**Undrained Cylindrical and Spherical Cavity Expansion in Elastic-Viscoplastic Soils**

<table>
<thead>
<tr>
<th>Journal:</th>
<th>Canadian Geotechnical Journal</th>
</tr>
</thead>
<tbody>
<tr>
<td>Manuscript ID</td>
<td>cgj-2020-0193.R1</td>
</tr>
<tr>
<td>Manuscript Type:</td>
<td>Article</td>
</tr>
<tr>
<td>Date Submitted by the Author:</td>
<td>13-Oct-2020</td>
</tr>
</tbody>
</table>
| Complete List of Authors: | Zhou, Hang; Chongqing University  
Wang, Zengliang; Chongqing University  
Liu, Hanlong; Chongqing University, School of Civil Engineering  
Shen, Hang; Chongqing University  
Ding, Xuanming; Chongqing University, |  |
| Keyword:                  | Elastic-viscoplastic, Cavity expansion, Numerical solution, Effective stress, Pore pressure |
| Is the invited manuscript for consideration in a Special Issue?: | Not applicable (regular submission) |
Undrained Cylindrical and Spherical Cavity Expansion in Elastic-Viscoplastic Soils

Hang Zhou*, Zengliang Wang, Hanlong Liu, Hang Shen, Xuanming Ding

Hang Zhou (Corresponding author)
Associate Professor
Key Laboratory of New Technology for the Construction of Cities in Mountain Areas, College of Civil Engineering, Chongqing University, Chongqing, 400045, China
E-mail: zh4412517@163.com
ABSTRACT

Various undrained cavity expansion solutions for elastic-plastic soil have been proposed previously. However, no solution has been presented for elastic-viscoplastic (EVP) soil until now. This paper presents a general solution method for solving the classical one-dimensional (1D) boundary value problem (BVP) for undrained cylindrical or spherical cavity expansion in EVP soil with an emphasis on the rate effect of soil. The solution method is summarized as three standard procedures: (a) obtaining the soil displacement and strain under incompressible conditions; (b) calculating the effective stress of soil through a suitable constitutive law; and (c) obtaining the pore pressure by numerically solving the stress equilibrium equation through the finite difference method (FDM) or other numerical solution techniques. The numerical algorithms for calculating the effective stress and pore pressure are very simple without any complex iteration processes, and they require little calculation time but provide high computational accuracy. In addition, some numerical results are given to investigate the influence of the cavity expansion velocity on the cavity expansion response. The proposed solution procedure is general and can be applied not only for the EVP model but also for other plasticity models, and the given EVP solution can be applied to interpret the rate effect of the CPT test and pile penetration.

KEYWORDS

Elastic-viscoplastic; Cavity expansion; Numerical solution; Effective stress; Pore pressure
INTRODUCTION

Cylindrical or spherical cavity expansion, which is a very simple theoretical analysis tool due to its simple mathematical structure (one-dimensional problem), is widely accepted in geotechnical engineering for interpreting problems such as in situ tests (Palmer 1972; Marchetti 1980; Salgado et al. 1997; Chang et al. 2001; Shuttle 2007; Tolooiyan and Gavin 2011; Suzuki, and Lehane 2015; Zhou et al. 2015; Mo et al. 2016; Liu et al. 2016; Silvestri 2018), pile penetration and bearing capacity (Randolph et al. 1979; Yasufuku and Hyde 1995; Mabsout et al. 1999; Manandhar and Yasufuku 2013; Liu et al. 2014; Zhang and Goh 2016; Li et al. 2017; Zhou et al. 2018a 2018b; Singh and Patra 2019; Luan 2019, 2020), tunnelling (Marshall and Mair 2011; Marshall 2012), pipeline (Xia and Moore 2006; Ngan et al. 2015), grouting (Au et a. 2006, 2007; Wang et al. 2010), and plant root growth (Siul Ruiz et al. 2015).

Solutions for cavity expansion can be mainly separated into two categories: perfectly drained and undrained expansion. The volumetric strain of soil changes during cavity expansion progress for perfectly drained conditions, while it is zero for undrained cavity expansion. This leads to the discrepancy in the solution procedure between drained and undrained problems. In fact, the undrained cavity expansion problem is easier to solve than the drained problem. The undrained cylindrical or spherical cavity expansion problem requires the soil deformation to satisfy the incompressible condition (or zero volumetric strain condition) as well as the symmetry characteristics of the 1D-BVP, which allows the kinematics (soil strain and displacement) of undrained cylindrical or spherical cavity expansion to be exactly determined. These factors indicate that the soil deformation of undrained cavity expansion is independent of the constitutive relations of soil. However, the kinematics of
undrained expansion are not established for drained cavity expansion, of which the soil deformation is related to the constitutive relations of soil. This paper mainly considered the undrained cavity expansion issue and proposed a general numerical and theoretical solution method for undrained cylindrical and spherical cavity expansion based on the fact that the nature that the soil deformation field is independent of the constitutive law.

Various undrained cavity expansion solutions for elastic-plastic soils, including the Tresca solution (Hill 1950; Shuttle 2007), Mohr-Coulomb (M-C) solution (Yu and Carter 2002; Carter et al. 1986), modified Cam Clay (MCC) solution (Cao et al. 2001; Chen et al. 2012 2019), bounding surface model (BSM) solution (Chen et al. 2016), and hypoplasticity model (HPM) solution (Osinov and Cudmani 2001; Ali et al. 2011), have previously been proposed. However, it is found that no solution is proposed for EVP soil, which is also very useful and may have a potential application for interpreting pile setup or CPT problems. Therefore, a theoretical solution for undrained cavity expansion in EVP soil with an emphasis on the rate effect of soil is necessary. In addition, in terms of the solution technique for undrained cavity expansion, the current solution technology is rather messy. From the author's viewpoint, the undrained cavity expansion problem can be solved with the following three standard procedures: (1) obtaining the soil displacement and strain through incompressible conditions; (2) calculating the effective stress of soil through a suitable constitutive law; and (3) obtaining the pore pressure by numerically solving the stress equilibrium equation through the finite difference method (FDM) or other numerical solution techniques. The solution procedure is general and can be applied not only for the EVP model but also for other plasticity models. The three procedures can be easily performed through a simple calculation program without any complex iteration process.
The aim of this paper is to propose a general numerical and theoretical solution for undrained cylindrical cavity expansion through the three standard procedures and to particularly produce a solution by incorporating an EVP constitutive model, proposed by Kelln et al. 2008, to consider the rate effect of soil. The EVP model is proposed based on the conventional MCC model and has a relatively simple theoretical framework that relates the viscoplastic strain rates to the specific volume and stress levels of soil. The numerical results of undrained cylindrical cavity expansion (CCE) and spherical cavity expansion (SCE) in EVP soil are given, and the influence of cavity expansion velocity is discussed in detail. The proposed solution provides a potential theoretical tool to evaluate the rate effect of pile penetration in soft soil so that the excess pore pressure induced by pile penetration can be captured. With an effective consolidation analysis through an EVP model, the whole pile setup, including the dissipation of excess pore pressure after pile penetration, thixotropic setup and creep or secondary compression of the soils, can be completely determined. However, this paper mainly focuses on the undrained cavity expansion process (pile penetration), while consolidation is not considered.

DEFINITION OF CCE AND SCE

LIST OF FIGURE CAPTIONS

Figure 1 describes the CCE (or SCE, which is not presented in the figure) problem in the infinite EVP soil domain. For CCE, the initial cylindrical cavity with radius $a_0$ uniformly expands under uniform internal pressure $\sigma_a$. The variable $a$ describes the cavity radius during the cavity expansion process. The cavity expands under a uniform speed with the cavity expansion velocity defined as $V = (a-a_0)/t$, where $t$ is the cavity expansion time. The initial lateral and vertical stress in the cavity expansion plane is defined as $\sigma_{h0}$ and $\sigma_{v0}$. For CCE, the initial lateral and vertical stress can be different since it does
not break the cylindrical symmetry condition. However, the initial stress for the SCE problem should be isotropic since a cross-anisotropic stress condition will break the spherical symmetry condition. For the convenience of comparative analysis, an isotropic stress condition is used for both CCE and SCE in the following analysis, but it should be clarified that the proposed solution method could effectively consider the cross-anisotropic stress condition for CCE. In addition, the initial pore pressure of the soil is defined as $u_0$. A cylindrical coordinate system ($r$-$\theta$-$z$) is used for CCE, while a spherical coordinate system ($r$-$\theta$-$\varphi$) can be used for SCE. These results are consistent with previous analyses of CCE and SCE.

For conventional undrained cavity expansion analysis, for instance, three different domains, namely, the critical state domain, plastic domain and elastic domain, occur around the cavity wall for MCC soil. Therefore, each region is solved separately, and the continuity conditions should be satisfied at the boundary of different regions. However, the EVP soil model proposed by Kelln et al. 2008) is proposed without a purely elastic zone, which means that viscoplastic strains develop for all states. In other words, viscoplastic strains occur during the whole cavity expansion process even though the expansion displacement is very small. In this case, the three zones in the conventional problem do not exist, and the conventional partition solution method is no longer suitable. In fact, the solution for EVP soil is simpler than the conventional elastic-plastic solution. One only needs to conduct the three standard procedures: (1) obtaining the soil displacement and strain through incompressible conditions; (2) calculating the effective stress of soil through a suitable constitutive law; and (3) obtaining the pore pressure by numerically solving the stress equilibrium equation through the finite difference method (FDM). Note that the second procedure can be directly performed without separating the soil into three
zones. In addition, a large enough solution domain (the excess pore pressure at the outside of the
domain is zero), of which the radius is defined as $r_c$, should be selected when conducting the third
procedure to solve the pore pressure of soil through the FDM. During the calculation for the EVP
model, only one boundary condition is the zero excess pore pressure condition, and the calculation is
more convenient than that of the conventional elastic-plastic solution where the boundary condition at
the boundary between the elastic and plastic domains need to be considered.

**KINEMATICS**

For undrained cylindrical or spherical cavity expansion problems, the soil volume change is zero (no
volumetric strain) throughout the whole expansion process. Furthermore, both cylindrical and
spherical issues are one-dimensional problems in which all the physical variables of the soil, including
the displacement, strain, stress and pore pressure, are only functions of the radial position, $r$. In this
case, the kinematics of the undrained cavity expansion, namely, the soil displacement and strain, can
be exactly determined. Subsequently, the effective stress of the soil can be calculated through a suitable
constitutive model of the soil. Moreover, the pore pressure can be computed by numerically or
analytically solving the stress equilibrium equation. Therefore, undrained cavity expansion has a
simple mathematical framework, and theoretical solutions can be conveniently determined.

For undrained cylindrical or spherical cavity expansion, the kinematic equation can be expressed as:

$$r^{m+1} - r_0 = a^{m+1} - a_0$$  \hspace{1cm} (1)

where $m=1$ for CCE and $m=2$ for SCE; $r_0$ and $r$ are the radial positions of a soil particle before and
after expansion; and $a_0$ and $a$ are the cavity radii before and after expansion. Equation (1) can be
further written in the following incremental form:
To consider the time effect, the cavity expansion velocity is assumed to be a constant value \( V \), which is consistent with practical problems, such as pile penetration and cone penetration tests, and is obtained as:

\[
V = \frac{a - a_0}{t}
\]  

This can be further simplified as:

\[
a = Vt + a_0
\]  

or:

\[
\delta a = V \delta t
\]  

Substituting Equations (4) and (5) into Equation (2) leads to

\[
\delta r = \left( \frac{a}{r} \right)^m V \delta t
\]  

The strain-displacement relation under large deformation conditions can be expressed as:

\[
\delta \varepsilon_r = -\frac{\partial \delta r}{\partial r}
\]

\[
\delta \varepsilon_\theta = \frac{\delta r}{r}
\]

\[
\delta \varepsilon_z = 0 \quad \text{for CCE and} \quad \delta \varepsilon_\phi = \delta \varepsilon_\phi \quad \text{for SCE}
\]  

Introducing Equation (6), one obtains:

\[
\delta \varepsilon_r = -\frac{\partial}{\partial r} \left[ \left( \frac{a}{r} \right)^m \right] V \delta t = \frac{m V a^m}{r^{m+1}} \delta t
\]

\[
\delta \varepsilon_\theta = \frac{V a^m}{r^{m+1}} \delta t
\]  

**EVP MODEL**

**General constitutive relations**
The EVP model utilized in this paper is proposed by Kelln et al. 2008). This model has a relatively simple theoretical framework that relates the viscoplastic strain rates to the specific volume and stress levels of soil. This section gives only a concise summary of the formulations and properties of the EVP model. In three-dimensional stress space, the plastic potential function of the EVP model is identical to that of the MCC model (see Figure 2), and it can be expressed as:

$$g = \frac{q^2}{p M^2} - p_m^p + p^' = 0$$

(12)

where $p^'$, $q$, and $M$ are identical to those in the MCC model, and $p_m^p$ is the size of the plastic potential passing through the current effective-stress state. The specific volume $\nu_m$ corresponding to $p_m^p$ on the particular unloading-reloading line is:

$$\nu_m = \nu - \kappa \ln \left( \frac{p_m^p}{p} \right)$$

(13)

where $\kappa$ is the slope of the unloading-reloading line as defined in the MCC model.

The elliptical yield surface is also adopted as:

$$f = \frac{q^2}{p M^2} - p_c^p + p^' = 0$$

(14)

where the hardening parameter $p_c^p$ is directly related to $p^'$ as:

$$p_c^p = \exp \left[ \left( \frac{1}{\lambda - \kappa} \right) \left( Z - \nu - \kappa \ln p^' \right) \right]$$

(15)

where $Z = N - (\lambda - \kappa) \ln R$ is the intercept on the viscoplastic limit line (vpl) at the assumed mean stress $p^' = 1$ kPa, $N$ is the intercept on the isotropic normal compression line at the assumed mean stress $p^' = 1$ kPa, $R$ is the overconsolidation ratio and defined the same as that by Chen et al. 2012, 2013).

Furthermore, the total strain increment is the sum of the elastic strain rate and viscoplastic strain
increment and can be written as:

\[
\delta e_{ij} = \delta e_{ij}^e + \delta e_{ij}^\nu
\]  \hspace{1cm} (16)

The elastic strain rate is described by Hooke’s law, and the viscoplastic strain rate is:

\[
\delta e_{ij}^\nu = S \frac{\partial g}{\partial \sigma_{ij}} \delta t = \frac{\psi}{\nu_m t_0} \exp \left( \frac{\nu_m - N}{\psi} \right) \left( p_m^i \right)^{\lambda} \frac{1}{\left| \frac{\partial g}{\partial \sigma^i} \right|} \frac{\partial g}{\partial \sigma_{ij}} \delta t
\]  \hspace{1cm} (17)

where \( S \) is the scalar multiplier, \( t_0 \) is the curve fitting parameter, \( \lambda \) is the slope of the isotropic normal compression line, and \( \psi \) is the slope of the secondary compression line.

**Incremental constitutive relations in the cylindrical coordinate system**

To produce a cylindrical cavity expansion solution, the constitutive relations should be rewritten in cylindrical coordinate system. The three viscoplastic strain rate components in the cylindrical coordinate system can be expressed as:

\[
\delta e_{rr}^\nu = S \frac{\partial g}{\partial \sigma_{rr}} \delta t = \frac{\psi}{\nu_m t_0} \exp \left( \frac{\nu_m - N}{\psi} \right) \left( p_m^i \right)^{\lambda} \frac{1}{\left| \frac{\partial g}{\partial \sigma_{rr}} \right|} \frac{\partial g}{\partial \sigma_{rr}} \delta t
\]  \hspace{1cm} (18)

\[
\delta e_{\theta\theta}^\nu = S \frac{\partial g}{\partial \sigma_{\theta\theta}} \delta t = \frac{\psi}{\nu_m t_0} \exp \left( \frac{\nu_m - N}{\psi} \right) \left( p_m^i \right)^{\lambda} \frac{1}{\left| \frac{\partial g}{\partial \sigma_{\theta\theta}} \right|} \frac{\partial g}{\partial \sigma_{\theta\theta}} \delta t
\]  \hspace{1cm} (19)

\[
\delta e_{zz}^\nu = S \frac{\partial g}{\partial \sigma_{zz}} \delta t = \frac{\psi}{\nu_m t_0} \exp \left( \frac{\nu_m - N}{\psi} \right) \left( p_m^i \right)^{\lambda} \frac{1}{\left| \frac{\partial g}{\partial \sigma_{zz}} \right|} \frac{\partial g}{\partial \sigma_{zz}} \delta t
\]  \hspace{1cm} (20)

From the yield surface of Equation (14), one obtains

\[
\frac{\partial g}{\partial \sigma_{rr}} = \frac{\partial g}{\partial \sigma_{\theta\theta}} \frac{\partial \sigma_{rr}}{\partial \sigma_{\theta\theta}} + \frac{\partial g}{\partial \sigma_{zz}} \frac{\partial \sigma_{rr}}{\partial \sigma_{zz}} = \left[ \frac{p}{3} \left( M^2 - \frac{q^2}{p^2} \right) + 3 \left( \sigma'_{rr} - p' \right) \right]
\]  \hspace{1cm} (21)

\[
\frac{\partial g}{\partial \sigma_{\theta\theta}} = \frac{\partial g}{\partial \sigma_{rr}} \frac{\partial \sigma_{\theta\theta}}{\partial \sigma_{rr}} + \frac{\partial g}{\partial \sigma_{zz}} \frac{\partial \sigma_{\theta\theta}}{\partial \sigma_{zz}} = \left[ \frac{p}{3} \left( M^2 - \frac{q^2}{p^2} \right) + 3 \left( \sigma'_{\theta\theta} - p' \right) \right]
\]  \hspace{1cm} (22)

\[
\frac{\partial g}{\partial \sigma_{zz}} = \frac{\partial g}{\partial \sigma_{rr}} \frac{\partial \sigma_{zz}}{\partial \sigma_{rr}} + \frac{\partial g}{\partial \sigma_{\theta\theta}} \frac{\partial \sigma_{zz}}{\partial \sigma_{\theta\theta}} = \left[ \frac{p}{3} \left( M^2 - \frac{q^2}{p^2} \right) + 3 \left( \sigma'_{zz} - p' \right) \right]
\]  \hspace{1cm} (23)
\begin{equation}
\frac{\partial g}{\partial p} = p \left( M^2 - \frac{q^2}{p^2} \right) \tag{24}
\end{equation}

\begin{equation}
p_m' = p' \left( 1 + \frac{q^2}{p^2 M^2} \right) \tag{25}
\end{equation}

Note that the three stress components in the cylindrical coordinate system, effective radial stress \( \sigma_r' \), effective circumferential stress \( \sigma_\theta' \), and effective vertical stress \( \sigma_z' \), are the three principal stresses:

\begin{equation}
p' = \frac{\sigma_r' + \sigma_\theta' + \sigma_z'}{3} \tag{26}
\end{equation}

\begin{equation}
q = \frac{1}{\sqrt{2}} \sqrt{\left( \sigma_r' - \sigma_\theta' \right)^2 + \left( \sigma_\theta' - \sigma_z' \right)^2 + \left( \sigma_z' - \sigma_r' \right)^2} \tag{27}
\end{equation}

Here, it should be noted that the viscoplastic strain rate is directly dependent on the stress rather than the stress increment, which is different from the elastic-plastic model. In addition, the three elastic strain rate components in the cylindrical coordinate system are consistent with the previous results. Therefore, the constitutive relation of the EVP model in the cylindrical coordinate system can be summarized as:

\begin{equation}
\begin{bmatrix}
\delta e_r \\
\delta e_\theta \\
\delta e_z
\end{bmatrix} =
\begin{bmatrix}
1/E & -v'/E & -v'/E \\
-v'/E & 1/E & -v'/E \\
-v'/E & -v'/E & 1/E
\end{bmatrix}
\begin{bmatrix}
\delta \sigma_r' \\
\delta \sigma_\theta' \\
\delta \sigma_z'
\end{bmatrix}
+ \begin{bmatrix}
S \frac{\partial g}{\partial \sigma_r'} \\
S \frac{\partial g}{\partial \sigma_\theta'} \\
S \frac{\partial g}{\partial \sigma_z'}
\end{bmatrix} \delta t \tag{28}
\end{equation}

Because the soil strain is given from the kinematics of undrained cavity expansion and one has to solve the soil stress, Equation (28) can be rewritten as:
Equation (29) allows the three effective stress components in the cylindrical coordinate system to be determined if the soil strain is given. Equation (29) can be easily solved through a simple incremental calculation process.

**Incremental constitutive relations in the spherical coordinate system**

To produce a spherical cavity expansion solution, all the physical variables should be written in a spherical coordinate system \((r, \theta, \phi)\). Due to the symmetry, \(\sigma_r' = \sigma_\theta'\) and \(\varepsilon_r' = \varepsilon_\theta'\). The mean effective stress \(p'\) and deviatoric stress \(q\) decrease to the following in the spherical coordinate system:

\[
p' = \frac{\sigma_r' + \sigma_\theta' + \sigma_\phi'}{3} \quad (30)
\]

\[
q = |\sigma_r' - \sigma_\theta'| \quad (31)
\]

In addition, the effective stress components in the spherical coordinate system can be determined as:

\[
\begin{bmatrix}
\delta\sigma_r' \\
\delta\sigma_\theta' \\
\delta\sigma_\phi'
\end{bmatrix} = \begin{bmatrix}
-E\left(1-\nu\right) & -E\nu' & -E\nu' \\
-2\nu^2+\nu'-1 & 2\nu^2+\nu'-1 & 2\nu^2+\nu'-1 \\
-E\nu' & -E\left(1-\nu\right) & E
\end{bmatrix} \begin{bmatrix}
\delta\varepsilon_r - \frac{\partial g}{\partial \sigma_r} \delta t \\
\delta\varepsilon_\theta - \frac{\partial g}{\partial \sigma_\theta} \delta t \\
\delta\varepsilon_\phi - \frac{\partial g}{\partial \sigma_\phi} \delta t
\end{bmatrix} \quad (32)
\]

**NUMERICAL SOLUTION**

**Effective stress**

Substituting the undrained cavity expansion-induced strain increments in Equations (10) and (11)
Equations (29) and (32) leads to:

\[
\begin{bmatrix}
\frac{\partial \sigma_r'}{\partial z} \\
\frac{\partial \sigma_\theta'}{\partial z} \\
\frac{\partial \sigma_z'}{\partial z}
\end{bmatrix} = 
\begin{bmatrix}
-\frac{E(1-\nu')}{2\nu'^2+v'-1} & -\frac{E\nu'}{2\nu'^2+v'-1} & -\frac{E\nu'}{2\nu'^2+v'-1} \\
-\frac{E\nu'}{2\nu'^2+v'-1} & \frac{E(1-\nu')}{2\nu'^2+v'-1} & -\frac{E\nu'}{2\nu'^2+v'-1} \\
-\frac{E\nu'}{2\nu'^2+v'-1} & -\frac{E\nu'}{2\nu'^2+v'-1} & \frac{E(1-\nu')}{2\nu'^2+v'-1}
\end{bmatrix}
\begin{bmatrix}
\frac{V_a}{r^2} \delta t \frac{\partial \sigma_r'}{\partial \sigma_r} - S \frac{\partial g}{\partial \sigma_r} \delta t \\
\frac{V_a}{r^2} \delta t \frac{\partial \sigma_\theta'}{\partial \sigma_\theta} - S \frac{\partial g}{\partial \sigma_\theta} \delta t \\
-S \frac{\partial g}{\partial \sigma_z} \delta t
\end{bmatrix}
\]

(33)

\[
\begin{bmatrix}
\frac{\partial \sigma_r'}{\partial z} \\
\frac{\partial \sigma_\theta'}{\partial z} \\
\frac{\partial \sigma_z'}{\partial z}
\end{bmatrix} = 
\begin{bmatrix}
-\frac{E(1-\nu')}{2\nu'^2+v'-1} & -\frac{2E\nu'}{2\nu'^2+v'-1} & -\frac{2E\nu'}{2\nu'^2+v'-1} \\
-\frac{E\nu'}{2\nu'^2+v'-1} & \frac{E}{2\nu'^2+v'-1} & -\frac{2E\nu'}{2\nu'^2+v'-1} \\
-\frac{E\nu'}{2\nu'^2+v'-1} & -\frac{E\nu'}{2\nu'^2+v'-1} & \frac{E(1-\nu')}{2\nu'^2+v'-1}
\end{bmatrix}
\begin{bmatrix}
\frac{2V_a^2}{r^3} \delta t - S \frac{\partial g}{\partial \sigma_r} \delta t \\
\frac{2V_a^2}{r^3} \delta t - S \frac{\partial g}{\partial \sigma_\theta} \delta t \\
\frac{2V_a^2}{r^3} \delta t - S \frac{\partial g}{\partial \sigma_z} \delta t
\end{bmatrix}
\]

(34)

Equations (33) and (34) are explicit incremental forms of the three effective stress components in the cylindrical and spherical coordinate systems, and they can be used to compute the effective stress induced by undrained cylindrical and spherical cavity expansion.

**Pore pressure**

Once the effective stress components are given, the pore pressure can be obtained by numerically solving the stress equilibrium equation through the finite difference method (FDM). The stress equilibrium equation ignoring body forces for cavity expansion can be written as:

\[
\frac{\partial \sigma_r'}{\partial r} + \frac{\partial u}{\partial r} + m \frac{\sigma_r'}{r} = 0
\]

(35)

where \( u \) is the pore pressure of the soil.

To perform an FDM analysis, one must first determine the computational domain and boundary conditions. Since the value of pore pressure at the cavity wall is unknown, one must pursue other conditions. In fact, it is known from the conventional cavity expansion analysis that the excess pore pressure should vanish at a certain radius \( r > r_c \). The value of \( r_c \) can be selected as a relatively large
value, such as 100a. Therefore, the boundary condition for the FDM calculation can be determined as:

$$u = u_0 \quad \text{at} \quad r = r_c$$  (36)

Then, the computational domain can be fixed from $r = a$ to $r = r_c$ (see Figure 3). Note that the direction of the FDM calculation should start from $r = r_c$ to $r = a$ since the pore pressure at the cavity wall is unknown. In this case, the computational domain can be uniformly discretized, as shown in Figure 3. The value of $\Delta r$ is the difference spacing, and the radial distance of the FDM points from $r = r_c$ can be expressed as:

$$r_i = i\Delta r$$  (37)

The effective stress components and pore pressure of the soil at each FDM point can be defined as:

$$u_i = u(r = r_i)$$  (38)
$$\sigma_{r,i} = \sigma_r(r = r_i)$$  (39)
$$\sigma_{\theta,i} = \sigma_\theta(r = r_i)$$  (40)

By using the difference approximation, Equation (35) can be discretized as:

$$\frac{\sigma_{r,i} - \sigma_{r,i+1}}{\Delta r} + \frac{u_i - u_{i+1}}{\Delta r} + m \frac{\sigma_{r,i} - \sigma_{\theta,i}}{r_i} \Delta r = 0$$  (41)

Rearranging Equation (41) leads to:

$$u_{i+1} = u_i + m \frac{\sigma_{r,i} - \sigma_{\theta,i}}{r_i} \Delta r \left( \sigma_{r,i+1} - \sigma_{r,i} \right)$$  (42)

This is an explicit FDM calculation without any iteration process and can be conveniently used. A relatively small value of $\Delta r$ would produce a high precision numerical solution for the pore pressure.

**NUMERICAL RESULTS**

*Pressure-expansion relations*
Pressure-expansion (P-E) relations, namely, the relation between the cavity wall pressure (or excess pore pressure) and the expanded cavity radius, are very significant in cavity expansion theory since they can be used to interpret the pressuremeter test, pile behaviour and so forth. In this section, some typical numerical results for P-E relations are selected for analysis. Since this paper focuses on the time-dependent soil behaviour, three values of the cavity expansion velocity, \( V = 0.005 \text{ m/s}, 0.05 \text{ m/s} \) and \( 0.2 \text{ m/s} \), are used to investigate the influence of the expansion velocity on the P-E relation. It is assumed that the soil is always undrained during the whole cavity expansion process under these three expansion velocities. In fact, Silva et al. 2006) investigated the effect of the penetration rate on piezocone tests in clay and found that the soil could be regarded as undrained when the penetration velocity, \( v_c \), is larger than 0.8 cm/s. Note that the relation between the cone penetration velocity and cavity expansion velocity is \( V = v \tan \alpha \) (\( \alpha \) is half of the cone apex angle and is equal to \( 30^\circ \)). Therefore, the cavity expansion velocity \( V \) should be larger than 0.005 cm/s to satisfy the undrained condition. The selected three values of cavity expansion velocity could be used for undrained cavity expansion analysis. In addition, the calculation parameters can be summarized as \( \nu = 0.3, \lambda = 0.4, \kappa = 0.03, \psi = 0.008, \tau^0 = 1, N = 3.3, \) and \( M = 1.1, \) which is consistent with Kelln et al. 2009). The initial stress is isotropic, and the values are \( \sigma_{\theta 0}^0 = \sigma_{r 0}^0 = \sigma_{\phi 0}^0 = \sigma_{r 0}^0 = u_0 = 100 \text{ kPa} \) for CCE and \( \sigma_{\theta 0}^0 = \sigma_{r 0}^0 = \sigma_{\phi 0}^0 = \sigma_{r 0}^0 = u_0 = 100 \text{ kPa} \) for SCE. Furthermore, three values for the overconsolidation ratio \( R = 1, 2, \) and 10, representing normally consolidated, lightly overconsolidated and highly overconsolidated soils, are used for comparison analysis.

Figure 4 to Figure 6 show the plots for the variation in the normalized cavity wall pressure (\( \Delta \sigma_a / p_0 \)), \( \Delta \sigma_a = \Delta \sigma_a^* + \Delta u_a \) and excess pore pressure (\( \Delta u_a / p_0 \)) with a normalized cavity radius under the
condition of different cavity expansion velocities for \( R \) = 1, 2 and 10, respectively. Both CCE and SCE are considered. The P-E relation sharply increases when \( a/a_0 < 2 \), and then the cavity wall pressure and excess pore pressure gradually approach the limit value. The values of the cavity wall pressure and excess pore pressure for SCE are larger than those for CCE for all cases. The value of the cavity wall pressure and excess pore pressure for SCE is larger than CCE for most cases except for the case of \( R = 10 \) at a relatively small cavity expansion. In addition, it is seen that increasing the overconsolidation ratio \( R \) leads to an increase in the cavity wall pressure and excess pore pressure. In particular, negative excess pore pressure occurs around the cavity wall for a high overconsolidation ratio \( R \). This finding is identical to that of Cao et al. 2001. Furthermore, it is clearly seen that the cavity expansion velocity has an influence on the P-E relation. The larger the cavity expansion velocity, the larger the cavity wall pressure and excess pore pressure will be, but the influence of the cavity expansion velocity on the excess pore pressure is not significant, particularly for highly overconsolidated soil. This finding indicates that the conventional undrained cavity expansion solution will underestimate the cavity wall pressure and excess pore pressure if the time-dependent behaviour of soil is not considered. Therefore, the proposed solution for the P-E relation overcomes this shortcoming.

**Stress distributions around the cavity wall**

In addition to the P-E relation, the stress distributions around the cavity wall are also presented here for analysis. The expanded cavity radius is selected as \( a/a_0 = 10 \). Figure 7 to Figure 9 plot the distributions of the effective stress components in the cylindrical (or spherical) coordinate system and excess pore pressure along the radial distance around the cavity wall for \( R = 1, 2, \) and 10 under different
cavity expansion velocities of $V = 0.005$ m/s, 0.05 m/s, and 0.2 m/s, respectively. It can be clearly seen that the cavity expansion velocity has an insignificant influence on the stress distribution. In fact, the cavity expansion velocity mainly affects the cavity wall pressure, and the influence gradually decreases with increasing radial distance to the cavity centre. In addition, it is found that SCE could generate a larger excess pore pressure than CCE near the cavity wall, while the latter (CCE) produces a larger disturbance zone. Negative excess pore pressure occurs in heavily overconsolidated soil with $R = 10$. The effective stress near the cavity wall tends to be a stable value, which means that the soil enters the critical state. The effective stress in the soil slightly away from the cavity wall increase or decreases as the radial distance increases and it tends to the initial value when the radial distance is larger than $50a$, where the soil is undisturbed.

*Effective stress paths for a soil particle at the cavity wall*

The produced cavity expansion solution could also capture the exact stress path for a soil particle around the cavity wall. In this section, the position at the cavity wall is selected for discussion. Figure 10 plots the effective stress path in the $p'-q$ plane for a soil particle at the cavity wall with two initial stress conditions including $p/p'_{0} = 1$, $q = 0$ (isotropic) and $p/p'_{0} = 1$, $q = 0.46$ (cross-anisotropic) for CCE. The value of $q$ is calculated from Equation (27). The cavity expanded from $a/a_{0}=1$ to $a/a_{0} = 10$. Three values of the cavity expansion velocity ($V = 0.005$ m/s, 0.05 m/s, and 0.2 m/s) and two values for the overconsolidation ratio ($R = 1$ and 5) are considered in the figure. For $R = 1$ soil, the mean effective stress decreases as the cavity expansion develops, while the deviatoric stress increases until reaching the critical state line (CSL). However, for $R = 5$ soil, the stress path is different from that of $R = 1$, and it first moves along a vertical line until the soil particle reaches the initial yield locus. Then,
the mean effective stress and deviatoric stress increase as the cavity expands until reaching the CSL. Moreover, the effective stress path is dependent on the cavity expansion velocity and presents an obvious rate effect, which cannot be captured by the previous cavity expansion solution. Figure 11 plots the effective stress path in the $p'-q$ plane for a soil particle at the spherical cavity wall with initial stress conditions ($p/p_0' = 1, q = 0$) for SCE. Four values of the overconsolidation ratio ($R = 1, 5, 8$ and $10$) are considered. Note that the isotropic initial stress is used for SCE. Similar patterns of the rate effect and effective stress path are also found for SCE.

**LIMIT PRESSURE OF CAVITY EXPANSION: EFFECT OF THE CAVITY EXPANSION VELOCITY**

The cavity expansion limit pressure is very significant in geotechnical engineering and can be used to evaluate the pile capacity and pressuremeter test. In this section, the effect of the cavity expansion velocity on the limit cavity wall pressure and excess pore pressure of CCE and SCE is investigated. The limit cavity wall pressure and excess pore pressure is normalized by a reference value of the limit pressure at the cavity expansion velocity of $V_0 = 0.005$ m/s. The cavity expansion velocity is normalized by the reference cavity expansion velocity of $V_0 = 0.005$ m/s. The influence of the soil properties (the soil’s compressibility parameter $\lambda$, viscoplasticity parameter $\psi$, slope of critical state line $M$, and overconsolidation ratio $R$) on the relation between the limit pressure and cavity expansion velocity is investigated through a series of parametric analyses. Figure 12 and Figure 13 respectively plots the variation of normalised limit cavity pressure and excess pore pressure with normalized cavity expansion velocity for CCE and SCE. It is found that the normalized limit cavity wall pressure of CCE and SCE both linearly increase as $\ln(V/V_0)$ (logarithm of normalized velocity) increases. Similar
pattern is found for the normalized limit cavity wall excess pore pressure, but some special cases should be noted that the normalized limit cavity wall excess pore pressure decreases as the normalized velocity increases for the heavily overconsolidated soil with $R = 6$ and 10. In addition, it can be clearly seen that the increasing of $\psi$ leads to the increasing of the normalized limit cavity wall pressure, while the increasing of $\lambda$, $M$ and $R$ leads to decreasing of the limit cavity wall pressure. An interesting phenomenon can be found that the relation between the normalized limit cavity wall excess pore pressure and velocity is not sensitive to the parameter of $M$. However, the normalized limit cavity wall excess pore pressure decreases as $\lambda$, $\psi$, and $R$ increase. The presented parametric analysis in Figure 12 and Figure 13 can also be regarded as a design chart for calculating limit cavity wall pressure and excess pore pressure under different cavity expansion velocity.

**CONCLUSIONS**

This paper presents a general and theoretical solution for undrained cylindrical and spherical cavity expansion in EVP soil with an emphasis on the rate effect of soil. A general solution method, including three standard procedures, is proposed for undrained cavity expansion. The method is suitable not only for the EVP soil model but also for other plasticity models. The numerical solution algorithms for calculating the effective stress and pore pressure are very simple without any complex iteration processes, requiring little calculation time and providing high computational accuracy. This solution procedure is suitable not only for the EVP soil model but also for other plasticity models. Some typical numerical results for undrained cylindrical and spherical cavity expansion are given. The results show that the cavity expansion velocity has an influence on the P-E relation. The larger the cavity expansion velocity, the larger the cavity wall pressure and excess pore pressure will be but the
influence of the cavity expansion velocity on the excess pore pressure is not significant, particularly for highly overconsolidated soil. In addition, it is found that the cavity expansion velocity has an insignificant influence on the stress distribution, which means that the cavity expansion velocity mainly affects the cavity wall pressure and that the influence gradually decreases with increasing radial distance to the cavity centre. The proposed solution provides a theoretical method for evaluating the rate effect for penetration problems such as the CPT test and pile penetration in soft soil, and it can also be extended to develop undrained cavity expansion solutions for other plasticity models.

ACKNOWLEDGEMENTS

The work is supported by the National Natural Science Foundation of China, Grant/Award Number: 51978105.

REFERENCES


LIST OF FIGURE CAPTIONS

Figure 1 Undrained cavity expansion in EVP soil

Figure 2 \( \nu \cdot \ln p' \) plane for EVP model

Figure 3 Mesh grid of the FDM

Figure 4 Variation in the (a) normalized cavity wall pressure and (b) normalized cavity wall excess pore pressure with a normalized cavity radius under the condition of different cavity expansion velocities for \( R=1 \)

Figure 5 Variation in the (a) normalized cavity wall pressure and (b) normalized cavity wall excess pore pressure with a normalized cavity radius under the condition of different cavity expansion velocities for \( R=2 \)

Figure 6 Variation in the (a) normalized cavity wall pressure and (b) normalized cavity wall excess pore pressure with a normalized cavity radius under the condition of different cavity expansion velocities for \( R=10 \)

Figure 7 Distributions of the effective stress components in the cylindrical (or spherical) coordinate system and excess pore pressure along the radial distance around the cavity wall for \( R = 1 \): (a) \( V = 0.005 \) m/s; (b) \( V = 0.05 \) m/s; and (c) \( V = 0.2 \) m/s

Figure 8 Distributions of the effective stress components in the cylindrical (or spherical) coordinate system and excess pore pressure along the radial distance around the cavity wall for \( R = 2 \): (a) \( V = 0.005 \) m/s; (b) \( V = 0.05 \) m/s; and (c) \( V = 0.2 \) m/s

Figure 9 Distributions of the effective stress components in the cylindrical (or spherical) coordinate system and excess pore pressure along the radial distance around the cavity wall for \( R = 10 \): (a) \( V = \)
0.005 m/s; (b) $V = 0.05$ m/s; and (c) $V = 0.2$ m/s

Figure 10 Effective stress path in the $p'$-$q$ plane for a soil particle at the cavity wall with conditions of
(a) initial stress ($p/p'_0 = 1$, $q = 0$) and (b) initial stress ($p/p'_0 = 1$, $q = 0.46$) for CCE

Figure 11 Effective stress path in the $p'$-$q$ plane for a soil particle at the cavity wall with initial stress
($p/p'_0 = 1$, $q = 0$) for (a) $R = 1$ and 5 and (b) $R = 8$ and 10 for SCE

Figure 12 Variation of normalised limit cavity pressure and excess pore pressure with normalized
cavity expansion velocity for CCE

Figure 13 Variation of normalised limit cavity pressure and excess pore pressure with normalized
cavity expansion velocity for SCE
Figure 1 Undrained cavity expansion in EVP soil
Figure 2 $\nu - \ln p'$ plane for EVP model

iso-ncl: isotropic normal compression line
vpl: viscoplastic limit line
Figure 3 Mesh grid of the FDM

Direction of differential calculation

Computational domain: \( r = r_c \)

No excess pore pressure domain: \( u = u_0 \) \((r > r_c)\)
Figure 4 Variation in the (a) normalized cavity wall pressure and (b) normalized cavity wall excess pore pressure with a normalized cavity radius under the condition of different cavity expansion velocities for $R=1$
Figure 5 Variation in the (a) normalized cavity wall pressure and (b) normalized cavity wall excess pore pressure with a normalized cavity radius under the condition of different cavity expansion velocities for $R=2$. 

© The Author(s) or their Institution(s)
Figure 6 Variation in the (a) normalized cavity wall pressure and (b) normalized cavity wall excess pore pressure with a normalized cavity radius under the condition of different cavity expansion velocities for $R=10$. 

\(\Delta \sigma' / \sigma_0\)

\(\Delta u / \sigma_0'\)
Figure 7 Distributions of the effective stress components in the cylindrical (or spherical) coordinate system and excess pore pressure along the radial distance around the cavity wall for $R = 1$: (a) $V = 0.005 \text{ m/s}$; (b) $V = 0.05 \text{ m/s}$; and (c) $V = 0.2 \text{ m/s}$
Figure 8 Distributions of the effective stress components in the cylindrical (or spherical) coordinate system and excess pore pressure along the radial distance around the cavity wall for $R = 2$: (a) $V = 0.005$ m/s; (b) $V = 0.05$ m/s; and (c) $V = 0.2$ m/s
Figure 9 Distributions of the effective stress components in the cylindrical (or spherical) coordinate system and excess pore pressure along the radial distance around the cavity wall for $R = 10$: (a) $V = 0.005$ m/s; (b) $V = 0.05$ m/s; and (c) $V = 0.2$ m/s
Figure 10 Effective stress path in the $p' - q$ plane for a soil particle at the cavity wall with conditions of (a) initial stress ($p'/p'_0 = 1, q = 0$) and (b) initial stress ($p'/p'_0 = 1, q = 0.46$) for CCE.
Figure 11 Effective stress path in the $p'-q$ plane for a soil particle at the cavity wall with initial stress $(p/p'_0 = 1, q = 0)$ for (a) $R = 1$ and 5 and (b) $R = 8$ and 10 for SCE.
Figure 12 Variation of normalised limit cavity pressure and excess pore pressure with normalized cavity expansion velocity for CCE
Figure 13 Variation of normalised limit cavity pressure and excess pore pressure with normalized cavity expansion velocity for SCE