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*Corrigendum*

# Unfoldings in Knot Theory

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In [N–R] the link at infinity of a hypersurface $V \subset \mathbb{C}^N$ defined by a "good" polynomial map $f: \mathbb{C}^N \to \mathbb{C}$ was used as an extended example. $f$ was called "good" if it had only isolated singularities and it was claimed that the link at infinity then always has a "Milnor fibration." This is incorrect (although we have found it to be a common misconception). To correct it, the definition of "good" must be modified as follows.

**Definition.** The fiber $f^{-1}(c)$ of $f$ is *regular* ("ordinaire" in [S]) if there exists a neighborhood $D$ of $c$ in $\mathbb{C}$ such that $f|f^{-1}(D): f^{-1}(D) \to D$ is a locally trivial $C^\infty$ fibration and it is *regular* at *infinity* if there exists a neighborhood $D$ of $c$ in $\mathbb{C}$ and a compact set $K$ in $\mathbb{C}^N$ such that $f|f^{-1}(D) - K: f^{-1}(D) - K \to D$ is a locally trivial $C^\infty$ fibration. The polynomial map $f: \mathbb{C}^N \to \mathbb{C}$ is *good* if every fiber is regular at infinity. "Regular" is equivalent to "regular at infinity and non-singular." Whether $f$ is good or not, it has at most finitely many irregular fibers. We denote by $\mathcal{K}(f, \infty)$ the link at infinity of any fiber which is regular at infinity; up to isotopy this is independent of the choice of the fiber.

*Example.* $f(x, y) = x^2y + x$ is a polynomial with no singularities. The fiber $f^{-1}(0)$ is not regular at infinity since it has three components at infinity while nearby fibers only have two. $\mathcal{K}(f, \infty)$ is the 2-component link consisting of an unknot together with a $(2, -1)$ cable on it (so both components are unknotted and they have mutual linking 2); it is not a fiberable link.

With the above correction to terminology (synonyms to "good" – e.g. "having only isolated singularities" – must also be replaced), the results of [N–R] remain correct, but the proof of 6.1 and the statement and proof of 7.1 need modification. From now on we assume ambient dimension $N = 2$. The statements are:

**6.1. Theorem** (Converse to Milnor Fibration). *If $N = 2$ and $\mathcal{K}(f, \infty)$ is a fiberable link then $f$ is good.*

**7.1. Lemma** (A Knot at Infinity is Good). *If $V \subset \mathbb{C}^2$ is a fiber of $f: \mathbb{C}^2 \to \mathbb{C}$ and is reduced and its link at infinity is a knot ($V$ is connected at infinity) then $f$ is good.*

We first give a careful construction of the link at infinity, extracted from [N]. Call a manifold pair $(\Sigma, L)$ *an abstract link at infinity* for $(\mathbb{C}^2, V)$ if $(\Sigma, L)\times[0,\infty)$ is diffeomorphic to a neighborhood of infinity for the pair $(\mathbb{C}^2, V)$. Any two abstract links at infinity for $(\mathbb{C}^2, V)$ are diffeomorphic, since they are homotopy equivalent as pairs and one can therefore apply Waldhausen [W].

Let $n$ be the degree of $f$. By a linear change of coordinates $w=(x,y)\in\mathbb{C}^2$ we can put $f(x,y)$ in the form

$$f(x,y)=x^n+f_{n-1}(y)x^{n-1}+\dots+f_0(y).$$

Since $f$ has only finitely many irregular fibers, their images are all contained in the interior of some sufficiently large disk $D^2(s)=\{z\in\mathbb{C}\mid |z|\leqq s\}$ about the origin $0\in\mathbb{C}$. Consider the polydisk $D(q,r)=\{(x,y)\in\mathbb{C}^2\mid |x|\leqq q,\ |y|\leqq r\}$.

**Lemma.** *For $s$ as above sufficiently large, $r$ sufficiently large with respect to $s$, and $q$ sufficiently large with respect to $r$ and $s$, the fibers $f^{-1}(z)$ for $z\in\partial D^2(s)$ intersect $\partial D(q,r)$ only in the part $|x|<q$, $|y|=r$, and do so transversely – in fact, they intersect each line $y=y_0$ with $|y_0|=r$ transversely.*

*Proof.* If, for given $r$ and $s$, $f^{-1}(D^2(s))$ intersected $\{|x|=q, |y|\leqq r\}$ non-trivially for arbitrarily large $q$, then $y=0$ would be a point at infinity of the fibers $f^{-1}(z)$. This is not so, so for large $q$, $f^{-1}(D^2(s))$ only meets the other part $\{|x|<q, |y|=r\}$ of $\partial D(q,r)$.

To see the transversality statement, consider $f(x,y)-z$ as a polynomial in $x$ with coefficients in $\mathbb{C}[y,z]$ and form its discriminant $\Delta\in\mathbb{C}[y,z]$ ($\Delta$ is a polynomial in the coefficients of $f$ which vanishes if and only if $f=0$ has multiple roots). Then the fiber $f^{-1}(z_0)$ is transverse to the line $y=y_0$ if and only if $\Delta(y_0,z_0)\neq 0$. In particular, the fiber $f^{-1}(z_0)$ is regular at infinity if $\Delta(y,z)\neq 0$ for each $z$ close to $z_0$ and each $y$ of sufficiently large absolute value. But this fails if and only if $z=z_0$ is tangent to $\Delta(y,z)$ at infinity. In homogeneous coordinates $(y,z,w)$ at infinity, this says that $z=w=0$ is a point of $\Delta=0$ and $z=z_0w$ is a tangent line to $\Delta=0$ at this point. This can only happen for finitely many $z_0$, so we choose our disk $D^2(s)$ to contain these values in its interior. □

Now choose $q$, $r$, and $s$, as in the above lemma. Let $D=f^{-1}(D^2(s))\cap D(q,r)$. Its boundary is piecewise-smooth and decomposes as $\partial D=S\cup E$ with

$$S=\partial D(q,r)\cap f^{-1}(D^2(s)),$$
$$E=D(q,r)\cap\partial(f^{-1}(D^2(s))).$$

$f$ restricts to a fibration of $E$ over a circle, and a typical fiber $F=f^{-1}(z)\cap E$ of $f|E$ satisfies: *the pair $(\partial D, \partial F)$ is an abstract link at infinity for* $(\mathbb{C}^2, V)$ (after smoothing the corner along $\partial S=\partial E$). To see that $(\mathbb{C}^2-\mathrm{int}(D),\ f^{-1}(z)-\mathrm{int}(F))$ is homeomorphic (diffeomorphic after smoothing corners) to $(\partial D,\partial F)\times[0,\infty)$ as desired, integrate along a suitable smooth vectorfield $v$ on $\mathbb{C}^2-\mathrm{int}(D)$ which is transversal inward on $\partial D$, is tangent to the fibers $f^{-1}(z)$ for $|y|\geqq r$ and $z\in\partial D^2(s)$, and whose $v$-derivative satisfies the following for some small $\varepsilon$: $v(|y|^2)\leqq -1$ when $|y|\geqq r-\varepsilon$ and $|f(x,y)|\leqq s+\varepsilon$, and $v(|f(x,y)|^2)\leqq -1$ otherwise. Such a vectorfield is easily constructed locally using the lemma, and a partition of unity then does it globally.

If $f$ is good, we can choose $r$ sufficiently large that *all* fibers $f^{-1}(z)$ with $z \in D^2(s)$ are transverse to $|y| = r$. Then $S$ is equivalent to a disk tubular neighborhood of the link at infinity, so the construction has given the link at infinity with its Milnor fibration.

*Proof of 6.1.* Suppose $f$ is not good. The lemma implies that for $|y_0| = r$ the intersection $S_0 = \{(x, y) | y = y_0\} \cap f^{-1}(D^2(s))$ is transverse and $f | S_0 : S_0 \to D^2(s)$ is a holomorphic branched cover with no singularities over $\partial D^2(s)$. On the other hand it certainly does have singularities over $\operatorname{int} D^2(s)$ (namely, near any fiber which is irregular at infinity). It follows that the inclusion $\partial F \subset S$ is not an isomorphism in homology. As in [N–R], this implies $\mathscr{K}(f, \infty)$ is not fiberable. □

*Proof of 7.1.* By Suzuki [S], the general fiber of $f$ is also connected at infinity, so $\mathscr{K}(f, \infty)$ is a knot. As described in [N–R], it is an iterated torus knot, hence fiberable, so $f$ is good by Theorem 6.1. □

## References

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