirected graph without negative circuits.* This result contains the well-known minimax theorems on minimum "odd joins" and maximum packings of "odd cuts" (namely, Lovász's theorem on half integer packings and its sharpening by Seymour and later by Frank and Tardos) and strengthens them by constructing a "canonical" maximum packing of odd cuts with favourable properties. This packing of odd cuts turns out then to be characteristic for *the structure of minimum odd joins.* Using these, a Gallai–Edmonds type structural description of minimum odd joins is worked out. (The generalization of the Kotzig–Lovàsz canonical partition will appear in a forthcoming paper.) Briefly, distances in ± 1-weighted graphs make it possible for us to treat some properties of matchings themselves in a more compact way, and to generalize them providing new results on some other interesting special cases of ± 1-weighted graphs as well. This technique is worked out in the present paper. © 1990 Academic Press, Inc.

1. INTRODUCTION

In this paper we investigate the distance function in weighted undirected graphs. Several results about different problems can be presented in this framework. Let us see some examples:

Berge's well-known improving path statement [3] about the maximum cardinality of a matching can be formulated in the following way: Let the edges of a matching be of length -1, and let all the other edges have

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length 1. This *matching is maximum if and only if the distance (the minimum weight of a path) between any two non-saturated points is at least* 2.

Similarly, in order to see how *the deletion of* 1 *or* 2 *given vertices decreases the matching number* (the cardinality of a maximum matching) it is enough to look at distances defined by appropriate weight functions. Such questions play an important role in the description of the matching-structure [17], and distances provide an elegant way to obtain this description [20].

A (Chinese) *postman tour* in a graph is a closed walk that covers every edge of the graph at least once [19]. It is also easy to see and well known from [19] that *a postman tour is of minimum cardinality, if and only if it covers each edge of G once or twice, and putting weight* -1 *on the edges covered twice and* 1 *on those covered once, there is no circuit with negative total weight in the graph* (see later).

If there is no negative circuit, then this weight function defines finite distances between the vertices of G. We shall see in this paper that through the examination of these distances we can get deeper in the structure of the Chinese Postman Problem, and of some special cases such as (weighted) matchings or plane multicommodity flows. As a consequence, e.g., the results on the matching structure can be generalized to weighted matchings as well, and the existence of integer flows can be characterized in some new cases (see [24], [25]).

Besides these results the general structure theorem we present provides a common formulation, and its proof gives a new proof, e.g., of Berge and Tutte's minimax theorem on matchings (and actually of the Gallai–Edmonds structure theorem [10, 4], see Theorem 5.1 below, or of the Kotzig–Lovàsz theorem [17]), of Edmonds and Johnson's "Chinese postman" minimax theorem [6] (and actually of the stronger forms proved by Lovász [14], Seymour [27], and Frank, Sebő, and Tardos [9]), and, of course, of all the consequences of these (e.g., the fractional sums of circuits theorem of Seymour [26] or the perfect matching polyhedron, cf. e.g. [17]). Thus distances in ± 1-weighted undirected graphs provide a general "language" which makes it possible for us to treat all these problems in a compact and unified way.

Actually, undirected distances will permit us to make a first step in the direction of Lovàsz and Plummer's problem of developing "a structure theory of T-joins similar to the Gallai–Edmonds structure theory for matchings." A second step, the generalization of the Kotzig-Lovàsz theorem about canonical partitions, and the related canonical decomposition complete the generalization of the structure theory of matchings, and extends its applications [20].

Finally, let us indicate how *plane multicommodity flows* relate to distances in undirected graphs. Profiting from the fact that they are easier to visualize, we also illustrate the main result and some ideas of the paper with the help of the special case of plane multicommodity flows.

Suppose we are given the planar graph G, a set $R \subseteq E(G)$ of edges, and we must *determine a path in* $E(G)\backslash R$ *between the endpoints of each* $r \in R$, *so that these paths are pairwise edge-disjoint.* Seymour [27] has solved the fractional relaxation of this problem by observing its *equivalence to the linear programming dual of the planar Chinese postman problem.* A trivial necessary condition for the existence of such paths is: *for every cut* C, $|C\backslash R| \geqslant |C \cap R|$. This is called the *cut condition.* This problem is unsolved in general, and by a theorem of Seymour, *if* G *is Eulerian* (all of its degrees are even) and planar, then *the cut condition is sufficient.* We shall indicate a simple constructive proof of Seymour's theorem that reflects the main ideas of the paper. Recently, this method made it possible for us to find some integrality results for non-Eulerian graphs as well [24].

So, from now on, suppose that G is Eulerian, and that the cut condition is satisfied.

Consider a fixed embedding of the graph in the plane, and write into each face f *the minimum cost* $\lambda(f)$ *of reaching the face from the infinite region, where the cost of traversing an edge of* $E(G)\backslash R$ *is* \$1, *and that of traversing an edge of* R *is* \$-1. (This latter condition should be interpreted as a gain. Of course $\lambda(f)$ can also be negative.) More precisely, we take the minimum of $p-q$ over all "dual paths" (paths of the planar dual), which join the infinite region to the given face, where p is the number of edges of $E(G)\backslash R$, and q is the number of edges of R which are crossed by the path. (We may even allow the repetition of edges, if, to prevent infinite gains, we forbid walks which cross request edges in both directions.)

First note that *the gain is not bounded,* if and only if there is a dual circuit of negative total cost, i.e., *if and only if the cut condition is not satisfied.* So, if our assumption, the cut condition, is satisfied, then $\lambda(f)$ is bounded. In this case, clearly, in order to reach a face with a minimum cost, we always have a dual path without any repetition of vertices.

(1.1) Let b be the face for which $\lambda(b)$ is minimum, and suppose b is not the infinite face. Then *the cycle that is the boundary of* b *contains exactly one edge of* R.

Proof. Take a dual path of minimum cost from the infinite face to b. The edge through which this path goes last is in R, otherwise the cost of the face that precedes b would be one less than that of b. This edge is on the boundary of b. If there were another edge of R on the boundary of b, then traversing it, we would get to a neighbouring face with less cost, a contradiction either with the cut condition or with the choice of b.

Statement (1.1) gives some hope that the boundary of b can participate in a flow. Fortunately even much more is true:

(1.2) *Deleting the boundary of b, the cut condition is still satisfied. Furthermore, the "united" face arising after the deletion inherits the λ of the neighbours of b (they all had the same λ), and if x is not the united face, then $\lambda(x)$ is unchanged.*

Statement (1.2) is the specialization of a key result of the paper (Lemma 3.7) to this special problem. Using this statement it is not difficult to develop an algorithm which finds the edge-disjoint paths. The details for an arbitrary plane multicommodity flow problem with consequences for the complexity of this problem are being worked out in [25]. (The computation of distances in undirected graphs is polynomially equivalent to the weighted matching problem, as it was remarked by Edmonds and Johnson [5], cf. also Lawler [13] and Barahona [2]. For a version which is advantageous in the algorithm sketched above, cf. [25 or 24].)

Statement (1.2) can be easily proved from (1.1) and the following "switching lemma" which is actually the heart of the proof.

(1.3) *If C is a tight cut, that is $|C\setminus R| = |C \cap R|$, then replacing R by $R\Delta C$, the cut condition still holds, and λ does not change.*

Statement (1.3) is the special case of (3.2) and (3.3).

The above example shows how distances will be used in the paper. Actually, it is *a good characterization of the existence of negative circuits and of distances* in undirected graphs that will yield the results. Let us see the simple and well-known analogous statement for *directed* graphs.

Let us call a directed graph *conservative*, if it does not contain a directed circuit with negative total weight. The *distance* of the ordered pair (x, y) is the minimum weight of a directed path from x to y.

Let G be a digraph, $x_0 \in V(G)$, and $w\colon E(G) \to \mathbb{Z}$. (G, w) is conservative if and only if there exists a potential, i.e., a function $\pi\colon V(G) \to \mathbb{Z}$ with the following property:

(1.4) For each directed edge $xy \in E(G)$, $\pi(y) - \pi(x) \leqslant w(xy)$, and $\pi(x_0) = 0$.

Moreover, if (G, w) is conservative, the distances $\lambda(x)$ from x_0 form a potential; for any potential π with $\pi(x_0) = 0$, we have $\lambda(x) \geqslant \pi(x)$ for each $x \in V(G)$.

It is natural to ask for a similar statement in the undirected case. A graph G with weight function $w\colon E(G) \to \mathbb{Z}$ is called *conservative* if for any circuit $C \subseteq E(G)$, $w(C) \geqslant 0$. (For $Y \subseteq E(G)$, $w(Y) := \sum \{w(e)\colon e \in Y\}$.) The *distance* of a and b is defined as $\lambda(a, b) := \lambda_w(a, b) := \lambda_{G,w}(a, b) := \min\{w(P)\colon P \text{ is an } (a, b) \text{ path}\}$. It will turn out that the essential property analogous to (1.4) in the undirected case is (1.5) below:

(1.5) For each edge $xy \in E(G)$: $|\pi(y)-\pi(x)| \leqslant w(xy)$, and $\pi(x_0)=0$. Furthermore, if D is a component of the "level-set" $\{x \in V(G): \lambda_w(x_0, x) \leqslant i\}$ for some integer i, then: $x_0 \in D$ implies that all edges which enter D have positive weight; $x_0 \notin D$ implies that there exists exactly one negative edge entering D.

The essence of the main result of the paper is that the distances $\lambda(x)$ from x_0 satisfy (1.5) (cf. Theorem 3.1, and see also its consequence for the weighted general case in Section 4). The first line of (1.5) (which is the same as that of (1.4)) is not sufficient to handle undirected distances; i.e., in an undirected graph it does not characterize conservativeness, and does not give a lower bound for distances from a fixed point!

Note that *the minimum weight path problem in undirected graphs cannot be reduced to the corresonding problem in directed graphs*: If we replace every edge by two parallel edges directed in opposite directions, then negative circuits arise. Undirected *distances do not satisfy the triangle inequality, and it is not true that a subpath of a minimum weight path has minimum weight. However, undirected distances also have some interesting properties. The study of these is one of the main goals of this paper.*

Finally, let us give an idea how conservative graphs apply to matchings. Clearly, if we are given an arbitrary graph and a matching, then putting -1 weights on the edges of this matching and $+1$ weights for the other edges we get a conservative graph. We shall see that *the distance in this conservative graph are the same for any choice of a maximum matching, and they reflect the matching-structure of this graph.* It might be already felt that the last line of (1.5) will correspond to the odd components of the Berge–Tutte (or Gallai–Edmonds) theorem.

In order to be able to speak more easily about the components of the level sets defined by a function, let us introduce the following notation used throughout the paper: Let $\pi\colon V(G) \to \mathbb{Z}$ be arbitrary, and define the family $\mathscr{D} := \mathscr{D}(\pi) := \mathscr{D}(G, \pi)$ to be the union of the families $\mathscr{D}^i$, where $\mathscr{D}^i$ consists of the vertex sets of the components of the graph induced by the *level set* $\{x \in V(G): \pi(x) \leqslant i\}$, $i = m, m+1, ..., M$, with $m := m(\pi) := m(G, \pi) = \min\{\pi(x) : x \in V(G)\}$, $M := M(\pi) := M(G, \pi) := \max\{\pi(x) : x \in V(G)\}$.

The notation $\mathscr{D}(\pi)$ will be used throughout the paper. Clearly, $\mathscr{D}(\pi)$ is a laminar family. (A family $\mathscr{H}$ of subsets of $V(G)$ is called *laminar* if $H_1, H_2 \in \mathscr{H}$ implies that either $H_1 \cap H_2 = \emptyset$ or $H_1 \subseteq H_2$, or $H_2 \subseteq H_1$.)

The paper contains the following: Section 2 is an introduction to the Chinese postman problem, in other words, to parity constrained graph factors. The goal of this section is to show that conservative weightings represent a hopeful equivalent language to investigate the structure of matchings and of their generalizations. Section 3 is a study of distances in conservative graphs. As a result the main theorem is proved. Section 4 deduces good characterizations of the conservativeness and distances in undirected graphs, and points out some consequences on the structure of the shortest paths. Section 5 is devoted to a structure theorem for the Chinese postman problem, which is a consequence of all the previous results. The application of the results to the matching-structure is also shown in Section 5.

2. Parity Constrained Factors

Let $G=(V, E)$ be a graph. The degree of $x \in V$ is denoted by $d_G(x)$. Given a function $t\colon V \to \mathbb{Z}$ ($\mathbb{Z}$ is the set of integers), $F \subseteq E$ will be called a *t-join* in G, if $d_F(x) \equiv t(x) \bmod 2$, for all $x \in V$. In the future "mod 2" will be deleted in the notation.

Obviously, G has a t-join if and only if for each connected component G' of G, $t(V(G')) \equiv 0$. ($V(G)$ is the vertex set and $E(G)$ the edge set of the graph G. $t(X) := \sum \{t(x) : x \in X\}$.)

Let us remark that in the literature t-joins are called T-joins, $T \subseteq V(G)$. A T-join is a t-join with $t(x)=1$ if $x \in T$, 0 otherwise. Our notation seems to be more convenient for our purposes. Of course, the only important factor is the parity of $t(x)$.

G is always supposed to be connected, and t (with maybe some index) will always denote an integer function on the vertices for which $t(V(G)) \equiv 0$ is satisfied; on the other hand, w will always denote a function on the edges. Finally, for any graph G and $w\colon E(G) \to \mathbb{R}$ let us introduce the notation $E^- := E^-(G, w) := \{e \in E(G)\colon w(e) < 0\}$ and $d^-(x) := d_{E_-}(x)$ for the whole paper.

We suppose from now on that a *path* does not contain a repetition of vertices. A *walk* may even contain a repetition of edges. Paths and walks are considered to be sets of edges, and if their two endpoints coincide, they are called *circuits* and *closed walks*, respectively. We allow loops and parallel edges in G.

Let us now see some examples for t-joins. Examples 3 and 4 and the definitions and notations they contain are used throughout the paper.

Example 1: Chinese Postman Tours. Obviously, a postman tour is designed by doubling certain edges of the graph so that all degrees become even. Thus a one-to-one correspondence is established between Chinese postman tours and d_G-joins, and in addition minimum (perhaps weighted) Chinese postman tours correspond to minimum (weighted) d_G-joins.

Since the results or the algorithms are not really more general for t-joins than for the Chinese postman problem, this latter term is often used for the subject of t-joins.

EXAMPLE 2: PATHS. For $a \neq b \in V(G)$ let $p^{a,b}(x) := 0$ if $x \notin \{a, b\}$ and $p^{a,b}(x) := 1$ if $x \in \{a, b\}$. A $p^{a,b}$-join is obviously the union of an (a, b) path and circuits, all pairwise edge-disjoint.

Obviously, $F \subseteq E(G)$ is a minimum weight $p^{a,b}$-join of a conservative graph if and only if it is the edge-disjoint union of a minimum weight (a, b) path and 0-weight circuits.

Paths as the simplest "joins" will play an important role in analyzing t-joins for arbitrary t. Actually, it turns out that *for arbitrary t, the minimum weight paths from a fixed vertex with respect to a certain ± 1 objective function contain all information about the set of minimum t-joins.*

EXAMPLE 3: MATCHINGS. Let G be a graph and let us add a point x_0 to its vertex set, and all edges $x_0 x$, $x \in V(G)$, to its edge set. Let $t(x) := 1$ for $x \in V(G)$, and $t(x_0) := |V(G)|$. To each matching of G there corresponds a t-join in the graph defined above, and the t-joins that correspond to maximum matchings are minimum.

Minimum weight perfect matchings can also be easily handled as minimum cardinality t-joins. Adding a big constant to the weights, the minimum weight **1**-joins will be exactly the minimum weight matchings, and the structure of the minimum weight t-join problem can be reduced to the cardinality case by subdividing each edge into as many edges as its weight.

This shows a major advantage of t-join problems compared to the special case of matchings: they are *closed under the contraction or the subdivision of edges* by new vertices, and consequently, for many purposes, it is enough to treat the cardinality case. As Lovàsz and Plummer [17, 8.0] remark, "many problems on matchings are just as interesting, and sometimes even more easily handled, when generalized to T-joins."

t-joins can also be applied to multicommodity flows [27] (cf. Introduction), and an application in physics has been shown in [1]. Such questions as deciding which edges of G are elements of all or none of the minimum t-joins have signification in physics. The results of this paper make it possible for one to get some insight into these questions [20].

Despite these, we do not emphasize any algorithmic aspects here, we only aim at giving simple proofs to the results. ([22] is devoted to an algorithmic approach.) However, a remark concerning the construction of plane multicommodity flows is included at the end of Section 3. According to a referee report, an alternative algorithmic proof of the main result can be given using Korach's postoptimality algorithm [11], which was worked out to determine maximum packings of odd cuts in bipartite graphs.

In the remainder of this section we present some minimax theorems on t-joins and show their relation to conservative graphs.

$B \subseteq E(G)$ is called a *t-cut* if $B = \delta(X)$ for some $X \subseteq V(G)$ with $t(X) \equiv 1$, where $\delta(X) := \delta_G(X) := \{xy \in E(G)\colon x \in X,\ y \notin X\}$ is the *coboundary* of X. Instead of $\delta(\{x\})$ we shall simply write $\delta(x)$. $G(X)$ will denote the *subgraph of G spanned by $X \subseteq V(G)$*.

Let $\nu = \nu(G, t)$ be the maximum number of disjoint t-cuts, and $\tau = \tau(G, t)$ the minimum cardinality of a t-join. Furthermore, let $\nu_2 = \nu_2(G, t)$ be the maximum cardinality of a family of t-cuts in G with the property that each $e \in E(G)$ is covered at most twice, i.e., contained in at most 2 of its elements. (A *family* is a list of sets where repetition is allowed, and its *cardinality* is the length of the list. A set of disjoint t-cuts is called a *packing*, and a family of t-cuts where each edge is covered at most twice is called a *2-packing* of t-cuts.) Obviously, $\tau \geqslant \nu_2/2 \geqslant \nu$. (Moreover, t-joins and t-cuts constitute a *blocking pair* of hypergraphs; i.e., the cut B is a minimal t-cut if and only if for all t-joins F: $|B \cap F| \geqslant 1$ and B has no proper subset with this property; F is a minimal t-join if and only if for all t-cuts B: $|F \cap B| \geqslant 1$ and F has no proper subset with this property. Actually, $|B \cap F|$ is odd for any pair of t-join and t-cut.) The following fundamental theorem is implicitly contained in Edmonds and Johnson [5], and a first proof was published by Lovàsz [14].

THEOREM 2.1 [14]. $\tau(G, t) = \nu_2(G, t)/2$.

The pair (K_4, t_4), where K_4 is the complete graph on 4 vertices, and $t_4(x) = 1$ for all $x \in V(K_4)$, is an example where $\tau(G, t) > \nu(G, t)$. Seymour proved the following:

THEOREM 2.2 [27]. *If G is bipartite, then $\tau(G, t) = \nu(G, t)$.*

For a first algorithmic proof of Theorem 2.1 or 2.2 see Korach [11]. Theorem 2.1 is easily proved to be a special case of Theorem 2.2: divide each $e \in E(G)$ by a new node v_e with $t(v_e) = 0$ and apply Theorem 2.2 to the resulting bipartite graph. This correspondence shows that it is not a restriction of generality to investigate t-joins and t-cuts only in bipartite graphs, and the statements even become stronger.

An easy computation using the equality of Theorem 2.2 (or 2.1) shows that each odd cut of a maximum (2-)packing contains one edge of each minimum t-join, and each edge of a minimum t-join is covered exactly once (twice) by any maximum (2-)packing.

It is not difficult to prove from these theorems the characterization of fractional sums of circuits [26], or of the perfect matching polyhedron [5], [17]. To prove the Berge–Tutte theorem it is better to use Theorem 2.4 below.

Let $\mathbf{0}$ be the function that takes the value 0 for all $x \in V(G)$. A $\mathbf{0}$-join is the edge-disjoint union of circuits. Thus conservativeness means that all **0**-joins are non-negative.

If F_1 is a t_1-join and F_2 is a t_2-join, then we shall often use the fact that $F_1 \Delta F_2$ is a $t_1 + t_2$-join. ($X \Delta Y := (X \setminus Y) \cup (Y \setminus X)$ is the symmetric difference of X and Y.) Thus, if both F_1 and F_2 are t-joins, then $F_1 \Delta F_2$ is a **0**-join.

The following simple fact was observed by Mei Gu Guan:

(2.3) A t-join $F \subseteq E(G)$ is minimum if and only if putting $w(e) = -1$ for $e \in F$ and $w(e) = 1$ for $e \notin F$, (G, w) is conservative.

Proof. If F is a minimum t-join and C is an arbitrary circuit, then $F \Delta C$ is a t-join and consequently $w(C) = |C \setminus F| - |C \cap F| = |F \Delta C| - |F| \geqslant 0$. Conversely, if (G, w) is conservative and F' is a t-join, then $F \Delta F'$ is a **0**-join, and $0 \leqslant w(F' \Delta F) = |F'| - |F|$. Q.E.D.

Through this remark any statement on t-joins has an equivalent reformulation in terms of conservative graphs. Reformulating the above theorems we arrive at a good characterization of conservativeness. We show how, for example, the reformulation of a stronger form of Theorem 2.1 due to Frank and Tardos can be carried out:

THEOREM 2.4 [9]. *Assume that G is bipartite with bipartition $\{A, B\}$, and let $q(X)$ $(X \subseteq V(G))$ denote the number of t-odd components of $G - X$. Then*

$$\tau = \max\left\{ \sum_{i=1}^{k} q(X_i) \colon \{X_1, ..., X_k\} \text{ is a partition of } A \right\}.$$

THEOREM 2.4′ [9]. *Assume that G is bipartite with bipartition $\{A, B\}$, and let $w \colon E(G) \to \{-1, 1\}$. (G, w) is conservative if and only if A has a partition $\{X_1, ..., X_k\}$ such that the coboundary of each component of $G - X_i$ contains at most one negative edge $(i = 1, ..., k)$.*

The proof of the equivalence of Theorems 2.4 and 2.4′ is left to the reader as an excercise (cf. [9]).

Besides giving an optimal packing of odd cuts in a particular form, the insight provided by Theorem 2.4 led to a simple direct proof (providing a simple proof of Seymour's theorems as well) *using* the distance function [9, 21]. However, in order to understand the role of distances in its full extent we need A. Frank's conjecture, which sharpens Theorem 2.4′ by *defining* an optimal partition with the help of distances. It is clearly implied by, and actually equivalent to, Theorem 3.1 or 4.1 below:

FRANK'S CONJECTURE [8]. *Assume that G is connected and bipartite, $w \colon E(G) \to \{-1, 1\}$, (G, w) is conservative, and $x_0 \in V(G)$. Let $i \in \mathbb{Z}$ and D*

be a component of the graph induced by the set $\{x \in V(G) : \lambda_w(x_0, x) \leqslant i\}$. *Then* $\delta(D)$ *contains at most one negative edge.*

Frank's conjecture easily implies Theorem 2.4′ and thus all the theorems presented in this section so far (and thus all their consequences):

The if part of Theorem 2.4′ is trivial since each circuit intersects each cut in an even number of edges at most one of which is of weight -1. To prove the only if part assume, e.g., that $x_0 \in A$ and define for fixed i, a partition $\mathscr{P}^i$ of the set $S^i := \{x \in V(G) : \lambda_w(x_0, x) = i\}$ whose classes are the intersections of S^i with the components of the graph G^i induced by $\{x \in V(G) : \lambda_w(x_0, x) \leqslant i\}$. Take the union of such partitions for all even i-s. Using the property of G^i stated in Frank's conjecture, we get that this union is a partition of A with the property stated in Theorem 2.4′.

It may be worth comparing the above theorems on the level of matchings: while we cannot prove the Berge–Tutte theorem easily from Theorem 2.1 or 2.2, it follows easily from Theorem 2.4 (cf. [9]). In addition, Frank's conjecture will provide the classes of the Gallai–Edmonds structure theorem (cf. Theorem 5.1), and more generally, it will provide a unique "canonical" maximum packing of odd cuts, depending only on (G, t).

3. Distances in Conservative Graphs

The goal of this section is to prove Frank's conjecture, which is a fundamental property of undirected distances. This is the main result of the paper. To facilitate its use we put it in the following more technical and detailed form:

THEOREM 3.1. *Assume that G is connected and bipartite, $w : E(G) \to \{-1, 1\}$, (G, w) is conservative, and $x_0 \in V(G)$. Let $\lambda(x) := \lambda_w(x_0, x)$. Then*

(1) $\lambda(x_0) = 0$

(2) $|\lambda(x) - \lambda(y)| = 1$ *for all* $xy \in E(G)$

(3) $|\delta(D) \cap E^-(G, w)| = 1$ *provided* $x_0 \notin D \in \mathscr{D}$,
$|\delta(D) \cap E^-(G, w)| = 0$ *provided* $x_0 \in D \in \mathscr{D}$, *where* $\mathscr{D} := \mathscr{D}(\lambda)$.

(This theorem has an inverse which is much easier, and will be discussed later. Together with its inverse, it yields a good characterization of conservativeness and distances in undirected graphs [cf. Theorem 4.1, and for arbitrary weights Theorem 4.4], and it is the basis of the notion of potentials, cf. Section 4.)

Let us fix the notation $\lambda(x) := \lambda_w(x_0, x)$ for the whole paper.

We first prove some simpler properties of the distance function. These properties are then used to prove the main property stated in Theorem 3.1 that actually implies all the others.

If w is a weight function and $Y \subseteq E(G)$, *let $w[Y]$ be the switched weight function defined as follows*: $w[Y](e) = -w(e)$ *if* $e \in Y$ *and* $w(e)$ *if* $e \notin Y$. The following two "switching lemmas" constitute the heart of this paper.

(3.2) Assume that (G, w) is conservative and $C \subseteq E(G)$, $w(C) = 0$. Then $(G, w[C])$ is also conservative, and $\lambda_{w[C]}(a, b) = \lambda_w(a, b)$ for all $a, b \in V(G)$.

Proof. Let P be an (a, b) path. ($a = b$ is allowed.) $w[C](P) = w(P \setminus C) - w(P \cap C) = w(P \Delta C) - w(C) = w(P \Delta C)$. But $P \Delta C$ is the edge-disjoint union of an (a, b) path and circuits. Since (G, w) is conservative, the circuits of G have non-negative weight, so if $a = b$, then we get $w(P \Delta C) \geqslant 0$, and if $a \neq b$, then $w(P \Delta C) \geqslant \lambda_w(a, b)$ follows. Thus $(G, w[C])$ is conservative, and $\lambda_{w[C]}(a, b) \geqslant \lambda_w(a, b)$. Apply this inequality to $w[C]$ instead of w and use $w[C][C] = w$ to get $\lambda_{w[C]}(a, b) = \lambda_w(a, b)$. Q.E.D.

(3.3) Assume that (G, w) is conservative, and $Q \subseteq E(G)$ is a w-minimum (x_0, x_0') path, $x_0 \neq x_0' \in V(G)$, $l := w(Q)$ $(= \lambda_w(x_0, x_0'))$. Then $(G, w[Q])$ is conservative, and

$$\forall x \in V(G): \qquad \lambda_{w[Q]}(x_0', x) = \lambda_w(x_0, x) - l.$$

Proof. Let P be an (x_0', x) path or a circuit. $w[Q](P) = w(P \setminus Q) - w(P \cap Q) = w(P \Delta Q) - l$.

— If P is a circuit, then $P \Delta Q$ is a $p^{x_0, x_0'}$-join and thus, using the conservativeness of (G, w), $w(P \Delta Q) \geqslant l$. Hence, $w[Q](P) \geqslant l - l = 0$, and the conservativeness of $(G, w[Q])$ is proved.

— If P is a w-minimum (x_0', x) path, then $P \Delta Q$ is a $p^{x_0', x} + p^{x_0, x_0'}$ join. Since $p^{x_0', x} + p^{x_0, x_0'} \equiv p^{x_0, x}$, $P \Delta Q$ is the union of an (x_0, x) path and disjoint circuits. Using the conservativeness of (G, w), $w[P \Delta Q] \geqslant \lambda_w(x_0, x)$, whence $\lambda_{w[Q]}(x_0', x) = w[Q](P) = w(P \Delta Q) - l \geqslant \lambda_w(x_0, x) - l$.

Apply this inequality for $w[Q]$ instead of w interchanging the role of x_0 and x_0' and use $w[Q][Q] = w$ and $w[Q](Q) = -l$ to get $\lambda_w(x_0, x) \geqslant \lambda_{w[Q]}(x_0', x) - (-l)$. ($Q$ is also a $w[Q]$-minimum (x_0, x_0') path: if Q' is an arbitrary (x_0, x_0') path, then $w[Q](Q') - w[Q](Q) = w(Q' \setminus Q) - w(Q \cap Q') + w(Q) = w(Q' \Delta Q) \geqslant 0$, since $Q' \Delta Q$ is a **0**-join.) Q.E.D.

Statement (3.3) means that $\lambda_{w[Q]}(x_0', x) - \lambda_w(x_0, x)$ is independent of x; thus, the "level-sets" of the distances from x_0 and x_0' are the same, and consequently, $\mathscr{D}(\lambda) = \mathscr{D}(\lambda')$, where $\lambda(x) := \lambda_w(x_0, x)$, $\lambda'(x) := \lambda_{w[Q]}(x_0', x)$.

The similarities in the claims and the proofs of (3.2) and (3.3) raise the question of whether they have a common generalization. The answer is that they do; they are two special cases of the following statement, which will be useful later:

(3.4)) If (G, w) is arbitrary and F_i is a t_i-join $(i=1, 2)$, then the following statements are equivalent:

(i) $F_1 \Delta F_2$ is a w-minimum t_1+t_2-join
(ii) $w[F_1 \Delta F_2]$ is conservative
(iii) F_1 is a $w[F_2]$-minimum t_1-join
(iv) F_2 is a $w[F_1]$-minimum t_2-join.

Proof. Copying the proof of (2.3) we get that F is a w-minimum t-join if and only if $w[F]$ is conservative. (Replace "$|\ |$" by "w", and "w" by "$w[F]$" in the proof of (2.3).) This implies the equivalence of (ii) with all the rest, since $w[F_1 \Delta F_2] = w[F_2][F_1] = w[F_1]][F_2]$. Q.E.D.

(To prove (3.2) and 3.3) from (3.4) note that in a conservative graph a 0-weight circuit is a w-minimum $\mathbf{0}$-join, and a w-minimum (a, b) path is a w-minimum $p^{a,b}$-join. Then use the trivial equality $w[X](Y) = w(X \Delta Y) - w(X)$, $X, Y \subseteq E(G)$. We preferred, however, to provide separate direct proofs as well.)

Now, we are getting closer to Theorem 3.1. We first show that λ satisfies (3) for the 1-element components $D = \{b\} \in \mathscr{D}$.

LEMMA 3.5. *If (G, w) is conservative, $w(e) \neq 0$, $\forall e \in E(G)$, and $b \in V(G)$ is such that $\lambda(x_0, b) = \min\{\lambda(x_0, y) : y \in V(G)\}$, then $d^-(b) = 1$ except if $b = x_0$, when $d^-(b) = 0$.*

Proof. If $b = x_0$, then $\forall y \in V(G)$: $\lambda(x_0, y) \geqslant \lambda(x_0, x_0) = 0$, whence $d^-(b) = d^-(x_0) = 0$. Assume that $b \neq x_0$, and let P be a w-minimum (x_0, b) path $(w(P) = \lambda(x_0, b))$. By the choice of b, $\forall x \in V(P)$: $w(P(x, b)) = w(P(x_0, b)) - w(P(x_0, x)) \leqslant 0$. Namely, $w(ab) < 0$ for the last edge ab of P. ($w(ab) \neq 0$ by hypothesis.) Suppose for a contradiction that $a'b \in E^-$, $a' \neq a$. $a' \notin V(P)$ since $a' \in V(P)$ would imply that $P(a', b) \cup a'b$ is a negative circuit. Thus $P \cup a'b$ is an (x_0, a') path, $w(P \cup a'b) < w(P)$, which is in contradiction with the choice of b. Q.E.D.

Note that the proof above is the same as the one we saw for plane multicommodity flows in the Introduction.

Statements (3.2) and (3.3) and Lemma 3.5 are summarized in the following statement:

LEMMA 3.6. *Let (G, w) be conservative, $w(e) \neq 0$, $\forall e \in E(G)$. Furthermore let $x_0, b \in V(G)$ be such that $\lambda(x_0, b) = \min\{\lambda(x_0, y) : y \in V(G)\}$. Let K be either a 0-weight circuit or a w-minimum path between x_0 and an arbitrary vertex $x_1 \neq b$ of G. If $|K \cap \delta(b)| \neq \varnothing$, then*

$$K \cap \delta(b) = \{e_1, e_2\}, \qquad \textit{where} \quad w(e_1) < 0,\, w(e_2) > 0.$$

Proof. By (3.2) and (3.3), resp., $(G, w[K])$ is conservative. Define $x_1 := x_0$ if K is a circuit. $\lambda_{w[K]}(x_1, b) = \min\{\lambda_{w[K]}(x_1, y) : y \in V(G)\}$, since $\lambda_{w[K]}(x_1, y)$ arises from $\lambda(x_0, y)$ by adding a constant independent of y. Consequently, we have by Lemma 3.5: $1 + 1 \geqslant d_{E^-(G, w)}(b) + d_{E^-(G, w[K])}(b) \geqslant |K \cap \delta(b)| = 2$, and equality must hold throughout. Q.E.D.

Note that Theorem 3.1 implies Lemma 3.6, and Lemma 3.5 is also an easy consequence of it. Of course, just conversely, we wish to prove Theorem 3.1 using these lemmas. Before proving Therem 3.1 we need one more step:

Shrinking $U \subseteq V(G)$ to u means replacing U by a single new vertex u and defining for each edge of the original graph an edge of the shrunken graph in the natural way. (Of course loops and parallel edges may emerge. The shrunken graph has the same number of edges as that of the original.) Shrinking the two endpoints of an edge is called the *contraction* of the edge. If we also delete the edge(s) induced by U we shall speak about identification.

If a function $t: V(G) \to \mathbb{Z}$ is given, then $t(u) := t(U)$. If a function $w: E(G) \to \mathbb{Z}$ is given, then it is inherited by the shrunken graph. The notations t and w *remain unchanged in the shrunken graph,* since it does not cause any misunderstanding. Moreover, if $u' \in U$, then u' will also be used as an alternative notation for u. The converse operation is *blowing up u to U.*

LEMMA 3.7. *Let G be connected and bipartite, and $w: E(G) \to \{-1, 1\}$. Assume that (G, w) is conservative, $x_0, b \in V(G)$, and $\lambda(x_0, b) = \min\{\lambda(x_0, y) : y \in V(G)\}$. Let $a_1 b, a_2 b \in E(G)$, $a_1 \neq a_2$. Then shrinking $\{a_1, a_2\}$ to a single point a, the resulting graph (G^*, w) is conservative and $\lambda_{G^*, w}(x_0, x) = \lambda_{G, w}(x_0, x)$ for all $x \in V(G)$.*

Proof. Let us first prove that (G^*, w) *is conservative and* $\lambda_{G^*, w}(x_0, x) \geqslant \lambda_{G, w}(x_0, x)$. Let K^* be either a circuit or an (x_0, x) path in G^* ($x \in V(G)$ is arbitrary). We have to prove that $w(K^*) \geqslant 0$ if K^* is a circuit, and $w(K^*) \geqslant \lambda_{G, w}(x_0, x)$ if it is an (x_0, x) path in G^*, i.e., it is enough to show that there exists in G a circuit K or an (x_0, x) path K, respectively, such that $w(K^*) \geqslant w(K)$. If $x = b$ and K is an arbitrary (x_0, x) path, then this is trivial, because $w(K^*) \geqslant w(K^* \backslash ab) - 1 \geqslant \lambda_{G, w}(x_0, a) - 1 = \lambda_{G, w}(x_0, b)$.

So suppose $x \neq b$. We can assume that $K^* \cap \delta(a) = \{ax_1, ax_2\}$, $a_1 x_1$, $a_2 x_2 \in E(G)$. (If $|K^* \cap \delta(a)| = \emptyset$, or if $|K^* \cap \delta(a)| = 2$ but in G either both edges of $K^* \cap \delta(a)$ have a_1 or both have a_2 as an endpoint, then K^* is a circuit or an (x_0, x) path resp. in G too, and $K := K^*$ is good.) But then $K := K^* \cup \{a_1 b, a_2 b\} \subseteq E(G)$ is a circuit or an (x_0, x) path, respectively.

— If $\{w(a_1 b), w(a_2 b)\} = \{-1, 1\}$, then $w(K^*) = w(K)$ and we are done.

— $w(a_1 b) = w(a_2 b) = -1$ cannot hold because $d^-(b) \leqslant 1$ by Lemma 3.5.

— Finally, if $w(a_1 b) = w(a_2 b) = 1$, then $w(K^*) = w(K) - 2$. But in this case we know from Lemma 3.6 that K can neither be a 0-weight circuit nor a minimum weight (x_0, x) path. Since G is bipartite, we have by parity: $w(K) \geqslant 2$ if K is a circuit, and $w(K) \geqslant \lambda_{G,w}(x_0, x) + 2$ if it is an (x_0, x) path. So $w(K^*) \geqslant 0$ or $w(K^*) \geqslant \lambda_{G,w}(x_0, x)$, resp., hold in any case.

$\lambda_{G^*,w^*}(x_0, x) \leqslant \lambda_{G,w}(x_0, x)$ remains to be proved. If we contract $\{a_1, a_2\}$ in a w-minimum (x_0, x) path $P \subseteq E(G)$, then the result is the edge-disjoint union of an (x_0, x) path P^* of G^* and perhaps a circuit of G^*. Since we have already proved the conservativeness of (G^*, w), we have $w(P^*) \leqslant w(P) = \lambda_{G,w}(x_0, x)$. Q.E.D.

Now we only have to put the results together, and Theorem 3.1 is proved:

Proof of Theorem 3.1. (1) is trivial. To prove (2) let $xy \in E(G)$. By parity $\lambda(x) \neq \lambda(y)$, say $\lambda(x) > \lambda(y)$. We prove that $\lambda(x) \leqslant \lambda(y) + 1$. Let $P \subseteq E(G)$ be a w-minimum (x_0, y) path. $w(xy) \leqslant 1$ and $-w(xy) \leqslant 1$ hold by hypothesis.

If $x \notin V(P)$, then set $P' := P \cup \{xy\}$. Clearly, $w(P') \leqslant w(P) + 1$.

If $x \in V(P)$, then $P' := P(x_0, x) = P \setminus P(x, y)$ is an (x_0, x) path:

if $xy \in P$, then $P(x, y) = xy$ and $w(P') \leqslant w(P) + 1$ immediately follows;

if $xy \notin P$, then $P(x, y) \cup xy$ is a circuit, and $-w(P(x, y)) \leqslant w(xy)$ follows by the conservativeness of (G, w), whence $w(P') = w(P) - w(P(x, y)) \leqslant w(P) + w(xy) \leqslant w(P) + 1$.

Since P' is an (x_0, x) path in any case, $\lambda(x) \leqslant w(P') \leqslant w(P) + 1 = \lambda(y) + 1$ and (2) is proved.

In order to prove (3) we proceed by induction on $|V(G)|$. For graphs consisting of one vertex the theorem is obvious. Let $b \in V(G)$, $\lambda(b) = \min\{\lambda(x) : x \in V(G)\}$.

— If $|\Gamma(b)| = 1$ $(\Gamma(b) := \{x \in V(G) : xb \in E(G)\})$, then $(G - b, w)$ is connected, bipartite, and conservative, and the statement for (G, w) follows by induction. (If $b = x_0$, then the "new center," that is the "new x_0" in $G - b$, is the element of $\Gamma(b)$.)

— If $|\Gamma(b)| \geqslant 2$, then choose $a_1 \neq a_2 \in \Gamma(b)$ and shrink $\{a_1, a_2\}$ to a. Let G^* be the contracted graph. Clearly, G^* is bipartite, and by Lemma 3.7 (G^*, w) is conservative. Furthermore, $|V(G^*)| < |V(G)|$, so, by induction, (3) holds for $\mathscr{D}^* := \mathscr{D}(G^*, \lambda^*)$, where $\lambda^*(x) := \lambda_{G^*,w}(x_0, x)$ $(x \in V(G^*))$. But the elements of $\mathscr{D} = \mathscr{D}(G, \pi)$ emerge by blowing up "a" to $\{a_1, a_2\}$ in the elements of $\mathscr{D}^*$, since $\lambda_{G,w}(x_0, x) = \lambda_{G^*,w}(x_0, x)$ by Lemma 3.7. (The connectedness of the components of the graph spanned by $\{x \in V(G^*): \lambda_{G^*,w}(x_0, x) \leqslant i\}$ is not broken when a is blown up since $a_1b, a_2b \in E(G)$.) Thus the coboundary of each $D \in \mathscr{D}$ contains the same number of negative edges as the coboundary of the corresponding element of $\mathscr{D}^*$. Q.E.D.

4. Potentials

In this section we develop potentials in undirected graphs, and describe the structure of shortest paths with their help.

First let G be bipartite, $w: E(G) \to \{-1, 1\}$ and $x_0 \in V(G)$. The function $\pi: V(G) \to \mathbb{Z}$ will be called a *potential in* (G, w) *centered at* x_0 if (1), (2), and (3) hold:

(1) $\pi(x_0) = 0$

(2) $|\pi(x) - \pi(y)| = 1$ for all $xy \in E(G)$

(3) $|\delta(D) \cap E^-(G, w)| = 1$ provided $x_0 \notin D \in \mathscr{D}$,
$|\delta(D) \cap E^-(G, w)| = 0$ provided $x_0 \in D \in \mathscr{D}$, where $\mathscr{D} := \mathscr{D}(\pi)$.

The following theorem, which is analogous to the corresponding statement on directed graphs (cf. statement after (1.3)), is nothing else but a reformulation of Theorem 3.1:

THEOREM 4.1. *Let G be connected and bipartite, $w: E(G) \to \{-1, 1\}$ and $x_0 \in V(G)$. Then (G, w) is conservative if and only if there exists a potential centered at x_0. Moreover, if (G, w) is conservative, then $\lambda(x) := \lambda_w(x_0, x)$ is a potential centered at x_0, and $\lambda(x) \geqslant \pi(x)$ for any potential π centered at x_0 and any $x \in V(G)$.*

Proof of Theorem 4.1. The essential only if part, and the fact that λ is a potential, is exactly Theorem 3.1. It is now worth checking the easy "if part" in detail to see how potentials work: Assume that π is a potential centered at x_0, and let $C \subseteq E(G)$ be an arbitrary circuit. Condition (2) implies that $\{\delta(D): D \in \mathscr{D}\}$ is a partition of $E(G)$, and hence $w(C) = \sum \{w(C \cap \delta(D)): D \in \mathscr{D}\}$. But $w(C \cap \delta(D)) \equiv |C \cap \delta(D)| \equiv 0$ and $|\delta(D) \cap E^-(G, w)| \leqslant 1$ imply $w(C \cap \delta(D)) \geqslant 0$ for any $D \in \mathscr{D}$, whence $w(C) \geqslant 0$ as stated above.

An informal way of describing this is the following: if an edge of a circuit goes from a level i down to level $i-1$, then, by (2), later another edge must come back from level $i-1$ to level i. Clearly, the two edges are in the same $\delta(D)$, and since $\delta(D)$ contains at most one negative edge, the contribution of these two edges is non-negative.

The same is true for a path if it starts from x_0 and first goes "under," and then comes "over," a level. If it leaves a level without coming back, then it leaves that component of the corresponding level-set which contains x_0, so it leaves on a positive edge. In this way it is easy to see the inequality $\lambda(x_0, x) \geqslant \pi(x)$. More formally:

Let P be a w-minimum (x_0, x) path, and for $a, b \in V(G)$ let $\mathscr{D}(a, b) := \{D \in \mathscr{D} : a \in D, \quad b \in D\}$, $\mathscr{D}(a, \bar{b}) := \mathscr{D}(\bar{b}, a) := \{D \in \mathscr{D} : a \in D, b \notin D\}$, $\mathscr{D}(\bar{a}, \bar{b}) := \{D \in \mathscr{D} : a \notin D, b \notin D\}$.

$$\begin{aligned}\lambda(x_0, x) = w(P) &= \sum \{w(P \cap \delta(D)) : D \in \mathscr{D}\}\\ &= \sum \{w(P \cap \delta(D)) : D \in \mathscr{D}(\bar{x}_0, x)\}\\ &\quad + \sum \{w(P \cap \delta(D)) : D \in \mathscr{D}(x_0, \bar{x})\}\\ &\quad + \sum \{w(P \cap \delta(D)) : D \in \mathscr{D}(\bar{x}_0, \bar{x})\}\\ &\quad + \sum \{w(P \cap \delta(D)) : D \in \mathscr{D}(x_0, x)\}\\ &\geqslant -|\mathscr{D}(\bar{x}_0, x)| + |\mathscr{D}(x_0, \bar{x})| + 0 + 0,\end{aligned}$$

because $w(P \cap \delta(D)) \geqslant \varepsilon$ with $\varepsilon = 0$ for the members of the third and fourth sum, with $\varepsilon = -1$ for those of the first, and $\varepsilon = 1$ for those of the second.

$$\begin{aligned}|\mathscr{D}(x_0, \bar{x})| - |\mathscr{D}(\bar{x}_0, x)| &= (|\mathscr{D}(x_0, \bar{x})| + |\mathscr{D}(x_0, x)|) - (\mathscr{D}(x_0, x) + \mathscr{D}(\bar{x}_0, x))\\ &= (M - \pi(x_0) + 1) - (M - \pi(x) + 1) = \pi(x).\end{aligned}$$

So $\lambda(x_0, x) \geqslant \pi(x)$. Q.E.D.

We have in addition that equality holds in the result if and only if equality holds throughout, i.e., $w(P \cap \delta(D)) = \varepsilon$. This determines the following structure:

(4.2) *If P is an (x_0, x) path and π is a potential centered at x_0, then the equality $w(P) = \pi(x)$ holds if and only if (a), (b), (c), and (d), are satisfied:*

(a) $P \cap \delta(D) = \{e\}, e \in E^-$, *provided* $D = \in \mathscr{D}(\bar{x}_0, x)$

(b) $P \cap \delta(D) = \{e\}, e \in E^+$, *provided* $D \in \mathscr{D}(x_0, \bar{x})$

(c) $P \cap \delta(D) = \{e_1, e_2\}$, $w(e_1) = -1$, $w(e_2) = 1$, *provided* $D \in \mathscr{D}(\bar{x}_0, \bar{x})$ *and* $P \cap \delta(D) \neq \varnothing$.

(d) $P \cap \delta(D) = \varnothing$, *provided* $D \in \mathscr{D}(x_0, x)$.

Moreover, if C is a circuit, $w(C) = 0$, then $C \cap \delta(D) \neq \varnothing$ $(D \in \mathscr{D})$ implies that

$$x_0 \notin D, \qquad C \cap \delta(D) = \{e_1, e_2\}, \qquad w(e_1) = -1, w(e_2) = 1.$$

Since by Theorem 4.1, λ is a potential, and $w(P^x) = \lambda(x)$ *holds for any w-minimum (x_0, x) path P^x*, (4.2) *applies for P^x and $\mathscr{D}(\lambda)$*, which is a crucial property of the potential λ.

We now extend the definition of potentials and Theorem 4.1 to arbitrary graphs and weights. This is a trivial technical matter but we shall need it later.

Assume that G is an arbitrary graph and $w: E(G) \to \mathbb{Z}$ is arbitrary. Contract the edges of weight 0 and subdivide each edge e, $w(e) \neq 0$ into $2\,|w(e)|$ edges of weight $w(e)/|w(e)|$ by adding $2\,|w(e)| - 1$ new points. Denote the result by (G', w'). Obviously, (G', w') is bipartite, ± 1 weighted, and it is conservative if and only if (G, w) is conservative. Moreover, $\lambda_{G',w'}(x, y) = 2\lambda_{G,w}(x, y)$ for all $x, y \in V(G)$.

We say that a function $\pi: V(G) \to \mathbb{Z}$ is a *potential centered at* x_0, if $\pi(x) = \pi(y)$ for $xy \in E(G)$, $w(xy) = 0$, and the function 2π can be extended to $V(G')$ so that the extended function $\pi': V(G') \to \mathbb{Z}$ is a potential in (G', w'). (π' is the extension of 2π, if $\pi'(x) = 2\pi(x)$ for all $x \in V(G)$.) It is easy to see that such an extension of the function 2π, provided it exists, is unique, and potentials can be defined directly in terms of (G, w) by converting properties (2) and (3) of π' into properties of π. We give this direct definition here for arbitrary G with ± 1 weights ((1), (2′), (3) below), for it will be used in Section 4. (For general weights (1.4) is the essential part, but it is not yet enough, see the details in [25]).

Given a function $\pi: V(G) \to \mathbb{Z}$ let us introduce the following notations: recall $m := m(\pi) := \min\{\pi(x): x \in V(G)\}$; $M := M(\pi) := \max\{\pi(x): x \in V(G)\}$; denote $D^i := D^i(\pi) := G(\{x \in V(G): \pi(x) \leqslant i\})$; $Q^i := Q^i(\pi) := D^i(\pi) \setminus \{xy \in E(G): \pi(x) = \pi(y) = i\}$; let $\mathscr{D}^i$ and $\mathscr{Q}^i$ consist of the vertex sets of components of D^i and Q^i resp., and $\mathscr{D} := \mathscr{D}(\pi) := \bigcup\{\mathscr{D}^i: m \leqslant i \leqslant M\}$, $\mathscr{Q} := \mathscr{Q}(\pi) := \bigcup\{\mathscr{Q}^i: m \leqslant i \leqslant M\}$. $\mathscr{R} := \mathscr{R}(\pi) := \mathscr{D}(\pi) \cup \mathscr{Q}(\pi)$, where the union is understood with multiplicity; i.e., if $R \in \mathscr{R}$ is an element of both $\mathscr{D}$ and $\mathscr{Q}$, then it is contained twice by the family $\mathscr{R}$. (For bipartite graphs $\mathscr{D} = \mathscr{Q}$, and $\mathscr{R}$ is twice this.)

Note that if $c \in \mathbb{Z}$, then $\mathscr{D}(\pi) = \mathscr{D}(\pi + c)$, $\mathscr{Q}(\pi) = \mathscr{Q}(\pi + c)$, $\mathscr{R}(\pi) = \mathscr{R}(\pi + c)$.

(4.3) *Let G be arbitrary and $w: E(G) \to \{-1, 1\}$. $\pi: V(G) \to \mathbb{Z}$ is a poten-*

tial centered at x_0 *if and only if* (1) *and* (3) *are satisfied by* $\mathscr{R} = \mathscr{R}(\pi)$ *instead of by* $\mathscr{D}$, *and* (2′) *holds*:

(2′) $|\pi(x) - \pi(y)| \leqslant 1$ provided $xy \in E(G)$, and if $xy \in E^-$, $\pi(x) = \pi(y) = i$, then x and y are in different elements of $\mathscr{Q}^i$ (i.e., in different components of Q^i).

Proof. For this special w (G', w') arises from (G, w) by putting a vertex v_e on each edge $e = xy \in E(G)$ and by defining $w'(xv_e) := w'(yv_e) := w(xy)$.

If π satisfies (1), (2′), and (3), then let $\pi'\colon V(G) \to \mathbb{Z}$, $\pi'(x) := 2\pi(x)$ if $x \in V(G)$, and $\pi'(v_e) := 1 + \min\{\pi'(x), \pi'(y)\}$ if $e = xy \in E(G)$. π' obviously satisfies (1), (2), and (3).

Conversely, assume that the extension π' of 2π is a potential in (G', w'); i.e., it satisfies (1), (2), and (3). We prove that (2′) holds for π. If $xy \in E(G)$, then by (2): $|\pi'(x) - \pi'(v_e)| = 1$, $|\pi'(y) - \pi'(v_e)| = 1$. $\pi'(v_e) = 1 + \min\{\pi'(x), \pi'(y)\}$, since otherwise $\pi'(v_e) = -1 + \min\{\pi'(x), \pi'(y)\}$, and $\{v_e\} \in \mathscr{D}(\pi')$, $|\delta(v_e) \cap E^-| \neq 1$ in contradiction with (3). So, $e = xy \in E^-$, $\pi(x) = \pi(y) = i$ implies $\pi'(v_e) = 2i + 1$. Since $\pi(x) = \pi'(x)/2$ for $x \in V(G)$, and $\mathscr{Q}^i = \{D \cap V(G)\colon D$ is the vertex set of a component of $G'(\{x \in V(G')\colon \pi'(x) \leqslant 2i\})\}$, (2′) follows. Q.E.D.

Theorem 3.1 holds now for arbitrary graphs and weights:

THEOREM 4.4. *Let* G *be an arbitrary connected graph,* $w\colon E(G) \to \mathbb{Z}$ *and* $x_0 \in V(G)$. *Then* (G, w) *is conservative if and only if there exists a potential centered at* x_0. *Moreover, if* (G, w) *is conservative, then* $\lambda(x) := \lambda_w(x_0, x)$ *is a potential and* $\lambda(x) \geqslant \pi(x)$ *for any potential* π *centered at* x_0 *and any* $x \in V(G)$.

Proof. If (G, w) is conservative, then (G', w') (see the definition of (G', w') earlier in this section, after (4.2)) is also conservative, and it is in addition bipartite. Applying Theorem 4.1 to (G', w'), one easily gets the claim. Q.E.D.

Similarly, (4.2) *holds true for arbitrary graphs, if* $\mathscr{D}(\pi)$ is replaced by $\mathscr{R}(\pi)$. (We just must apply (4.2) for (G', w').)

Note that the laminar system $\mathscr{R}(\pi)$ can be stored in a compact way, and that *the properties* (1), (2′), *and* (3) *can be checked in linear time in function of the number of edges.*

Remark. Lovàsz [14] proved Theorem 2.1 with the additional statement that there exists a maximum 2-packing of odd cuts which is laminar.

If $F \subseteq E(G)$ is a minimum t-join, then $(G, \mathbf{1}[F])$ is conservative (see (2.2)) and fixing $x_0 \in V(G)$ arbitrarily, according to Theorem 4.4, $\lambda(x) := \lambda_{\mathbf{1}[F]}(x_0, x)$ is a potential centered at x_0. Moreover, $\mathbf{1}[F]$ is ± 1-weighted and consequently (1), (2′), and (3) apply for $\mathscr{R}(\lambda)$. Properties (1), (2′), and (3) imply that $\{\delta(R): x_0 \notin R \in \mathscr{R}(\lambda)\}$ is a maximum 2-packing of odd cuts. Besides being laminar, $\mathscr{R}(\lambda)$ has some additional properties:

(1) $\mathscr{R}(\lambda) = \mathscr{D}(\lambda) \cup \mathscr{Q}(\lambda)$ and each $D \in \mathscr{D}(\lambda)$ is *partitioned* by elements of $\mathscr{Q}(\lambda)$.

(2) If $e \in E(G)$ is contained in some $\delta(R)$ ($R \in \mathscr{R}(\lambda)$), then *either* 1 element of $\delta(\mathscr{D})$ and 1 of $\delta(\mathscr{Q})$ *or* 2 elements of $\delta(\mathscr{Q})$ contain it, and *there is no other possibility.* (If $\mathscr{H} \subseteq 2^V(G)$, then $\delta(\mathscr{H}) := \{\delta(H): H \in \mathscr{H}\}$.)

(3) $\delta(\mathscr{D})$ is a packing of odd cuts; i.e., any two different elements in it are disjoint. (This is an obvious cosequence of 2.)

The following corollary of Theorem 4.4 is irrelevant in this paper, but it may be interesting for its own sake:

COROLLARY 4.5. *Let (G, w) be conservative and $x_0, x \in V(G)$. Denote by $\mathscr{P}(x_0, x)$ and by $\Pi(x_0)$ the set of all (x_0, x) paths and the set of all potentials centered at x_0, resp. Then for all $x \in V(G)$:*

$$\min\{w(P): P \in \mathscr{P}(x_0, x)\} = \max\{\pi(x): \pi \in \Pi(x_0)\}.$$

This is just a minimax reformulation of Theorem 4.4 (or 3.1). Note that by Theorem 4.4 *the maximum on the right hand side is satisfied by one and the same potential for every $x \in V(G)$*, namely, by $\lambda(x) = \lambda_w(x_0, x)$. This property characterizes λ.

Let us now see some applications. First let us remark that by a well-known elementary construction of Edmonds and Johnson's [6 or 13], the minimum t-join problem and the computation of the distance of two points in a weighted conservative undirected graph can be reduced to a weighted matching algorithm. Thus *the potential $\pi(x) := \lambda_w(x_0, x)$ can be determined by executing $|V(G)|$ weighted matching algorithms*, independently of one another, for instance, *parallely*. Thus, *knowing only the statement of Theorem* 4.4 *we have an algorithm to find a half-integer* (*integer*) *packing of odd cuts in* (*bipartite*) *graphs, which, e.g., implies a half integer* (*integer*) *packing of multicommodity flows in* (*Eulerian*) *planar graphs, with the same parallel complexity as that of matching algorithms.* This is quite surprising in view of the sophisticated methods of Edmonds and Johnson [6] and Barahona *et al* [1] for finding a (fractional) optimal packing of t-cuts, of Barahona [2] and Korach [11] for finding a maximum integer packing for bipartite graphs, and of Matsumoto *et al.* [18] for solving the plane multicommodity flow problem. The details of our approach are described in [25]. ([22] concerns only the cardinality case: it gives an algorithmic proof of Theorem 3.1. The subdivision of edges is not a polynomial reduction, and thus cannot be permitted in algorithms!)

Finally we apply (4.3) and Theorem 4.4 to prove an interesting recent result of Korach and Penn [12]. For the simplicity of notations we state it only in the unweighted case:

THEOREM 4.6 [12]. *If F is a minimum t-join, and its components are $F_1, ..., F_k$, then*

(a) *$\exists$ family $\mathscr{C}$ of pairwise disjoint t-cuts such that every $C \in \mathscr{C}$ contains exactly one edge of F, and all edges of F_i $(i = 1, ..., k)$ are contained in some $C \in \mathscr{C}$, except perhaps for one edge of F_i for $i \geqslant 2$.*

(b) $\tau(G, t) - (k-1) \leqslant \nu(G, t) \leqslant \tau(G, t)$.

Proof. Condition (b) follows immediately from (a). To prove (a) define $w(e) := -1$ if $e \in F$, and $w(e) := 1$ otherwise. Let $x_0 \in V(F_1)$, and $\lambda(x) := \lambda_w(x_0, x)$. By Theorem 4.4 λ is a potential. By (2′) and (3) we have

(4.7) *$\forall x \in V(F_i)$, there is at most one $y \in V(F_i)$ such that $xy \in E(F_i)$, and $\lambda(y) \geqslant \lambda(x)$ $(i = 1, ..., k)$.*

From (4.7) it straightforwardly follows that $T(F_i) := \{x \in V(F_i): \lambda(x) = \max_{y \in V(Fi)} \lambda(y)\}$ $(i = 1, ..., k)$ has at most 2 elements. If it has 2 elements, then they are joined by an edge of F_i, and all edges xy in $E(F_i)$ but this one must satisfy $|\lambda(x) - \lambda(y)| = 1$. Thus $\lambda(x) - \lambda(y) = 0$ can hold for at most one edge in each F_i, and for none of the edges of F_1 because of $T(F_1) = \{x_0\}$. This means that $\delta(\mathscr{D})$ contains all edges of F_1, and all or all but one edge of each F_i $(i = 1, ..., k)$. Since $\delta(\mathscr{D})$ is a packing of odd cuts by property (3) of $\mathscr{R}(\lambda)$ in the remark above, $\mathscr{C} := \delta(\mathscr{D})$ satisfies (a).

Q.E.D.

The proof provides a trivial algorithm to construct a packing which satisfies the bound if the distances have already been computed (cf. the algorithmic remark above). As Korach and Penn [12] have remarked, Theorem 4.6 implies easily that in a plane multicommodity flow problem all the requests can be "almost" satisfied with an integer flow. Again, the proof immediately gives the construction of such a flow.

Further applications of Theorem 4.4 are shown in [20, 23, 24, 25].

5. THE STRUCTURE THEOREM

Using the main result of the paper (i.e., Theorem 3.1 or rather Theorem 4.4 which is an equivalent formulation, and using the remark (4.3)), we derive here a "structure theorem" for t-joins. We do not know any definition of a structure theorem; we state the example of the Gallai–Edmonds theorem instead, and show by this example what we mean.

We mention, however, that our goal is to describe the set of minimum t-joins in a short way, by decomposing an arbitrary (G, t) pair into (G, t) pairs with a simple set of minimum t-joins.

Let $D(G) := \{x \in V(G) : G$ has a maximum matching contained in $G - x\}$. Let the graph induced by $D(G)$ have $k = k(G)$ components, and denote these by $D_1(G), \ldots, D_k(G)$. Morerover $A(G) := \{x \in V(G) \setminus D(G) : \exists y \in D(G),\ xy \in E(G)\}$, and $C(G) := V(G) \setminus (A(G) \cup D(G))$. A graph is called *factor-critical*, if $G - x$ has a perfect matching for any $x \in V(G)$.

THEOREM 5.1 [4, 10, 17].

(a) *The components of the graph induced by $D(G)$ are factor-critical.*

(b) *The graph induced by $C(G)$ has a perfect matching.*

(c) *Any $X \subseteq A(G)$ is connected by an edge to at least $|X| + 1$ components of $D(G)$.*

(d) *$F \subseteq E(G)$ is a maximum matching if and only if it is the union of a perfect matching of $C(G)$, perfect matchings of $D_i(G) - x_i$ $(x_i \in D_i(G),$ $i = 1, \ldots, k)$, and a matching of all points of $A(G)$ with points of the set $\{x_1, \ldots, x_k\}$.*

(e) *If F is a maximum matching, $|F| = \frac{1}{2}(|V(G)| + |A(G)| - k(G))$.*

Of course the Berge–Tutte theorem is contained in Theorem 5.1 (cf. (e)). The main virtue of Theorem 5.1 is that it defines one "canonical" Tutte-set A depending only on the graph.

Furthermore, in Theorem 5.1 "all properties" of the sets $A(G)$, $C(G)$, and $D(G)$ are listed in the sense that the partition $\{A, C, D\}$ of $V(G)$ satisfies (a), (b), and (c) if and only if $A = A(G)$, $C = C(G)$, and $D = D(G)$ (see [17]). The essentially unique way of doing a matching-decomposition, i.e. of having (d), is also provided by Theorem 5.1. Lovász and Plummer [15, 16, 17] refine Theorem 5.1 by further investigating $C(G)$ and $D(G)$. This refined analysis is also based on Theorem 5.1.

In contrast to the previous two sections which can be viewed to be dealing with minimum t-joins (conservative graphs and minimum t-joins correspond to each other by (2.3)), in this section the (G, t) *pair itself* and the *set of minimum t-joins* are investigated. For this purpose invariants of the (G, t) pair are welcome. The following invariant is a crucial consequence of (3.2):

(5.2) Let $V(G)$ be arbitrary and $t: V(G) \to \{0, 1\}$, $t(V(G)) \equiv 0 \bmod 2$. Assume that F_1, F_2 are w-minimum t-joins. Then $(G, w[F_i])$ $(i = 1, 2)$ are conservative, and

$$\lambda_{w[F_1]}(x, y) = \lambda_{w[F_2]}(x, y) \qquad \forall x, y \in V(G).$$

Proof. $(G, w[F_i])$ $(i=1, 2)$ is conservative by the weighted version of (2.3). (This is easy to prove in the same way as (2.3), cf. (3.4) "(i) $\Rightarrow$ (ii),") We know that $F_1 \Delta F_2$ is the disjoint union of circuits; denote these by $C_1, ..., C_k$. Clearly $w[F_1](C_i) = -w[F_2](C_i)$ $(i = 1, ..., k)$, and so $w[F_1](C_i) = w[F_2](C_i) = 0$. Starting from $w[F_1]$, and applying (3.2) k times, we get $\lambda_{w[F_1]}(x, y) = \lambda_{w[F_1][C_1] \cdots [C_k]}(x, y)$, $\forall x, y \in V(G)$. But $w[F_1][C_1] \cdots [C_k] = w[F_k]$. Q.E.D.

Statement (5.2) means that *to each function t there belongs a specific distance function.* Since w-minimum t-joins with arbitrary weight functions can easily be reduced to the weight function **1**, in the following we restrict ourselves to this weight function. (**1** is the weight function that is 1 for each edge.) Although the arguments apply for arbitrary weights, the notations and some claims become considerably simpler for this case.

Define $\lambda_t(x, y) := \lambda_{G,t}(x, y) := \lambda_{\mathbf{1}[F]}(x, y)$ for all $x, y \in V(G)$, where F is a minimum (cardinality) t-join. The definition is correct since $\mathbf{1}[F]$ is conservative and by (5.2) $\lambda_{\mathbf{1}[F]}(x, y) = \lambda_{\mathbf{1}[F']}(x, y)$ if F' is another minimum t-join. Thus the definition of λ_t does not deend on which minimum t-join F was chosen. (From now on, (5.2) will be used without reference.) Surprisingly enough, *if we want to determine the distance of two points in a ± 1 weighted conservative graph, it is enough to know the parity of the number of negative edges adjacent to each point* instead of the edge-weights.

The function λ_t will be our main tool in the investigation of (G, t) pairs. We apply the following method:

(*) (1) Fix an arbitrary minimum t-join F.

(*) (2) Examine the distance function in the conservative graph $(G, \mathbf{1}[F])$.

(3) Use $\lambda_t = \lambda_{\mathbf{1}[F]}$ and deduce consequences for (G, t).

The following proof can also be viewed to have such a scheme. Let $t^{a,b}(x) :\equiv t + p^{a,b} \cdot (t^{a,b}(x) \equiv t(x)$ if $x \notin \{a, b\}$ and $t(x)+1$ if $x \in \{a, b\}$.)

(5.3) $\lambda_t(a, b) = \tau(G, t^{a,b}) - \tau(G, t)$.

Proof. Let $F \subseteq E(G)$ be a minimum t-join and $P \subseteq E(G)$ a $\mathbf{1}[F]$-minimum (a, b) path. $\lambda_t(a, b) = \lambda_{\mathbf{1}[F]}(a, b) = \mathbf{1}[F](P) = |P \backslash F| - |P \cap F| = |F \Delta P| - |F|$. But $|F| = \tau(G, t)$ and by "(3.4)(iii) $\Rightarrow$ (i)" $|F \Delta P| = \tau(G, t^{a,b})$. Q.E.D.

Note that (5.3) is strongly related to (3.3). Statement (5.3) will be a tool to achieve (3) of (*), namely, to convert some properties of λ_t into properties of (G, t).

The result of the analysis of (G, t) pairs will be a "structure theorem," analogous to the Gallai–Edmonds theorem, and in fact containing

Theorem 5.1. The splitting up of (G, t) will be determined by $\mathscr{R}(\lambda)$, $\lambda(x) := \lambda_{\mathbf{1}[F]}(x_0, x)$, $x_0, x \in V(G)$, where F is a minimum t-join. ($\mathscr{R}$ is defined in Section 4.)

The role played by x_0 here would be disturbing in a theorem that is intended to depend only on G and t. We change the set of allowed (G, t) pairs, in order to obtain pairs whose structural decomposition does not depend on x_0:

Let $t\colon V(G) \to \{0, 1\}$ be such that $t(V(G)) \equiv 1$. Set $t^a(x) := t(x)$ if $x \neq a$ and $t^a(x) := t(x) + 1$ if $x = a$. The "structure" of the pair (G, t) will mean "the structure of (G, t^a) $(a \in V(G))$, with $x_0 := a$" which turns out to be *independent of* a. Set $\pi^a(x) := \lambda_{t^a}(a, x)$ for a, $x \in V(G)$.

(5.4) Let G be connected, $t(V(G)) \equiv 1$, and a, $b \in V(G)$. Then

(a) $\forall x \in V(G)\colon \pi^a(x) - \pi^b(x) = \pi^a(b) = -\pi^b(a)$

(b) $\mathscr{D}(\pi^a) = \mathscr{D}(\pi^b)$, $\mathscr{Q}(\pi^a) = \mathscr{Q}(\pi^b)$, $\mathscr{R}(\pi^a) = \mathscr{R}(\pi^b)$.

Proof. Let F be a minimum t^a-join, set $w := \mathbf{1}[F]$, and assume that Q is a w-minimum (a, b) path. $\pi^a(x) = \lambda_w(a, x)$ by definition, and $\pi^b(x) = \lambda_{w[Q]}(b, x)$, since $w[Q] = \mathbf{1}[F][Q] = \mathbf{1}[F \Delta Q]$, and $F \Delta Q$ is a minimum $t^a + p^{a,b} \equiv t^b$-join by (3.4)(iv) $\Rightarrow$ (i). Applying (3.3), $\pi^a(x) - \pi^b(x) = \lambda_w(a, x) - \lambda_{w[Q]}(b, x) = \lambda_w(a, b) = \pi^a(b)$. Interchanging the roles of (a) and (b), $\pi^b(x) - \pi^a(x) = \pi^b(a)$ and (a) is proved. Since (a) means that $\pi^a(x)$ and $\pi^b(x)$ only differ by a constant (a number independent of x), we have (b) as an immediate consequence. Q.E.D.

The pair (G, t) will be called a *tower* if G is connected and $t(V(G)) \equiv 1$. Let $\mathscr{D}_t := \mathscr{D}(\pi^a)$, $\mathscr{Q}_t := \mathscr{Q}(\pi^a)$, $\mathscr{R}_t := \mathscr{R}(\pi^a)$ ($a \in V(G)$ is arbitrary). (These notations are introduced before (4.3).) We have by (5.4)(b) that the definitions of $\mathscr{D}_t$, $\mathscr{Q}_t$, and $\mathscr{R}_t$ do not depend on the element a. Recall that according to Theorem 4.4, distances in ± 1-weighted undirected graphs satisfy (1), (2′), (3) (cf. (4.3)). This will be used without any more reference.

We now define the two fundamental classes of towers that will be used as "bricks" in splitting up an arbitrary tower, and characterize them with their distance function:

The tower (G, t) will be called *factor-critical*, if each $x \in V(G)$ satisfies $t(x) \equiv 1$ and $\tau(G, t^x) = (|V(G)| - 1)/2$. It will be called *comb-critical*, if there exists a stable set $B \subseteq V(G)$ for which $B \cup \Gamma(B) = V(G)$, and each $b \in B$ satisfies $t(b) \equiv 1$ and $\tau(G, t^b) = |B| - 1$. The elements of B are called the *teeth* of the comb-critical tower.

The following two lemmas explain that factor- and comb-critical towers are the simplest towers from the point of view of distances.

LEMMA 5.5. *The following statements are equivalent*:

(i) (G, t) *is factor-critical.*

(ii) $\pi^a(x)=0$ *for all* $a, x \in V(G)$.

(iii) *There exists* $a \in V(G)$ *such that* $\pi^a(x)=0$ *for all* $x \in V(G)$.

Proof. (i) $\Rightarrow$ (ii) follows from (5.3): $\pi^a(x)=\lambda_{t^a}(a, x)=\tau(G, t^x)-\tau(G, t^a)=0$. (ii) $\Rightarrow$ (iii) is trivial. To prove (iii) $\Rightarrow$ (i) suppose that $a \in V(G)$ is such that $\pi^a(x)=0$ for all $x \in V(G)$. Let F be a minimum t^a-join and $w := \mathbf{1}[F]$. Since $\lambda_w(a, x)=0$ for all $x \in V(G)$, we have by Lemma 3.5, $d_F(a)=d^-(a)=0$, and $d_F(x)=d^-(x)=1$ if $x \neq a$. Thus F is a perfect matching of $G-a$. Consequently, for all $x \in V(G)$, $t(x)\equiv 1$, which implies $\tau(G, t^x) \geqslant |V(G)|-1)/2$. Furthermore, if P is an (a, x) path with $\mathbf{1}[F](P)=0$, then $F \Delta P$ is a t^x-join, $|F \Delta P|=|V(G)|-1)/2$, whence $F \Delta P$ is a perfect matching of $G-x$. Q.E.D.

Factor-critical towers (G, t) are a very special type of tower: their minimum t^a-joins are perfect matchings of $G-a$, and consequently *G is a factor-critical graph.* Factor-critical graphs are investigated in [17] in detail. Their ear decomposition is a clear description of all of their minimum t^x-joins, for arbitrary $x \in V(G)$.

LEMMA 5.6. *Suppose* $B \subseteq V(G)$ *is a stable set, and* $A := \Gamma(B)=V(G) \backslash B$. *The following statements are equivalent*:

(i) (G, t) *is comb-critical with the elements of* B *as teeth.*

(ii) $\forall a \in A: \pi^a(x)=0$ *if* $x \in A$, *and* $\pi^a(x)=-1$ *if* $x \in B$.

(iii) $\exists a \in A: \pi^a(x)=0$ *if* $x \in A$, *and* $\pi^a(x)=-1$ *if* $x \in B$.

Proof. To prove (i) $\Rightarrow$ (ii) we show that $\tau(G, t^a)=|B|$ for all $a \in A$, provided (G, t) is comb-critical. Statement (5.3) then implies (ii), similarly to the previous proof. $\tau(G, t^a) \geqslant |B|$ is obvious since $t^a(b) \equiv 1$ $(b \in B)$, and B is a stable set. Fix $a \in A$ and choose $b \in B$ such that $ab \in E(G)$, and let F be a minimum t^b-join. $|F|=|B|-1$ by definition, and $F' := F \cup ab$ is a t^a-join, $|F'|=|B|$.

(ii) $\Rightarrow$ (iii) is trivial. To prove (iii) $\Rightarrow$ (i) let $a \in V(G)$, and assume that F is a minimum t^a-join. Apply Lemma 3.5 to $w := \mathbf{1}[F]$: $d_F(b)=d_-(b)=1$ for all $b \in B$. Consequently $t(b) \equiv 1$, and $\tau(G, t^b) \geqslant |B|-1$ for all $b \in B$. But $|F|=\sum \{d_F(b): b \in B\}=|B|$, and if P is an (a, b) path with $w(P)=-1$, then $F' := F \Delta P$ is a t^b-join, $|F'|=|B|-1$. Q.E.D.

Note that the edges induced by A do not participate in any minimum weight path; i.e., only the bipartite graph with bipartition $\{A, B\}$ is important.

Comb-critical towers are also a very special type of tower: minimum t^x-joins have degree 1 in each vertex $b \in B$, $b \neq x$. They are investigated in [20]. They also have an ear decomposition which describes clearly the set of their minimum t-joins.

Note that both factor-critical and comb-critical towers are "extreme towers" in the sense that the cardinality of their minimum t^x-joins satisfies with equality the trivial lower bound arising from the number of odd vertices in (G, t^x). They constitute the *simplest* and the *simplest bipartite* towers, respectively: factorcritical towers are exactly the "*one-level*" *towers*; it is easy to see that comb-critical towers are exactly the "*two-level*" *bipartite towers*, and those we get by joining two non-teeth vertices of such a tower.

We shall need the following technical lemma in the proof of the structure theorem. It might be of some interest for its own sake since it is a consolation for the fact that in undirected graphs, subpaths of minimum weight paths are not necessarily of minimum weight. (This is a reason why minimum weight paths in undirected graphs have a more complicated structure than those in directed graphs.) The following lemma says that this is true, however, for certain subpaths.

Given a conservative graph (G, w) $w: E(G) \to \{-1, 1\}$ and a potential $\pi := V(G) \to \mathbb{Z}$ centered at $x_0 \in V(G)$, let us call $r \in V(G)$ the *root* of R $(x_0 \notin R \in \mathscr{R}(\pi))$, if $|\delta(D) \cap E^-(G, w)| = \{sr\}$, $s \notin R$, $r \in R$. Each R, $x_0 \notin R \in \mathscr{R}$ has a unique root that will be denoted by $r(R)$. These roots play an important role: for example, if $x \in R$, then (4.2) claims that for any w-minimum (x_0, x) path P, $r := r(R) \in V(P)$, $P(x_0, r) \cap E(G(D)) = \varnothing$, $P(r, x) \subseteq E(G(D))$.

LEMMA 5.7. *Let G be an arbitrary graph, $w: E(G) \to \{-1, 1\}$ and $x_0 \in V(G)$. Assume that (G, w) is conservative and $\lambda(x) := \lambda_{G,w}(x_0, x)$.*

(a) *If $x_0 \notin R \in \mathscr{R}(\lambda)$, then $\lambda_{G(D),w}(r(R), x) = \lambda(x) - \lambda(r(R))$ for any $x \in R$.*

(b) *If we contract $R \in \mathscr{R}(\lambda)$ such that $x_0 \notin R$ to a single point r after having deleted the edges of $G(R)$, the resulting graph (G^*, w) is conservative, and*

$$\lambda_{G^*,w}(x_0, x) = \lambda_{G,w}(x_0, x) \qquad \textit{for all} \quad x \notin R$$

and

$$\lambda_{G^*,w}(x_0, r) = \lambda_{G,w}(x_0, r(R))$$

Proof. In order to prove (a), note that the restriction of $\lambda - \lambda(r(R))$ to R is a potential in $(G(R), w)$ centered at $r(R)$. Let $x \in R$, and let P be a w-minimum (x_0, x) path. By Theorem 4.4, $w(P(r(R), x)) \geqslant \lambda(x) - \lambda(r(R))$, and by the remark preceding the lemma $\lambda(x) = w(P) = w(P(x_0, r(R))) +$

$w(P(r(R), x)) \geqslant \lambda(r(R)) + \lambda(x) - \lambda(r(R)) = \lambda(x)$. Thus equality holds throughout, and (a) is proved.

Now let $\lambda^*(x) := \lambda(x)$ for $x \notin R$ and $\lambda^*(r) := \lambda(r(R))$. Clearly, λ^* is a potential centered at x_0 in (G^*, w). Thus (G^*, w) is conservative, and by Theorem 4.4,

$$\lambda_{G^*,w}(x_0, x) \geqslant \lambda^*(x) = \lambda_{G,w}(x_0, x) \qquad \text{for all} \quad x \notin R$$

and

$$\lambda_{G^*,w}(x_0, r) \geqslant \lambda^*(r) = \lambda_{G,w}(x_0, r(R)).$$

(We have applied the trivial part of Theorem 4.4.) Now let P be an (x_0, x) path, $x \notin R$ or $x = r(R)$. We construct an (x_0, x) path P^* of G^* and we prove $w(P^*) \leqslant w(P)$. If $P \cap \delta(R) = \varnothing$, then $P^* := P$ will do. If $P \cap \delta(R) \neq \varnothing$, then by (4.2)(c) $P \cap \delta(R) = \{e_1, e_2\}$, $e_1 \in E^-$, $e_2 \in E^+$. $e_1 =: sr$, $s \notin R$, $r = r(R)$, and $e_2 =: pq$, $p \notin R$, $q \in R$. Since the restriction of $\lambda - \lambda(r(R))$ to R is a potential centered at $r(R)$, $w(P(r, q)) \geqslant \lambda(q) - \lambda(r) = 0$. Thus, for $P^* := P \backslash P(r, q)$ we have that P^* is an (x_0, x) path in G^*, and $w(P^*) \leqslant w(P)$. Q.E.D.

We have arrived now at the structure theorem. Let $a \in V(G)$, and define the *top* of the tower (G, t) as $T_t := \{x \in V(G) : \pi^a(x) = \max\{\pi^a(y) : y \in V(G)\}\}$. By (5.4)(a), $\pi^a(x)$ and $\pi^b(x)$ differ only by a constant independent of x, and so T_t does not depend on a. Specially, if $a \in T_t$, then $\pi^a(x) \leqslant \pi^a(a) = 0$ for all $x \in V(G)$.

Let $a \in T_t$ and define $\pi_t(x) := \pi^a(x)$. The definition of $\pi_t(x)$ is independent of the choice of a: if $a, b \in T_t$, then $\pi^a(x) - \pi^b(x) = \pi^a(b) = -\pi^b(a)$ by (5.4)(a), and $\pi^a(b) \leqslant 0$, $\pi^b(a) \leqslant 0$ imply $\pi^a(b) = \pi^b(a) = 0$, whence $\pi^a(x) = \pi^b(x)$ follows for all $x \in V(G)$ (from 5.4(a)). Clearly, $m(\pi_t) \leqslant 0 = M(\pi_t)$. Let $m_t := m(\pi_t) = \min\{\pi_t(x) : x \in V(G)\}$. Of course $\mathscr{D}_t = \mathscr{D}(\pi_t)$, $\mathscr{Q}_t = \mathscr{Q}(\pi_t)$, $\mathscr{R}_t = \mathscr{R}(\pi_t)$ hold true.

If $E' \subseteq E(G)$ and H is a subgraph of G, then we shall sometimes write $E' \cap H$ instead of $E' \cap E(H)$. If $\mathscr{H}$ is a family of subsets of $V(G)$ and $X \subseteq V(G)$, then we say that $H \in \mathscr{H}$ is an X-maximal element of $\mathscr{H}$, if $H \subseteq X$ and there is no $H' \in \mathscr{H}$ with $H \subseteq H' \subseteq X$, $H \neq H'$. If $\mathscr{H}$ is laminar, then clearly, the X-maximal elements are pairwise disjoint for any $X \subseteq V(G)$.

THEOREM 5.8. *Let (G, t) be a tower. Then*:

(a) *If $D \in \mathscr{D}_t$, then the identification in $(G(D), t)$ of all D-maximal elements of $\mathscr{Q}_t$ results in a factor-critical tower $(G^*(D), t)$.*

(b *If $Q \in \mathscr{Q}_t$, then the identification in $(G(Q), t)$ of all Q-maximal elements of $\mathscr{D}_t$ results in a comb-critical tower $(G^*(Q), t)$. The teeth are the contracted Q-maximal elements.*

(c) *If $x \in V(G)$ and $F \subseteq E(G)$ is a minimum t^x-join, then*:

If $x \in R \in \mathscr{R}_t$, then $F \cap \delta(R) = \emptyset$, and $F \cap G^*(R)$ is a minimum t^x-join of $G^*(R)$.

If $x \notin R \in \mathscr{R}_t$, then $F \cap \delta(R) = \{sr\}$, $s \notin R$, $r \in R$, and $F \cap G^*(R)$ is a minimum t^r-join of $G^*(R)$.

(d) *$\{\delta(R) : x \notin R \in \mathscr{R}_t\}$ is a maximum 2-packing of t^x-cuts and* $\tau(G, t^x) = 1/2\,|\mathscr{R}_t| + \pi_t(x) - 1$ $(\forall x \in V(G))$.

Proof. If F is a minimum t-join, then $(G, \mathbf{1}[F])$ is conservative, and π^x is a potential centered at x in $(G, \mathbf{1}[F])$. (First (2.3), then Theorem 4.4 was applied.) So, by (4.3) (1), (2′) and (3) hold. Condition (c) is just a reformulation of (3), and the first part of (d) is an easy consequence of (2′) and (3).

Now fix $x_0 \in T_t$ and a minimum t^{x_0}-join F_0. Recall that $\forall x \in V(G)$: $\pi_t(x) = \pi^{x_0}(x) = \lambda_{\mathbf{1}[F_0]}(x_0, x) \leqslant 0$. Note that only two elements of $\mathscr{R}_t$ contain x_0: $D_0 := V(D^0)$ $(= V(G))$ and a component Q_0 of the graph $Q^0 = G - \{xy \in E(G):\ \pi_t(x) = \pi_t(y) = 0\}$. Since (c) claims that $\delta(R)$ $(R \in \mathscr{R}_t \setminus \{D_0, Q_0\})$ contains one edge of F_0 and (2′) implies that each edge $e \in F_0$ is contained in 2 elements of $\delta(R)$, we have $\tau(G, t^{x_0}) = \frac{1}{2}(|\mathscr{R}_t| - 2)$. By (5.3), (applying it to t^{x_0} instead of t and $a = x_0$, $b = x$), $\tau(G, t^x) = \tau(G, t^{x_0}) + \pi_t(x)$ and (d) is proved.

Let us add a new vertex x_0' to G with the only edge $x_0'x_0$, $w(x_0'x_0) = -1$. (This is merely a technical step: formally we need $x_0' \notin D_0$, $x_0' \notin Q_0$ to apply Lemma 5.7, and it is convenient that D_0 and Q_0 also have a root, $r(D_0) = r(Q_0) = x_0$.)

We shall use the following trivial consequences of the definitions of $\mathscr{D}_t$ and $\mathscr{Q}_t$: *if $D \in \mathscr{D}_t$, then the D-maximal elements of $\mathscr{Q}_t$ partition D, and $\pi_t(r(D)) = \pi_t(r(Q))$ provided $Q \in \mathscr{Q}_t$ is D-maximal; if $Q \in \mathscr{Q}_t$, and $D \in \mathscr{D}_t$ is Q-maximal, then $\pi_t(r(D)) = \pi_t(r(Q)) - 1$; if $Q \in \mathscr{Q}_t$, and $x \in Q$ is not contained in a Q-maximal element of $\mathscr{D}_t$, then $\pi_t(x) = \pi_t(r(Q))$.*

To prove (a), let $D \in \mathscr{D}_t$, and denote by r^* the contraction of that D-maximal element of $\mathscr{Q}_t$ which contains $r(D)$. Contract the D-maximal elements of $\mathscr{Q}_t$ one by one, and apply Lemma 5.7(b) each time. Then apply Lemma 5.7(a) to get $\lambda_{G^*(D), w}(r^*, x) = 0$ for all $x \in V(G^*(D))$.

Statement (a) follows now from "Lemma 5.5(iii) $\Rightarrow$ (i)" as applied to $G^*(D)$ and $a := r$.

The proof of (b) is similar: contract the Q-maximal elements of $\mathscr{D}_t$ one by one, and apply Lemma 5.7(b) each time; then apply Lemma 5.7(a) to get that $\lambda_{G^*(Q), w}(r, x) = -1$ if x is the contraction of a Q-maximal element of $\mathscr{D}$. If $x \in Q$ is not contained in any Q-maximal element of $\mathscr{D}_t$, then Lemma 5.7(a) implies that $\lambda_{G^*(Q), w}(r, x) = 0$. Statement (b) follows now from "Lemma 5.6(iii) $\Rightarrow$ (i)." (Clearly, the contracted elements form a stable set B in $G^*(D)$, and any other vertex of $G^*(D)$ has a neighbour in B.)

Q.E.D.

The only towers that are not split up by this theorem are the factor-critical and comb-critical towers, which in turn, have ear decompositions (cf. [17, 20]).

Let us sketch now the **proof of Theorem 5.1**, using Theorem 5.8: Add the point x_0 to the vertex set of G together with all edges x_0x ($x \in V(G)$), and denote the result by G'.

Let $t(x) := 1$ if $x \in V(G)$ and $t(x_0) := |V(G)| + 1$. (G', t) is a tower. If M is a matching, then $F_M := M \cup \{x_0y : y$ is not covered by $M\}$ is a t^{x_0}-join. There is a one-to one correspondence between alternating paths augmenting M and negative circuits in $(G', \mathbf{1}[F_M])$: it follows that M is a maximum matching if and only if F_M is a minimum t^{x_0}-join.

It is easy to see that $x_0 \in T_t$, $m_t = -1$ and $D(G) = \{x \in V(G): \pi_t(x) = -1\} = V(D^{-1})$. Theorem 5.8(a) states that the components of $D(G)$ are factor-critical and Theorem 1.2(a) is proved.

The component Q_0 of Q^0 that contains x_0 is easily seen to consist of $D(G)$ and its neighbours. Contracting the components of $D(G)$ ($= V(D^{-1})$) in Q_0, we get a comb-critical tower by Theorem 5.8(b). This is equivalent to Theorem 5.1(c).

The other components of Q^0 are 1-element components. Theorem 5.8(a) states that contracting Q_0 to a single vertex q_0 we get a factor-critical tower. Thus $G - q_0$ has a perfect matching, and Therem 5.1(b) is proved. Theorem 5.1(d) and (e) are immediate consequences of Theorem 5.8(c) and (d), respectively.

Clearly, since we saw in the Introduction that weighted matchings can be reduced to minimum t-joins, Theorem 5.8 is also *a generalization of the Gallai–Edmonds structure theorem for weighted matchings.*

Note that the characterization of $\pi(x)$ as the maximal potential (cf. Theorem 4.4) yields, in this special case, the following well-known characterization of $A(G)$ (from [17]): *$A(G)$ is an extreme set (i.e., a set with $\frac{1}{2}(|V(G)| + |A| - q(A)) = \max\{|F| : F$ is a matching$\}$, where $q(A)$ is the number of odd components of $G - A$) and it has the property that the union of the odd components of $G - A(G)$ is contained in the union of the odd components of $G - A$ for any extreme set A.*

It is important to remark (but we find it too long to state it precisely) that the properties listed in Theorem 5.8 are "complete" in the sense that they provide *one unique* decomposition of the graph G in the following way:

> Starting from a factor-critical tower and blowing up its vertices to comb-critical towers, then blowing up the teeth of the comb-critical towers to factor-critical towers then blowing up the vertices of the new factor-critical towers to comb-critical towers, etc., results in a tower (G, t) with these graphs as $(G^*(D), t)$, $D \in \mathscr{D}_t$. (If a vertex d is blown up to D, the adjacent edges can be distributed arbitrarily among the vertices of D.)

Moreover, a minimum t^x-join $(x \in V(G))$ can be constructed by taking the union of an arbitrary minimum t^r-join of each factorcritical and comb-critical tower of the construction. ($r \in D$ is determined by the previously constructed odd joins.)

Conversely, by Theorem 5.8 all graphs are constructed in this way, and all of their minimum t-joins arise by some choice in the step by step construction.

In short: *Given a tower (G, t) there exists a unique laminar system that satisfies* (a) *and* (b) *of Theorem* 5.8, *and essentially one way of building up all minimum t^x-joins $(x \in V(G))$.*

Finally we remark that $(*)$ can be used to treat and generalize some other concepts and results of the structure theory of matchings set forth in [15], [16], or [17]. An example: the decomposition provided by *Theorem* 5.1 *is refined in* [15] *on the basis of a theorem of Kotzig and Lovász*; recently, on the basis of the present paper, *the Kotzig–Lovász theorem has been generalized to conservative graphs* [20], and the decomposition of (G, t) pairs has been refined. (Then, the decomposition has been applied, e.g., to determine the dimension of minimum t-joins, generalizing Edmonds, Lovász's, and Pulleyblank's [7] result.) It has also been used to derive new conditions for the existence of integer plane multicommodity flows [24]. Distances often help to simplify proofs concerning matchings themselves.

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