Unhinging the Surfaces of Higher-Order Topological Insulators and Superconductors

Apoorv Tiwari,[1,2] Ming-Hao Li,[3] B. A. Bernevig,[4] Titus Neupert,[1] and S. A. Parameswaran[5]
[1]Department of Physics, University of Zurich, Winterthurerstrasse 190, 8057 Zurich, Switzerland
[2]Condensed Matter Theory Group, Paul Scherrer Institute, CH-5232 Villigen PSI, Switzerland
[3]Department of Physics, ETH Zurich, 8093 Zurich, Switzerland
[4]Department of Physics, Princeton University, Princeton, New Jersey 08540, USA
[5]Rudolf Peierls Centre for Theoretical Physics, Clarendon Laboratory, University of Oxford, Oxford OX1 3PU, United Kingdom

(Received 29 July 2019; published 27 January 2020)

We show that the chiral Dirac and Majorana hinge modes in three-dimensional higher-order topological insulators (HOTIs) and superconductors (HOTSCs) can be gapped while preserving the protecting $\mathsf{C}_{2n}\mathcal{T}$ symmetry upon the introduction of non-Abelian surface topological order. In both cases, the topological order on a single side surface breaks time-reversal symmetry, but appears with its time-reversal conjugate on alternating sides in a $\mathsf{C}_{2n}\mathcal{T}$ preserving pattern. In the absence of the HOTI/HOTSC bulk, such a pattern necessarily involves gapless chiral modes on hinges between $\mathsf{C}_{2n}\mathcal{T}$-conjugate domains. However, using a combination of K-matrix and anyon condensation arguments, we show that on the boundary of a 3D HOTI/HOTSC these topological orders are fully gapped and hence "anomalous." Our results suggest that new patterns of surface and hinge states can be engineered by selectively introducing topological order only on specific surfaces.

DOI: 10.1103/PhysRevLett.124.046801

Introduction.—A defining aspect of topological phases of matter is the bulk-boundary correspondence. This predicts the existence of gapless excitations on the boundary of an insulating phase from the bulk electronic structure alone, irrespective of boundary details. Initially, it was believed that the correspondence inevitably requires gapless surface excitations as long as system and boundary both respect the protecting symmetries of the bulk topological phase. This would imply, for example, that a three-dimensional (3D) electronic topological insulator (TI), protected by time-reversal ($\mathcal{T}$) and $\mathsf{U}(1)$ charge conservation symmetry always hosts a surface Dirac fermion if $\mathcal{T} \ltimes \mathsf{U}(1)$ is respected. However, there is another possibility [1–9]: the 3D TI surface can be fully gapped with $\mathcal{T} \ltimes \mathsf{U}(1)$ symmetry intact, if it hosts a topologically ordered state [10–15], i.e., an intrinsically interacting phase with emergent fractionalized excitations. Thus, the complete bulk-boundary correspondence for a 3D TI states that a symmetry-preserving surface *either* carries a gapless Dirac fermion *or* the appropriate surface topological order (STO). Both these surface terminations cancel the bulk anomaly arising from the $\mathbf{E} \cdot \mathbf{B}$ electromagnetic response, although only the former has been experimentally observed. As a corollary, the STO cannot be realized with the same symmetries in a purely 2D system. This generalized bulk-boundary correspondence also applies to other 3D topological phases such as $\mathcal{T}$-symmetric topological superconductors (TSCs) [16–23].

A different type of bulk-boundary correspondence emerges in higher-order topological insulators and superconductors (HOTIs/HOTSCs) [24–37]. These bulk-gapped phases of matter carry topologically protected boundary modes on corners or hinges, instead of surfaces (in 3D). Such protection requires a spatial symmetry that maps between patches of the surface, making the interplay of topology and crystal symmetry [38–43] central to the study of HOTIs/HOTSCs.

In this Letter, we generalize the higher-order bulk-boundary correspondence to include the possibility of STO. Specifically, we study 3D topological insulators and superconductors with chiral hinge modes—the HOTI/HOTSC analogs of integer quantum Hall states or $\mathsf{p} + i\mathsf{p}$ superconductors. For concreteness, we consider cases where the protecting symmetry is $\mathsf{C}_{2n}\mathcal{T}$, i.e., the product of a $(2n)$-fold rotation and time-reversal $\mathcal{T}$. In other words, $\mathcal{T}$ and C_{2n} are individually broken but their product remains unbroken. (Here n is a positive integer, and $n \leq 3$ for any 3D space group). Nontrivial HOTI/HOTSC phases with these symmetries support chiral fermionic modes on each of $2n$ hinges in a C_{2n}-symmetric geometry with open boundary conditions in the rotation plane. Such phases have a $\mathbb{Z}_2$ topological classification: while a single chiral fermionic mode is stable and symmetry protected in the noninteracting limit, *two* chiral Dirac or Majorana modes on each hinge can be gapped out by pasting copies of the integer quantum Hall phase with $\nu = \pm 1$ (for the HOTI) or $\mathsf{p} \pm i\mathsf{p}$ 2D topological superconductors (for the HOTSC) in alternating fashion on the surfaces while preserving $\mathsf{C}_{2n}\mathcal{T}$ symmetry. It is natural to ask: can these modes be gapped while preserving symmetry in an interacting system?

We answer this question in the affirmative by constructing symmetry-preserving STOs that "unhinge" the gapless modes on the HOTI/HOTSC surfaces. In the HOTI case, we leverage the K-matrix formulation of coupled Luttinger liquids to show that the hinge is gapped. For the HOTSC we cannot use this method, but instead map the question to an auxiliary anyon condensation problem. We close with a discussion of why the resulting $\mathsf{C}_{2n}\mathcal{T}$ STOs we construct are *anomalous*—in that they can be fully gapped only on the surface of a HOTI/HOTSC—and identify directions for future work.

Higher order TI.—We begin by constructing a symmetry-preserving STO for the $\mathsf{C}_{2n}\mathcal{T}$ HOTI. Since we are discussing insulators, in addition to $\mathsf{C}_{2n}\mathcal{T}$ we must impose $\mathsf{U}(1)$ charge conservation symmetry (implicit in the noninteracting classification [26]), otherwise the hinge could be simply gapped by depositing $p \pm ip$ superconductors on alternating surfaces. Each fermionic hinge mode carries $\mathsf{U}(1)$ electric charge $q = +1$ (in units of the electron charge e) and has chiral central charge $c_- = 1$ [44,45]. These, respectively, quantify the chiral hinge transport of charge and heat. In order to respect $\mathsf{C}_{2n}\mathcal{T}$ symmetry, we must impose an alternating pattern of topological order $\mathcal{A}$ and its $\mathcal{T}$-conjugate $\bar{\mathcal{A}}$ on adjacent side surfaces; however, the STO on the top and bottom surface (that we denote $\mathcal{A}_\mathcal{T}$) should preserve $\mathsf{C}_{2n}\mathcal{T}$. In order for the side STOs to cancel the contribution of the hinge, the Hall conductance $\sigma_{xy}^{\mathcal{A}} = -\sigma_{xy}^{\bar{\mathcal{A}}} = \frac{1}{2}$ in units of e^2/h and the chiral central charge $c_-^{\mathcal{A}} = -c_-^{\bar{\mathcal{A}}} = \frac{1}{2}$. Thus, $\mathcal{A}$, $\bar{\mathcal{A}}$ must be chiral and non-Abelian. The same constraints emerge when constructing STO for TIs [5,7], where a close cousin of the Pfaffian topological order [46–48] known as the $\mathcal{T}$ Pfaffian was constructed. Notably, as it has $c_- \neq 0$ the $\mathcal{T}$ Pfaffian necessarily breaks $\mathcal{T}$ when realized in a purely 2D system, but it can preserve $\mathcal{T}$ on the 2D surface of a 3D TI [7].

A fully gapped surface termination for the HOTI can be constructed by taking the top or bottom STO $\mathcal{A}_\mathcal{T}$ to be the $\mathcal{T}$ Pfaffian, and the side STO $\mathcal{A}$ to be the 2D $\mathcal{T}$-breaking phase with chiral edge modes that has the same anyon content as the $\mathcal{T}$ Pfaffian, and $\bar{\mathcal{A}}$ the $\mathcal{T}$ conjugate of $\mathcal{A}$. To motivate this choice, we note that the free-fermion $\mathsf{C}_{2n}\mathcal{T}$ HOTI emerges upon introducing $\mathcal{T}$-breaking gaps (denoted $m_\pm$, where the sign indicates that of the $\mathcal{T}$ breaking) on alternating surfaces of a first-order TI in a $\mathsf{C}_{2n}\mathcal{T}$-preserving manner [Fig. 1(a) depicts a $\mathsf{C}_2\mathcal{T}$ example]. The top and bottom surfaces then each host a single 2D Dirac fermion. By imposing $\mathcal{A}_\mathcal{T}$ on the top and bottom surfaces we gap out the surface Dirac fermion while preserving $\mathsf{C}_{2n}\mathcal{T}$; however, this introduces modes with $|c_-| = |q| = 1/2$ on the top and bottom hinges between $\mathcal{A}_\mathcal{T}$ and $m_\mp$, which combine with the side hinges in a "wire frame" pattern [Fig. 1(b)]. The edges between the $\mathcal{T}$ Pfaffian and the time-reversal-breaking region $m_\pm$ are, respectively, identical to those between its 2D analogs $\mathcal{A}, \bar{\mathcal{A}}$ and vacuum [5].

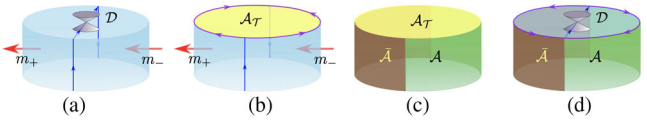

FIG. 1. Possible surface terminations of $\mathsf{C}_2\mathcal{T}$ HOTI/HOTSC. (a) The underlying free-fermion phase has $\mathcal{T}$-breaking surface gaps $m_\pm$ in a $\mathsf{C}_2\mathcal{T}$ pattern, leading to chiral modes on side hinges and a 2D Dirac or Majorana fermion $\mathcal{D}$ stabilized by $\mathsf{C}_2\mathcal{T}$ on the gapless top surface. (b) Non-Abelian STO $\mathcal{A}_\mathcal{T}$ only on the top surface leads to a wire frame of chiral modes on all hinges. (c) Adding $\mathcal{T}$-breaking 2D analogs $\mathcal{A}, \bar{\mathcal{A}}$ of $\mathcal{A}_\mathcal{T}$ on $\mathsf{C}_2\mathcal{T}$-related sides fully gaps the boundary. (d) Non-Abelian STO only on the sides yields chiral edge modes coexisting with a 2D Dirac or Majorana fermion on the top surfaces. (Bottom surfaces follow a similar pattern, omitted for clarity).

Accordingly, we may gap the top and bottom hinges by adding $\mathcal{A}, \bar{\mathcal{A}}$ to the m_- and m_+ surfaces, respectively, as this yields the necessary pattern of counterpropagating modes. Finally, the boundary between $\mathcal{A}$, $\bar{\mathcal{A}}$ oriented as in Fig. 1(c) carries $c_- = q = -1$, which cancels the side hinges. (We can shrink gapless top and bottom regions to a set of 1D chiral modes that slice across them, while preserving $\mathsf{C}_{2n}\mathcal{T}$. For $n = 1$ this leaves one chiral mode that encircles the sample, and the analysis is just that for the side hinge. For $n > 1$ the surface chiral mode pattern is more complicated. Introducing $\mathcal{A}_\mathcal{T}$ makes our approach n independent.)

Before explicitly verifying the hinge gapping, we review some properties of the $\mathcal{T}$ Pfaffian and its 2D $\mathcal{T}$-breaking analogs. These all have identical bulk anyon content: a subset of the product of topological quantum field theories (TQFTs) $\mathsf{U}(1)_8 \times \overline{\mathrm{Ising}}$ with anyon types $1_j, \psi_j$ (with $j = 0, 2, 4, 6$) and σ_j (with $j = 1, 3, 5, 7$), and braiding and fusion rules derived from the direct product theory [49]. This is a spin TQFT [59–61] containing a charge 1 "transparent" fermion, ψ_4 that braids trivially with all other particles. In conventional TQFTs, such particles are identified with vacuum, but this is precluded here as ψ_4 is a fermion; instead it is identified with the physical electron. The vacuum of a spin TQFT is "graded" by fermion parity, meaning that only those anyons in $\mathsf{U}(1)_8 \times \overline{\mathrm{Ising}}$ that braid trivially with ψ_4 are retained (see Table I). A TQFT with these anyons is necessarily chiral and can be realized in a $\mathcal{T}$-preserving

TABLE I. Anyons a in the $\mathcal{T}$ Pffaffian and 2D analogs and their topological spin $e^{i\theta_a}$, $\mathsf{U}(1)$ charge Q_a (units of e), time-reversal partner $\mathcal{T}_a$, and "Kramers sign" $\mathcal{T}_a^2$ (where applicable).

$a \to$	1_0	ψ_0	1_2	ψ_2	1_4	ψ_4	1_6	ψ_6	σ_1	σ_3	σ_5	σ_7
$e^{i\theta_a}$	1	-1	$-i$	i	1	-1	$-i$	i	1	-1	-1	1
Q_a	0	0	1/2	1/2	1	1	3/2	3/2	1/4	3/4	5/4	7/4
$\mathcal{T}(a)$	1_0	ψ_0	ψ_2	1_2	1_4	ψ_4	ψ_6	1_6	σ_1	σ_3	σ_5	σ_7
$\mathcal{T}_a^2$	1	1			-1	-1			1	-1	-1	1

manner only on the surface of a 3D TI, where it is termed the $\mathcal{T}$ Pfaffian (our choice of $\mathcal{A}_\mathcal{T}$). On the 3D TI surface, $\mathcal{T}$ interchanges $1_2 \leftrightarrow \psi_2$ and $1_6 \leftrightarrow \psi_6$, and squares to -1 on ψ_4; all other anyons are $\mathcal{T}$ invariant [62]. While $\mathcal{A}_\mathcal{T}$ cannot have an edge with vacuum, it has a chiral edge with $\mathcal{T}$-breaking regions $m_\pm$ on the TI surface.

$\mathcal{T}$-breaking TQFTs with identical anyon content *can* be realized in two dimensions with chiral edges to vacuum: these are the 2D analogs $\mathcal{A}, \bar{\mathcal{A}}$ of the $\mathcal{T}$ Pfaffian. The edges all share the same Lagrangian [5],

$$\mathcal{L}^{\mathsf{a}}_{\pm} = \frac{2}{4\pi}\partial_x\phi^{\mathsf{a}}(\partial_t \mp v\partial_x)\phi^{\mathsf{a}} + i\psi^{\mathsf{a}}(\partial_t \pm v'\partial_x)\psi^{\mathsf{a}}, \quad (1)$$

consisting of a chiral $\mathsf{U}(1)$ boson ϕ^{a} and a counterpropagating chiral Majorana fermion ψ^{a}, where $\pm$ denotes the sign of both c_- and q. (We adopt a Lagrangian description to conveniently describe chiral modes.) We label edge fields between $\mathcal{A}$, $\bar{\mathcal{A}}$ and vacuum by $\mathsf{a} = \mathsf{A}, \bar{\mathsf{A}}$, and those between the $\mathcal{T}$-breaking side surfaces $m_\pm$ and $\mathcal{A}_\mathcal{T}$ by $\mathsf{a} = \mathsf{m}_\pm$. Additionally we enforce a $\mathbb{Z}_2^{\mathsf{a}}$ gauge symmetry $\psi^{\mathsf{a}} \mapsto -\psi^{\mathsf{a}}$, $\phi^{\mathsf{a}} \mapsto \phi^{\mathsf{a}} + (\pi/2)$, which identify $\psi^{\mathsf{a}}e^{-2i\phi^{\mathsf{a}}}$ as the edge electron operator [23,49]. Any top/bottom hinge is a 'composite' of the edges between $\mathcal{A}$ (or $\bar{\mathcal{A}}$) and vacuum, and between $\mathcal{A}_\mathcal{T}$ and m_- (or m_+), and is hence described by $\mathcal{L}^{\mathsf{A}}_s + \mathcal{L}^{\mathsf{m}_-}_{-s}$ (or $\mathcal{L}^{\bar{\mathsf{A}}}_s + \mathcal{L}^{\mathsf{m}_+}_{-s}$), with $s = \pm$. The two theories in each sum are mutually $\mathcal{T}$-conjugate (i.e., acting with $\mathcal{T}$ on one yields the other), so $c_- = q = 0$, and can be gapped without breaking $\mathsf{U}(1)$ symmetry. At each side hinge, the bulk HOTI contributes a chiral mode

$$\mathcal{L}^{h}_{\pm} = \frac{1}{4\pi}\partial_x\varphi(\partial_t \mp u\partial_x)\varphi. \quad (2)$$

We next observe that the effective Lagrangian at a single side hinge [see Fig. 2(a)] that includes the chiral modes from both the HOTI bulk and from $\mathcal{A}, \bar{\mathcal{A}}$ takes the form $\mathcal{L} = \mathcal{L}^{\mathsf{A}}_- + \mathcal{L}^{\bar{\mathsf{A}}}_+ + \mathcal{L}^{h}_-$. Since $\mathsf{A}, \bar{\mathsf{A}}$ are $\mathcal{T}$ conjugates, $\mathcal{L}^{\mathsf{A}}_- + \mathcal{L}^{\bar{\mathsf{A}}}_+$ is really just two copies of $\mathcal{L}^{\mathsf{A}}_-$. The two Majorana modes therefore copropagate with each other and with the hinge mode φ, but counterpropagate relative to the chiral boson fields $\phi^{\mathsf{A}}, \phi^{\bar{\mathsf{A}}}$. Therefore, we may combine $\psi^{\mathsf{A}}, \psi^{\bar{\mathsf{A}}}$ into a single chiral Dirac fermion, that we then bosonize into a compact chiral neutral boson via $e^{i\phi} \sim \psi^{\mathsf{a}} + i\psi^{\bar{\mathsf{a}}}$. This series of manipulations recasts the edge as a coupled Luttinger-liquid theory [10,11] described by the K-matrix $K = \mathrm{diag}(1, -2, -2, 1)$ in the boson basis $\Phi \coloneqq (\phi, \phi^{\mathsf{A}}, \phi^{\bar{\mathsf{A}}}, \varphi)^{\mathsf{T}}$, where the coefficients follow from Eqs. (1) and (2). The $\mathsf{U}(1)$ electric charges of the boson fields are captured by the vector $q = (0, 1, 1, 1)^{\mathsf{T}}$. The combined theory has vanishing Hall conductance $\sigma_{xy} = q^{\mathsf{T}}K^{-1}q = 0$, and the chiral central charge $c_- = \mathrm{signature}(K) = 0$, meaning there is no immediate obstruction (i.e., due to Hall or thermal Hall responses) to gapping the hinge theory $\mathcal{L}$. We do so by adding $\Delta\mathcal{L} = \sum_i \lambda_i \cos[\ell_i^{\mathsf{T}}\Phi + \alpha_i]$ and driving all the λ_i to strong coupling [8,63–65]. The combination of fields $\ell_i^{\mathsf{T}}\Phi$ must (i) correspond to bosonic nonchiral edge operators which is true if $\ell_i^{\mathsf{T}}K^{-1}\ell_i = 0$; (ii) be nonfractional, i.e., $\ell_i \in K\mathbb{Z}^4$; (iii) be charge neutral so that the gapped phase preserves $\mathsf{U}(1)$, requiring $\ell_i^{\mathsf{T}}K^{-1}q = 0$. Finally the $\mathbb{Z}_2^{\mathsf{A}} \times \mathbb{Z}_2^{\bar{\mathsf{A}}}$ gauge symmetry must also be satisfied. First, we condense $\ell_1 = (0, 4, 4, 4)^{\mathsf{T}}$; this locks the two independent gauge transformations to act together as $\phi \mapsto \phi + \pi$, $\phi^{\mathsf{A}(\bar{\mathsf{A}})} \mapsto \phi^{\mathsf{A}(\bar{\mathsf{A}})} \pm \pi/2$ [49]. This lets us condense $\ell_2 = (2, 2, -2, 0)^{\mathsf{T}}$ which is invariant under this unbroken subgroup of $\mathbb{Z}_2^{\mathsf{A}} \times \mathbb{Z}_2^{\bar{\mathsf{A}}}$. Since $\ell_{1,2}$ satisfy all the above criteria and $\ell_1^{\mathsf{T}}K^{-1}\ell_2 = 0$, they can simultaneously flow to strong coupling, leading to a symmetric, gapped, nondegenerate edge.

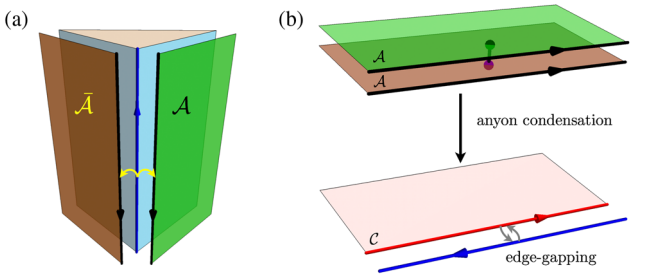

FIG. 2. Constructing symmetry-preserving gapped side surfaces of a $\mathsf{C}_{2n}\mathcal{T}$-HOTI/HOTSC. (a) 2D STO $\mathcal{A}$ and its $\mathcal{T}$-conjugate $\bar{\mathcal{A}}$ are imposed on alternate surfaces, so that their edge modes combine with the hinge mode yield a fully gapped boundary. $\mathcal{A}$, $\bar{\mathcal{A}}$ are 2D analogs of the $\mathcal{T}$-symmetric STO $\mathcal{A}_\mathcal{T}$ for the "parent" first-order HOTI/HOTSC. (b) By folding across the hinge, this can also be viewed as the process of condensing some subset of anyons in $\mathcal{A} \times \mathcal{A}$ to yield a chiral topological phase with no bulk anyons ($\mathcal{C}$) whose edge mode then gaps out the hinge mode of the bulk HOTI/HOTSC.

Higher order TSC.—We now consider the $\mathsf{C}_{2n}\mathcal{T}$-symmetric HOTSC that hosts an alternating pattern of $c_- = \frac{1}{2}$ Majorana hinge modes. In analogy with the HOTI, to construct an STO we should start with the "parent" first-order topological phase, namely, the $\nu = 1$ class DIII TSC, whose surface hosts a single Majorana cone in the free-fermion limit. However, the STO for this phase is complicated [23]. A simpler route is to recognize that only the *parity* of ν is relevant to the $\mathsf{C}_{2n}\mathcal{T}$-HOTSC: we can change hinge chiral central charge in multiples of $1/2$ by gluing $\mathsf{p} \pm i\mathsf{p}$ superconductors to alternating side surface in a $\mathsf{C}_{2n}\mathcal{T}$-preserving manner (i.e., it suffices that $c_-^{\mathcal{A}} = -c_-^{\bar{\mathcal{A}}} = \frac{1}{4} \bmod \frac{1}{2}$). Since a pure surface perturbation changes $\nu \to \nu + 2$, we can instead consider a related $\mathsf{C}_{2n}\mathcal{T}$ HOTSC obtained by decorating the $\nu = 3$ DIII first-order TSC with $\mathcal{T}$-breaking domains $\mathsf{m}_\pm$ on side surfaces, yielding a chiral hinge mode with three Majoranas ($|c_-| = 3/2$). The $\nu = 3$ STO in class DIII is the

$\mathsf{SO}(3)_6$ TQFT, which may be viewed as the integer spin sector of the $\mathsf{SO}(3)_6$ theory [16,20]. Similar reasoning as in the HOTI case suggests that we should take this as the topological order $\mathcal{A}_\mathcal{T}$ for the top or bottom surfaces, and then pattern its 2D $\mathcal{T}$-breaking analogs $\mathcal{A}$ and $\bar{\mathcal{A}}$ in a $\mathsf{C}_{2n}\mathcal{T}$-preserving fashion on the side surface. It will be convenient to also glue three copies of $\mathsf{p}+i\mathsf{p}$ superconductors in a $\mathsf{C}_{2n}\mathcal{T}$-preserving pattern on the side surfaces. We now show that the side hinge is gapped; then, by Kirchoff's law for edge modes, we can infer that the top and bottom hinges are gapped. A single $\mathsf{SO}(3)_6$ edge is described by a chiral Wess-Zumino-Witten theory with $c_- = 9/4$, so the side hinge is more complicated and unlike the HOTI case cannot be rewritten in terms of chiral bosons. Therefore, we cannot use the K-matrix approach and need some other strategy to proceed. One route is via "conformal embedding" [23]. Here we instead use anyon condensation to infer the edge structure.

We first impose periodic boundary conditions along the C_{2n} axis, to focus only on the alternating pattern of side STOs $\mathcal{A}, \bar{\mathcal{A}}$. The question of gappability now reduces to (i) determining the hinge mode between $\mathcal{T}$-conjugate topological orders $\mathcal{A}, \bar{\mathcal{A}}$, and (ii) showing that it can gap the hinge modes contributed by the combination of the bulk HOTI and the 3 additional $\mathsf{p}\pm i\mathsf{p}$ states decorating the side surfaces. Step (i) may be further simplified by "folding" $\bar{\mathcal{A}}$ across the hinge which maps the boundary between $\mathcal{A}$ and $\bar{\mathcal{A}}$ to an edge between $\mathcal{A}\times\mathcal{A}$ and the vacuum [see Fig. 2(b)]. We can infer the minimal edge theory by condensing a maximal subset of anyons in the bulk of the folded theory $\mathcal{A}\times\mathcal{A}$.

We first validate this approach for the HOTI. We denote anyons in $\mathcal{A}\times\mathcal{A}$ by elements in the set $\{1_j^{\mathsf{A}}, \psi_j^{\mathsf{A}}, \sigma_j^{\mathsf{A}}\} \times \{1_j^{\bar{\mathsf{A}}}, \psi_j^{\bar{\mathsf{A}}}, \sigma_j^{\bar{\mathsf{A}}}\}$ (see Table I; we label anyons in the second copy of $\mathcal{A}$ by $\bar{\mathsf{A}}$, to indicate their origin in $\bar{\mathcal{A}}$ before folding). Following Ref. [7], we perform a two-step condensation procedure. First, we condense the bosons $\{1_2^{\mathsf{A}}\psi_6^{\bar{\mathsf{A}}}, \psi_2^{\mathsf{A}}1_6^{\bar{\mathsf{A}}}, \psi_6^{\mathsf{A}}1_2^{\bar{\mathsf{A}}}, 1_6^{\mathsf{A}}\psi_2^{\bar{\mathsf{A}}}, 1_4^{\mathsf{A}}1_4^{\bar{\mathsf{A}}}, \psi_0^{\mathsf{A}}\psi_0^{\bar{\mathsf{A}}}, \psi_4^{\mathsf{A}}\psi_4^{\bar{\mathsf{A}}}\}$. This confines all sectors in $\mathcal{A}\times\mathcal{A}$ whose topological spin is not a good quantum number, leaving only the Abelian anyons $\{1_0^{\mathsf{A}}1_0^{\bar{\mathsf{A}}}, \psi_0^{\mathsf{A}}1_0^{\bar{\mathsf{A}}}, \psi_4^{\mathsf{A}}1_0^{\bar{\mathsf{A}}}, 1_4^{\mathsf{A}}1_0^{\bar{\mathsf{A}}}\} \simeq \{1_0^{\mathsf{A}}, \psi_0^{\mathsf{A}}, \psi_4^{\mathsf{A}}, 1_4^{\mathsf{A}}\}$ and the non-Abelian anyons $\sigma_1^{\mathsf{A}}\sigma_3^{\bar{\mathsf{A}}}$ and $\sigma_1^{\mathsf{A}}\sigma_7^{\bar{\mathsf{A}}}$. Crucially, the non-Abelian anyon sectors split into two Abelian anyons each in the condensed theory. Therefore the condensed theory contains eight Abelian anyons, four of which are charge neutral while the remaining four carry charge $+1$ [49]. The neutral anyons correspond to the toric code topological order [12]. The charged anyons correspond to a copy of the toric code obtained from the neutral anyons by fusing with the physical electron ψ_4^{A}. Next, we condense the "e-particle" in the charge-neutral copy of the toric code. This gaps out the entire theory except for $\{1_0^{\mathsf{A}}, \psi_4^{\mathsf{A}}\}$. The surviving sectors correspond to a bulk theory whose edge has a single chiral fermionic mode with unit $\mathsf{U}(1)$ charge (since $c_- = 1$ is unchanged by condensation). We then use this to gap the counterpropagating hinge mode of the bulk HOTI [49]. Note that no additional surface decorations were needed in this case.

We now turn to the HOTSC case where $\mathcal{A}$ corresponds to the $\mathsf{SO}(3)_6$ TQFT, which contains four anyons labeled $j = \{0,1,2,3\}$ with topological spin $\{+1,+i,-i,-1\}$, respectively. The surface of the 3D class DIII TSC, admits a time-reversal symmetric realization of $\mathsf{SO}(3)_6$ wherein $\mathcal{T}$ exchanges the anyons $j=1$ and $j=2$, leaves $j=0$ invariant, and squares to -1 on $j=3$, which is identified with the physical electron. As in the HOTI case we label the anyons in the folded theory $\mathcal{A}\times\mathcal{A}$ (equivalent to operators on the hinge or domain wall between $\mathcal{A}$ and $\bar{\mathcal{A}}$) by $(j^{\mathsf{A}}, j^{\bar{\mathsf{A}}}) \in \{0,1,2,3\}\times\{0,1,2,3\}$. $\mathcal{A}\times\mathcal{A}$ contains four mutually local bosons with labels $\{(00),(33),(21),(12)\}$. Condensing these four bosons confines all remaining anyons except for $\{(03),(30),(11),(22)\}$. In the condensed theory these are all fermions and may be identified with a single fermionic sector, which we denote $\mathfrak{f}$. We can verify [49] that $\mathfrak{f}$ is neutral and local, i.e., braids trivially with itself. The domain wall between $\mathcal{A}$ and $\bar{\mathcal{A}}$ thus reduces to a local neutral fermion with $c_- = \frac{9}{2}$ (recall condensation preserves c_-). We combine this with the 9 noninteracting Majorana modes ($3+3$ from $\mathsf{p}\pm i\mathsf{p}$ SCs decorating adjacent side surfaces, and 3 from the $\nu = 3$ HOTSC bulk) to fully gap the side hinge.

Discussion.—We have constructed fully gapped $\mathsf{C}_{2n}\mathcal{T}$-preserving STOs for HOTI/HOTSCs, exemplifying the generalized higher-order bulk-boundary correspondence. The STOs are anomalous and cannot be realized in strictly two dimensions. For instance, imposing STO only on the top surface [Fig. 1(b)] yields a chiral mode pattern that is impossible on any orientable 2D manifold, but is consistent on a HOTI surface because of the hinges. Similarly, if we consider the $\mathsf{C}_{2n}\mathcal{T}$-preserving alternating pattern of $\mathcal{T}$-breaking orders on the side surfaces only (with, e.g., periodic boundary conditions along z), we see that in two dimensions these would host gapless modes at every hinge, but these are canceled by those from the bulk when the same pattern is realized on the 3D HOTI/HOTSC side surface. This also gives us insight into the $\mathsf{C}_{2n}\mathcal{T}$-preserving gapless surface state present on the top or bottom surfaces of the HOTI: by gapping only the side surfaces with STOs, we see that the top or bottom surfaces host a chiral Dirac or Majorana in their 2D bulk, but also have a characteristic $\mathsf{C}_{2n}\mathcal{T}$-preserving pattern of edge modes [Fig. 1(d)]; this warrants further study. Junction structures—e.g., the wire frame, where imposing STO only on the top or bottom surfaces yields a symmetric "beam splitter" dividing a noninteracting chiral mode into two intrinsically interacting ones—are natural with the lower symmetry of HOTIs/HOTSCs, offering a promising line of investigation.

Although so far most predicted HOTIs/HOTSCs are weakly interacting, they likely have a rich set of interacting counterparts similar to the topological Kondo and Mott insulators proposed in the first-order case. For example, a natural way to break $\mathcal{T}$ while preserving $\mathsf{C}_{2n}\mathcal{T}$ is to trigger surface magnetic order, which requires interactions. Our results are likely relevant to experiments in the strongly correlated regime where interactions can gap out the hinge modes, leaving only the more subtle signatures of higher-order topology described here. Furthermore, our ideas generalize to analogous higher order symmetry-protected topological phases (HOSPTs) in bosonic or spin systems that lack a "free" limit. For instance, perturbing the bosonic class DIII TSC [1] with time-reversal breaking in a $\mathsf{C}_{2n}\mathcal{T}$-preserving manner yields a bosonic $\mathsf{C}_{2n}\mathcal{T}$ HOSPT. The relevant STO is obtained by taking $\mathcal{A}_{\mathcal{T}}$ to be the "3-fermion $\mathbb{Z}_2$" state [2] that cancels the bulk anomaly of the first-order DIII TSC and $\mathcal{A}$, $\bar{\mathcal{A}}$ its $\mathcal{T}$-breaking 2D analogs. Extensions to second-order SPTs protected by inversion [66] and to third-order 3D SPTs with gapless corner modes, are avenues for future work.

We are grateful to D. Aasen, M. Barkeshli, L. Fidkowski, F. Pollmann, A. C. Potter, A. Nahum, S. Ramamurthy, and S. H. Simon for useful discussions, and are especially grateful to M. Cheng for very useful discussions and correspondence on Ref. [23]. S. A. P. and T. N. thank the Max Planck Institute for the Structure and Dynamics of Matter for hospitality during the initiation of this work. We acknowledge support from the European Research Council (ERC) under the European Union Horizon 2020 Research and Innovation Programme [Grant Agreements No. 757867-PARATOP (T. N.) and No. 804213-TMCS (S. A. P.)] and the Marie Sklodowska Curie Grant Agreement No. 701647 (A. T.). B. A. B. acknowledges support from the Department of Energy de-sc0016239, Simons Investigator Award, the Packard Foundation, the Schmidt Fund for Innovative Research, NSF EAGER Grants No. DMR-1643312, No. ONR-N00014-14-1-0330, and NSF-MRSEC No. DMR-1420541.

[1] A. Vishwanath and T. Senthil, Phys. Rev. X **3**, 011016 (2013).
[2] F. J. Burnell, X. Chen, L. Fidkowski, and A. Vishwanath, Phys. Rev. B **90**, 245122 (2014).
[3] M. A. Metlitski, C. L. Kane, and M. P. A. Fisher, Phys. Rev. B **88**, 035131 (2013).
[4] M. A. Metlitski, C. L. Kane, and M. P. A. Fisher, Phys. Rev. B **92**, 125111 (2015).
[5] P. Bonderson, C. Nayak, and X.-L. Qi, J. Stat. Mech. (2013) P09016.
[6] C. Wang, A. C. Potter, and T. Senthil, Phys. Rev. B **88**, 115137 (2013).
[7] X. Chen, L. Fidkowski, and A. Vishwanath, Phys. Rev. B **89**, 165132 (2014).
[8] M. Barkeshli, C.-M. Jian, and X.-L. Qi, Phys. Rev. B **88**, 235103 (2013).
[9] X. Chen, F. J. Burnell, A. Vishwanath, and L. Fidkowski, Phys. Rev. X **5**, 041013 (2015).
[10] X.-G. Wen, *Quantum Field Theory of Many-Body Systems: From the Origin of Sound to an Origin of Light and Electrons* (Oxford University Press, New York, 2004).
[11] X.-G. Wen, Adv. Phys. **44**, 405 (1995).
[12] A. Kitaev, Ann. Phys. (Berlin) **303**, 2 (2003).
[13] M. A. Levin and X.-G. Wen, Phys. Rev. B **71**, 045110 (2005).
[14] A. Kitaev, Ann. Phys. (Berlin) **321**, 2 (2006), January Special Issue.
[15] A. Kitaev and J. Preskill, Phys. Rev. Lett. **96**, 110404 (2006).
[16] L. Fidkowski, X. Chen, and A. Vishwanath, Phys. Rev. X **3**, 041016 (2013).
[17] C. Wang and T. Senthil, Phys. Rev. B **89**, 195124 (2014).
[18] M. A. Metlitski, L. Fidkowski, X. Chen, and A. Vishwanath, arXiv:1406.3032.
[19] M. Barkeshli, P. Bonderson, C.-M. Jian, M. Cheng, and K. Walker, arXiv:1612.07792 [Commun. Math. Phys. (to be published)].
[20] C. Wang and M. Levin, Phys. Rev. Lett. **119**, 136801 (2017).
[21] Y. Tachikawa and K. Yonekura, Prog. Theor. Exp. Phys. **2017**, 033B04 (2017).
[22] Y. Tachikawa and K. Yonekura, Phys. Rev. Lett. **119**, 111603 (2017).
[23] M. Cheng, Phys. Rev. Lett. **120**, 036801 (2018).
[24] W. A. Benalcazar, B. A. Bernevig, and T. L. Hughes, Science **357**, 61 (2017).
[25] W. A. Benalcazar, B. A. Bernevig, and T. L. Hughes, Phys. Rev. B **96**, 245115 (2017).
[26] F. Schindler, A. M. Cook, M. G. Vergniory, Z. Wang, S. S. P. Parkin, B. A. Bernevig, and T. Neupert, Sci. Adv. **4**, eaat0346 (2018).
[27] Z. Song, Z. Fang, and C. Fang, Phys. Rev. Lett. **119**, 246402 (2017).
[28] J. Langbehn, Y. Peng, L. Trifunovic, F. von Oppen, and P. W. Brouwer, Phys. Rev. Lett. **119**, 246401 (2017).
[29] S. A. Parameswaran and Y. Wan, Physics **10**, 132 (2017).
[30] E. Khalaf, Phys. Rev. B **97**, 205136 (2018).
[31] M. Geier, L. Trifunovic, M. Hoskam, and P. W. Brouwer, Phys. Rev. B **97**, 205135 (2018).
[32] L. Trifunovic and P. W. Brouwer, Phys. Rev. X **9**, 011012 (2019).
[33] F. Schindler, Z. Wang, M. G. Vergniory, A. M. Cook, A. Murani, S. Sengupta, A. Y. Kasumov, R. Deblock, S. Jeon, I. Drozdov, H. Bouchiat, S. Guéron, A. Yazdani, B. A. Bernevig, and T. Neupert, Nat. Phys. **14**, 918 (2018).
[34] S. Imhof, C. Berger, F. Bayer, J. Brehm, L. W. Molenkamp, T. Kiessling, F. Schindler, C. H. Lee, M. Greiter, T. Neupert, and R. Thomale, Nat. Phys. **14**, 925 (2018).
[35] Y. You, T. Devakul, F. J. Burnell, and T. Neupert, Phys. Rev. B **98**, 235102 (2018).
[36] A. Rasmussen and Y.-M. Lu, arXiv:1809.07325.
[37] S. A. A. Ghorashi, X. Hu, T. L. Hughes, and E. Rossi, Phys. Rev. B **100**, 020509 (2019).
[38] H. Song, S.-J. Huang, L. Fu, and M. Hermele, Phys. Rev. X **7**, 011020 (2017).
[39] H. Isobe and L. Fu, Phys. Rev. B **92**, 081304 (2015).

[40] S. Jiang and Y. Ran, Phys. Rev. B **95**, 125107 (2017).
[41] S.-J. Huang, H. Song, Y.-P. Huang, and M. Hermele, Phys. Rev. B **96**, 205106 (2017).
[42] R. Thorngren and D. V. Else, Phys. Rev. X **8**, 011040 (2018).
[43] R. Kobayashi and K. Shiozaki, arXiv:1901.06195.
[44] P. Francesco, P. Mathieu, and D. Sénéchal, *Conformal Field Theory* (Springer Science & Business Media, New York, 2012).
[45] P. Ginsparg, arXiv:hep-th/9108028.
[46] E. Fradkin, C. Nayak, A. Tsvelik, and F. Wilczek, Nucl. Phys. **B516**, 704 (1998).
[47] P. Fendley, M. P. A. Fisher, and C. Nayak, Phys. Rev. B **75**, 045317 (2007).
[48] W. Bishara and C. Nayak, Phys. Rev. B **77**, 165302 (2008).
[49] See Supplemental Material at http://link.aps.org/supplemental/10.1103/PhysRevLett.124.046801 for (i) discussion of gauge symmetry in the $\mathcal{T}$ Pfaffian and (ii) details of anyon condensation in HOTI and HOTSC, which includes Refs. [50–58].
[50] F. A. Bais and J. K. Slingerland, Phys. Rev. B **79**, 045316 (2009).
[51] L. Kong, Nucl. Phys. **B886**, 436 (2014).
[52] T. Neupert, H. He, C. von Keyserlingk, G. Sierra, and B. A. Bernevig, Phys. Rev. B **93**, 115103 (2016).
[53] A. Kapustin and N. Saulina, arXiv:1012.0911.
[54] B. Bakalov and A. A. Kirillov, *Lectures on Tensor Categories and Modular Functors* (American Mathematical Society, College Park, MD, 2001), Vol. 21.
[55] G. Moore and N. Seiberg, *Physics, Geometry and Topology* (Springer, New York, 1990), pp. 263–361.
[56] E. Verlinde, Nucl. Phys. **B300**, 360 (1988).
[57] A. Kapustin, arXiv:1403.1467.
[58] A. Kapustin, arXiv:1404.6659.
[59] P. Bruillard, C. Galindo, T. Hagge, S.-H. Ng, J. Y. Plavnik, E. C. Rowell, and Z. Wang, J. Math. Phys. (N.Y.) **58**, 041704 (2017).
[60] L. Bhardwaj, D. Gaiotto, and A. Kapustin, J. High Energy Phys. 04 (2017) 96.
[61] D. Aasen, E. Lake, and K. Walker, arXiv:1709.01941.
[62] $\mathcal{A}_{\mathcal{T}}$ is the $\mathcal{T}$ Pfaffian$_+$, the STO of the free-fermion TI. A distinct $\mathcal{T}$ Pfaffian$_-$ with sign-reversed topological spins and $\mathcal{T}^2$ actions (where defined) for $1_{2,6}, \psi_{2,6}$, and σ_j yields an STO for an intrinsically interacting HOTI.
[63] F. D. M. Haldane, Phys. Rev. Lett. **74**, 2090 (1995).
[64] M. Levin, Phys. Rev. X **3**, 021009 (2013).
[65] J. C. Wang and X.-G. Wen, Phys. Rev. B **91**, 125124 (2015).
[66] A. Tiwari, M. Li, T. Neupert, B. A. Bernevig, and S. A. Parameswaran (to be published).