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Unicity of Meromorphic Function and its Derivative

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Abstract. In this paper, we deal with the uniqueness problems of meromorphic functions that share a small function with its derivative and improve some results of Yang, Yu, Lahiri, and Zhang, also answer some questions of T. D. Zhang and W. R. Lü.

1. Introduction and main results

In this article, by meromorphic functions we shall always mean meromorphic functions in the complex plane. we are going to mainly use the basic notation of Nevanlinna Theory, (see [1], [3], [2]) such as $T(r,f)$, $N(r,f)$, $m(r,f)$, $\overline{N}(r,f)$ and $S(r,f) = o(T(r,f))$. Let $f(z)$ and $g(z)$ denote two non-constant meromorphic functions, and let $a(z)$ be a meromorphic function. If $f(z) - a(z)$ and $g(z) - a(z)$ have the same zeros with the same multiplicities(ignoring multiplicities), then we say that $f(z)$ and $g(z)$ share $a(z)$ CM(IM). Let k be a positive integer. We denote by $N_{k)}(r, 1/f-a)$ the counting function for the zeros of $f-a$ with multiplicity $\leq k$, and by $\overline{N}_{k)}(r, 1/f-a)$ the corresponding one for which the multiplicity is not counted. Let $N_{(k}(r, 1/f-a)$ be the counting function for the zeros of $f-a$ with multiplicity $\geq k$, and $\overline{N}_{(k}(r, 1/f-a)$ be the corresponding one for which the multiplicity is not counted. Set $N_k(r, 1/f-a) = \overline{N}(r, 1/f-a) + \overline{N}_{(2}(r, 1/f-a) + \cdots + \overline{N}_{(k}(r, 1/f-a)$. And we define

$$\delta_p(a,f) = 1 - \limsup_{r \to +\infty} \frac{N_p(r, 1/f-a)}{T(r,f)}.$$

Obviously, $1 \geq \Theta(a,f) \geq \delta_p(a,f) \geq \delta(a,f) \geq 0$.

In 1996, Brück(see [6]) posed the following conjecture.

Conjecture. Let f be a non-constant entire function such that the hyper-order $\sigma_2(f)$ of f is not a positive integer and $\sigma_2(f) < \infty$. If f and f' share a finite value a CM, then $\dfrac{f'-a}{f-a} = c$, where c is nonzero constant.

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In [6], Brück proved under additional hypothesis that the conjecture holds when $a = 1$, in fact he proved:

Theorem A. *Let f be non-constant entire funciton. If f and f' share the value 1 CM and if $N(r, 1/f') = S(r, f)$, then $(f' - 1) \setminus (f - 1) \equiv c$ for some constant $c \in \mathbb{C} \setminus \{0\}$.*

After that many people extended Theorem A and obtained many excellent results, such as L. Z. Yang[7], Q. C. Zhang[8] and Yu[9].

Theorem B([8]). *Let f be a non-constant meromorphic function and k be a positive integer. Suppose that f and $f^{(k)}$ share 1 CM and*

$$2\overline{N}(r, f) + \overline{N}(r, 1/f') + N(r, 1/f^{(k)}) < (\lambda + o(1))T(r, f^{(k)}),$$

for $r \in I$, where I is a set of infinite linear measure and λ satisfies $0 < \lambda < 1$, then $(f^{(k)} - 1) \setminus (f - 1) \equiv c$ for some constant $c \in \mathbb{C} \setminus \{0\}$.

Theorem C([1]). *Let f be a non-constant meromorphic function and $a(z)(\not\equiv 0, \infty)$ be a small function with respect to f, If*
(1) f and g have no common poles,
(2) $f - a$ and $f^{(k)} - a$ share the value 0 CM,
(3) $4\delta(0, f) + 2(k + 8)\Theta(\infty, f) > 2k + 19$,
then $f \equiv f^{(k)}$, where k is a positive integer.

T.D.Zhang and W.R.Lü[10] recently considered the problem of a meromorphic function sharing one small function with its k-th derivative and proved the following two theorems.

Theorem D. *Let $k(\geq 1)$, $n(\geq 1)$ be integers and f be a non-constant meromorphic function. Also let $a \equiv a(z)(\not\equiv 0, \infty)$ be a meromorphic functions such that $T(r, a) = S(r, f)$, as $r \rightharpoonup \infty$. Suppose that f^n and $f^{(k)}$ share $a(z)$ IM and*

$$4\overline{N}(r, f) + 2\overline{N}(r, \frac{1}{(f^n/a)'}) + 2N_2(r, \frac{1}{f^{(k)}}) + \overline{N}(r, \frac{1}{f^{(k)}}) < (\lambda + o(1))T(r, f^{(k)}),$$

or f^n and $f^{(k)}$ share $a(z)$ CM and

$$2\overline{N}(r, f) + \overline{N}(r, \frac{1}{(f^n/a)'}) + N_2(r, \frac{1}{f^{(k)}}) < (\lambda + o(1))T(r, f^{(k)}),$$

for $0 < \lambda < 1$, $r \in I$, here I is a set of infinite linear measure, then $\dfrac{f^{(k)} - a}{f^n - a} \equiv c$ for some constant $c \in \mathbb{C} \setminus \{0\}$.

Theorem E. *Let $k(\geq 1)$, $n(\geq 1)$ be integers and f be a non-constant meromorphic function. Also let $a(z)(\not\equiv 0, \infty)$ be a small function with respect to f. If f^n and $f^{(k)}$ share $a(z)$ IM and*

$$(2k + 6)\Theta(\infty, f) + 4\Theta(0, f) + 2\delta_{k+2}(0, f) > 2k + 12 - n,$$

or f^n *and* $f^{(k)}$ *share* $a(z)$ *CM and*

$$(k+3)\Theta(\infty, f) + 2\Theta(0, f) + \delta_{k+2}(0, f) > k + 6 - n,$$

then $f^n \equiv f^{(k)}$.

At the end of [10] T.D.Zhang and W.R.Lü asked a question: What will happen if f^n and $[f^{(k)}]^m$ share a small function? We consider the problem and get our theorems as follow.

Theorem 1. *Let* $k(\geq 1)$, $n(\geq 1)$, $m(\geq 1)$ *be integers and* f *be a non-constant meromorphic function. Also let* $a(z)(\not\equiv 0, \infty)$ *be a small function with respect to* f. *If* f^n *and* $[f^{(k)}]^m$ *share* $a(z)$ *IM, and*

$$\begin{aligned}(1.1)\qquad &\frac{1}{m}[4\overline{N}(r, f) + 2\overline{N}(r, \frac{1}{(f^n/a)'}) + 2N_2(r, \frac{1}{f^{(k)}}) + \overline{N}(r, \frac{1}{f^{(k)}})]\\ &< (\lambda + o(1))T(r, f^{(k)}),\end{aligned}$$

or f^n *and* $[f^{(k)}]^m$ *share* $a(z)$ *CM, and*

$$(1.2)\qquad \frac{1}{m}[2\overline{N}(r, f) + \overline{N}(r, \frac{1}{(f^n/a)'}) + N_2(r, \frac{1}{f^{(k)}})] < (\lambda + o(1))T(r, f^{(k)}),$$

for $0 < \lambda < 1$, $r \in I$, *and* I *is a set of infinite linear measure. Then*$([f^{(k)}]^m - a) \setminus (f^n - a) \equiv c$ *for some constant* $c \in \mathbb{C} \setminus \{0\}$.

Theorem 2. *Let* $k(\geq 1)$, $n(\geq 1)$, $m(\geq 1)$ *be integers and* f *be a non-constant meromorphic function. Also let* $a(z)(\not\equiv 0, \infty)$ *be a small function with respect to* f. *If* f^n *and* $[f^{(k)}]^m$ *share* $a(z)$ *IM and*

$$(1.3)\qquad (2k+6)\Theta(\infty, f) + 3\Theta(0, f) + 2\delta_{k+2}(0, f) > 2k + 11 - n,$$

or f^n *and* $[f^{(k)}]^m$ *share* $a(z)$ *CM and*

$$(1.4)\qquad (k+3)\Theta(\infty, f) + \delta_2(0, f) + 2\delta_{k+2}(0, f) > k + 5 - n,$$

then $f^n \equiv [f^{(k)}]^m$.

Though we use the standard notations and definitions of the value distribution theory, we will explain some definitions and notations which are used in the paper.

Definition 1.1. Let F and G be two meromorphic functions defined in $\mathbb{C}$, assume F and G share 1 IM, let z_0 be a zero of $F-1$ with multiplicity p and a zero of $G-1$ with multiplicity q. We denote by $N_E^{1)}(r, 1/F-1)$ the counting function of the zeros of $F-1$ where $p = q = 1$; by $N_E^{(2}(r, 1/F-1)$ the counting funciton of zeros of $F-1$ where $p = q \geq 2$. We denote by $N_L(r, 1/F-1)$ the counting function of the zeros of $F-1$ where $p>q\geq 1$; each point is counted according to its multiplicity, and $\overline{N}_L(r,1/F-1)$ denote its reduced form. In the same way, we can define $N_E^{1)}(r,1/G-1)$, $N_E^{(2}(r,1/G-1)$, $\overline{N}_L(r,1/G-1)$ and so on.

Definition 1.2. In this paper $N_0(r,\dfrac{1}{F'})$ denotes the counting function of the zeros of F' which are not the zeros of F and $F-1$, and $\overline{N}_0(r,\dfrac{1}{F'})$ denote its reduced form. In the same way, we can define $N_0(r,\dfrac{1}{G'})$ and $\overline{N}_0(r,\dfrac{1}{G'})$.

2. Some lemmas

In this section we present some lemmas which will be needed in the sequel. Let F, G be two non-constant meromorphic functions defined in $\mathbb{C}$. We shall denote by H the following function.

$$H=(\frac{F''}{F'}-2\frac{F'}{F-1})-(\frac{G''}{G'}-2\frac{G'}{G-1}). \tag{2.1}$$

Lemma 2.1([1]). *Let f be a meromorphic function and a is a finite complex number. Then*
(i) $T(r,\dfrac{1}{f-a})=T(r,f)+O(1)$,
(ii) $m(r,\dfrac{f^{(k)}}{f^{(l)}})=S(r,f)$ *for* $k>l\geq 0$,
(iii) $T(r,f)\leq \overline{N}(r,f)+\overline{N}(r,\dfrac{1}{f-a_1(z)})+\overline{N}(r,\dfrac{1}{f-a_2(z)})+S(r,f)$,
where $a_1(z)$, $a_2(z)$ are two meromorphic functions such that $T(r,a_i)=S(r,f)$, (i=1, 2).

Lemma 2.2(see page 354 in [3])**.** *Let F, G be two nonconstant meromorphic functions defined in $\mathbb{C}$. If $H\not\equiv 0$, F and G sharing* 1 *IM. Then*

$$N(r,H)\leq \overline{N}(r,F)+\overline{N}_{(2}(r,\frac{1}{F})+\overline{N}_{(2}(r,\frac{1}{G})+\overline{N}_L(r,\frac{1}{F-1})$$
$$+\overline{N}_L(r,\frac{1}{G-1})+\overline{N}_0(r,\frac{1}{F'})+\overline{N}_0(r,\frac{1}{G'})+S(r,f).$$

If F and G sharing 1 *CM. Then*

$$N(r,H)\leq \overline{N}(r,F)+\overline{N}_{(2}(r,\frac{1}{F})+\overline{N}_{(2}(r,\frac{1}{G})+\overline{N}_0(r,\frac{1}{F'})+\overline{N}_0(r,\frac{1}{G'})+S(r,f).$$

Lemma 2.3([11]). *Let f be a non-constant meromorphic function and n be a* *positive integer.* $P(f) = a_n f^n + a_{n-1} f^{n-1} + \cdots + a_1 f$, *where* a_i *are meromorphic functions such that* $T(r, a_i) = S(r, f)$ $(i = 1, 2, ..., n)$ *and* $a_n \not\equiv 0$. *Then*

$$T(r.P(f)) = nT(r, f) + S(r, f).$$

Lemma 2.4([5]). *Let* f *be a non-constant meromorphic function,* k *be a positive integer. Then*

$$N_p(r, \frac{1}{f^{(k)}}) \le N_{(p+k)}(r, \frac{1}{f}) + k\overline{N}(r, f) + S(r, f).$$

3. Proof of Theorem 1

Let $F = \frac{f^n}{a}$, $G = \frac{[f^{(k)}]^m}{a}$, then

$$F - 1 = \frac{f^n - a}{a}, \; G - 1 = \frac{[f^{(k)}]^m - a}{a}. \tag{3.1}$$

From the definitions of F and G, we get

$$N_E^{1)}(r, \frac{1}{F-1}) = N_E^{1)}(r, \frac{1}{G-1}) + S(r, f), \; N_E^{(2}(r, \frac{1}{F-1}) = N_E^{(2}(r, \frac{1}{G-1}) + S(r, f), \tag{3.2}$$

$$\overline{N}_L(r, \frac{1}{F-1}) \le \overline{N}(r, \frac{1}{F}) + \overline{N}(r, F) + S(r, F), \tag{3.3}$$

$$\begin{aligned} \overline{N}(r, \frac{1}{F-1}) &= \overline{N}(r, \frac{1}{G-1}) + S(r, F) \\ &= N_E^{1)}(r, \frac{1}{F-1}) + N_E^{(2}(r, \frac{1}{F-1}) \\ &\quad + \overline{N}_L(r, \frac{1}{F-1}) + \overline{N}_L(r, \frac{1}{G-1}) + S(r, F). \end{aligned} \tag{3.4}$$

We will distinguish two cases below.

Case 1 $H \not\equiv 0$. From (2.1) it is easy to see that $m(r, H) = S(r, f)$.

Subcase 1.1. Suppose that f^n and $[f^{(k)}]^m$ share $a(z)$ IM. According to (3.1), F and G share 1 IM except the zeros and poles of $a(z)$. By (3.1), we have

$$\overline{N}(r, F) = \overline{N}(r, f) + S(r, f), \; \overline{N}(r, G) = \overline{N}(r, f) + S(r, f). \tag{3.5}$$

Let z_0 be a common zero of $F - 1$ and $G - 1$, but $a(z_0) \ne 0, \; \infty$. Through a simple calculation we know that z_0 is a zero of H, so

$$N_E^{1)}(r, \frac{1}{F-1}) \le N(r, \frac{1}{H}) + S(r, f) \le T(r, H) + S(r, f) \le N(r, H) + S(r, f). \tag{3.6}$$

From(3.4)-(3.6), lemma 2.1 and lemma 2.2, we have

$$(3.7)\quad \begin{aligned}\overline{N}(r,\frac{1}{G-1}) &\le \overline{N}(r,F)+2\overline{N}_L(r,\frac{1}{F-1})+2\overline{N}_L(r,\frac{1}{G-1})+\overline{N}_{(2}(r,\frac{1}{F})\\ &\quad+\overline{N}_{(2}(r,\frac{1}{G})+\overline{N}_E^{(2}(r,\frac{1}{F-1})+\overline{N}_0(r,\frac{1}{F'})\\ &\quad+\overline{N}_0(r,\frac{1}{G'})+S(r,f)\\ &\le \overline{N}(r,f)+2\overline{N}(r,\frac{1}{F'})+2\overline{N}_L(r,\frac{1}{G-1})+\overline{N}_{(2}(r,\frac{1}{G})\\ &\quad+\overline{N}_0(r,\frac{1}{G'})+S(r,f).\end{aligned}$$

It follows by the second fundamental theorem, and (3.5), we get

$$\begin{aligned}T(r,G) &\le \overline{N}(r,G)+\overline{N}(r,\frac{1}{G})+\overline{N}(r,\frac{1}{G-1})-N_0(r,\frac{1}{G'})+S(r,G)\\ &\le 2\overline{N}(r,f)+2\overline{N}(r,\frac{1}{F'})+2\overline{N}(r,\frac{1}{G'})+\overline{N}(r,\frac{1}{G})+S(r,f).\end{aligned}$$

By lemma 2.4, we can get

$$T(r,G)\le 4\overline{N}(r,f)+2\overline{N}(r,\frac{1}{(f^n/a)'})+2N_2(r,\frac{1}{f^{(k)}})+\overline{N}(r,\frac{1}{f^{(k)}})+S(r,f),$$

and by lemma 2.3 we have

$$T(r,f^{(k)})\le \frac{1}{m}[4\overline{N}(r,f)+2\overline{N}(r,\frac{1}{(f^n/a)'})+2N_2(r,\frac{1}{f^{(k)}})+\overline{N}(r,\frac{1}{f^{(k)}})]+S(r,f),$$

which contradicts (1.1).

Subcase 1.2. Suppose that f^n and $[f^{(k)}]^m$ share $a(z)$ CM.

Noting that $N_{1)}(r,1/F-1)=N_{1)}(r,1/G-1)+S(r,f)$. let z_0 be a common zero of $F-1$ and $G-1$, but $a(z_0)\neq 0,\ \infty$. by a simple calculation, we can still get $H(z_0)=0$. Therefore

$$(3.8)\quad N_{1)}(r,\frac{1}{F-1})\le N(r,\frac{1}{H})+S(r,f)\le N(r,H)+S(r,f).$$

Noting that $N_{1)}(r,\frac{1}{F-1})=N_{1)}(r,\frac{1}{G-1})+S(r,f)$, by (3.4) and (3.8), we can deduce

$$(3.9)\quad \begin{aligned}\overline{N}(r,\frac{1}{G-1}) &\le \overline{N}(r,F)+\overline{N}_{(2}(r,\frac{1}{F})+\overline{N}_{(2}(r,\frac{1}{G})+\overline{N}_0(r,\frac{1}{F'})\\ &\quad+\overline{N}_0(r,\frac{1}{G'})+\overline{N}_{(2}(r,\frac{1}{F-1})+S(r,f).\end{aligned}$$

By the second fundamental theorem, (3.5) and lemma (2.2), we have

$$\begin{aligned} T(r,G) &\leq \overline{N}(r,G)+\overline{N}(r,\frac{1}{G})+\overline{N}(r,\frac{1}{G-1})-N_0(r,\frac{1}{G'})+S(r,G) \\ &\leq 2\overline{N}(r,f)+N_2(r,\frac{1}{G})+\overline{N}(r,\frac{1}{F'})+S(r,f). \end{aligned}$$

Then by lemma 2.4 we have

$$mT(r,f^{(k)})\leq 2\overline{N}(r,f)+\overline{N}(r,1/(f^n/a)')+N_2(r,\frac{1}{f^{(k)}})+S(r,f).$$

This contradicts (1.2).

Case 2. $H\equiv 0$. Integration yields

$$\frac{1}{F-1}\equiv\frac{A}{G-1}+B. \tag{3.10}$$

where $A,\ B$ are constants and $A\neq 0$. It is easy to see that F and G share 1 CM. Now we claim $B=0$.

If $\overline{N}(r,f)\neq S(r,f)$, then by (3.10), we get $B=0$. So our claim holds. Hence we can assume

$$\overline{N}(r,f)=S(r,f). \tag{3.11}$$

since $B\neq 0$, then we can rewrite (3.10) as

$$\frac{1}{F-1}\equiv\frac{B(G-1+A/B)}{G-1}.$$

So

$$\overline{N}(r,\frac{1}{G-1+A/B})=\overline{N}(r,F)=S(r,f). \tag{3.12}$$

If $A\neq B$, by the second fundamental theorem and (3.12) we have

$$\begin{aligned} T(r,G) &\leq \overline{N}(r,G)+\overline{N}(r,\frac{1}{G})+\overline{N}(r,\frac{1}{G-1+A/B})+S(r,f) \\ &\leq \overline{N}(r,\frac{1}{G})+S(r,f)\leq T(r,G)+S(r,f). \end{aligned}$$

Hence

$$T(r,G)=\overline{N}(r,\frac{1}{G})+S(r,f),\quad \textit{i.e.},$$

$$mT(r,f^{(k)})=\overline{N}(r,\frac{1}{f^{(k)}})+S(r,f).$$

This is a contradiction with (1.1) and (1.2). If $A = B$, then from (3.1) we get $\frac{1}{F-1} = \frac{AG}{G-1}$. Rewrite it as

$$-\frac{a^2}{f^m(Af^n - a - aA)} \equiv \frac{[f^{(k)}]^m}{f^m}.$$

So by lemma 2.4, we have

$$\begin{aligned}(m+n+1)T(r,f) &= T(r,(\frac{f^{(k)}}{f})^m) + S(r,f)\\ &\leq m[N(r,\frac{1}{f}) + k\overline{N}(r,f)] + S(r,f) \leq mT(r,f) + S(r,f).\end{aligned}$$

This implies $T(r,f) = S(r,f)$ since $n,\ m \geq 1$. which is a contradiction. Hence our claim is right. So $(G-1)/(F-1) = A$. Theorem 1 is thus completely proved.

4. Proof of Theorem 2

The proof of Theorem 2 is partly similar to the proof of Theorem 1, Let F and G defined as Theorem 1, hence we have (3.1)-(3.5). We still distinguish two cases.

Case 1 $H \not\equiv 0$.

Subcase 1.1. Suppose that f^n and $[f^{(k)}]^m$ share $a(z)$ IM, then we can still get (3.6), and by (3.4)-(3.6) and lemma 2.2 we have (3.7).

then by the second fundamental theorem, (3.5) and (3.7), we have

$$\begin{aligned}T(r,F) &\leq \overline{N}(r,F) + \overline{N}(r,\frac{1}{F}) + \overline{N}(r,\frac{1}{F-1}) - N_0(r,\frac{1}{F'}) + S(r,F)\\ &\leq 2\overline{N}(r,f) + 2\overline{N}(r,\frac{1}{G'}) + 2\overline{N}(r,\frac{1}{F'}) + \overline{N}(r,\frac{1}{F}) + S(r,f).\end{aligned}$$

Applying lemma 2.3 and lemma 2.4 to the above inequality, we get

$$nT(r,f) \leq (2k+6)\overline{N}(r,f) + 3\overline{N}(r,\frac{1}{f}) + 2N_{k+2}N(r,\frac{1}{f}) + S(r,f).$$

This implies

$$(2k+6)\Theta(\infty,f) + 3\Theta(0,f) + 2\delta_{k+2}(0,f) \leq 2k + 11 - n.$$

This contradicts (1.3)

Subcase 1.2. Suppose that f^n and $[f^{(k)}]^m$ share $a(z)$ CM. From the above discuss, we can easily obtain $N_{1)}(r,\frac{1}{F-1}) = N_{1)}(r,\frac{1}{G-1}) + S(r,f)$, by lemma 2.1, we can deduce

$$\begin{aligned}\overline{N}(r,\frac{1}{F-1}) \leq\ & \overline{N}(r,F) + \overline{N}_{(2}(r,\frac{1}{F}) + \overline{N}_{(2}(r,\frac{1}{G}) + \overline{N}_0(r,\frac{1}{F'})\\ &+ \overline{N}_0(r,\frac{1}{G'}) + \overline{N}_{(2}(r,\frac{1}{G-1}) + S(r,f).\end{aligned} \tag{4.1}$$

So use the second fundamental theorem again and we get

$$nT(r,f) \le [(k+5)-(k+3)\Theta(\infty,f)+\delta_2(0,f)+\delta_{k+2}(0,f)]T(r,f)+S(r,f).$$

This is a contradiction.

Case 2. $H \equiv 0$

Similarly, we can also get (3.10). Next we claim $B=0$. If $\overline{N}(r,f) \neq S(r,f)$, then it follows that $B=0$ from (3.10). Hence, we may assume that (3.11) holds. If $B \neq 0,\ B \neq -1$. then

$$\frac{A}{G-1} \equiv -\frac{BF-(B+1)}{F-1},$$

and so

$$N(r,G) = \overline{N}(r,\frac{1}{F-(B+1)/B}).$$

Again by second fundamental theorem and (4.5) we have

$$T(r,F) = \overline{N}(r,\frac{1}{F}) + S(r,f). \quad i.e.,$$

$$nT(r,f) \le \overline{N}(r,\frac{1}{F}) + S(r,f) \le T(r,f)+S(r,f). \tag{4.2}$$

If $n \ge 2$, then we have $T(r,f)=S(r,f)$. This is a contradiction. If $n=1$, then we have $\overline{N}(r,\frac{1}{f}) = T(r,f)+S(r,f)$, and it follows that $\Theta(0,f)=0$ and from (3.11), (1.3), (1.4) we may deduce $\delta_{k+2}(0,f) > 1$, it is impossible. So we can assume $B=-1$, then we can get

$$\frac{[f^{(k)}]^m}{a} - (A+1) \equiv -A\cdot a\cdot\frac{1}{f^n}. \tag{4.3}$$

Therefore, by(4.5), we get

$$nT(r,f) = T(r,[f^{(k)}]^m) + S(r,f) \le mT(r,f)+S(r,f). \tag{4.4}$$

It shows that $n \le m$.

If $A=-1$, by (4.3), we have $f\cdot[f^{(k)}]^m \equiv a^2$, which with the above inequality may lead to $n=m=1$, from Theorem E. We can get our claim. If $A \neq -1$. By second fundamental theorem, lemma 2.3, (3.11) and (4.3) we have

$$\begin{aligned} T(r,[f^{(k)}]^m) &\le \overline{N}(r,\frac{1}{[f^{(k)}]^m - a(A+1)}) + \overline{N}(r,\frac{1}{f^{(k)}}) + S(r,f) \\ &\le k\overline{N}(r,f) + N_{k+2}(r,\frac{1}{f}) + S(r,f) \le T(r,f)+S(r,f). \end{aligned}$$

which with (4.4) may deduce $m=1$, thus from Theorem E, we can get our claim. Hence our claim $B=0$ holds.

Next we will proof $A = 1$. From (3.12) we have $G - 1 \equiv A(F - 1)$ then

$$\overline{N}(r, \frac{1}{G}) = \overline{N}(r, \frac{1}{F + 1/A - 1}).$$

If $A \neq 1$, by second fundamental theorem, we have

$$T(r, F) \leq \overline{N}(r, F) + \overline{N}(r, \frac{1}{F}) + \overline{N}(r, \frac{1}{G}) + S(r, f). \quad \textit{i.e.},$$

$$nT(r, f) \leq \overline{N}(r, f) + \overline{N}(r, \frac{1}{f}) + N_{k+2}(r, \frac{1}{f}) + S(r, f).$$

It implies

$$\Theta(\infty, f) + \Theta(0, f) + \delta_{k+2}(0, f) \leq 3 - n. \tag{4.5}$$

Combining (4.5) with (1.3) yields $\Theta(0, f) > 1$, since $n \geq 1$. This is a contradiction. Hence $A = 1$ and $f^n \equiv [f^{(k)}]^m$. Now Theorem 2 has been completely proved.

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