BULL. AUSTRAL. MATH. SOC. 30D35
VOL. 52 (1995) [215-224]

UNICITY THEOREMS FOR MEROMORPHIC OR ENTIRE FUNCTIONS II

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In 1976, Gross posed the question "can one find two (or possibly even one) finite sets S_j $(j = 1, 2)$ such that any two entire functions f and g satisfying $E_f(S_j) = E_g(S_j)$ for $j = 1, 2$ must be identical?", where $E_f(S_j)$ stands for the inverse image of S_j under f. In this paper, we show that there exists a finite set S with 11 elements such that for any two non-constant meromorphic functions f and g the conditions $E_f(S) = E_g(S)$ and $E_f(\{\infty\}) = E_g(\{\infty\})$ imply $f \equiv g$. As a special case this also answers the question posed by Gross.

1. INTRODUCTION

By a meromorphic function we shall always mean a meromorphic function in the complex plane. It is assumed that the reader is familiar with the notation of Nevanlinna Theory (see, for example, [3]). We use E to denote any set of positive real numbers of finite linear measure, not necessarily the same at each occurrence. For any nonconstant meromorphic function f, we denote by $S(r, f)$ any quantity satisfying

$$S(r, f) = o(T(r, f)) \quad (r \to \infty, r \notin E).$$

For any set S and any meromorphic function f let

$$E_f(S) = \bigcup_{a \in S} \{z \mid f(z) - a = 0\},$$

where each zero of $f - a$ with multiplicity m is repeated m times in $E_f(S)$.

In 1976, Gross proved [1] that there exist three finite sets S_j $(j = 1, 2, 3)$ such that any two entire functions f and g satisfying $E_f(S_j) = E_g(S_j)$ for $j = 1, 2, 3$ must be identical. In the same paper Gross posed the following question (see [1, Question 6]): Can one find two (or possibly even one) finite sets S_j $(j = 1, 2)$ such that any two non-constant entire functions f and g satisfying $E_f(S_j) = E_g(S_j)$ for $j = 1, 2$ must be identical?

The present author proved the following result which is partial answer of the above question.

Received 3 November 1994
Project supported by the National Natural Science Foundation of PRC.

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https://doi.org/10.1017/S0004972700014635 Published online by Cambridge University Press

Theorem A. (see [7, Theorem 3]). *Let $S_1 = \{w \mid w^n - 1 = 0\}$, $S_2 = \{a, b\}$, where n (> 6) is a positive integer, a and b are constants such that $ab \neq 0$, $a^n \neq b^n$, $a^{2n} \neq 1$, $b^{2n} \neq 1$ and $a^n b^n \neq 1$. Suppose that f and g are nonconstant entire functions satisfying $E_f(S_j) = E_g(S_j)$ for $j = 1, 2$. Then $f \equiv g$.*

For the meromorphic case, the present author proved a corresponding theorem, which is the generalisation of Theorem A.

Theorem B. (see [7, Theorem 2]). *Let S_1 and S_2 be defined as in Theorem A, and let $S_3 = \{\infty\}$. Suppose that f and g are nonconstant meromorphic functions satisfying $E_f(S_j) = E_g(S_j)$ for $j = 1, 2, 3$. Then $f \equiv g$.*

The set S such that for any two nonconstant entire functions f and g the condition $E_f(S) = E_g(S)$ implies $f \equiv g$ is called a unique range set (URS in brief) of entire functions (see [2]). In 1982, Gross and Yang proved the following result.

Theorem C. (see [2, Theorem 3]). *The set $S = \{w \mid e^w + w = 0\}$ is a URS of entire functions.*

Note that the set $S = \{w \mid e^w + w = 0\}$ contains an infinite number of elements and so Theorem C does not answer the question posed by Gross.

In this paper we give a positive answer to Gross' question. In fact, we prove more generally the following theorem.

Theorem 1. *Let $S = \{w \mid w^n + aw^{n-m} + b = 0\}$, where n and m are two positive integers such that n and m have no common factors and $n \geqslant 2m + 5$, a and b are two nonzero constants such that the algebric equation $w^n + aw^{n-m} + b = 0$ has no multiple roots. Then the set S is a URS of entire functions.*

By Theorem 1, we immediately obtain the following corollary.

Corollary 1. *Let $S = \{w \mid w^7 + aw^6 + b = 0\}$, where a and b are two non-zero constants such that $b \neq -6^6(a/7)^7$. Then the set S is a URS of entire functions with 7 elements.*

For meromorphic functions, we have the following result, which is an extension of Theorem 1.

Theorem 2. *Let $S = \{w \mid w^n + aw^{n-m} + b = 0\}$, where n and m are two positive integers such that $m \geqslant 2$, $n \geqslant 2m + 7$ with n and m having no common factors, a and b are two nonzero constants such that the algebric equation $w^n + aw^{n-m} + b = 0$ has no multiple roots. Suppose that f and g are nonconstant meromorphic functions satisfying $E_f(S) = E_g(S)$ and $E_f(\{\infty\}) = E_g(\{\infty\})$. Then $f \equiv g$.*

By Theorem 2, we immediately obtain the following corollary.

Corollary 2. *Let $S = \{w \mid w^{11} + aw^9 + b = 0\}$, where a and b are two non-zero constants such that $b^2 \neq -2^2 9^9 (a/11)^{11}$. Then for any two nonconstant meromorphic*

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functions f *and* g, *the conditions* $E_f(S) = E_g(S)$ *and* $E_f(\{\infty\}) = E_g(\{\infty\})$ *imply* $f \equiv g$.

2. Some lemmas

The following lemmas will be needed in the proof of our theorems.

Lemma 1. (see [6]). *Let* f *and* g *be two nonconstant meromorphic functions, and let* c_1, c_2 *and* c_3 *be three nonzero constants. If*

$$c_1 f + c_2 g = c_3,$$

then

$$T(r, f) < \overline{N}\left(r, \frac{1}{f}\right) + \overline{N}\left(r, \frac{1}{g}\right) + \overline{N}(r, f) + S(r, f).$$

Lemma 2. (see [4]). *Let* $f_1, f_2, \ldots, f_n$ *be linearly independent meromorphic functions satisfying*

$$\sum_{j=1}^{n} f_j = 1.$$

Then for $k = 1, 2, \ldots, n$ *we have*

$$T(r, f_k) < \sum_{j=1}^{n} N\left(r, \frac{1}{f_j}\right) + N(r, f_k) + N(r, D) - \sum_{j=1}^{n} N(r, f_j) - N\left(r, \frac{1}{D}\right) + o(T(r)) \quad (r \notin E),$$

where D *denotes the Wronskian of the functions* $f_1, f_2, \ldots, f_n$ *and* $T(r)$ *denotes the maximum of* $T(r, f_j)$, $j = 1, 2, \ldots, n$.

Lemma 3. (see [5]). *Let* f *be a nonconstant meromorphic function, and let* $P(f)$ *be a polynomial in* f *of the form*

$$P(f) = a_0 f^n + a_1 f^{n-1} + \ldots + a_{n-1} f + a_n,$$

where a_0 $(\neq 0)$, $a_1, \ldots, a_n$ *are constants. Then*

$$T(r, P(f)) = nT(r, f) + S(r, f).$$

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3. Proof of Theorem 2

Let $w_1, w_2, \ldots, w_n$ be the roots of the equation $w^n + aw^{n-m} + b = 0$. Since $E_f(S) = E_g(S)$ and $E_f(\{\infty\}) = E_g(\{\infty\})$, we have from Nevanlinna's second fundamental theorem

$$\begin{aligned}(n-1)T(r, g) &< \sum_{j=1}^{n} N\left(r, \frac{1}{g - w_j}\right) + N(r, g) + S(r, g)\\ &= \sum_{j=1}^{n} N\left(r, \frac{1}{f - w_j}\right) + N(r, f) + S(r, g)\\ &\leqslant (n+1)T(r, f) + S(r, g).\end{aligned}$$

Thus

$$T(r, g) < \frac{n+1}{n-1}T(r, f) + S(r, g). \tag{1}$$

In the same manner as above, we have

$$T(r, f) < \frac{n+1}{n-1}T(r, g) + S(r, f). \tag{2}$$

Again by $E_f(S) = E_g(S)$ and $E_f(\{\infty\}) = E_g(\{\infty\})$, we obtain

$$\frac{f^n + af^{n-m} + b}{g^n + ag^{n-m} + b} = e^h, \tag{3}$$

where h is an entire function. From Lemma 3, (1) and (3) we have

$$\begin{aligned}T(r, e^h) &\leqslant T(r, f^n + af^{n-m} + b) + T(r, g^n + ag^{n-m} + b) + O(1)\\ &= nT(r, f) + nT(r, g) + S(r, f)\\ &< \frac{2n^2}{n-1}T(r, f) + S(r, f).\end{aligned} \tag{4}$$

Let us put

$$f_1 = -\frac{1}{b}f^{n-m}(f^m + a), \tag{5}$$

$$f_2 = e^h, \tag{6}$$

$$f_3 = \frac{1}{b}g^{n-m}(g^m + a)e^h, \tag{7}$$

and let $T(r)$ denote the maximum of $T(r, f_j)$, $j = 1, 2, 3$. From (3), (5), (6) and (7), we obtain

$$f_1 + f_2 + f_3 = 1. \tag{8}$$

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From (1), (4), (5), (6) and (7) we have

$$T(r) = O(T(r, f)) \quad (r \notin E). \tag{9}$$

Next, we need the following lemma.

Lemma 4. *f_1, f_2 and f_3 are linearly dependent.*

Proof: Suppose that f_1, f_2 and f_3 are linearly independent. Applying Lemma 2 to the functions f_j $(j = 1, 2, 3)$, from (8) and (9) we have

(10)
$$T(r, f_1) < \sum_{j=1}^{3} N\left(r, \frac{1}{f_j}\right) - N\left(r, \frac{1}{D}\right) + N(r, D) - N(r, f_2) - N(r, f_3) + S(r, f),$$

where

$$D = \begin{vmatrix} f_1 & f_2 & f_3 \\ f_1' & f_2' & f_3' \\ f_1'' & f_2'' & f_3'' \end{vmatrix}. \tag{11}$$

From (5), (6) and (7) we have

$$\sum_{j=1}^{3} N\left(r, \frac{1}{f_j}\right) = (n-m)N\left(r, \frac{1}{f}\right) + N\left(r, \frac{1}{f^m + a}\right) + (n-m)N\left(r, \frac{1}{g}\right) + N\left(r, \frac{1}{g^m + a}\right).$$

By looking at the zeros of f and g, from (5), (6), (7) and (11) we see that

$$N\left(r, \frac{1}{D}\right) \geqslant (n-m)N\left(r, \frac{1}{f}\right) - 2\overline{N}\left(r, \frac{1}{f}\right) + (n-m)N\left(r, \frac{1}{g}\right) - 2\overline{N}\left(r, \frac{1}{g}\right). \tag{13}$$

From (8) and (11) we get

$$D = \begin{vmatrix} f_2' & f_3' \\ f_2'' & f_3'' \end{vmatrix}. \tag{14}$$

Since f_2 is entire, from (7) and (14) we have

$$N(r, D) - N(r, f_2) - N(r, f_3) \leqslant N(r, f_3'') - N(r, f_3) = 2\overline{N}(r, f_3) = 2\overline{N}(r, g) = 2\overline{N}(r, f). \tag{15}$$

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From Lemma 3, (5), (10), (12), (13) and (15) we deduce

$$\begin{aligned} nT(r, f) &< 2\overline{N}\left(r, \frac{1}{f}\right) + N\left(r, \frac{1}{f^m + a}\right) + 2\overline{N}\left(r, \frac{1}{g}\right) \\ &\quad + N\left(r, \frac{1}{g^m + a}\right) + 2\overline{N}(r, f) + S(r, f) \\ &< (4 + m)T(r, f) + (2 + m)T(r, g) + S(r, f). \end{aligned}$$

From this and (1) we obtain

$$nT(r, f) < \left(2m + 6 + \frac{2(2 + m)}{n - 1}\right)T(r, f) + S(r, f). \tag{16}$$

Since $n \geqslant 2m + 7$, (16) is a contradiction, which proves Lemma 4. □

Next we proceed to prove Theorem 2.

By Lemma 4 we know that f_1, f_2 and f_3 are linearly dependent. Then there exist three constants c_1, c_2 and c_3, at least one of which is not zero, such that

$$c_1 f_1 + c_2 f_2 + c_3 f_3 = 0. \tag{17}$$

If $c_1 = 0$, from (17) we have $c_2 \neq 0$, $c_3 \neq 0$ and

$$f_3 = -\frac{c_2}{c_3} f_2.$$

Hence, from (6) and (7) we obtain

$$g^n + ag^{n-m} = -bc_2/c_3,$$

which is impossible. Thus $c_1 \neq 0$ and

$$f_1 = -\frac{c_2}{c_1} f_2 - \frac{c_3}{c_1} f_3. \tag{18}$$

Now combining (8) and (18) we get

$$\left(1 - \frac{c_2}{c_1}\right) f_2 + \left(1 - \frac{c_3}{c_1}\right) f_3 = 1. \tag{19}$$

We discuss the following three cases.

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CASE I. Assume $c_1 \neq c_2$ and $c_1 \neq c_3$. From (6), (7) and (19) we have

$$-\frac{1}{b}\left(1-\frac{c_3}{c_1}\right)g^{n-m}(g^m+a)+c^{-h}=1-\frac{c_2}{c_1}. \tag{20}$$

From (20) we know that g is a entire function. By Lemma 1, Lemma 3 and (20) we obtain

$$\begin{aligned} nT(r,g) &< \overline{N}\left(r,\frac{1}{g^{n-m}(g^m+a)}\right)+S(r,g) \\ &= \overline{N}\left(r,\frac{1}{g}\right)+\overline{N}\left(r,\frac{1}{g^m+a}\right)+S(r,g) \\ &< (1+m)T(r,g)+S(r,g), \end{aligned}$$

which is impossible.

CASE II. Assume $c_1 = c_2$. From (19) we have $c_1 \neq c_3$ and

$$f_3 = \frac{c_1}{c_1 - c_3}. \tag{21}$$

From (7) and (21) we get

$$g^{n-m}(g^m+a)=\frac{bc_1}{c_1-c_3}e^{-h}. \tag{22}$$

Let $a_1, a_2, \ldots, a_m$ be the roots of equation $w^m + a = 0$. From (22) we know that ∞, 0, $a_1, a_2, \ldots, a_m$ are Picard exceptional values of g, which is impossible.

CASE III. Assume $c_1 = c_3$. From (19) we have $c_1 \neq c_2$ and

$$f_2 = \frac{c_1}{c_1 - c_2}$$

that is

$$e^h = \frac{c_1}{c_1 - c_2}. \tag{23}$$

From (5), (7), (8) and (23) we get

$$-f^{n-m}(f^m+a)+\frac{c_1}{c_1-c_2}g^{n-m}(g^m+a)=\frac{bc_2}{c_2-c_1}. \tag{24}$$

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If $c_2 \neq 0$, by Lemma 1 and Lemma 3, we have from (24),

$$\begin{aligned} nT(r, f) &< \overline{N}\left(r, \frac{1}{f^{n-m}(f^m+a)}\right) + \overline{N}\left(r, \frac{1}{g^{n-m}(g^m+a)}\right) + \overline{N}(r, f) + S(r, f) \\ &= \overline{N}\left(r, \frac{1}{f}\right) + \overline{N}\left(r, \frac{1}{f^m+a}\right) + \overline{N}\left(r, \frac{1}{g}\right) \\ &\quad + \overline{N}\left(r, \frac{1}{g^m+a}\right) + \overline{N}(r, f) + S(r, f) \\ &< (2+m)T(r, f) + (1+m)T(r, g) + S(r, f). \end{aligned}$$

From this and (1) we obtain

$$nT(r, f) < \left(3 + 2m + \frac{2(1+m)}{n-1}\right)T(r, f) + S(r, f),$$

which is impossible. Thus $c_2 = 0$. From (24) we deduce

$$f^n - g^n = -a\left(f^{n-m} - g^{n-m}\right). \tag{25}$$

If $f \not\equiv g$, from (25) we obtain

$$g^m = -\frac{a(H-v)\left(H-v^2\right)\ldots\left(H-v^{n-m-1}\right)}{(H-u)(H-u^2)\ldots(H-u^{n-1})}, \tag{26}$$

where $H = f/g$, $u = \exp((2\pi i)/n)$ and $v = \exp((2\pi i)/(n-m))$. From (26) we know that H is a nonconstant meromorphic function. Since n and m have no common factors, we have $u^j \neq v^k$ ($j = 1, 2, \ldots, n-1$; $k = 1, 2, \ldots, n-m-1$). Suppose that z_j is a zero of $H - u^j$ of order p_j. From (26) we have $p_j \geqslant m \geqslant 2$. Thus

$$\overline{N}\left(r, \frac{1}{H-u^j}\right) \leqslant \frac{1}{m}N\left(r, \frac{1}{H-u^j}\right) \leqslant \frac{1}{2}T(r, H) + O(1). \tag{27}$$

By the second fundamental theorem, from (27) we obtain

$$\begin{aligned} (n-3)T(r, H) &< \sum_{j=1}^{n-1}\overline{N}\left(r, \frac{1}{H-u^j}\right) + S(r, H) \\ &< \frac{n-1}{2}T(r, H) + S(r, H), \end{aligned}$$

which is impossible. Hence, $f \equiv g$.

This completes the proof of Theorem 2. ∎

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4. Supplement to Theorem 2

It is reasonable to ask: What can be said if $m = 1$ in Theorem 2? In this section, we prove the following theorem, which is a supplement to Theorem 2.

Theorem 3. *Let $S = \{w \mid w^n + aw^{n-1} + b = 0\}$, where $n > 8$, and a and b are two nonzero constants such that the algebraic equation $w^n + aw^{n-1} + b = 0$ has no multiple roots. Suppose that f and g are two distinct nonconstant meromorphic functions satisfying $E_f(S) = E_g(S)$ and $E_f(\{\infty\}) = E_g(\{\infty\})$. Then*

$$f = -\frac{aH\left(H^{n-1} - 1\right)}{H^n - 1} \quad \text{and} \quad g = -\frac{a\left(H^{n-1} - 1\right)}{H^n - 1},$$

where H is a nonconstant meromorphic function.

Proof: Proceeding as in the proof of Theorem 2, we have

$$f^n - g^n = -a\left(f^{n-1} - g^{n-1}\right). \tag{28}$$

Noting $f \not\equiv g$, from (28) we obtain

$$g = -\frac{a\left(H^{n-1} - 1\right)}{H^n - 1}, \tag{29}$$

where $H = f/g$. From (29) we know that H is a nonconstant meromorphic function. Thus, from (29) we have

$$f = -\frac{aH\left(H^{n-1} - 1\right)}{H^n - 1}.$$

This completes the proof of Theorem 3. □

5. Proof of Theorem 1

If $f \not\equiv g$, noting $N(r, f) = N(r, g) = 0$ and proceeding as in the proof of Theorem 2, we can obtain (26). Since g is a nonconstant entire function, from (26) we know that u^j $(j = 1, 2, \ldots, n-1)$ are Picard exceptional values of H, which is impossible. Thus $f \equiv g$, which proves Theorem 1. □

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