

UNFOLDINGS OF HOLOMORPHIC FOLIATIONS

XAVIER GÓMEZ-MONT

Abstract

The objective of this paper is to give a criterium for an unfolding of a holomorphic foliation with singularities to be holomorphically trivial.

1. Introduction

An unfolding is a holomorphic foliation with singularities \mathcal{F} in the complex manifold M , with $\pi : M \rightarrow \Delta_1 \subset \mathbb{C}$ a smooth holomorphic map, defining a family of foliations \mathcal{F}_t in the family of complex manifolds $M_t = \pi^{-1}(t)$, obtained by intersecting the leaves of \mathcal{F} with M_t (see definitions 2.1 and 3.1), such that the leaves of \mathcal{F} are not contained in the fibres of π . The leaves of \mathcal{F} have one more dimension than the leaves of \mathcal{F}_t , and the problem we are addressing in this paper is to give a criterium to determine when the family of foliations \mathcal{F}_t is trivial, that is, they are all biholomorphically equivalent. The technique that we will use to analyse this problem, will consist in measuring the obstructions to lifting the vector field $\frac{\partial}{\partial t}$ on Δ_1 to a holomorphic vector field Y tangent to the foliation \mathcal{F} . If π is a proper map, or a germ of a map, then the 1-parameter group obtained by integrating Y gives the triviality of the unfolding.

We begin by showing in Proposition 2.2 that the local existence of a lifting of $\frac{\partial}{\partial t}$ is equivalent to the triviality of the extension class of the normal sheaves (i.e. the normal sheaf of the family of foliations \mathcal{F}_t is a direct sum of the normal sheaf of the foliation \mathcal{F} by the structure sheaf). We do not need the integrability of the foliation for this first step. In Theorem 3.3 we use the integrability conditions, and the assumption that the foliation is full, to show that the triviality of the extension class of the conormal sheaves is equivalent to the local triviality of the unfolding. We globalize this criterium in Theorem 4.1.

Having formulated the triviality of the unfolding in terms of a triviality of an extension class, we use a spectral sequence to divide this second problem into two problems: The first is local and concentrated in the singularities of the foliation, and the second is global and measured by a first cohomology group. In section 5 we apply the above procedure to foliations by curves. We show that the local obstructions vanish if the depth of the singularity set is at least 3, and that the global obstructions are measured by a cohomology group of an invertible sheaf, which may then be analysed by standard vanishing theorems (Cartan–Serre's Theorem B or Kodaira–Nakano). In particular we obtain:

Theorem 5.3. *Let $X_0 : \mathcal{L}_0 \rightarrow \Theta_{M_0}$ be a foliation by curves with isolated singularities in the compact manifold M_0 . Assume that M_0 has dimension at least 3 and $H^1(M_0, \mathcal{L}_0) = 0$; then every unfolding of X_0 is trivial.*

We finish by analysing the case of foliations by curves in surfaces:

Theorem 5.5. [G-O] *Let $X_0 : \mathcal{L}_0 \rightarrow \Theta_{M_0}$ be a foliation by curves with isolated singularities in the compact complex 2-dimensional manifold M_0 . Assume that $H^1(M_0, \mathcal{L}_0) = 0$ and that every singular point has multiplicity 1 and non-vanishing trace; then every unfolding of X_0 is trivial.*

The method we present in this paper is a generalization of an idea of Ilasenko [I]. He originally applied it to foliations by curves in the complex projective plane tangent to a line, where after taking affine coordinates of the complement of the line he applied Noether's AF+BG Theorem (see [G-H] p. 703) instead of carrying out the cohomological computation.

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2. Local unfoldings of distributions

Let

$$\pi : \mathbb{C} \times \mathbb{C}^n \rightarrow \mathbb{C}, \quad \pi(t, z) = t$$

with $t \in \mathbb{C}$ and $z = (z_1, \dots, z_n) \in \mathbb{C}^n$, be the projection of $\mathbb{C} \times \mathbb{C}^n$ to the first factor. π induces a ring homomorphism

$$\pi^* : \mathcal{O}_1 \rightarrow \mathcal{O}_{n+1}$$

from the ring of germs of holomorphic functions at 0 in \mathbb{C} to the ring of holomorphic functions at $\mathbf{0} = (0, 0)$ in \mathbb{C}^{n+1} . If we denote by Ω_{n+1} and Ω_1 the (free) modules of germs of holomorphic 1-forms on \mathbb{C}^{n+1} and \mathbb{C} , respectively, and by $\pi^*\Omega_1$ the \mathcal{O}_{n+1} -module $\Omega_1 \otimes_{\mathcal{O}_1} \mathcal{O}_{n+1}$, then the pull back of 1-forms under π extends to an exact sequence of \mathcal{O}_{n+1} -modules

$$(2.1) \quad 0 \rightarrow \pi^*\Omega_1 \rightarrow \Omega_{n+1} \xrightarrow{\rho} \Omega_\pi \rightarrow 0$$

where the right hand term is the (free) \mathcal{O}_{n+1} -module of relative differential 1-forms to π and ρ is the projection map.

An element ϑ of Ω_{n+1} may be written as $\vartheta = b(t, z)dt + \sum_{j=1}^n a_j(t, z)dz_j$ and it may be interpreted as describing a possibly singular distribution of codimension one planes in some open neighbourhood of $\mathbf{0}$ in \mathbb{C}^{n+1} . Its projection

$\rho(\vartheta)$ may be interpreted as a 1-parameter family of distributions defined in neighbourhoods of 0 in \mathbb{C}^n by $\sum_{j=1}^n a_j(t, z) dz_j$ and it describes the family of distributions obtained by intersecting the distribution defined by ϑ with the hyperplanes tangent to the π -fibres.

Definition 2.1. A germ of a family of (singular holomorphic) distributions is an injective \mathcal{O}_{n+1} -module map $\bar{\omega}$ defined from an \mathcal{O}_{n+1} -module F into Ω_π , and giving rise to the short exact sequence of \mathcal{O}_{n+1} -modules:

$$(2.2) \quad 0 \longrightarrow F \xrightarrow{\bar{\omega}} \Omega_\pi \longrightarrow \Omega_\pi/\bar{\omega}(F) \longrightarrow 0$$

An unfolding of a germ of a family of distributions $\bar{\omega} : F \longrightarrow \Omega_\pi$ is a lifting of $\bar{\omega}$ to an \mathcal{O}_{n+1} morphism $\omega : F \longrightarrow \Omega_{n+1}$. We will also call an injective morphism $\omega : F \longrightarrow \Omega_{n+1}$ a germ of a (singular holomorphic) distribution and if $\bar{\omega} = \rho \circ \omega$ is an injective \mathcal{O}_{n+1} -module map, we will call ω a germ of an unfolding with $\bar{\omega}$ as underlying family of distributions.

Remark: The requirement that $\bar{\omega}$ is an injective map means that the image of ω does not contain elements of the form $b(t, z) dt$, or equivalently, that the planes defined by the unfolding are not all contained in the tangent planes to the π -fibres.

An unfolding gives rise to an exact commutative diagram of \mathcal{O}_{n+1} -modules:

$$(2.3) \quad \begin{array}{ccccccc} & & & 0 & & 0 & \\ & & & \downarrow & & \downarrow & \\ & & & F & = & F & \\ & & & \downarrow \omega & & \downarrow \bar{\omega} & \\ 0 & \longrightarrow & \pi^* \Omega_1 & \longrightarrow & \Omega_{n+1} & \xrightarrow{\rho} & \Omega_\pi & \longrightarrow & 0 \\ & & \parallel & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \pi^* \Omega_1 & \longrightarrow & \Omega_{n+1}/\omega(F) & \longrightarrow & \Omega_\pi/\bar{\omega}(F) & \longrightarrow & 0 \\ & & & & \downarrow & & \downarrow & & \\ & & & & 0 & & 0 & & \end{array}$$

Proposition 2.2. Let $\bar{\omega} : F \longrightarrow \Omega_\pi$ be a family of distributions, then:

1) There is a one to one correspondence between the unfoldings of F and the elements in $\text{Hom}(F, \pi^* \Omega_1) \cong F^*$.

2) The map which associates to each unfolding $\eta \oplus \bar{\omega}$ the extension class of $\Omega_\pi/\bar{\omega}(F)$ by $\pi^* \Omega_1$ in the lower row of (2.3) is the coboundary morphism δ of a long exact sequence of Ext-groups.

3) If the above extension class $\delta(\eta)$ is 0, then we may find a germ of a holomorphic vector field tangent to the distribution of the form

$$(2.4) \quad \hat{X} = \frac{\partial}{\partial t} - \sum_{j=1}^n X_j(t, z) \frac{\partial}{\partial z_j}$$

Proof: Since the middle vertical sequence splits, $\Omega_{n+1} = \pi^*\Omega_1 \oplus \Omega_\pi$, we may write $\omega = \eta \oplus \bar{\omega}$, where $\eta \in \text{Hom}(F, \pi^*\Omega_1) \cong F^*$. This proves part 1.

The long exact sequence of Ext-modules associated to the functor $\text{Hom}(\cdot, \pi^*\Omega_1)$ applied to the right hand column in (2.3) is:

$$(2.5) \quad \begin{array}{ccccccc} 0 & \longrightarrow & \text{Hom}(\Omega_\pi/\bar{\omega}(F), \pi^*\Omega_1) & \longrightarrow & \text{Hom}(\Omega_\pi, \pi^*\Omega_1) & \xrightarrow{\gamma} & \\ & & & & & & \\ \text{Hom}(F, \pi^*\Omega_1) & \xrightarrow{\delta} & \text{Ext}^1(\Omega_\pi/\bar{\omega}(F), \pi^*\Omega_1) & \xrightarrow{\epsilon} & \text{Ext}^1(\Omega_\pi, \pi^*\Omega_1) & \longrightarrow & \end{array}$$

The kernel of ϵ corresponds to those extensions that induce the trivial extension of Ω_π by $\pi^*\Omega_1$, and these extensions are described as images of δ by means of elements $\eta \in \text{Hom}(F, \pi^*\Omega_1)$. This proves part 2.

If $\delta(\eta) = 0$, then by the exactness of (2.5), we may find an extension $\tilde{\eta} : \Omega_\pi \rightarrow \pi^*\Omega_1$ of $\eta : F \rightarrow \pi^*\Omega_1$; that is, there is a relative vector field

$$X = \sum_{j=1}^n X_j \frac{\partial}{\partial z_j} \in \Theta_\pi = \text{Hom}(\Omega_\pi, \mathcal{O}_{n+1}) \cong \text{Hom}(\Omega_\pi, \pi^*\Omega_1)$$

such that if for $f \in F$ we have $\bar{\omega}(f) = \sum a_j(f) dz_j$ then

$$\begin{aligned} \omega(f) &= \eta(f) dt + \sum a_j(f) dz_j = \\ &= \left(\sum a_j(f) dz_j \right) \circ \left(\sum X_j \frac{\partial}{\partial z_j} \right) dt + \sum a_j(f) dz_j \end{aligned}$$

where \circ evaluates a 1-form on a vector field through the natural duality. Hence we have

$$\omega(f) = \left(\sum a_j(f) X_j \right) dt + \sum a_j(f) dz_j$$

Define now \tilde{X} as in (2.4) using the previously obtained components X_j , and then

$$\omega(f) \circ \tilde{X} = \sum a_j(f) X_j - \sum a_j(f) X_j = 0$$

Hence \tilde{X} is tangent to the distribution defined by $\omega(f)$ for every $f \in F$, so \tilde{X} is tangent to the distribution defined by ω . This finishes the proof of the proposition. ■

To further exploit the vector field (2.4), we need to impose the integrability conditions to the distribution. Although the statement (Theorem 3.3) is local, the proof we present is semi-local; so we will begin the next section with global definitions of foliations and unfoldings of foliations.

3. Local unfoldings of foliations

Let $\pi : M \rightarrow \mathbb{C}$ be a smooth holomorphic map between the complex manifold M of dimension $n+1$ and the complex line \mathbb{C} . The fibers of $\pi^{-1}(t) = M_t$ of π are smooth complex manifolds of dimension n . If M_0 is compact, we will also assume that π is a proper map onto its image. Overall, we will be interested in a neighbourhood of M_0 , so we consider \mathbb{C} as a pointed space $(\mathbb{C}, 0)$, and we shrink the neighbourhoods around M_0 .

We will denote by

$$\mathcal{O}_M, \Omega_M, \Theta_M, \Omega_\pi, \Theta_\pi$$

the sheaves on M of germs of holomorphic functions, 1-forms, vector fields, relative 1-forms, relative vector fields, and

$$\Omega_M^* = \bigoplus_{p=0}^{n+1} \bigwedge^p \Omega_M, \quad \Omega_\pi^* = \bigoplus_{p=0}^n \bigwedge^p \Omega_\pi \quad \text{and} \quad \pi^* \Omega_{\mathbb{C}} = \Omega_{\mathbb{C}} \otimes_{\mathcal{O}_{\mathbb{C}}} \mathcal{O}_M$$

where the first two are the sheaves of exterior algebras of differential and relative differential 1-forms, respectively, with derivatives d and d_π , where d is a \mathbb{C} -linear map and d_π is $\mathcal{O}_{\mathbb{C}}$ -linear. Given a subset $A \subset \Omega^*$, we will denote by $A \cdot \Omega^*$ the sheaf of ideals generated by the elements of A .

We obtain an exact sequence of sheaves on M , as in (2.1)

$$(3.1) \quad 0 \rightarrow \pi^* \Omega_{\mathbb{C}} \rightarrow \Omega_M \xrightarrow{\rho} \Omega_\pi \rightarrow 0$$

Definition 3.1. A family of distributions (or a distribution) on M is an injective \mathcal{O}_M -morphism $\bar{\omega} : \mathcal{F} \rightarrow \Omega_\pi$ (respectively, an injective \mathcal{O}_M -morphism $\omega : \mathcal{F} \rightarrow \Omega_M$) from the coherent sheaf \mathcal{F} on M . The singular set $\text{Sing}(\bar{\omega})$ (or $\text{Sing}(\omega)$) of the family of distributions (or of the distribution) is the set of points where $\Omega_\pi / \bar{\omega}(\mathcal{F})$ (or $\Omega_M / \omega(\mathcal{F})$) is not locally free.

A family of foliations (or a foliation, or an unfolding of a family of foliations) is a family of distributions $\bar{\omega} : \mathcal{F} \rightarrow \Omega_\pi$ such that on the non-singular points of $\omega(\mathcal{F})$ we have $d_\pi(\omega(\mathcal{F})) \subset \bar{\omega}(\mathcal{F}) \cdot \Omega_\pi^*$ (respectively, a distribution $\omega : \mathcal{F} \rightarrow \Omega_M$ such that on the non-singular points of $\omega(\mathcal{F})$ we have $d(\omega(\mathcal{F})) \subset \omega(\mathcal{F}) \cdot \Omega_M^*$, and for an unfolding we require that ω is a foliation and the morphism of \mathcal{O}_M -sheaves $\bar{\omega} = \rho \circ \omega : \mathcal{F} \rightarrow \Omega_\pi$ is injective).

We also say in the preceding definition that the germs of distributions are *involutive*. We will always consider an unfolding of a family of foliations as being involutive. An unfolding on M gives rise to an exact commutative diagram of \mathcal{O}_M -modules:

$$\begin{array}{ccccccc}
 & & & 0 & & 0 & \\
 & & & \downarrow & & \downarrow & \\
 & & & \mathcal{F} & = & \mathcal{F} & \\
 & & & \downarrow \omega & & \downarrow \bar{\omega} & \\
 (3.2) & 0 & \longrightarrow & \pi^* \Omega_{\mathbb{C}} & \longrightarrow & \Omega_M & \xrightarrow{\rho} & \Omega_{\pi} & \longrightarrow & 0 \\
 & & & \parallel & & \downarrow & & \downarrow & & \\
 & 0 & \longrightarrow & \pi^* \Omega_{\mathbb{C}} & \longrightarrow & \Omega_M / \omega(\mathcal{F}) & \longrightarrow & \Omega_{\pi} / \bar{\omega}(\mathcal{F}) & \longrightarrow & 0 \\
 & & & & & \downarrow & & \downarrow & & \\
 & & & & & 0 & & 0 & &
 \end{array}$$

The process of restriction to the fibre M_t is carried out by tensoring with $\otimes_{\mathcal{O}_{\mathbb{C}}} \mathbb{C}_t$, where $\mathbb{C}_t = \mathcal{O}_{\mathbb{C}}/m_t$ and m_t is the sheaf of ideals in $\mathcal{O}_{\mathbb{C}}$ associated to the point $t \in \mathbb{C}$. If $\bar{\omega} : \mathcal{F} \rightarrow \Omega_{\pi}$ is a family of foliations, then its restriction to M_t is

$$\bar{\omega}_t : \mathcal{F} \otimes_{\mathcal{O}_{\mathbb{C}}} \mathbb{C}_t \rightarrow \Omega_{\pi} \otimes_{\mathcal{O}_{\mathbb{C}}} \mathbb{C}_t \cong \Omega_{M_t}$$

which may be considered as an \mathcal{O}_{M_t} -morphism, and after dividing by its kernel (if it is not injective) it defines a foliation on M_t .

Given $\bar{\omega}_0 : \mathcal{F}_0 \rightarrow \Omega_{M_0}$ a foliation on M_0 , its *trivial extension to a family of foliations* is defined on $\pi : M_0 \times \mathbb{C} \rightarrow \mathbb{C}$ by

$$(3.3) \quad \bar{\omega} = \bar{\omega}_0 \otimes_{\mathbb{C}} id : \mathcal{F}_0 \otimes_{\mathbb{C}} \mathcal{O}_{\mathbb{C}} \rightarrow \Omega_{M_0} \otimes_{\mathbb{C}} \mathcal{O}_{\mathbb{C}} \cong \Omega_{\pi}$$

and its *trivial extension as an unfolding* by further defining $\omega = \eta \oplus \bar{\omega}$ with $\eta = 0$.

If $\bar{\omega} : \mathcal{F} \rightarrow \Omega_{\pi}$ (or $\omega : \mathcal{F} \rightarrow \Omega_{n+1}$) is a germ of a family of foliations (or a foliation or an unfolding), then it extends to a family of foliations (or a foliation or an unfolding) in a neighbourhood of 0. This may be realized by extending a set of generators of \mathcal{F} . By coherence, this extension is unique in a perhaps smaller neighbourhood of 0.

Definition 3.2. Two unfoldings $\omega_j : \mathcal{F}_j \rightarrow \Omega_{M_j}$, $j = 1, 2$ (or families of foliations $\omega_j : \mathcal{F}_j \rightarrow \Omega_{\pi_j}$) are *isomorphic* if there is a biholomorphism $\Phi : M^1 \rightarrow M^2$ over \mathbb{C} (i.e. it induces the identity on \mathbb{C}) such that $\omega_1(\mathcal{F}_1) = \Phi^* \omega_2(\mathcal{F}_2)$, where Φ^* is the natural action of Φ on 1-forms (respectively $\bar{\omega}_1(\mathcal{F}_1) = \Phi^* \bar{\omega}_2(\mathcal{F}_2)$). We say that an *unfolding* $\omega : \mathcal{F} \rightarrow \Omega_M$ is *trivial* over 0 (or that a family of foliations $\bar{\omega}$ is *trivial* over 0) if ω (or $\bar{\omega}$, respectively) is isomorphic to the trivial extension of the restriction of ω to M_0 in some neighbourhood of the fibers over $0 \in \mathbb{C}$. A *germ of an unfolding at* $0 \in \mathbb{C} \times \mathbb{C}^n$ *is trivial* if there is a neighbourhood of 0 where the unfolding is trivial.

The *singular set* $Sing(\bar{\omega})$ of a family of foliations $\bar{\omega} : \mathcal{F} \rightarrow \Omega_{\pi}$ is defined as the set of points where $\Omega_{\pi} / \bar{\omega}(\mathcal{F})$ is not locally free. The *singular set* $Sing(\omega)$

of an unfolding or of a foliations $\omega : \mathcal{F} \rightarrow \Omega_M$ is defined as the set of points where $\Omega_M/\omega(\mathcal{F})$ is not locally free. By the Theorem of Frobenius (see [Ch]), we may find at a non-singular point of $\bar{\omega}$ (or of ω) coordinates (t, w_1, \dots, w_n) such that $\bar{\omega}(\mathcal{F}) \subset \Omega_\pi$ (respectively $\omega(\mathcal{F}) \subset \Omega_M$) is generated by dw_1, \dots, dw_q . With these coordinate charts, that are called *foliated coordinate charts*, we may define on $M - \text{Sing}(\bar{\omega})$ (or $M - \text{Sing}(\omega)$) a geometric foliation $\text{Fol}(\omega)$ (or $\text{Fol}(\bar{\omega})$) that gives a decomposition of $M - \text{Sing}(\bar{\omega})$ into a disjoint union of connected complex submanifolds, which in general are only immersed and called the *leaves of the foliation*, that are locally defined by $w_j = K_j, j = 1, \dots, q$.

We may associate to a family of foliations $\bar{\omega} : \mathcal{F} \rightarrow \Omega_\pi$ a foliation on the total space M which has the same leaves as $\text{Fol}(\bar{\omega})$. This is done by taking the inverse image of $\bar{\omega}$ in (3.1) to obtain an exact commutative diagram:

$$\begin{array}{ccccccc}
 & & & 0 & & 0 & \\
 & & & \downarrow & & \downarrow & \\
 & 0 & \longrightarrow & \pi^*\Omega_M & \longrightarrow & \tilde{\mathcal{F}} & \longrightarrow & \mathcal{F} & \longrightarrow & 0 \\
 & & & \parallel & & \downarrow \bar{\omega} & & \downarrow \bar{\omega} & & \\
 (3.4) & 0 & \longrightarrow & \pi^*\Omega_M & \longrightarrow & \Omega_M & \xrightarrow{\rho} & \Omega_\pi & \longrightarrow & 0 \\
 & & & & & \downarrow & & \downarrow & & \\
 & & & & & \Omega_M/\tilde{\omega}(\tilde{\mathcal{F}}) & = & \Omega_\pi/\bar{\omega}(\mathcal{F}) & & \\
 & & & & & \downarrow & & \downarrow & & \\
 & & & & & 0 & & 0 & &
 \end{array}$$

An \mathcal{O}_{n+1} -submodule G of Ω_{n+1} (or of Ω_π) is *full* if G is equal to its double dual G^{**} , where the duality is the one between 1-forms and vector fields, (or between relative 1-forms and relative vector fields, respectively). We will say that a germ of a family of foliations $\bar{\omega} : F \rightarrow \Omega_\pi$ (or a germ of a foliation $\omega : F \rightarrow \Omega_{n+1}$) is *full* if $\bar{\omega}(F) \subset \Omega_\pi$ (or $\omega(F) \subset \Omega_{n+1}$) is a full submodule. Similarly an \mathcal{O}_M -submodule of $\Omega_M, \dots, \text{etc.}$ is *full* if the stalk over every point of M is full. The fullness condition is useful, since a full foliation is completely determined by the foliation at the non-singular points.

Theorem 3.3. *Let $\omega : F \rightarrow \Omega_{n+1}$ be a germ of a full unfolding of a family of foliations $\bar{\omega} : F \rightarrow \Omega_\pi$. ω is a trivial unfolding if and only if there is an \mathcal{O}_{n+1} -module isomorphism*

$$(3.5) \quad \Omega_{n+1}/\omega(F) \cong \pi^*\Omega_1 \oplus \Omega_\pi/\bar{\omega}(F)$$

Proof: If ω is a trivial unfolding, then we may find a neighbourhood and a biholomorphism over π such that ω has the form $\omega = 0 \oplus \bar{\omega}_0$, where $\bar{\omega}_0$ is the induced foliation in $\{0\} \times \mathbb{C}^n$, as in (3.3). In these coordinates, the extension class in Proposition 2.2 is $\delta(0) = 0$; hence we have (3.5).

Conversely, assume that (3.5) holds. This condition means that the extension class in Proposition 2.2 is 0, and so by Proposition 2.2 we may find a germ of a holomorphic vector field \tilde{X} as in (2.4) tangent to the distribution. Extend everything to a neighbourhood of 0.

By integrating \tilde{X} with initial conditions on $\{0\} \times \mathbb{C}^n$ we obtain a biholomorphism over \mathbb{C} . Using this biholomorphism as a change of coordinates, we may assume that \tilde{X} is $\frac{\partial}{\partial t}$. In the following argument we will make use of the integrability conditions. We will first argue at the nonsingular points of the foliation, and then use the fullness hypothesis to extend the conclusion to the singular points.

Restrict to an open set $U = \Delta_1 \times \Delta_n$ of $\mathbb{C} \times \mathbb{C}^n$ which is the product of small disks around the origin. Let V be a connected open subset in $U - \text{Sing}(\omega)$ such that for every $z \in \Delta_n$ we have that $V \cap \pi_2^{-1}\pi_2(z)$ is connected, where $\pi_2: \Delta_1 \times \Delta_n \rightarrow \Delta_n$ is the projection to the second factor. We claim that $\pi_2^{-1}\pi_2(V)$ is contained in $U - \text{Sing}(\omega)$ and that the unfolding there is the pullback of a non-singular foliation in $\pi_2(V)$.

To see this, observe that $\text{Fol}(\omega)$ in V is non-singular and tangent to $\frac{\partial}{\partial t}$, so if we intersect V with a set of the form $\{t\} \times \Delta_n$, it will induce a non-singular foliation there, of the same codimension. Project it to $\pi_2(V)$. These foliations coincide for distinct values of t , since the foliation is tangent to $\frac{\partial}{\partial t}$ and $V \cap \pi_2^{-1}(z)$ is connected. By pulling back this foliation, we obtain two foliations in $\pi_2^{-1}\pi_2(V)$, one of them non-singular, the other one full, and they coincide on an open non-empty subset. By the principle of analytic continuation, they coincide at the non-singular points of $\text{Fol}(\omega)$; but since ω is full, actually they coincide. This proves the claim.

If we denote by $\tilde{\omega}$ the trivial extension as an unfolding to the restriction of ω to $\{0\} \times \Delta_n$, as in (3.3), the preceding argument shows that $\text{Fol}(\omega)$ and $\text{Fol}(\tilde{\omega})$ have the same singular set, and that they agree at the non-singular points. Since ω is full by hypothesis, this implies that we have the sheaf inclusion

$$(3.6) \quad \tilde{\omega}((\mathcal{F} \otimes_{\mathcal{O}_U} \mathcal{C}_0) \otimes_{\mathbb{C}} \mathcal{O}_{\mathbb{C}}) \subset \omega(\mathcal{F})$$

We must still show that they coincide. Let ϑ be an element in the right hand side of (3.6). Expand ϑ as a power series in t

$$(3.7) \quad \vartheta(t, z) = \sum_{j=1}^n \vartheta_j(z) t^j$$

where the $\vartheta_j(z)$ are holomorphic 1-forms on Δ_n . We claim that each ϑ_j belongs to the left hand side of (3.6). To see this, observe that ϑ_0 is the restriction of ϑ to $0 \times \Delta_n$. Hence by definition it lies on the left hand side of (3.6). Now $t^{-1}(\vartheta - \vartheta_0)$

belongs to the right hand side of (3.6) on the complement of $\{0\} \times \Delta_\pi$, so by fullness also on all of U . Repeating the argument for this new 1-form, we obtain that ϑ_1 also belongs to the left hand side of (3.6). By induction then, all ϑ_j belong to the left hand side of (3.6). By properties of closures under convergence in compact sets of sections of a coherent sheaf, see [G-R], we then have that the sum (3.7) also lies in the left hand side of (3.6). This proves equality, and hence the theorem. ■

Proposition 3.4. *Let $\omega : \mathcal{F} \rightarrow \Omega_M$ be an unfolding of a family of foliations $\bar{\omega} : \mathcal{F} \rightarrow \Omega_\pi$. Then*

1) $Sing(\omega) \subset Sing(\bar{\omega})$

2) $Sing(\bar{\omega}) - Sing(\omega)$ is the set of points p of $M - Sing(\omega)$ such that the leaf of $Fol(\omega)$ through p is tangent to $M_{\pi(p)}$ at p .

Proof: 1) We prove that $M - Sing(\bar{\omega}) \subset M - Sing(\omega)$. At a non-singular point of $\bar{\omega}$ we have that $(\Omega_\pi/\bar{\omega}(\mathcal{F}))_p$ is locally free, but since all extensions of locally free modules by locally free modules are trivial (i.e. $Ext^1((\Omega_\pi/\bar{\omega}(\mathcal{F}))_p, (\pi^*\Omega_C)_p) = 0$), we also have by (2.3) that $(\Omega_M/\omega(\mathcal{F}))_p$ is free. Hence $p \in M - Sing(\omega)$.

2) Let $p \in M - Sing(\omega)$. Then \mathcal{F}_p is locally free at p , and so is $\tilde{\mathcal{F}}_p$ in (3.4). The map $\tilde{\omega} : \tilde{\mathcal{F}} \rightarrow \Omega_M$ in (3.4) has the same cokernel sheaf as $\bar{\omega} : \mathcal{F} \rightarrow \Omega_\pi$, so we are looking for a criterium under which $(\Omega_M/\tilde{\omega}(\tilde{\mathcal{F}}))_p$ is free. Let $dt, \vartheta_1, \dots, \vartheta_q$ be a basis for $\tilde{\omega}(\tilde{\mathcal{F}})_p$. The point p is non-singular for $Fol(\tilde{\omega})$ if and only if $dt(p), \vartheta_1(p), \dots, \vartheta_q(p)$ are linearly independent as elements in the cotangent space to M at p . Since $p \notin Sing(\omega)$, we have that $\vartheta_1(p), \dots, \vartheta_q(p)$ are linearly independent; so p is non-singular for $Fol(\tilde{\omega})$ if and only if the leaf of $Fol(\omega)$ through p is not tangent to $M_{\pi(p)}$ at p . This proves the proposition. ■

4. Global unfoldings of foliations

Let $\pi : M \rightarrow \Delta_1$ be a proper smooth holomorphic map onto the the unit disk Δ_1 in \mathbb{C} , and let $\omega : \mathcal{F} \rightarrow \Omega_M$ be an unfolding of a family of holomorphic foliations $\bar{\omega} : \mathcal{F} \rightarrow \Omega_\pi$. The lower row in (3.2) shows that $\Omega_M/\omega(\mathcal{F})$ is an extension of $\Omega_\pi/\bar{\omega}(\mathcal{F})$ by $\pi^*\Omega_{\Delta_1}$. Recall (see [G-H] p.705 or [H-2] p.237) that these extension classes are classified by elements in the global $Ext^1(M; \Omega_\pi/\bar{\omega}(\mathcal{F}), \pi^*\Omega_{\Delta_1})$ group.

Theorem 4.1. *Let $\pi : M \rightarrow \Delta_1$ be a proper and smooth morphism onto the unit disk Δ_1 in \mathbb{C} , and let $\omega : \mathcal{F} \rightarrow \Omega_M$ be a full unfolding of a family of foliations $\bar{\omega} : \mathcal{F} \rightarrow \Omega_\pi$. ω is a trivial unfolding over $0 \in \Delta_1$ if and only if after shrinking the disk Δ_1 there is an \mathcal{O}_M -sheaf isomorphism*

$$(4.1) \quad \Omega_M/\omega(\mathcal{F}) \cong \pi^*\Omega_{\Delta_1} \oplus \Omega_\pi/\bar{\omega}(\mathcal{F})$$

Proof. If ω is a trivial unfolding over 0, then we may find a neighbourhood of M_0 and a biholomorphism over π such that ω has the form $\omega = 0 \oplus \bar{\omega}_0$, where $\bar{\omega}_0$ is the induced foliation in M_0 , as in (3.3). After applying this isomorphism, we clearly have (4.1).

Conversely, assume that (4.1) holds. There is a long exact sequence of global Ext groups associated to the functor $Hom(M; -, \pi^*\Omega_{\Delta_1})$ applied to the right hand column in (3.2):

$$(4.2) \quad \begin{array}{ccccc} 0 & \longrightarrow & Hom(M; \Omega_\pi/\bar{\omega}(\mathcal{F}), \pi^*\Omega_{\Delta_1}) & \longrightarrow & Hom(M; \Omega_\pi, \pi^*\Omega_{\Delta_1}) \\ & \nearrow & Hom(M; \mathcal{F}, \pi^*\Omega_{\Delta_1}) & \xrightarrow{\delta} & Ext^1(M; \Omega_\pi/\bar{\omega}(\mathcal{F}), \pi^*\Omega_{\Delta_1}) \\ & \xrightarrow{\epsilon} & Ext^1(M; \Omega_\pi, \pi^*\Omega_{\Delta_1}) & \longrightarrow & \end{array}$$

There is a canonical element $\xi \in Ext^1(M; \Omega_\pi, \pi^*\Omega_{\Delta_1})$ which represents Ω_M as the extension in the middle row of (3.2). Recalling that $Ext^1(M; \Omega_\pi, \pi^*\Omega_{\Delta_1})$ is isomorphic to $H^1(M, \Theta_\pi)$, we see that this element is the Kodaira–Spencer class of the family of varieties $\{M_t\}$ (see [K–S]). Using the isomorphism between this extension and Ω_M , we see that $\epsilon^{-1}(\xi)$ in (4.2) represent all the unfoldings of $\bar{\omega}$ as a family of distributions, as in diagram (3.2).

The hypothesis (4.1) is then equivalent, after shrinking Δ_1 to a smaller disk, to having the unfolding ω represented in $Ext^1(M; \Omega_\pi/\bar{\omega}(\mathcal{F}), \pi^*\Omega_{\Delta_1})$ by the 0 element. Since $\epsilon(0) = 0$, we deduce that the family π is the trivial family: $M = \Delta_1 \times M_0$. In this case, by the commutativity of (4.2), we may represent all unfoldings by means of elements of $Hom(M; \mathcal{F}, \pi^*\Omega_{\Delta_1})$. Since by hypothesis the unfolding ω is inducing the trivial extension, again by the exactness of (4.2) we may find an element

$$X \in H^0(M, \Theta_\pi) \cong Hom(M; \Omega_\pi, \pi^*\Omega_{\Delta_1})$$

The holomorphic vector field $\bar{X} = \frac{\partial}{\partial t} - X$ is, as in the proof of Proposition 2.2 and Theorem 3.3, tangent to the unfolding ω and its integral 1-parameter group sends π -fibres to π -fibres. We may then use this vector field, as in the proof of Proposition 2.2 and Theorem 3.3, to show that the unfolding is trivial. This proves the Theorem. ■

In order to compute if an unfolding induces a trivial extension (4.1), we may use one of the spectral sequences associated to the global Ext construction, which gives an exact sequence:

$$(4.3) \quad \begin{array}{ccccc} 0 & \longrightarrow & H^1(M, \mathcal{T}) & \longrightarrow & Ext^1(M; \Omega_\pi/\bar{\omega}(\mathcal{F}), \pi^*\Omega_{\Delta_1}) \\ & & & \xrightarrow{\rho} & H^0(M; \mathcal{E}xt^1(\Omega_\pi/\bar{\omega}(\mathcal{F}), \pi^*\Omega_{\Delta_1})) \end{array}$$

where \mathcal{T} is the sheaf of germs of relative holomorphic vector fields tangent to the family of foliations $\bar{\omega}$, whose stalk over $p \in M$ is

$$\mathcal{T}_p = \{Y \in \Theta_{\pi,p}/\bar{\omega}(f) \circ X = 0, \forall f \in \mathcal{F}_p\}$$

It is the annihilator subsheaf of \mathcal{F} under the natural pairing of Ω_π and Θ_π . The sheaf $\mathcal{E}xt^1(\Omega_\pi/\bar{\omega}(\mathcal{F}), \pi^*\Omega_{\Delta_1})$ has as stalks the local $\mathcal{E}xt^1$ discussed in section 2, and it has support on $Sing(\bar{\omega})$. The map ρ associates to an extension e the local invariants $\rho(e)_p$ that measure the non-triviality of the extensions at the points p of M . The kernel of ρ consists of those extensions that are locally a direct sum at every point of M , and are parametrized by $H^1(M, \mathcal{T})$.

5. Unfoldings of holomorphic foliations by curves

In this section we obtain a criterium for a foliation by curves to have only trivial unfoldings.

Definition 5.1. A family of foliations by curves in $\pi : M \rightarrow \Delta_1 \subset \mathbb{C}$ may be defined by a holomorphic map $X : \mathcal{L} \rightarrow \Theta_\pi$ from an invertible sheaf \mathcal{L} on M to the relative tangent bundle Θ_π , such that X does not vanish identically along a π -fibre. The coherent sheaf \mathcal{F} defined as the kernel of the sheaf map obtained by dualising X :

$$(5.1) \quad 0 \rightarrow \mathcal{F} \xrightarrow{\bar{\omega}} \Omega_\pi \xrightarrow{X^*} \mathcal{L}^*$$

defines the family of foliations associated to X , in the sense of definition 3.1. By an *unfolding of the family of foliations by curves X* , we understand an unfolding of $\bar{\omega}$. An *unfolding of a foliation by curves $X_0 : \mathcal{L}_0 \rightarrow \Theta_{M_0}$* in the manifold M_0 is an unfolding of any family of foliations by curves $X : \mathcal{L} \rightarrow \Theta_\pi$ on M , $\pi : M \rightarrow \Delta_1 \subset \mathbb{C}$ a smooth holomorphic map such that $\pi^{-1}(0) = M_0$ and $X|_{M_0} = X_0$. If M_0 is compact, we assume that the map π is proper.

The *singular set S* of the family of foliations by curves defined by (5.1) is the subspace of M defined locally by the ideal $(X_1, \dots, X_n) \subset \mathcal{O}_U$, where

$$(5.2) \quad X = \sum_{j=1}^n X_j(t, z) \frac{\partial}{\partial z_j}$$

is an expression of X in local coordinates of $U \subset M$, after having trivialized \mathcal{L} , $\mathcal{L} = \mathcal{O}_U$. We may form two short exact sequences from (5.1):

$$(5.3) \quad 0 \rightarrow \mathcal{F} \xrightarrow{\bar{\omega}} \Omega_\pi \rightarrow \Omega_\pi/\bar{\omega}(\mathcal{F}) \rightarrow 0$$

and

$$(5.4) \quad 0 \rightarrow \Omega_\pi/\bar{\omega}(\mathcal{F}) \rightarrow \mathcal{L}^* \rightarrow \text{Cok} X^* \rightarrow 0$$

where $\Omega_\pi/\bar{\omega}(\mathcal{F}) = \text{Im}X^*$ is locally isomorphic to the ideal generated by $(X_1, \dots, X_n) \subset \mathcal{O}_U = \mathcal{L}|_U$, where we are using the expression (5.2). $\text{Cok}X^*$ is a sheaf supported on S .

Recall that a complex analytic subspace $V \subset M$ in the complex manifold M has *depth* at least r at p if we may find germs f_1, \dots, f_r of holomorphic functions at p belonging to the ideal defining V such that they form a regular sequence, namely for $j = 2, \dots, r$ we have that f_j is not a divisor of zero in $\mathcal{O}_p/(f_1, \dots, f_{j-1})$, (see [G-H], p. 688, or [H], p. 36).

Theorem 5.2. *Let $X : \mathcal{L} \rightarrow \Theta_\pi$ define a family of holomorphic foliations by curves $\bar{\omega} : \mathcal{F} \rightarrow \Omega_\pi$ in $\pi : M \rightarrow \Delta_1 \subset \mathbb{C}$. If the singular set $S = \{X = 0\}$ has depth at least 3 in M , then:*

$$(5.5) \quad \text{Ext}^1(\Omega_\pi/\bar{\omega}(\mathcal{F}), \pi^*\Omega_{\Delta_1}) = 0$$

and

$$(5.6) \quad \text{Ext}^1(M; \Omega_\pi/\bar{\omega}(\mathcal{F}), \pi^*\Omega_{\Delta_1}) = H^1(M, \mathcal{L})$$

If we further assume that π is a proper map and $H^1(M_0, \mathcal{L}_0) = 0$, where \mathcal{L}_0 is the restriction of \mathcal{L} to $M_0 = \pi^{-1}(0)$, then every unfolding of X is trivial.

Proof: Let U be an open subset of M where the foliation is defined by the vector field X as in (5.2). Denote by \mathcal{I} the sheaf of ideals on U generated by (X_1, \dots, X_n) . The hypothesis that S has depth at least 3 is equivalent (see [H], p.39) that for $j = 0, 1, 2$ we have on U

$$\text{Ext}_{\mathcal{O}_U}^j(\mathcal{O}_U/\mathcal{I}, \mathcal{O}_U) = 0.$$

Consider the long exact sequence of Ext -sheaves associated to the functor $\text{Hom}(*, \mathcal{O}_U)$ applied to

$$0 \rightarrow \mathcal{I} \rightarrow \mathcal{O}_U \rightarrow \mathcal{O}_U/\mathcal{I} \rightarrow 0.$$

From it, and the above observation, we deduce that $\text{Ext}^1(\mathcal{I}, \mathcal{O}_U) = 0$. This proves (5.5). We deduce now (5.6) from (5.5) and (4.3).

The last assertion follows from (5.6), since by upper-semicontinuity of cohomology for flat families of sheaves, (see [H-2], p. 288 and 250) and the hypothesis $H^1(M_0, \mathcal{L}_0) = 0$ implies $H^1(M, \mathcal{L}) = 0$. ■

Theorem 5.3. *Let $X_0 : \mathcal{L}_0 \rightarrow \Theta_{M_0}$ be a foliation by curves with isolated singularities in the compact manifold M_0 . Assume that M_0 has dimension at least 3 and $H^1(M_0, \mathcal{L}_0) = 0$; then every unfolding of X_0 is trivial.*

Proof: Let $X : \mathcal{L} \rightarrow \Theta_\pi$ be a family of foliations by curves that extend X_0 , and let X_t be its restriction to M_t . The singular set S of X has dimension 1, since its intersection with the fibres M_t is a finite number of points ([G-H], p. 662). Hence S is locally a complete intersection, and so Theorem 5.2 applies. ■

As a particular case we obtain:

Corollary 5.4. *Let X_0 be a homogeneous polynomial vector field inducing a foliation by curves in the projective space $\mathbb{C}P^n$, $n \geq 3$, with isolated singularities; then any unfolding of X_0 is trivial.*

Proof: By a Theorem of Serre ([H-2], p. 225) for any invertible sheaf \mathcal{L}_0 on $\mathbb{C}P^n$, $n \geq 2$, we have $H^1(\mathbb{C}P^n, \mathcal{L}_0) = 0$. Hence Corollary 5.3 applies. ■

We say that a singularity p of a foliation by curves $X_0 : \mathcal{L}_0 \rightarrow \Theta_{M_0}$ in M_0 has *non-vanishing trace* if the trace of the linear part at p of a vector field Y_0 defining the foliation in a neighbourhood of p is non-zero; i.e., if $Y_0 = \sum_{j=1}^n Y_j(z) \frac{\partial}{\partial z_j}$, then $\sum_{j=1}^n \frac{\partial Y_j}{\partial z_j}(p_0) \neq 0$. We also say that the singularity p above has *multiplicity 1* if the ideal generated by Y_1, \dots, Y_n in the local ring $\mathcal{O}_{M_0, p}$ of germs of holomorphic functions on M_0 at p is the maximal ideal.

Theorem 5.5. *Let $X_0 : \mathcal{L}_0 \rightarrow \Theta_{M_0}$ be a foliation by curves with isolated singularities in the compact complex 2-dimensional manifold M_0 , such that every singular point has multiplicity 1, non-vanishing trace, and we have $H^1(M_0, \mathcal{L}_0) = 0$; then every unfolding of X_0 is trivial.*

We will first prove the following:

Lemma 5.6. *Let X_0 be a germ of a holomorphic vector field in $(\mathbb{C}^2, 0)$, having 0 as a critical point of multiplicity 1 and having non-vanishing trace. Then, any germ of an unfolding of X_0 as a foliation is trivial.*

Proof: A germ of a family of foliations by curves may be described by a germ of a family of vector fields $X_t = B(z_1, z_2, t) \frac{\partial}{\partial z_1} - A(z_1, z_2, t) \frac{\partial}{\partial z_2}$. Since by hypothesis X_0 has multiplicity 1, we may after a change of coordinates assume that 0 is the only singular point of X_t and it has multiplicity 1. This may also be stated by saying that the ideal generated by A and B in $\mathcal{O}_{\mathbb{C}^3, 0}$ is (z_1, z_2) .

A germ of an unfolding of the family of foliations X_t may be described by a germ of a holomorphic 1-form $\omega = A(z_1, z_2, t) dz_1 + B(z_1, z_2, t) dz_2 + C(z_1, z_2, t) dt$ which satisfies the integrability condition $\omega \wedge d\omega = 0$. Writing out the integrability condition, we obtain:

$$\omega \wedge d\omega = \left[A \left(-\frac{\partial B}{\partial t} + \frac{\partial C}{\partial z_2} \right) - B \left(-\frac{\partial A}{\partial t} + \frac{\partial C}{\partial z_1} \right) + C \left(-\frac{\partial A}{\partial z_2} + \frac{\partial B}{\partial z_1} \right) \right] dz_1 \wedge dz_2 \wedge dt = 0$$

Restricting now to the singular set S of the family of foliations X_t , defined by $A = B = 0$, we obtain on S the equality $C \left(-\frac{\partial A}{\partial z_2} + \frac{\partial B}{\partial z_1} \right)$. Observe that the factor within the parenthesis is the trace of X_t , and since we are assuming that it is non-zero, we obtain that $C = 0$ restricted to S . In terms of ideals, this means that C is a section of the ideal defining S (since S is reduced), so that we may find germs of holomorphic functions P and Q in $(\mathbb{C}^3, 0)$ such that $C = PA + QB$. Now, define a germ of a holomorphic vector field in $(\mathbb{C}^3, 0)$ by

$Y = -P \frac{\partial}{\partial x_1} - Q \frac{\partial}{\partial x_2} + \frac{\partial}{\partial t}$. We then have that $\omega(Y) = 0$, which means that Y is a vector field tangent to the unfolding. Arguing now as we did in Theorem 3.3, we deduce that the germ of the unfolding is trivial. This proves the Lemma. ■

Proof of the Theorem: Let ω be an unfolding of the foliation by curves X_0 . By Theorem 4.1, this unfolding is trivial if and only if the associated extension class e in $\text{Ext}^1(M; \Omega_\pi/\bar{\omega}(\mathcal{F}), \pi^*\Omega_{\Delta_1})$ is 0. We use the sequence (4.3) to show that it is 0.

First, $\rho(e) \in H^0(M; \text{Ext}^1(\Omega_\pi/\bar{\omega}(\mathcal{F}), \pi^*\Omega_{\Delta_1}))$ is 0, since by Lemma 5.6 the unfolding is locally trivial as a germ at every singular point, and this section measures exactly this local obstructions. But then by the exactness of (4.3) we have that e is in the image of $H^1(M, \mathcal{T})$. By the upper semicontinuity of cohomology groups in flat families (see [H-2], p. 288) we have $H^1(M_t, \mathcal{L}_t) = 0$, and then using the degenerate spectral sequence associated to the higher direct image functors of π (see [H-2], p.250), we have that $H^1(M, \mathcal{T}) = 0$. This then shows that $e = 0$, and so the unfolding is trivial. This proves the Theorem. ■

Corollary 5.7. *Let X_0 be a homogeneous polynomial vector field inducing a foliation by curves in the projective plane $\mathbb{C}P^2$, with only singularities of multiplicity 1 and non-vanishing trace; then any unfolding of X_0 is trivial.*

Proof: By a Theorem of Serre ([H-2], p. 225) for any invertible sheaf \mathcal{L}_0 on $\mathbb{C}P^2$, we have $H^1(\mathbb{C}P^2, \mathcal{L}_0) = 0$. Hence Theorem 5.5 applies. ■

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CIMAT, A.P. 402
Guanajuato 36000
MÉXICO

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