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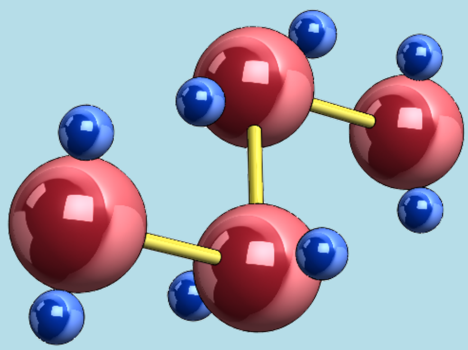

UNIDIRECTIONAL LARGE-AMPLITUDE OSCILLATORY SHEAR FLOW OF HUMAN BLOOD

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ABSTRACT

Blood is non-Newtonian suspension of red and white cells, platelets, fibrinogen and cholesterols in Newtonian plasma. To assess its non-Newtonian behaviors, this work considers a newly proposed blood test, unidirectional large-amplitude oscillatory shear flow (*ud*LAOS). In the laboratory, we generate this experiment by superposing LAOS onto steady shear flow in such a way that the shear rate never changes sign. It is thus intended to best represent the unidirectional pulsatile flow in veins and arteries. To model human blood, we consider the simplest model that can predict infinite-shear viscosity, the corotational Jeffreys fluid. We arrive at an exact analytical expression for the shear stress response of this model fluid. We discover fractional harmonics comprising the transient part of the shear stress response, and both integer and fractional harmonics, the alternant part. By *fractional*, we mean that these occur at frequencies other than integer multiples of the superposed oscillation frequency. We generalize the corotational Jeffreys fluid to multimode to best represent three blood samples from three healthy but different donors. To further improve our model predictions, we consider the multimode Oldroyd 8-constant framework, which contains the corotational Jeffreys fluid as special case. In other words, by advancing from the multimode corotational Jeffreys fluid to the multimode Oldroyd 8-constant framework, five more model parameters are added yielding better predictions. We find that the multimode corotational Jeffreys fluid adequately describes the steady shear viscosity functions measured for three different healthy donors. We further find that adding two more specific nonlinear constants to the multimode corotational Jeffreys fluid also adequately describes the behaviors of these same bloods in *ud*LAOS. This new Oldroyd 5-constant model may find usefulness in monitoring health through *ud*LAOS.

Keywords: hemorheology, Oldroyd 8-constant framework, Pipkin map.

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CONTENTS

TABLES

FIGURES

I INTRODUCTION

Blood is considered to be non-Newtonian as a whole. It is characterized by its thixotropic viscoelastic and plastic (TEVP) behaviour [1], governed by blood microstructure and biochemistry [2,3]. By volume, human blood comprises 55% Newtonian plasma, and 45% cells, which impart the non-Newtonian properties [2]. Within the plasma, there are proteins, water and other solutes that include electrolytes, nutrients, gases and waste. The protein contains 54% albumins, 38% globulins, 7% fibrinogen and 1% of other compounds [2]. Of the cellular component, red blood cells (RBC) account for most; the RBC contents primarily determine hemorheology [10,4,5]. Several parameters affect hemorheology ranging from health conditions such as diabetes [6], inherited sickle-cell disease [7], malaria [8], and certainly, blood biochemical [9]. Since cardiovascular problems are attributable to irregular hemorheology [10,11], it is of great interest to pursue a comprehensive constitutive model that can provide molecular insights into blood flow.

Unidirectional large-amplitude oscillatory shear flow (*ud*LAOS) experiment was conceived recently to study thixotropic fluids, both experimentally (fumed silica in paraffin oil and polyisobutylene) and theoretically [12]. The experiment superposes steady shear flow onto large-amplitude oscillatory shear flow (LAOS) (Row h. of FIG. 4.3-1 of [51]):

$$\dot{\gamma} = \dot{\gamma}_s + \dot{\gamma}^0 \cos\omega t \tag{1}$$

where $\dot{\gamma}_s$ and $\dot{\gamma}^0$ are shear rate amplitudes of steady shear and LAOS respectively. To ensure that the superposed experiment is one directional, the following criterion must obtain:

$$\dot{\gamma}_s \geq \dot{\gamma}^0 \tag{2}$$

For the simplest case when $\dot{\gamma}_s = \dot{\gamma}^0$, Eq. (1) becomes [9]:

$$\dot{\gamma} = \dot{\gamma}^0 \left(1 + \cos\omega t\right) \tag{3}$$

which is the focused special case that we consider in this paper. Using the longest relaxation time, Eq. (3) can be nondimensionalized to:

$$\lambda_1 \dot{\gamma} = \mathrm{Wi}\left(1 + \cos \mathrm{De}\, t/\lambda_1\right) \tag{4}$$

where $\mathrm{Wi} \equiv \lambda_1 \dot{\gamma}^0$ and $\mathrm{De} \equiv \lambda_1 \omega$. In this experiment, the velocity profile is given by:

$$\mathbf{v} = \left[\ \dot{\gamma}^0 y\left(1 + \cos\tau\right) \quad 0 \quad 0\ \right] \tag{5}$$

where $\tau \equiv \omega t$. Figure 1 depicts Eq. (5).

There are several important representations used in studing nonlinear rheology. Lissajous loops are the most fundamental approach to observe fluid nonlinearity through their distortions from linear ellipticity (see Depiction of Nonlinearities of [13]; [14]). The Pipkin map can be used to indicate the nonlinear regime in experimental-condition space, $(\mathrm{De}, \mathrm{Wi})$. Lissajous loops can be miniaturized and mapped into cells in Pipkin space to arrive at the *Ewoldt grid* (see §Ewoldt Grids of [15]), which is now widely used [16,17,18,19] to provide a phenomenal view of how complex fluids reveal their nonlinearity in oscillatory shear flow. These diagrams have been exploited in

LAOS to elucidate fluid nonlinearities, but not in *ud*LAOS. This paper discusses this potential in Section III.

To study hemorheology, some use blood biochemical and microstructures to model blood [5,20,21,22,23,24]. These molecular models can provide direct insights, but computational cost is generally high. An alternative and more popular approach is continuum modelling, where relations between blood biochemical and model parameters can be determined afterward [25,26,27,28]. There are a number of review articles on blood continua that are mindful of hemorheology [10,29,30]. These continuum constitutive equations range from one-dimensional [25,31,32] to three-dimensional [33,34]. However, only a few models have proven useful to study blood in *ud*LAOS [9,25,33]. These *ud*LAOS continuum models show promise, but they contain many fitted parameters, and these best-fits to experimental data sets are not necessarily unique. As a result, when there is more than one best-fit, it is hard to associate these parameters with the fluid physics.

This paper employs the Oldroyd 8-constant (O8) framework to model nonlinear hemorheology. This framework is given by (Eq. (8.1-2) of [51]; [35]):

$$\begin{aligned}&\boldsymbol{\tau}+\lambda_1\frac{\mathscr{D}\boldsymbol{\tau}}{\mathscr{D}t}\underline{+\tfrac{1}{2}\mu_0(\mathrm{tr}\,\boldsymbol{\tau})\dot{\boldsymbol{\gamma}}-\tfrac{1}{2}\mu_1\{\boldsymbol{\tau}\cdot\dot{\boldsymbol{\gamma}}+\dot{\boldsymbol{\gamma}}\cdot\boldsymbol{\tau}\}+\tfrac{1}{2}\nu_1(\boldsymbol{\tau}:\dot{\boldsymbol{\gamma}})\boldsymbol{\delta}}\\ &\quad=-\eta_0\left(\dot{\boldsymbol{\gamma}}+\lambda_2\frac{\mathscr{D}\dot{\boldsymbol{\gamma}}}{\mathscr{D}t}\underline{-\mu_2\{\dot{\boldsymbol{\gamma}}\cdot\dot{\boldsymbol{\gamma}}\}+\tfrac{1}{2}\nu_2(\dot{\boldsymbol{\gamma}}:\dot{\boldsymbol{\gamma}})\boldsymbol{\delta}}\right)\end{aligned} \tag{6}$$

where Table I defines the corotational derivative $\mathscr{D}/\mathscr{D}t$, extra stress tensor $\boldsymbol{\tau}$, rate-of-strain tensor $\dot{\boldsymbol{\gamma}}$ and vorticity tensor $\boldsymbol{\omega}$. The underlined terms in Eq. (6) add nonlinearity to the model, where the three on the left are all possible tensorial products of $\boldsymbol{\tau}$ and $\dot{\boldsymbol{\gamma}}$, and the two terms on the right are all possible terms that are quadratic in $\dot{\boldsymbol{\gamma}}$. This framework has been studied extensively, both in theory [36] and in engineering practice [37,38,39]. This framework has been successfully applied to capture rheology of polymer melts and solutions [40,41,42], and wormlike micelles [43]. Most importantly, specifically for polymer melts, when generalized to multimode, this model provides good nonlinear predictions.

II EXACT SHEAR STRESS

After subjecting the O8 framework to the velocity field given in Eq. (5), we find the following governing equations for stress responses in *ud*LAOS:

$$\begin{bmatrix}d\breve{\tau}_{yx}/d\tau\\ d\breve{N}_1/d\tau\\ d\breve{N}_2/d\tau\\ d\breve{\tau}_{zz}/d\tau\end{bmatrix}=\begin{bmatrix}\begin{gathered}-\tfrac{1}{\mathrm{De}}\breve{\tau}_{yx}-\tfrac{1}{2}\left[1+\breve{\mu}_0-\breve{\mu}_1\right]\tfrac{\mathrm{Wi}}{\mathrm{De}}(1+\cos\tau)\breve{N}_1-\left[\breve{\mu}_0-\breve{\mu}_1\right]\tfrac{\mathrm{Wi}}{\mathrm{De}}(1+\cos\tau)\breve{N}_2\\ -\left[\tfrac{3}{2}\breve{\mu}_0-\breve{\mu}_1\right]\tfrac{\mathrm{Wi}}{\mathrm{De}}(1+\cos\tau)\breve{\tau}_{zz}-\tfrac{\mathrm{Wi}}{\mathrm{De}}(1+\cos\tau)+\breve{\lambda}_2\,\mathrm{Wi}\sin\tau\end{gathered}\\ 2\tfrac{\mathrm{Wi}}{\mathrm{De}}(1+\cos\tau)\breve{\tau}_{yx}-\tfrac{1}{\mathrm{De}}\breve{N}_1+2\breve{\lambda}_2\tfrac{\mathrm{Wi}^2}{\mathrm{De}}(1+\cos\tau)^2\\ -(1-\breve{\mu}_1)\tfrac{\mathrm{Wi}}{\mathrm{De}}(1+\cos\tau)\breve{\tau}_{yx}-\tfrac{1}{\mathrm{De}}\breve{N}_2-\left(\breve{\lambda}_2-\breve{\mu}_2\right)\tfrac{\mathrm{Wi}^2}{\mathrm{De}}(1+\cos\tau)^2\\ -\breve{\nu}_1\tfrac{\mathrm{Wi}}{\mathrm{De}}(1+\cos\tau)\breve{\tau}_{yx}-\tfrac{1}{\mathrm{De}}\breve{\tau}_{zz}-\breve{\nu}_2\tfrac{\mathrm{Wi}^2}{\mathrm{De}}(1+\cos\tau)^2\end{bmatrix} \tag{7}$$

which has yet to yield to an exact solution. However, we can arrive at an exact solution if we reduce the framework to its special case, the corotational Jeffreys (CJ) fluid, by setting $\mu_0 = \mu_1 = \mu_2 = \nu_1 = \nu_2 = 0$, to get:

$$\begin{bmatrix} d\breve{\tau}_{yx}/d\tau \\ d\breve{N}_1/d\tau \end{bmatrix} = \begin{bmatrix} -\frac{1}{\mathrm{De}}\breve{\tau}_{yx} - \frac{1}{2}\frac{\mathrm{Wi}}{\mathrm{De}}(1+\cos\tau)\breve{N}_1 - \frac{\mathrm{Wi}}{\mathrm{De}}(1+\cos\tau) + \breve{\lambda}_2\,\mathrm{Wi}\sin\tau \\ 2\frac{\mathrm{Wi}}{\mathrm{De}}(1+\cos\tau)\breve{\tau}_{yx} - \frac{1}{\mathrm{De}}\breve{N}_1 + 2\breve{\lambda}_2\frac{\mathrm{Wi}^2}{\mathrm{De}}(1+\cos\tau)^2 \end{bmatrix} \tag{8}$$

for which we will use to derive the exact solution later in this section.

To get the exact shear stress expression from the corotational Jeffreys fluid, we employ the method of Section **4.** Of [42], that is, the method of Kovacic for the transient part [44] and **Theorem 10.** on p. 711 of Section 9.7 of [45] for the alternant part, to find:

$$\breve{\tau}_{yx} = \begin{bmatrix} \begin{pmatrix} -\frac{\mathrm{Wi}}{\mathrm{De}}(I_1 - I_{1,0}) - \frac{\mathrm{Wi}}{\mathrm{De}}(I_3 - I_{3,0}) + \breve{\lambda}_2\,\mathrm{Wi}(I_5 - I_{5,0}) \\ +\breve{\lambda}_2\frac{\mathrm{Wi}^2}{\mathrm{De}}\left(\frac{3}{2}(I_2 - I_{2,0}) + 2(I_4 - I_{4,0}) + \frac{1}{2}(I_7 - I_{7,0})\right) \end{pmatrix} e^{\frac{-\tau}{\mathrm{De}}}\cos\frac{\mathrm{Wi}}{\mathrm{De}}(\tau + \sin\tau) \\ + \begin{pmatrix} -\frac{\mathrm{Wi}}{\mathrm{De}}(I_2 - I_{2,0}) - \frac{\mathrm{Wi}}{\mathrm{De}}(I_4 - I_{4,0}) + \breve{\lambda}_2\,\mathrm{Wi}(I_6 - I_{6,0}) \\ -\breve{\lambda}_2\frac{\mathrm{Wi}^2}{\mathrm{De}}\left(\frac{3}{2}(I_1 - I_{1,0}) + 2(I_3 - I_{3,0}) + \frac{1}{2}(I_8 - I_{8,0})\right) \end{pmatrix} e^{\frac{-\tau}{\mathrm{De}}}\sin\frac{\mathrm{Wi}}{\mathrm{De}}(\tau + \sin\tau) \end{bmatrix} \tag{9}$$

which we call the *compact form*, and wherein, the integrals:

$$I_1 \equiv \int e^{\frac{\tau'}{\mathrm{De}}}\cos\frac{\mathrm{Wi}}{\mathrm{De}}(\tau' + \sin\tau')\,d\tau' \tag{10}$$

$$I_2 \equiv \int e^{\frac{\tau'}{\mathrm{De}}}\sin\frac{\mathrm{Wi}}{\mathrm{De}}(\tau' + \sin\tau')\,d\tau' \tag{11}$$

$$I_3 \equiv \int e^{\frac{\tau'}{\mathrm{De}}}\cos\tau'\cos\frac{\mathrm{Wi}}{\mathrm{De}}(\tau' + \sin\tau')\,d\tau' \tag{12}$$

$$I_4 \equiv \int e^{\frac{\tau'}{\mathrm{De}}}\cos\tau'\sin\frac{\mathrm{Wi}}{\mathrm{De}}(\tau' + \sin\tau')\,d\tau' \tag{13}$$

$$I_5 \equiv \int e^{\frac{\tau'}{\mathrm{De}}}\sin\tau'\cos\frac{\mathrm{Wi}}{\mathrm{De}}(\tau' + \sin\tau')\,d\tau' \tag{14}$$

$$I_6 \equiv \int e^{\frac{\tau'}{\mathrm{De}}}\sin\tau'\sin\frac{\mathrm{Wi}}{\mathrm{De}}(\tau' + \sin\tau')\,d\tau' \tag{15}$$

$$I_7 \equiv \int e^{\frac{\tau'}{\mathrm{De}}}\cos 2\tau'\sin\frac{\mathrm{Wi}}{\mathrm{De}}(\tau' + \sin\tau')\,d\tau' \tag{16}$$

$$I_8 \equiv \int e^{\frac{\tau'}{\mathrm{De}}}\cos 2\tau'\cos\frac{\mathrm{Wi}}{\mathrm{De}}(\tau' + \sin\tau')\,d\tau' \tag{17}$$

are evaluated in Appendix VIIIb. From the compact form, we can extract its alternant part:

$$\begin{aligned} \breve{\tau}_{yx} &= \left(-\frac{\mathrm{Wi}}{\mathrm{De}}I_1 - \frac{\mathrm{Wi}}{\mathrm{De}}I_3 + \breve{\lambda}_2\,\mathrm{Wi}\,I_5 + \breve{\lambda}_2\frac{\mathrm{Wi}^2}{\mathrm{De}}\left(\frac{3}{2}I_2 + 2I_4 + \frac{1}{2}I_7\right)\right)e^{\frac{-\tau}{\mathrm{De}}}\cos\frac{\mathrm{Wi}}{\mathrm{De}}(\tau + \sin\tau) \\ &\quad + \left(-\frac{\mathrm{Wi}}{\mathrm{De}}I_2 - \frac{\mathrm{Wi}}{\mathrm{De}}I_4 + \breve{\lambda}_2\,\mathrm{Wi}\,I_6 - \breve{\lambda}_2\frac{\mathrm{Wi}^2}{\mathrm{De}}\left(\frac{3}{2}I_1 + 2I_3 + \frac{1}{2}I_8\right)\right)e^{\frac{-\tau}{\mathrm{De}}}\sin\frac{\mathrm{Wi}}{\mathrm{De}}(\tau + \sin\tau) \end{aligned} \tag{18}$$

and its transient part:

$$\breve{\tau}_{yx,0} = -e^{\frac{-\tau}{\mathrm{De}}}\begin{bmatrix} \left(-\frac{\mathrm{Wi}}{\mathrm{De}}I_{1,0} - \frac{\mathrm{Wi}}{\mathrm{De}}I_{3,0} + \breve{\lambda}_2\,\mathrm{Wi}\,I_{5,0} + \breve{\lambda}_2\frac{\mathrm{Wi}^2}{\mathrm{De}}\left(\frac{3}{2}I_{2,0} + 2I_{4,0} + \frac{1}{2}I_{7,0}\right)\right)\cos\frac{\mathrm{Wi}}{\mathrm{De}}\left(\tau + \sin\tau\right) \\ +\left(-\frac{\mathrm{Wi}}{\mathrm{De}}I_{2,0} - \frac{\mathrm{Wi}}{\mathrm{De}}I_{4,0} + \breve{\lambda}_2\,\mathrm{Wi}\,I_{6,0} - \breve{\lambda}_2\frac{\mathrm{Wi}^2}{\mathrm{De}}\left(\frac{3}{2}I_{1,0} + 2I_{3,0} + \frac{1}{2}I_{8,0}\right)\right)\sin\frac{\mathrm{Wi}}{\mathrm{De}}\left(\tau + \sin\tau\right) \end{bmatrix} \tag{19}$$

where $I_{1,0}$ denotes $\lim_{\tau\to 0} I_{1,0}$. Eq. (9) [or Eqs. (18) and (19)] is not in the form a Fourier series, but rather contains trig function of trig function, specifically, $\cos\frac{\mathrm{Wi}}{\mathrm{De}}\left(\tau+\sin\tau\right)$ and $\sin\frac{\mathrm{Wi}}{\mathrm{De}}\left(\tau+\sin\tau\right)$. We find this compact form to be intrinsically beautiful.

By Fourier transform of the compact form, we then get, for the alternant response:

$$\breve{\tau}_{yx} = \alpha_0 + \sum_{m=1}^{\infty}\left[\alpha_m + \beta_m + \gamma_m + \delta_m\right] \tag{20}$$

where:

$$\begin{aligned} \alpha_m \equiv \frac{J_{2m}}{2}\sum_{k=0}^{\infty} & \begin{bmatrix} \sum_{j=1}^{2}\begin{bmatrix} \left(C_1^{(2j)} - S_2^{(2j)}\right)\cos\left(2k+2m+j\right)\tau + \left(C_1^{(2j)} + S_2^{(2j)}\right)\cos\left(2k-2m+j-2\frac{\mathrm{Wi}}{\mathrm{De}}\right)\tau \\ +\left(C_2^{(2j)} + S_1^{(2j)}\right)\sin\left(2k+2m+j\right)\tau + \left(-C_2^{(2j)} + S_1^{(2j)}\right)\sin\left(2k-2m+j-2\frac{\mathrm{Wi}}{\mathrm{De}}\right)\tau \end{bmatrix} \\ +\sum_{j=1}^{3}\begin{bmatrix} \left(C_1^{(2j-1)} + S_2^{(2j-1)}\right)\cos\left(2k-2m+j-1\right)\tau + \left(C_1^{(2j-1)} - S_2^{(2j-1)}\right)\cos\left(2k+2m+j-1+2\frac{\mathrm{Wi}}{\mathrm{De}}\right)\tau \\ +\left(-C_2^{(2j-1)} + S_1^{(2j-1)}\right)\sin\left(2k-2m+j-1\right)\tau + \left(C_2^{(2j-1)} + S_1^{(2j-1)}\right)\sin\left(2k+2m+j-1+2\frac{\mathrm{Wi}}{\mathrm{De}}\right)\tau \end{bmatrix} \end{bmatrix} \\ +\frac{J_{2m}}{2}\sum_{k=1}^{\infty} & \begin{bmatrix} \sum_{j=3}^{7}\begin{bmatrix} \left(C_1^{(2j)} - S_2^{(2j)}\right)\cos\left(2k+2m+j-6\right)\tau + \left(C_1^{(2j)} + S_2^{(2j)}\right)\cos\left(2k-2m+j-6-2\frac{\mathrm{Wi}}{\mathrm{De}}\right)\tau \\ +\left(C_2^{(2j)} + S_1^{(2j)}\right)\sin\left(2k+2m+j-6\right)\tau + \left(-C_2^{(2j)} + S_1^{(2j)}\right)\sin\left(2k-2m+j-6-2\frac{\mathrm{Wi}}{\mathrm{De}}\right)\tau \end{bmatrix} \\ +\sum_{j=4}^{8}\begin{bmatrix} \left(C_1^{(2j-1)} + S_2^{(2j-1)}\right)\cos\left(2k-2m+j-7\right)\tau + \left(C_1^{(2j-1)} - S_2^{(2j-1)}\right)\cos\left(2k+2m+j-7+2\frac{\mathrm{Wi}}{\mathrm{De}}\right)\tau \\ +\left(-C_2^{(2j-1)} + S_1^{(2j-1)}\right)\sin\left(2k-2m+j-7\right)\tau + \left(C_2^{(2j-1)} + S_1^{(2j-1)}\right)\sin\left(2k+2m+j-7+2\frac{\mathrm{Wi}}{\mathrm{De}}\right)\tau \end{bmatrix} \end{bmatrix} \end{aligned} \tag{21}$$

$$\begin{aligned} \beta_m \equiv \frac{J_{2m}}{2}\sum_{k=0}^{\infty} & \begin{bmatrix} \sum_{j=1}^{2}\begin{bmatrix} \left(C_1^{(2j)} - S_2^{(2j)}\right)\cos\left(2k-2m+j\right)\tau + \left(C_1^{(2j)} + S_2^{(2j)}\right)\cos\left(2k+2m+j-2\frac{\mathrm{Wi}}{\mathrm{De}}\right)\tau \\ +\left(C_2^{(2j)} + S_1^{(2j)}\right)\sin\left(2k-2m+j\right)\tau + \left(-C_2^{(2j)} + S_1^{(2j)}\right)\sin\left(2k+2m+j-2\frac{\mathrm{Wi}}{\mathrm{De}}\right)\tau \end{bmatrix} \\ +\sum_{j=1}^{3}\begin{bmatrix} \left(C_1^{(2j-1)} + S_2^{(2j-1)}\right)\cos\left(2k+2m+j-1\right)\tau + \left(C_1^{(2j-1)} - S_2^{(2j-1)}\right)\cos\left(2k-2m+j-1+2\frac{\mathrm{Wi}}{\mathrm{De}}\right)\tau \\ +\left(-C_2^{(2j-1)} + S_1^{(2j-1)}\right)\sin\left(2k+2m+j-1\right)\tau + \left(C_2^{(2j-1)} + S_1^{(2j-1)}\right)\sin\left(2k-2m+j-1+2\frac{\mathrm{Wi}}{\mathrm{De}}\right)\tau \end{bmatrix} \end{bmatrix} \\ +\frac{J_{2m}}{2}\sum_{k=1}^{\infty} & \begin{bmatrix} \sum_{j=3}^{7}\begin{bmatrix} \left(C_1^{(2j)} - S_2^{(2j)}\right)\cos\left(2k-2m+j-6\right)\tau + \left(C_1^{(2j)} + S_2^{(2j)}\right)\cos\left(2k+2m+j-6-2\frac{\mathrm{Wi}}{\mathrm{De}}\right)\tau \\ +\left(C_2^{(2j)} + S_1^{(2j)}\right)\sin\left(2k-2m+j-6\right)\tau + \left(-C_2^{(2j)} + S_1^{(2j)}\right)\sin\left(2k+2m+j-6-2\frac{\mathrm{Wi}}{\mathrm{De}}\right)\tau \end{bmatrix} \\ +\sum_{j=4}^{8}\begin{bmatrix} \left(C_1^{(2j-1)} + S_2^{(2j-1)}\right)\cos\left(2k+2m+j-7\right)\tau + \left(C_1^{(2j-1)} - S_2^{(2j-1)}\right)\cos\left(2k-2m+j-7+2\frac{\mathrm{Wi}}{\mathrm{De}}\right)\tau \\ +\left(-C_2^{(2j-1)} + S_1^{(2j-1)}\right)\sin\left(2k+2m+j-7\right)\tau + \left(C_2^{(2j-1)} + S_1^{(2j-1)}\right)\sin\left(2k-2m+j-7+2\frac{\mathrm{Wi}}{\mathrm{De}}\right)\tau \end{bmatrix} \end{bmatrix} \end{aligned} \tag{22}$$

$$\gamma_m \equiv \frac{J_{2m-1}}{2}\sum_{k=0}^{\infty}\left[\begin{aligned}&\sum_{j=1}^{2}\left[\begin{aligned}&\left(-C_1^{(2j)}+S_2^{(2j)}\right)\cos\left(2k-2m+j+1\right)\tau+\left(-C_1^{(2j)}-S_2^{(2j)}\right)\cos\left(2k+2m+j-1-2\tfrac{\mathrm{Wi}}{\mathrm{De}}\right)\tau\\&+\left(-C_2^{(2j)}-S_1^{(2j)}\right)\sin\left(2k-2m+j+1\right)\tau+\left(C_2^{(2j)}-S_1^{(2j)}\right)\sin\left(2k+2m+j-1-2\tfrac{\mathrm{Wi}}{\mathrm{De}}\right)\tau\end{aligned}\right]\\&+\sum_{j=1}^{3}\left[\begin{aligned}&\left(-C_1^{(2j-1)}-S_2^{(2j-1)}\right)\cos\left(2k+2m+j-2\right)\tau+\left(-C_1^{(2j-1)}+S_2^{(2j-1)}\right)\cos\left(2k-2m+j+2\tfrac{\mathrm{Wi}}{\mathrm{De}}\right)\tau\\&+\left(C_2^{(2j-1)}-S_1^{(2j-1)}\right)\sin\left(2k+2m+j-2\right)\tau+\left(-C_2^{(2j-1)}-S_1^{(2j-1)}\right)\sin\left(2k-2m+j+2\tfrac{\mathrm{Wi}}{\mathrm{De}}\right)\tau\end{aligned}\right]\end{aligned}\right]$$

$$+\frac{J_{2m-1}}{2}\sum_{k=1}^{\infty}\left[\begin{aligned}&\sum_{j=3}^{7}\left[\begin{aligned}&\left(-C_1^{(2j)}+S_2^{(2j)}\right)\cos\left(2k-2m+j-5\right)\tau+\left(-C_1^{(2j)}-S_2^{(2j)}\right)\cos\left(2k+2m+j-7-2\tfrac{\mathrm{Wi}}{\mathrm{De}}\right)\tau\\&+\left(-C_2^{(2j)}-S_1^{(2j)}\right)\sin\left(2k-2m+j-5\right)\tau+\left(C_2^{(2j)}-S_1^{(2j)}\right)\sin\left(2k+2m+j-7-2\tfrac{\mathrm{Wi}}{\mathrm{De}}\right)\tau\end{aligned}\right]\\&+\sum_{j=4}^{8}\left[\begin{aligned}&\left(-C_1^{(2j-1)}-S_2^{(2j-1)}\right)\cos\left(2k+2m+j-8\right)\tau+\left(-C_1^{(2j-1)}+S_2^{(2j-1)}\right)\cos\left(2k-2m+j-6+2\tfrac{\mathrm{Wi}}{\mathrm{De}}\right)\tau\\&+\left(C_2^{(2j-1)}-S_1^{(2j-1)}\right)\sin\left(2k+2m+j-8\right)\tau+\left(-C_2^{(2j-1)}-S_1^{(2j-1)}\right)\sin\left(2k-2m+j-6+2\tfrac{\mathrm{Wi}}{\mathrm{De}}\right)\tau\end{aligned}\right]\end{aligned}\right] \tag{23}$$

$$\delta_m \equiv \frac{J_{2m-1}}{2}\sum_{k=0}^{\infty}\left[\begin{aligned}&\sum_{j=1}^{2}\left[\begin{aligned}&\left(C_1^{(2j)}-S_2^{(2j)}\right)\cos\left(2k+2m+j-1\right)\tau+\left(C_1^{(2j)}+S_2^{(2j)}\right)\cos\left(2k-2m+j+1-2\tfrac{\mathrm{Wi}}{\mathrm{De}}\right)\tau\\&+\left(C_2^{(2j)}+S_1^{(2j)}\right)\sin\left(2k+2m+j-1\right)\tau+\left(-C_2^{(2j)}+S_1^{(2j)}\right)\sin\left(2k-2m+j+1-2\tfrac{\mathrm{Wi}}{\mathrm{De}}\right)\tau\end{aligned}\right]\\&+\sum_{j=1}^{3}\left[\begin{aligned}&\left(C_1^{(2j-1)}+S_2^{(2j-1)}\right)\cos\left(2k-2m+j\right)\tau+\left(C_1^{(2j-1)}-S_2^{(2j-1)}\right)\cos\left(2k+2m+j-2+2\tfrac{\mathrm{Wi}}{\mathrm{De}}\right)\tau\\&+\left(-C_2^{(2j-1)}+S_1^{(2j-1)}\right)\sin\left(2k-2m+j\right)\tau+\left(C_2^{(2j-1)}+S_1^{(2j-1)}\right)\sin\left(2k+2m+j-2+2\tfrac{\mathrm{Wi}}{\mathrm{De}}\right)\tau\end{aligned}\right]\end{aligned}\right]$$

$$+\frac{J_{2m-1}}{2}\sum_{k=1}^{\infty}\left[\begin{aligned}&\sum_{j=3}^{7}\left[\begin{aligned}&\left(C_1^{(2j)}-S_2^{(2j)}\right)\cos\left(2k+2m+j-7\right)\tau+\left(C_1^{(2j)}+S_2^{(2j)}\right)\cos\left(2k-2m+j-5-2\tfrac{\mathrm{Wi}}{\mathrm{De}}\right)\tau\\&+\left(C_2^{(2j)}+S_1^{(2j)}\right)\sin\left(2k+2m+j-7\right)\tau+\left(-C_2^{(2j)}+S_1^{(2j)}\right)\sin\left(2k-2m+j-5-2\tfrac{\mathrm{Wi}}{\mathrm{De}}\right)\tau\end{aligned}\right]\\&+\sum_{j=4}^{8}\left[\begin{aligned}&\left(C_1^{(2j-1)}+S_2^{(2j-1)}\right)\cos\left(2k-2m+j-6\right)\tau+\left(C_1^{(2j-1)}-S_2^{(2j-1)}\right)\cos\left(2k+2m+j-8+2\tfrac{\mathrm{Wi}}{\mathrm{De}}\right)\tau\\&+\left(-C_2^{(2j-1)}+S_1^{(2j-1)}\right)\sin\left(2k-2m+j-6\right)\tau+\left(C_2^{(2j-1)}+S_1^{(2j-1)}\right)\sin\left(2k+2m+j-8+2\tfrac{\mathrm{Wi}}{\mathrm{De}}\right)\tau\end{aligned}\right]\end{aligned}\right] \tag{24}$$

and for the transient response:

$$\breve{\tau}_{yx,0} = -e^{\frac{-\tau}{\mathrm{De}}}\left(\alpha_{0,0}+\sum_{m=1}^{\infty}\left[\alpha_{m,0}+\beta_{m,0}+\gamma_{m,0}+\delta_{m,0}\right]\right) \tag{25}$$

$$\alpha_{m,0} \equiv J_{2m}\left[\left(\sum_{k=0}^{\infty}\sum_{i=1}^{5}C_1^{(i)}+\sum_{k=1}^{\infty}\sum_{i=6}^{15}C_1^{(i)}\right)\cos\left(2m+\tfrac{\mathrm{Wi}}{\mathrm{De}}\right)\tau+\left(\sum_{k=0}^{\infty}\sum_{i=1}^{5}C_2^{(i)}+\sum_{k=1}^{\infty}\sum_{i=6}^{15}C_2^{(i)}\right)\sin\left(2m+\tfrac{\mathrm{Wi}}{\mathrm{De}}\right)\tau\right] \tag{26}$$

$$\beta_{m,0} \equiv J_{2m}\left[\left(\sum_{k=0}^{\infty}\sum_{i=1}^{5}C_1^{(i)}+\sum_{k=1}^{\infty}\sum_{i=6}^{15}C_1^{(i)}\right)\cos\left(2m-\tfrac{\mathrm{Wi}}{\mathrm{De}}\right)\tau-\left(\sum_{k=0}^{\infty}\sum_{i=1}^{5}C_2^{(i)}+\sum_{k=1}^{\infty}\sum_{i=6}^{15}C_2^{(i)}\right)\sin\left(2m-\tfrac{\mathrm{Wi}}{\mathrm{De}}\right)\tau\right] \tag{27}$$

$$\gamma_{m,0} \equiv J_{2m-1}\left[-\left(\sum_{k=0}^{\infty}\sum_{i=1}^{5}C_1^{(i)}+\sum_{k=1}^{\infty}\sum_{i=6}^{15}C_1^{(i)}\right)\cos\left(2m-1-\tfrac{\mathrm{Wi}}{\mathrm{De}}\right)\tau+\left(\sum_{k=0}^{\infty}\sum_{i=1}^{5}C_2^{(i)}+\sum_{k=1}^{\infty}\sum_{i=6}^{15}C_2^{(i)}\right)\sin\left(2m-1-\tfrac{\mathrm{Wi}}{\mathrm{De}}\right)\tau\right] \tag{28}$$

$$\delta_{m,0} \equiv J_{2m-1}\left[\left(\sum_{k=0}^{\infty}\sum_{i=1}^{5}C_1^{(i)}+\sum_{k=1}^{\infty}\sum_{i=6}^{15}C_1^{(i)}\right)\cos\left(2m-1+\tfrac{\mathrm{Wi}}{\mathrm{De}}\right)\tau+\left(\sum_{k=0}^{\infty}\sum_{i=1}^{5}C_2^{(i)}+\sum_{k=1}^{\infty}\sum_{i=6}^{15}C_2^{(i)}\right)\sin\left(2m-1+\tfrac{\mathrm{Wi}}{\mathrm{De}}\right)\tau\right] \tag{29}$$

where $J_\nu \equiv J_\nu(\mathrm{Wi}/\mathrm{De})$ is the νth-order Bessel function of first kind of the argument Wi/De. Appendix VIIIb provides $C_1^{(i)}$, $C_2^{(i)}$, $S_1^{(i)}$ and $S_2^{(i)}$. The transient response, given in Eq. (25) [with Eqs. (26)–(29)], can be superposed onto the alternant one, by Eq. (20) [with Eqs. (21)–(24)], to form the startup response, which is the main result of this work.

By contrast with LAOS, where only odd harmonics are observed [46,47], our main results uncover fractional harmonics in the shear stress response to *ud*LAOS. By *fractional*, we mean that these occur at frequencies that are not integer multiples of the superposed oscillation frequency. See, for instance, the $\cos\left(2k-2m+j-2\frac{\mathrm{Wi}}{\mathrm{De}}\right)\tau$ or $\sin\left(2k-2m+j-2\frac{\mathrm{Wi}}{\mathrm{De}}\right)\tau$ terms in Eq. (21) in which $-2\frac{\mathrm{Wi}}{\mathrm{De}}$ is added to the integer multiple of the superposed oscillation frequency $2k-2m+j$. When $-2\frac{\mathrm{Wi}}{\mathrm{De}}$ is not an integer, we thus get fractional harmonics. We find this result to be both beautiful and fascinating. Whereas for the alternant part (Eq. (20) [with Eqs. (21)–(24)]), we get both integer and fractional harmonics, for the transient part, we get just the fractional harmonics (Eq. (25) [with Eqs. (26)–(29)]).

Since the cause of fractional harmonics is Wi/De, these vanish from both parts for unidirectional small-amplitude oscillatory shear flow (*ud*SAOS), where $\mathrm{Wi}\to 0$. We will discuss this in Section III. We alert the reader that LAOS is not a limiting case of *ud*LAOS. This is why the alternance of our main result does not reduce to the exact solution for the corotational Jeffreys fluid in LAOS (Eq. (65) of [48] with $\mu_0 = \mu_1 = \mu_2 = \nu_1 = \nu_2 = 0$).

For consistency, we first check our main result with finite difference calculations, applying an adaptive Runge-Kutta numerical scheme [49,50] to solve Eqs. (8). We coded Eq. (8), subject to the stress-free initial condition, using the *ode45* scheme in MATLAB (Version R2019a) on a Macbook Pro (3.1 GHz Intel Core i7 processor with 16GB 1867 MHz DDR3 memory) employing the High Sierra operating system (Version 10.13.6) operating system. For each point in Figure 2 through Figure 5, we find such an evaluation to consume less than a millisecond of CPU time. Figure 2 through Figure 4 show good consistency between the alternant part of our exact solution, Eq. (20) [with Eqs. (21)–(24)], and the corresponding finite difference calculations, as it must. Figure 5 shows good consistency between the startup response of our exact solution, Eqs. (20) and (25) [with Eqs. (21)–(24) and Eqs. (26)–(29)], and the corresponding finite difference calculations, as it must.

III PIPKIN AND EWOLDT DIAGRAMS

On the limiting behaviors consistency check of our main result, we first consider *ud*SAOS, when $\mathrm{Wi}\to 0$. Evaluating the limit as $\mathrm{Wi}\to 0$ in our main result, we find:

$$\lim_{\mathrm{Wi}\to 0}\frac{\breve{\tau}_{yx}}{\mathrm{Wi}} = -1-\frac{1+\breve{\lambda}_2\,\mathrm{De}^2}{1+\mathrm{De}^2}\cos\tau-\frac{\left(1-\breve{\lambda}_2\right)\mathrm{De}}{1+\mathrm{De}^2}\sin\tau \tag{30}$$

Eq. (30) poses the form of the zero-shear solution superposes onto the small-amplitude oscillatory shearing solution (Eq. (7.3-29) of [51]). For low frequency consistency check, $\mathrm{De}\to 0$, we follow the finite difference scheme given at the end of Section II to construct Figure 6. This figure shows good agreement between the two approaches.

To illustrate the comprehensive shear stress responses from the corotational Jeffreys fluid, We depict the Ewoldt grid in Figure 7 using our main result [Eq. (20) with Eqs. (21)–(24)]. This figure shows loop ellipticity along the De-axis signifying SAOS predominance (*i.e.* linear viscoelasticity), whereas loops along Wi-axis shows steady shear predominance considering the loop area is zero. The departures from linear responses increase with Wi.

IV APPLICATIONS: HEMORHEOLOGY

This section applies the shear stress exact solution for the CJ fluid derived in Section II [Eq. (20) with Eqs. (21)–(24)] to predict the nonlinear rheology of blood in *ud*LAOS experiments. We also show the potential of the multimode O8 framework, which contains the CJ fluid as special case, to predict the nonlinear behaviour. Two rheological experiments were conducted on three blood samples: (i) steady shear and (ii) *ud*LAOS; detailed information of these two experiments can be found in Section III.B. of [9], and biochemistry of three samples are tabulated in TABLE I. Of [9].

Complex fluids are often multimode, ranging from polymer melts [16,40,41], polymer solutions [52,53,54] and blood. By contrast, some wormlike micellar solutions are not multimode [43]. The essence of multimode fluids is infinitely discretized model parameters, for instance relaxation and retardation times. We thus first generalize our O8 constitutive equation, given in Eq. (6) to multimode framework:

$$\begin{aligned}\boldsymbol{\tau}_k+\lambda_{1,k}\frac{\mathscr{D}\boldsymbol{\tau}_k}{\mathscr{D}t}+\frac{1}{2}\mu_{0,k}(\mathrm{tr}\,\boldsymbol{\tau}_k)\dot{\boldsymbol{\gamma}}-\frac{1}{2}\mu_{1,k}\{\boldsymbol{\tau}_k\cdot\dot{\boldsymbol{\gamma}}+\dot{\boldsymbol{\gamma}}\cdot\boldsymbol{\tau}_k\}+\frac{1}{2}\nu_{1,k}(\boldsymbol{\tau}_k:\dot{\boldsymbol{\gamma}})\boldsymbol{\delta}\\ =-\eta_k\left(\dot{\boldsymbol{\gamma}}+\lambda_{2,k}\frac{\mathscr{D}\dot{\boldsymbol{\gamma}}}{\mathscr{D}t}-\mu_{2,k}\{\dot{\boldsymbol{\gamma}}\cdot\dot{\boldsymbol{\gamma}}\}+\frac{1}{2}\nu_{2,k}(\dot{\boldsymbol{\gamma}}:\dot{\boldsymbol{\gamma}})\boldsymbol{\delta}\right)\end{aligned} \tag{31}$$

where:

$$\boldsymbol{\tau}=\sum_{k=1}^{\infty}\boldsymbol{\tau}_k \tag{32}$$

so that the bulk shear stress response is given by:

$$\tau_{yx}=\sum_{k=1}^{\infty}\tau_{yx,k} \tag{33}$$

Essentially, each of the 8 constants is replaced by its *k*th element of the discrete spectrum following the following *Spriggs relations* (Eq. (6.1-14) and (6.1-15) of [51]):

$$\eta_k\equiv\frac{k^{-\alpha}}{\zeta(\alpha)}\eta_0,\ \lambda_{1,k}=k^{-\alpha}\lambda_1,\ \lambda_{2,k}=k^{-\alpha}\lambda_2,\ \mu_{0,k}=k^{-\alpha}\mu_0,\ \mu_{1,k}=k^{-\alpha}\mu_1,\ \mu_{2,k}=k^{-\alpha}\mu_2 \tag{34}$$

$$\nu_{1,k}=k^{-\alpha}\nu_1,\ \nu_{2,k}=k^{-\alpha}\nu_2$$

where $\zeta(\alpha)$ is the Riemann zeta function of the Spriggs exponent α (Eq. (6.2–11a) of [51]). The Spriggs exponent is ranges from about 2 to 4, but sometimes it is reported slightly outside of this range [16]. For a single relaxation time fluid, $\alpha=\infty$.

Concerning the model fitting protocol, the multimode CJ contains four model parameters ($\alpha,\eta_0,\lambda_1,\lambda_2$) all of which can be readily and uniquely fitted to the steady shear experiment. These three fitted parameters are then used in predictions of the *ud*LAOS response. The multimode O8, however, comprises nine parameters (α,η_0,λ_1,

$\lambda_2, \mu_0, \mu_1, \mu_2, \nu_1, \nu_2$), four of which ($\alpha, \eta_0, \lambda_1, \lambda_2$) are fitted to the steady shear experiment, and rest are fitted to the linear experimental condition of *ud*LAOS response, $\omega = 10\,\mathrm{rad/s}$ and $\dot{\gamma}^0 = 10\,\mathrm{s}^{-1}$. We follow the method in the "Results" section of [55] to minimize the residual sum of squares (RSS) in these fittings. These nine fitted parameters are then used to predict the nonlinear *ud*LAOS response.

For the steady shear viscosity function, multimode O8 yields:

$$\frac{\eta(\dot{\gamma})}{\eta_0} = \frac{1}{\zeta(\alpha)} \sum_{k=1}^{\infty} \left[k^{-\alpha} \frac{k^{2\alpha} + \left(\breve{\lambda}_2 + \breve{\mu}_0\left(\breve{\mu}_2 - \frac{3}{2}\breve{\nu}_2\right) - \breve{\mu}_1\left(\breve{\mu}_2 - \breve{\nu}_2\right)\right)\left(\lambda_1\dot{\gamma}\right)^2}{k^{2\alpha} + \left(1 + \breve{\mu}_0\left(\breve{\mu}_1 - \frac{3}{2}\breve{\nu}_1\right) - \breve{\mu}_1\left(\breve{\mu}_1 - \breve{\nu}_1\right)\right)\left(\lambda_1\dot{\gamma}\right)^2} \right] \tag{35}$$

For multimode CJ, by assigning $\mu_0 = \mu_1 = \mu_2 = \nu_1 = \nu_2 = 0$, Eq. (35) becomes:

$$\frac{\eta(\dot{\gamma})}{\eta_0} = \frac{1}{\zeta(\alpha)} \sum_{k=1}^{\infty} \left[k^{-\alpha} \frac{k^{2\alpha} + \breve{\lambda}_2\left(\lambda_1\dot{\gamma}\right)^2}{k^{2\alpha} + \left(\lambda_1\dot{\gamma}\right)^2} \right] \tag{36}$$

which agrees with Eq. (185) of [52], as it must. Following the fitting protocol discussed above, we fit both Eqs. (35) and (36) to the measured steady shear viscosity, reported in **FIG. 3.** Of [9] as well as the Supplementary Materials therein. Figure 8 shows the fitted multimode CJ and O8 to the reported steady shear data. The minimized residual sum of squares of both fluid models are tabulated in Table IV.

For *ud*LAOS, multimode CJ yields the following shear stress response:

$$\tau_{yx} = \sum_{k=1}^{\infty} \frac{\eta_k}{\lambda_{1,k}} \left(\alpha_{0,k} + \sum_{m=1}^{\infty} \left[\alpha_{m,k} + \beta_{m,k} + \gamma_{m,k} + \delta_{m,k} \right] \right) \tag{37}$$

where $\alpha_{m,k}, \beta_{m,k}, \gamma_{m,k}, \delta_{m,k}$ are calculated by replacing O8 parameters with their *k*th spectral elements in [Eqs. (21)–(24)]. The shear stress response from multimode O8 is arrived at by means of finite difference calculations following the protocol given immediately after Eq. (24). For this, we need the set of governing equations for the shear stress responses:

$$\begin{bmatrix} d\tau_{yx,k}/d\tau \\ dN_{1,k}/d\tau \\ dN_{2,k}/d\tau \\ d\tau_{zz,k}/d\tau \end{bmatrix} = \frac{\eta_k}{\lambda_{1,k}} \begin{bmatrix} \begin{array}{c} -\frac{1}{\mathrm{De}}\breve{\tau}_{yx} - \frac{1}{2}\left[1 + \breve{\mu}_0 - \breve{\mu}_1\right]\frac{\mathrm{Wi}}{\mathrm{De}}\left(1+\cos\tau\right)\breve{N}_1 - \left[\breve{\mu}_0 - \breve{\mu}_1\right]\frac{\mathrm{Wi}}{\mathrm{De}}\left(1+\cos\tau\right)\breve{N}_2 \\ -\left[\frac{3}{2}\breve{\mu}_0 - \breve{\mu}_1\right]\frac{\mathrm{Wi}}{\mathrm{De}}\left(1+\cos\tau\right)\breve{\tau}_{zz} - \frac{\mathrm{Wi}}{\mathrm{De}}\left(1+\cos\tau\right) + \breve{\lambda}_2\,\mathrm{Wi}\sin\tau \end{array} \\ 2\frac{\mathrm{Wi}}{\mathrm{De}}\left(1+\cos\tau\right)\breve{\tau}_{yx} - \frac{1}{\mathrm{De}}\breve{N}_1 + 2\breve{\lambda}_2\frac{\mathrm{Wi}^2}{\mathrm{De}}\left(1+\cos\tau\right)^2 \\ -\left(1-\breve{\mu}_1\right)\frac{\mathrm{Wi}}{\mathrm{De}}\left(1+\cos\tau\right)\breve{\tau}_{yx} - \frac{1}{\mathrm{De}}\breve{N}_2 - \left(\breve{\lambda}_2 - \breve{\mu}_2\right)\frac{\mathrm{Wi}^2}{\mathrm{De}}\left(1+\cos\tau\right)^2 \\ -\breve{\nu}_1\frac{\mathrm{Wi}}{\mathrm{De}}\left(1+\cos\tau\right)\breve{\tau}_{yx} - \frac{1}{\mathrm{De}}\breve{\tau}_{zz} - \breve{\nu}_2\frac{\mathrm{Wi}^2}{\mathrm{De}}\left(1+\cos\tau\right)^2 \end{bmatrix} \tag{38}$$

Table I summarizes all dimensional variables and symbols, and Table II gathers the dimensionless ones. Figure 9 through Figure 11 depict predictions from both multimode CJ and O8 comparing with the reported *ud*LAOS measurements; good predictions are observed. The improvement of the multimode O8 over the multimode CJ is clearly illustrated in by the minimized RSS in Table IV. Overall, the predictions from both models fall slightly under the measured values, and loop skewness is observed when $\mathrm{Wi/De}$ is high, in Figure 10 and Figure 11.

V CONCLUSION

We assessed nonlinear hemorheological behaviors using unidirectional large-amplitude oscillatory shear flow (*ud*LAOS), conceived expressly for blood analysis. We generate this experiment by superposing LAOS onto steady shear flow, and in particular, by equating the characteristic shear rates in both experiments (*i.e.* $\dot{\gamma}_s = \dot{\gamma}^0$). It is thus intended to best represent the flow in veins and arteries since virtually no backflow is observed in physiological pulsatile blood flow. To model human blood, we considered the simplest constitutive model that can predict infinite-shear viscosity, the corotational Jeffrey (CJ) fluid. To gain theoretical insights, we derived an exact analytical expression for shear stress response of this model.

By contrast with LAOS, where only odd harmonics are observed [46,47], our main results uncover fractional harmonics in the shear stress response to *ud*LAOS. By *fractional*, we mean that these occur at frequencies that are not integer multiples of the superposed oscillation frequency. We find this discovery to be both beautiful and fascinating. Whereas for the alternant part (Eq. (20) [with Eqs. (21)–(24)]), we get both integer and fractional harmonics, for the transient part, we get just the fractional harmonics (Eq. (25) [with Eqs. (26)–(29)]).

We generalized the CJ model to multimode for best representation of three blood samples from three different healthy donors. To further improve our model predictions, we consider the multimode Oldroyd 8-constant (O8) framework, which contains the CJ fluid as its special case.

We find that the multimode CJ fluid adequately describes the steady shear viscosity functions measured for three different healthy donors. We further find that adding two more specific nonlinear constants to the multimode CJ fluid also adequately describes the behaviors of these same bloods in *ud*LAOS. This new Oldroyd 5-constant model may find usefulness in monitoring health through *ud*LAOS. Of course, we understand that unhealthy blood may, or may not, require more than five of the Oldroyd 8-constant parameters.

Inserting our new result for the shear stress response, [Eq. (20) with Eqs. (21)–(24)], into Eq. (8)(b) is the path toward deepening our understanding of the normal stress difference responses in *ud*LAOS, and thus, in physiological pulsatile blood flow. We leave this exploration for another day.

VI SUPPLEMENTARY MATERIAL

Section I of the Supplementary Material provides the detailed derivation of the stress governing equation for both the Oldroyd 8-constant framework, Eq. (7), and its special case, the corotational Jeffreys fluid, Eq. (8). Section II of the Supplementary Material provides the detailed derivation of our main results, for the alternant part, Eq. (20) [with Eqs. (21)–(24)], and the transient part, Eq. (25) [with Eqs. (26)–(29)].

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VIII APPENDICES

a $I_1 - I_8$ and $I_{1,0} - I_{8,0}$

$$I_1 = \mathrm{De}\, e^{\frac{\tau}{\mathrm{De}}} \left[\begin{array}{l} \sum_{k=0}^{\infty} \left[\phi_1^{(1)} \cos\left(2k + \frac{\mathrm{Wi}}{\mathrm{De}}\right)\tau + \psi_1^{(1)} \sin\left(2k + \frac{\mathrm{Wi}}{\mathrm{De}}\right)\tau \right] \\ + \sum_{k=1}^{\infty} \left[\begin{array}{l} \phi_1^{(2)} \cos\left(2k - 1 - \frac{\mathrm{Wi}}{\mathrm{De}}\right)\tau + \phi_1^{(3)} \cos\left(2k - 1 + \frac{\mathrm{Wi}}{\mathrm{De}}\right)\tau + \phi_1^{(4)} \cos\left(2k - \frac{\mathrm{Wi}}{\mathrm{De}}\right)\tau \\ + \psi_1^{(2)} \sin\left(2k - 1 - \frac{\mathrm{Wi}}{\mathrm{De}}\right)\tau + \psi_1^{(3)} \sin\left(2k - 1 + \frac{\mathrm{Wi}}{\mathrm{De}}\right)\tau + \psi_1^{(4)} \sin\left(2k - \frac{\mathrm{Wi}}{\mathrm{De}}\right)\tau \end{array} \right] \end{array} \right] \tag{39}$$

$$I_2 = \mathrm{De}\, e^{\frac{\tau}{\mathrm{De}}} \left[\begin{array}{l} \sum_{k=0}^{\infty} \left[\phi_2^{(1)} \cos\left(2k + \frac{\mathrm{Wi}}{\mathrm{De}}\right)\tau + \psi_2^{(1)} \sin\left(2k + \frac{\mathrm{Wi}}{\mathrm{De}}\right)\tau \right] \\ + \sum_{k=1}^{\infty} \left[\begin{array}{l} \phi_2^{(2)} \cos\left(2k - 1 - \frac{\mathrm{Wi}}{\mathrm{De}}\right)\tau + \phi_2^{(3)} \cos\left(2k - 1 + \frac{\mathrm{Wi}}{\mathrm{De}}\right)\tau + \phi_2^{(4)} \cos\left(2k - \frac{\mathrm{Wi}}{\mathrm{De}}\right)\tau \\ + \psi_2^{(2)} \sin\left(2k - 1 - \frac{\mathrm{Wi}}{\mathrm{De}}\right)\tau + \psi_2^{(3)} \sin\left(2k - 1 + \frac{\mathrm{Wi}}{\mathrm{De}}\right)\tau + \psi_2^{(4)} \sin\left(2k - \frac{\mathrm{Wi}}{\mathrm{De}}\right)\tau \end{array} \right] \end{array} \right] \tag{40}$$

$$I_3 = \frac{\mathrm{De}\, e^{\frac{\tau}{\mathrm{De}}}}{2} \left[\begin{array}{l} \sum_{k=0}^{\infty} \left[\begin{array}{l} \phi_3^{(1)} \cos\left(2k + 1 - \frac{\mathrm{Wi}}{\mathrm{De}}\right)\tau + \phi_3^{(2)} \cos\left(2k + 1 + \frac{\mathrm{Wi}}{\mathrm{De}}\right)\tau \\ + \psi_3^{(1)} \sin\left(2k + 1 - \frac{\mathrm{Wi}}{\mathrm{De}}\right)\tau + \psi_3^{(2)} \sin\left(2k + 1 + \frac{\mathrm{Wi}}{\mathrm{De}}\right)\tau \end{array} \right] \\ + \sum_{k=1}^{\infty} \left[\begin{array}{l} \phi_3^{(3)} \cos\left(2k - 2 - \frac{\mathrm{Wi}}{\mathrm{De}}\right)\tau + \phi_3^{(4)} \cos\left(2k - 2 + \frac{\mathrm{Wi}}{\mathrm{De}}\right)\tau + \phi_3^{(5)} \cos\left(2k - 1 - \frac{\mathrm{Wi}}{\mathrm{De}}\right)\tau \\ + \phi_3^{(6)} \cos\left(2k - 1 + \frac{\mathrm{Wi}}{\mathrm{De}}\right)\tau + \phi_3^{(7)} \cos\left(2k - \frac{\mathrm{Wi}}{\mathrm{De}}\right)\tau + \phi_3^{(8)} \cos\left(2k + \frac{\mathrm{Wi}}{\mathrm{De}}\right)\tau \\ + \psi_3^{(3)} \sin\left(2k - 2 - \frac{\mathrm{Wi}}{\mathrm{De}}\right)\tau + \psi_3^{(4)} \sin\left(2k - 2 + \frac{\mathrm{Wi}}{\mathrm{De}}\right)\tau + \psi_3^{(5)} \sin\left(2k - 1 - \frac{\mathrm{Wi}}{\mathrm{De}}\right)\tau \\ + \psi_3^{(6)} \sin\left(2k - 1 + \frac{\mathrm{Wi}}{\mathrm{De}}\right)\tau + \psi_3^{(7)} \sin\left(2k - \frac{\mathrm{Wi}}{\mathrm{De}}\right)\tau + \psi_3^{(8)} \sin\left(2k + \frac{\mathrm{Wi}}{\mathrm{De}}\right)\tau \end{array} \right] \end{array} \right] \tag{41}$$

$$I_4 = \frac{\mathrm{De}\, e^{\frac{\tau}{\mathrm{De}}}}{2} \left[\begin{array}{l} \sum_{k=0}^{\infty} \left[\begin{array}{l} \phi_4^{(1)} \cos\left(2k + 1 - \frac{\mathrm{Wi}}{\mathrm{De}}\right)\tau + \phi_4^{(2)} \cos\left(2k + 1 + \frac{\mathrm{Wi}}{\mathrm{De}}\right)\tau \\ + \psi_4^{(1)} \sin\left(2k + 1 - \frac{\mathrm{Wi}}{\mathrm{De}}\right)\tau + \psi_4^{(2)} \sin\left(2k + 1 + \frac{\mathrm{Wi}}{\mathrm{De}}\right)\tau \end{array} \right] \\ + \sum_{k=1}^{\infty} \left[\begin{array}{l} \phi_4^{(3)} \cos\left(2k - 2 - \frac{\mathrm{Wi}}{\mathrm{De}}\right)\tau + \phi_4^{(4)} \cos\left(2k - 2 + \frac{\mathrm{Wi}}{\mathrm{De}}\right)\tau + \phi_4^{(5)} \cos\left(2k - 1 - \frac{\mathrm{Wi}}{\mathrm{De}}\right)\tau \\ + \phi_4^{(6)} \cos\left(2k - 1 + \frac{\mathrm{Wi}}{\mathrm{De}}\right)\tau + \phi_4^{(7)} \cos\left(2k - \frac{\mathrm{Wi}}{\mathrm{De}}\right)\tau + \phi_4^{(8)} \cos\left(2k + \frac{\mathrm{Wi}}{\mathrm{De}}\right)\tau \\ + \psi_4^{(3)} \sin\left(2k - 2 - \frac{\mathrm{Wi}}{\mathrm{De}}\right)\tau + \psi_4^{(4)} \sin\left(2k - 2 + \frac{\mathrm{Wi}}{\mathrm{De}}\right)\tau + \psi_4^{(5)} \sin\left(2k - 1 - \frac{\mathrm{Wi}}{\mathrm{De}}\right)\tau \\ + \psi_4^{(6)} \sin\left(2k - 1 + \frac{\mathrm{Wi}}{\mathrm{De}}\right)\tau + \psi_4^{(7)} \sin\left(2k - \frac{\mathrm{Wi}}{\mathrm{De}}\right)\tau + \psi_4^{(8)} \sin\left(2k + \frac{\mathrm{Wi}}{\mathrm{De}}\right)\tau \end{array} \right] \end{array} \right] \tag{42}$$

$$I_5=\frac{\mathrm{De}\,e^{\frac{\tau}{\mathrm{De}}}}{2}\left[\begin{array}{l}\displaystyle\sum_{k=0}^{\infty}\left[\begin{array}{l}\phi_5^{(1)}\cos\left(2k+1-\tfrac{\mathrm{Wi}}{\mathrm{De}}\right)\tau+\phi_5^{(2)}\cos\left(2k+1+\tfrac{\mathrm{Wi}}{\mathrm{De}}\right)\tau\\+\psi_5^{(1)}\sin\left(2k+1-\tfrac{\mathrm{Wi}}{\mathrm{De}}\right)\tau+\psi_5^{(2)}\sin\left(2k+1+\tfrac{\mathrm{Wi}}{\mathrm{De}}\right)\tau\end{array}\right]\\+\displaystyle\sum_{k=1}^{\infty}\left[\begin{array}{l}+\phi_5^{(3)}\cos\left(2k-2-\tfrac{\mathrm{Wi}}{\mathrm{De}}\right)\tau+\phi_5^{(4)}\cos\left(2k-2+\tfrac{\mathrm{Wi}}{\mathrm{De}}\right)\tau+\phi_5^{(5)}\cos\left(2k-1-\tfrac{\mathrm{Wi}}{\mathrm{De}}\right)\tau\\+\phi_5^{(6)}\cos\left(2k-1+\tfrac{\mathrm{Wi}}{\mathrm{De}}\right)\tau+\phi_5^{(7)}\cos\left(2k-\tfrac{\mathrm{Wi}}{\mathrm{De}}\right)\tau+\phi_5^{(8)}\cos\left(2k+\tfrac{\mathrm{Wi}}{\mathrm{De}}\right)\tau\\+\psi_5^{(3)}\sin\left(2k-2-\tfrac{\mathrm{Wi}}{\mathrm{De}}\right)\tau+\psi_5^{(4)}\sin\left(2k-2+\tfrac{\mathrm{Wi}}{\mathrm{De}}\right)\tau+\psi_5^{(5)}\sin\left(2k-1-\tfrac{\mathrm{Wi}}{\mathrm{De}}\right)\tau\\+\psi_5^{(6)}\sin\left(2k-1+\tfrac{\mathrm{Wi}}{\mathrm{De}}\right)\tau+\psi_5^{(7)}\sin\left(2k-\tfrac{\mathrm{Wi}}{\mathrm{De}}\right)\tau+\psi_5^{(8)}\sin\left(2k+\tfrac{\mathrm{Wi}}{\mathrm{De}}\right)\tau\end{array}\right]\end{array}\right] \tag{43}$$

$$I_6=\frac{\mathrm{De}\,e^{\frac{\tau}{\mathrm{De}}}}{2}\left[\begin{array}{l}\displaystyle\sum_{k=0}^{\infty}\left[\begin{array}{l}\phi_6^{(1)}\cos\left(2k+1-\tfrac{\mathrm{Wi}}{\mathrm{De}}\right)\tau+\phi_6^{(2)}\cos\left(2k+1+\tfrac{\mathrm{Wi}}{\mathrm{De}}\right)\tau\\+\psi_6^{(1)}\sin\left(2k+1-\tfrac{\mathrm{Wi}}{\mathrm{De}}\right)\tau+\psi_6^{(2)}\sin\left(2k+1+\tfrac{\mathrm{Wi}}{\mathrm{De}}\right)\tau\end{array}\right]\\+\displaystyle\sum_{k=1}^{\infty}\left[\begin{array}{l}\phi_6^{(3)}\cos\left(2k-2-\tfrac{\mathrm{Wi}}{\mathrm{De}}\right)\tau+\phi_6^{(4)}\cos\left(2k-2+\tfrac{\mathrm{Wi}}{\mathrm{De}}\right)\tau+\phi_6^{(5)}\cos\left(2k-1-\tfrac{\mathrm{Wi}}{\mathrm{De}}\right)\tau\\+\phi_6^{(6)}\cos\left(2k-1+\tfrac{\mathrm{Wi}}{\mathrm{De}}\right)\tau+\phi_6^{(7)}\cos\left(2k-\tfrac{\mathrm{Wi}}{\mathrm{De}}\right)\tau+\phi_6^{(8)}\cos\left(2k+\tfrac{\mathrm{Wi}}{\mathrm{De}}\right)\tau\\+\psi_6^{(3)}\sin\left(2k-2-\tfrac{\mathrm{Wi}}{\mathrm{De}}\right)\tau+\psi_6^{(4)}\sin\left(2k-2+\tfrac{\mathrm{Wi}}{\mathrm{De}}\right)\tau+\psi_6^{(5)}\sin\left(2k-1-\tfrac{\mathrm{Wi}}{\mathrm{De}}\right)\tau\\+\psi_6^{(6)}\sin\left(2k-1+\tfrac{\mathrm{Wi}}{\mathrm{De}}\right)\tau+\psi_6^{(7)}\sin\left(2k-\tfrac{\mathrm{Wi}}{\mathrm{De}}\right)\tau+\psi_6^{(8)}\sin\left(2k+\tfrac{\mathrm{Wi}}{\mathrm{De}}\right)\tau\end{array}\right]\end{array}\right] \tag{44}$$

$$I_7=\frac{\mathrm{De}\,e^{\frac{\tau}{\mathrm{De}}}}{2}\left[\begin{array}{l}\displaystyle\sum_{k=0}^{\infty}\left[\begin{array}{l}\phi_7^{(1)}\cos\left(2k+2-\tfrac{\mathrm{Wi}}{\mathrm{De}}\right)\tau+\phi_7^{(2)}\cos\left(2k+2+\tfrac{\mathrm{Wi}}{\mathrm{De}}\right)\tau\\+\psi_7^{(1)}\sin\left(2k+2-\tfrac{\mathrm{Wi}}{\mathrm{De}}\right)\tau+\psi_7^{(2)}\sin\left(2k+2+\tfrac{\mathrm{Wi}}{\mathrm{De}}\right)\tau\end{array}\right]\\+\displaystyle\sum_{k=1}^{\infty}\left[\begin{array}{l}\phi_7^{(3)}\cos\left(2k-3-\tfrac{\mathrm{Wi}}{\mathrm{De}}\right)\tau+\phi_7^{(4)}\cos\left(2k-3+\tfrac{\mathrm{Wi}}{\mathrm{De}}\right)\tau+\phi_7^{(5)}\cos\left(2k-2-\tfrac{\mathrm{Wi}}{\mathrm{De}}\right)\tau\\+\phi_7^{(6)}\cos\left(2k-2+\tfrac{\mathrm{Wi}}{\mathrm{De}}\right)\tau+\phi_7^{(7)}\cos\left(2k+1-\tfrac{\mathrm{Wi}}{\mathrm{De}}\right)\tau+\phi_7^{(8)}\cos\left(2k+1+\tfrac{\mathrm{Wi}}{\mathrm{De}}\right)\tau\\+\psi_7^{(3)}\sin\left(2k-3-\tfrac{\mathrm{Wi}}{\mathrm{De}}\right)\tau+\psi_7^{(4)}\sin\left(2k-3+\tfrac{\mathrm{Wi}}{\mathrm{De}}\right)\tau+\psi_7^{(5)}\sin\left(2k-2-\tfrac{\mathrm{Wi}}{\mathrm{De}}\right)\tau\\+\psi_7^{(6)}\sin\left(2k-2+\tfrac{\mathrm{Wi}}{\mathrm{De}}\right)\tau+\psi_7^{(7)}\sin\left(2k+1-\tfrac{\mathrm{Wi}}{\mathrm{De}}\right)\tau+\psi_7^{(8)}\sin\left(2k+1+\tfrac{\mathrm{Wi}}{\mathrm{De}}\right)\tau\end{array}\right]\end{array}\right] \tag{45}$$

$$I_8=\frac{\mathrm{De}\,e^{\frac{\tau}{\mathrm{De}}}}{2}\left[\begin{array}{l}\displaystyle\sum_{k=0}^{\infty}\left[\begin{array}{l}\phi_8^{(1)}\cos\left(2k+2-\tfrac{\mathrm{Wi}}{\mathrm{De}}\right)\tau+\phi_8^{(2)}\cos\left(2k+2+\tfrac{\mathrm{Wi}}{\mathrm{De}}\right)\tau\\+\psi_8^{(1)}\sin\left(2k+2-\tfrac{\mathrm{Wi}}{\mathrm{De}}\right)\tau+\psi_8^{(2)}\sin\left(2k+2+\tfrac{\mathrm{Wi}}{\mathrm{De}}\right)\tau\end{array}\right]\\+\displaystyle\sum_{k=1}^{\infty}\left[\begin{array}{l}\phi_8^{(3)}\cos\left(2k-3-\tfrac{\mathrm{Wi}}{\mathrm{De}}\right)\tau+\phi_8^{(4)}\cos\left(2k-3+\tfrac{\mathrm{Wi}}{\mathrm{De}}\right)\tau+\phi_8^{(5)}\cos\left(2k-2-\tfrac{\mathrm{Wi}}{\mathrm{De}}\right)\tau\\+\phi_8^{(6)}\cos\left(2k-2+\tfrac{\mathrm{Wi}}{\mathrm{De}}\right)\tau+\phi_8^{(7)}\cos\left(2k+1-\tfrac{\mathrm{Wi}}{\mathrm{De}}\right)\tau+\phi_8^{(8)}\cos\left(2k+1+\tfrac{\mathrm{Wi}}{\mathrm{De}}\right)\tau\\+\psi_8^{(3)}\sin\left(2k-3-\tfrac{\mathrm{Wi}}{\mathrm{De}}\right)\tau+\psi_8^{(4)}\sin\left(2k-3+\tfrac{\mathrm{Wi}}{\mathrm{De}}\right)\tau+\psi_8^{(5)}\sin\left(2k-2-\tfrac{\mathrm{Wi}}{\mathrm{De}}\right)\tau\\+\psi_8^{(6)}\sin\left(2k-2+\tfrac{\mathrm{Wi}}{\mathrm{De}}\right)\tau+\psi_8^{(7)}\sin\left(2k+1-\tfrac{\mathrm{Wi}}{\mathrm{De}}\right)\tau+\psi_8^{(8)}\sin\left(2k+1+\tfrac{\mathrm{Wi}}{\mathrm{De}}\right)\tau\end{array}\right]\end{array}\right] \tag{46}$$

$$I_{1,0}\equiv\lim_{\tau\to0}I_1=\mathrm{De}\left[\sum_{k=0}^{\infty}\phi_1^{(1)}+\sum_{k=1}^{\infty}\left(\phi_1^{(2)}+\phi_1^{(3)}+\phi_1^{(4)}\right)\right] \tag{47}$$

$$I_{2,0}\equiv\lim_{\tau\to0}I_2=\mathrm{De}\left[\sum_{k=0}^{\infty}\phi_2^{(1)}+\sum_{k=1}^{\infty}\left(\phi_2^{(2)}+\phi_2^{(3)}+\phi_2^{(4)}\right)\right] \tag{48}$$

$$I_{3,0}\equiv\lim_{\tau\to0}I_3=\frac{\mathrm{De}}{2}\left[\sum_{k=0}^{\infty}\left(\phi_3^{(1)}+\phi_3^{(2)}\right)+\sum_{k=1}^{\infty}\left(\phi_3^{(3)}+\phi_3^{(4)}+\phi_3^{(5)}+\phi_3^{(6)}+\phi_3^{(7)}+\phi_3^{(8)}\right)\right] \tag{49}$$

$$I_{4,0} \equiv \lim_{\tau\to 0} I_4 = \frac{\mathrm{De}}{2}\left[\sum_{k=0}^{\infty}\left(\phi_4^{(1)}+\phi_4^{(2)}\right)+\sum_{k=1}^{\infty}\left(\phi_4^{(3)}+\phi_4^{(4)}+\phi_4^{(5)}+\phi_4^{(6)}+\phi_4^{(7)}+\phi_4^{(8)}\right)\right] \tag{50}$$

$$I_{5,0} \equiv \lim_{\tau\to 0} I_5 = \frac{\mathrm{De}}{2}\left[\sum_{k=0}^{\infty}\left(\phi_5^{(1)}+\phi_5^{(2)}\right)+\sum_{k=1}^{\infty}\left(\phi_5^{(3)}+\phi_5^{(4)}+\phi_5^{(5)}+\phi_5^{(6)}+\phi_5^{(7)}+\phi_5^{(8)}\right)\right] \tag{51}$$

$$I_{6,0} \equiv \lim_{\tau\to 0} I_6 = \frac{\mathrm{De}}{2}\left[\sum_{k=0}^{\infty}\left(\phi_6^{(1)}+\phi_6^{(2)}\right)+\sum_{k=1}^{\infty}\left(\phi_6^{(3)}+\phi_6^{(4)}+\phi_6^{(5)}+\phi_6^{(6)}+\phi_6^{(7)}+\phi_6^{(8)}\right)\right] \tag{52}$$

$$I_{7,0} \equiv \lim_{\tau\to 0} I_7 = \frac{\mathrm{De}}{2}\left[\sum_{k=0}^{\infty}\left(\phi_7^{(1)}+\phi_7^{(2)}\right)+\sum_{k=1}^{\infty}\left(\phi_7^{(3)}+\phi_7^{(4)}+\phi_7^{(5)}+\phi_7^{(6)}+\phi_7^{(7)}+\phi_7^{(8)}\right)\right] \tag{53}$$

$$I_{8,0} \equiv \lim_{\tau\to 0} I_8 = \frac{\mathrm{De}}{2}\left[\sum_{k=0}^{\infty}\left(\phi_8^{(1)}+\phi_8^{(2)}\right)+\sum_{k=1}^{\infty}\left(\phi_8^{(3)}+\phi_8^{(4)}+\phi_8^{(5)}+\phi_8^{(6)}+\phi_8^{(7)}+\phi_8^{(8)}\right)\right] \tag{54}$$

$\phi_1^{(i)}-\phi_8^{(i)}$ and $\psi_1^{(i)}-\psi_8^{(i)}$ are defined in Appendix VIIIb.

b $C_1^{(i)}, C_2^{(i)}, S_1^{(i)}$ $S_2^{(i)}$, $\phi_1^{(i)}-\phi_8^{(i)}$ and $\psi_1^{(i)}-\psi_8^{(i)}$

$$C_1^{(i)} \equiv \begin{bmatrix}
\left(-\mathrm{Wi}\phi_1^{(1)}+\tfrac{3}{2}\breve{\lambda}_2\mathrm{Wi}^2\phi_2^{(1)}\right)_{i=1},\left(-\tfrac{1}{2}\mathrm{Wi}\phi_3^{(1)}+\breve{\lambda}_2\mathrm{Wi}^2\phi_4^{(1)}+\tfrac{1}{2}\breve{\lambda}_2\mathrm{De}\,\mathrm{Wi}\phi_5^{(1)}\right)_2 \\
\left(-\tfrac{1}{2}\mathrm{Wi}\phi_3^{(2)}+\breve{\lambda}_2\mathrm{Wi}^2\phi_4^{(2)}+\tfrac{1}{2}\breve{\lambda}_2\mathrm{De}\,\mathrm{Wi}\phi_5^{(2)}\right)_3,\left(\tfrac{1}{4}\breve{\lambda}_2\mathrm{Wi}^2\phi_7^{(1)}\right)_4,\left(\tfrac{1}{4}\breve{\lambda}_2\mathrm{Wi}^2\phi_7^{(2)}\right)_5, \\
\left(\tfrac{1}{4}\breve{\lambda}_2\mathrm{Wi}^2\phi_7^{(3)}\right)_6,\left(\tfrac{1}{4}\breve{\lambda}_2\mathrm{Wi}^2\phi_7^{(4)}\right)_7,\left(-\tfrac{1}{2}\mathrm{Wi}\phi_3^{(3)}+\breve{\lambda}_2\mathrm{Wi}^2\phi_4^{(3)}+\tfrac{1}{2}\breve{\lambda}_2\mathrm{De}\,\mathrm{Wi}\phi_5^{(3)}+\tfrac{1}{4}\breve{\lambda}_2\mathrm{Wi}^2\phi_7^{(5)}\right)_8 \\
\left(-\tfrac{1}{2}\mathrm{Wi}\phi_3^{(4)}+\breve{\lambda}_2\mathrm{Wi}^2\phi_4^{(4)}+\tfrac{1}{2}\breve{\lambda}_2\mathrm{De}\,\mathrm{Wi}\phi_5^{(4)}+\tfrac{1}{4}\breve{\lambda}_2\mathrm{Wi}^2\phi_7^{(6)}\right)_9 \\
\left(-\mathrm{Wi}\phi_1^{(2)}+\tfrac{3}{2}\breve{\lambda}_2\mathrm{Wi}^2\phi_2^{(2)}-\tfrac{1}{2}\mathrm{Wi}\phi_3^{(5)}+\breve{\lambda}_2\mathrm{Wi}^2\phi_4^{(5)}+\tfrac{1}{2}\breve{\lambda}_2\mathrm{De}\,\mathrm{Wi}\phi_5^{(5)}\right)_{10} \\
\left(-\mathrm{Wi}\phi_1^{(3)}+\tfrac{3}{2}\breve{\lambda}_2\mathrm{Wi}^2\phi_2^{(3)}-\tfrac{1}{2}\mathrm{Wi}\phi_3^{(6)}+\breve{\lambda}_2\mathrm{Wi}^2\phi_4^{(6)}+\tfrac{1}{2}\breve{\lambda}_2\mathrm{De}\,\mathrm{Wi}\phi_5^{(6)}\right)_{11} \\
\left(-\mathrm{Wi}\phi_1^{(4)}+\tfrac{3}{2}\breve{\lambda}_2\mathrm{Wi}^2\phi_2^{(4)}-\tfrac{1}{2}\mathrm{Wi}\phi_3^{(7)}+\breve{\lambda}_2\mathrm{Wi}^2\phi_4^{(7)}+\tfrac{1}{2}\breve{\lambda}_2\mathrm{De}\,\mathrm{Wi}\phi_5^{(7)}\right)_{12} \\
\left(-\tfrac{1}{2}\mathrm{Wi}\phi_3^{(8)}+\breve{\lambda}_2\mathrm{Wi}^2\phi_4^{(8)}+\tfrac{1}{2}\breve{\lambda}_2\mathrm{De}\,\mathrm{Wi}\phi_5^{(8)}\right)_{13},\left(\tfrac{1}{4}\breve{\lambda}_2\mathrm{Wi}^2\phi_7^{(7)}\right)_{14},\left(\tfrac{1}{4}\breve{\lambda}_2\mathrm{Wi}^2\phi_7^{(8)}\right)_{15}
\end{bmatrix} \tag{55}$$

$$S_1^{(i)} \equiv \begin{bmatrix}
\left(-\mathrm{Wi}\psi_1^{(1)}+\tfrac{3}{2}\breve{\lambda}_2\mathrm{Wi}^2\psi_2^{(1)}\right)_{i=1},\left(-\tfrac{1}{2}\mathrm{Wi}\psi_3^{(1)}+\breve{\lambda}_2\mathrm{Wi}^2\psi_4^{(1)}+\tfrac{1}{2}\breve{\lambda}_2\mathrm{De}\,\mathrm{Wi}\psi_5^{(1)}\right)_2, \\
\left(-\tfrac{1}{2}\mathrm{Wi}\psi_3^{(2)}+\breve{\lambda}_2\mathrm{Wi}^2\psi_4^{(2)}+\tfrac{1}{2}\breve{\lambda}_2\mathrm{De}\,\mathrm{Wi}\psi_5^{(2)}\right)_3,\left(\tfrac{1}{4}\breve{\lambda}_2\mathrm{Wi}^2\psi_7^{(1)}\right)_4,\left(\tfrac{1}{4}\breve{\lambda}_2\mathrm{Wi}^2\psi_7^{(2)}\right)_5, \\
\left(\tfrac{1}{4}\breve{\lambda}_2\mathrm{Wi}^2\psi_7^{(3)}\right)_6,\left(\tfrac{1}{4}\breve{\lambda}_2\mathrm{Wi}^2\psi_7^{(4)}\right)_7,\left(-\tfrac{1}{2}\mathrm{Wi}\psi_3^{(3)}+\breve{\lambda}_2\mathrm{Wi}^2\psi_4^{(3)}+\tfrac{1}{2}\breve{\lambda}_2\mathrm{De}\,\mathrm{Wi}\psi_5^{(3)}+\tfrac{1}{4}\breve{\lambda}_2\mathrm{Wi}^2\psi_7^{(5)}\right)_8, \\
\left(-\tfrac{1}{2}\mathrm{Wi}\psi_3^{(4)}+\breve{\lambda}_2\mathrm{Wi}^2\psi_4^{(4)}+\tfrac{1}{2}\breve{\lambda}_2\mathrm{De}\,\mathrm{Wi}\psi_5^{(4)}+\tfrac{1}{4}\breve{\lambda}_2\mathrm{Wi}^2\psi_7^{(6)}\right)_9, \\
\left(-\mathrm{Wi}\psi_1^{(2)}+\tfrac{3}{2}\breve{\lambda}_2\mathrm{Wi}^2\psi_2^{(2)}-\tfrac{1}{2}\mathrm{Wi}\psi_3^{(5)}+\breve{\lambda}_2\mathrm{Wi}^2\psi_4^{(5)}+\tfrac{1}{2}\breve{\lambda}_2\mathrm{De}\,\mathrm{Wi}\psi_5^{(5)}\right)_{10}, \\
\left(-\mathrm{Wi}\psi_1^{(3)}+\tfrac{3}{2}\breve{\lambda}_2\mathrm{Wi}^2\psi_2^{(3)}-\tfrac{1}{2}\mathrm{Wi}\psi_3^{(6)}+\breve{\lambda}_2\mathrm{Wi}^2\psi_4^{(6)}+\tfrac{1}{2}\breve{\lambda}_2\mathrm{De}\,\mathrm{Wi}\psi_5^{(6)}\right)_{11}, \\
\left(-\mathrm{Wi}\psi_1^{(4)}+\tfrac{3}{2}\breve{\lambda}_2\mathrm{Wi}^2\psi_2^{(4)}-\tfrac{1}{2}\mathrm{Wi}\psi_3^{(7)}+\breve{\lambda}_2\mathrm{Wi}^2\psi_4^{(7)}+\tfrac{1}{2}\breve{\lambda}_2\mathrm{De}\,\mathrm{Wi}\psi_5^{(7)}\right)_{12}, \\
\left(-\tfrac{1}{2}\mathrm{Wi}\psi_3^{(8)}+\breve{\lambda}_2\mathrm{Wi}^2\psi_4^{(8)}+\tfrac{1}{2}\breve{\lambda}_2\mathrm{De}\,\mathrm{Wi}\psi_5^{(8)}\right)_{13},\left(\tfrac{1}{4}\breve{\lambda}_2\mathrm{Wi}^2\psi_7^{(7)}\right)_{14},\left(\tfrac{1}{4}\breve{\lambda}_2\mathrm{Wi}^2\psi_7^{(8)}\right)_{15}
\end{bmatrix} \tag{56}$$

$$
C_2^{(i)} \equiv \begin{bmatrix}
\left(-\tfrac{3}{2}\breve{\lambda}_2 \,\mathrm{Wi}^2 \phi_1^{(1)} - \mathrm{Wi}\phi_2^{(1)}\right)_{i=1}, \left(-\breve{\lambda}_2 \,\mathrm{Wi}^2 \phi_3^{(1)} - \tfrac{1}{2}\mathrm{Wi}\phi_4^{(1)} + \tfrac{1}{2}\breve{\lambda}_2 \,\mathrm{De}\,\mathrm{Wi}\phi_6^{(1)}\right)_2, \\
\left(-\breve{\lambda}_2 \,\mathrm{Wi}^2 \phi_3^{(2)} - \tfrac{1}{2}\mathrm{Wi}\phi_4^{(2)} + \tfrac{1}{2}\breve{\lambda}_2 \,\mathrm{De}\,\mathrm{Wi}\phi_6^{(2)}\right)_3, \left(-\tfrac{1}{4}\breve{\lambda}_2 \,\mathrm{Wi}^2 \phi_8^{(1)}\right)_4, \left(-\tfrac{1}{4}\breve{\lambda}_2 \,\mathrm{Wi}^2 \phi_8^{(2)}\right)_5, \\
\left(-\tfrac{1}{4}\breve{\lambda}_2 \,\mathrm{Wi}^2 \phi_8^{(3)}\right)_6, \left(-\tfrac{1}{4}\breve{\lambda}_2 \,\mathrm{Wi}^2 \phi_8^{(4)}\right)_7, \left(-\breve{\lambda}_2 \,\mathrm{Wi}^2 \phi_3^{(3)} - \tfrac{1}{2}\mathrm{Wi}\phi_4^{(3)} + \tfrac{1}{2}\breve{\lambda}_2 \,\mathrm{De}\,\mathrm{Wi}\phi_6^{(3)} - \tfrac{1}{4}\breve{\lambda}_2 \,\mathrm{Wi}^2 \phi_8^{(5)}\right)_8, \\
\left(-\breve{\lambda}_2 \,\mathrm{Wi}^2 \phi_3^{(4)} - \tfrac{1}{2}\mathrm{Wi}\phi_4^{(4)} + \tfrac{1}{2}\breve{\lambda}_2 \,\mathrm{De}\,\mathrm{Wi}\phi_6^{(4)} - \tfrac{1}{4}\breve{\lambda}_2 \,\mathrm{Wi}^2 \phi_8^{(6)}\right)_9, \\
\left(-\tfrac{3}{2}\breve{\lambda}_2 \,\mathrm{Wi}^2 \phi_1^{(2)} - \mathrm{Wi}\phi_2^{(2)} - \breve{\lambda}_2 \,\mathrm{Wi}^2 \phi_3^{(5)} - \tfrac{1}{2}\mathrm{Wi}\phi_4^{(5)} + \tfrac{1}{2}\breve{\lambda}_2 \,\mathrm{De}\,\mathrm{Wi}\phi_6^{(5)}\right)_{10}, \\
\left(-\tfrac{3}{2}\breve{\lambda}_2 \,\mathrm{Wi}^2 \phi_1^{(3)} - \mathrm{Wi}\phi_2^{(3)} - \breve{\lambda}_2 \,\mathrm{Wi}^2 \phi_3^{(6)} - \tfrac{1}{2}\mathrm{Wi}\phi_4^{(6)} + \tfrac{1}{2}\breve{\lambda}_2 \,\mathrm{De}\,\mathrm{Wi}\phi_6^{(6)}\right)_{11}, \\
\left(-\tfrac{3}{2}\breve{\lambda}_2 \,\mathrm{Wi}^2 \phi_1^{(4)} - \mathrm{Wi}\phi_2^{(4)} - \breve{\lambda}_2 \,\mathrm{Wi}^2 \phi_3^{(7)} - \tfrac{1}{2}\mathrm{Wi}\phi_4^{(7)} + \tfrac{1}{2}\breve{\lambda}_2 \,\mathrm{De}\,\mathrm{Wi}\phi_6^{(7)}\right)_{12}, \\
\left(-\breve{\lambda}_2 \,\mathrm{Wi}^2 \phi_3^{(8)} - \tfrac{1}{2}\mathrm{Wi}\phi_4^{(8)} + \tfrac{1}{2}\breve{\lambda}_2 \,\mathrm{De}\,\mathrm{Wi}\phi_6^{(8)}\right)_{13}, \left(-\tfrac{1}{4}\breve{\lambda}_2 \,\mathrm{Wi}^2 \phi_8^{(7)}\right)_{14}, \left(-\tfrac{1}{4}\breve{\lambda}_2 \,\mathrm{Wi}^2 \phi_8^{(8)}\right)_{15}
\end{bmatrix} \tag{57}
$$

$$
S_2^{(i)} \equiv \begin{bmatrix}
\left(-\tfrac{3}{2}\breve{\lambda}_2 \,\mathrm{Wi}^2 \psi_1^{(1)} - \mathrm{Wi}\psi_2^{(1)}\right)_{i=1}, \left(-\breve{\lambda}_2 \,\mathrm{Wi}^2 \psi_3^{(1)} - \tfrac{1}{2}\mathrm{Wi}\psi_4^{(1)} + \tfrac{1}{2}\breve{\lambda}_2 \,\mathrm{De}\,\mathrm{Wi}\psi_6^{(1)}\right)_2, \\
\left(-\breve{\lambda}_2 \,\mathrm{Wi}^2 \psi_3^{(2)} - \tfrac{1}{2}\mathrm{Wi}\psi_4^{(2)} + \tfrac{1}{2}\breve{\lambda}_2 \,\mathrm{De}\,\mathrm{Wi}\psi_6^{(2)}\right)_3, \left(-\tfrac{1}{4}\breve{\lambda}_2 \,\mathrm{Wi}^2 \psi_8^{(1)}\right)_4, \left(-\tfrac{1}{4}\breve{\lambda}_2 \,\mathrm{Wi}^2 \psi_8^{(2)}\right)_5, \\
\left(-\tfrac{1}{4}\breve{\lambda}_2 \,\mathrm{Wi}^2 \psi_8^{(3)}\right)_6, \left(-\tfrac{1}{4}\breve{\lambda}_2 \,\mathrm{Wi}^2 \psi_8^{(4)}\right)_7, \left(-\breve{\lambda}_2 \,\mathrm{Wi}^2 \psi_3^{(3)} - \tfrac{1}{2}\mathrm{Wi}\psi_4^{(3)} + \tfrac{1}{2}\breve{\lambda}_2 \,\mathrm{De}\,\mathrm{Wi}\psi_6^{(3)} - \tfrac{1}{4}\breve{\lambda}_2 \,\mathrm{Wi}^2 \psi_8^{(5)}\right)_8, \\
\left(-\breve{\lambda}_2 \,\mathrm{Wi}^2 \psi_3^{(4)} - \tfrac{1}{2}\mathrm{Wi}\psi_4^{(4)} + \tfrac{1}{2}\breve{\lambda}_2 \,\mathrm{De}\,\mathrm{Wi}\psi_6^{(4)} - \tfrac{1}{4}\breve{\lambda}_2 \,\mathrm{Wi}^2 \psi_8^{(6)}\right)_9, \\
\left(-\tfrac{3}{2}\breve{\lambda}_2 \,\mathrm{Wi}^2 \psi_1^{(2)} - \mathrm{Wi}\psi_2^{(2)} - \breve{\lambda}_2 \,\mathrm{Wi}^2 \psi_3^{(5)} - \tfrac{1}{2}\mathrm{Wi}\psi_4^{(5)} + \tfrac{1}{2}\breve{\lambda}_2 \,\mathrm{De}\,\mathrm{Wi}\psi_6^{(5)}\right)_{10}, \\
\left(-\tfrac{3}{2}\breve{\lambda}_2 \,\mathrm{Wi}^2 \psi_1^{(3)} - \mathrm{Wi}\psi_2^{(3)} - \breve{\lambda}_2 \,\mathrm{Wi}^2 \psi_3^{(6)} - \tfrac{1}{2}\mathrm{Wi}\psi_4^{(6)} + \tfrac{1}{2}\breve{\lambda}_2 \,\mathrm{De}\,\mathrm{Wi}\psi_6^{(6)}\right)_{11}, \\
\left(-\tfrac{3}{2}\breve{\lambda}_2 \,\mathrm{Wi}^2 \psi_1^{(4)} - \mathrm{Wi}\psi_2^{(4)} - \breve{\lambda}_2 \,\mathrm{Wi}^2 \psi_3^{(7)} - \tfrac{1}{2}\mathrm{Wi}\psi_4^{(7)} + \tfrac{1}{2}\breve{\lambda}_2 \,\mathrm{De}\,\mathrm{Wi}\psi_6^{(7)}\right)_{12}, \\
\left(-\breve{\lambda}_2 \,\mathrm{Wi}^2 \psi_3^{(8)} - \tfrac{1}{2}\mathrm{Wi}\psi_4^{(8)} + \tfrac{1}{2}\breve{\lambda}_2 \,\mathrm{De}\,\mathrm{Wi}\psi_6^{(8)}\right)_{13}, \left(-\tfrac{1}{4}\breve{\lambda}_2 \,\mathrm{Wi}^2 \psi_8^{(7)}\right)_{14}, \left(-\tfrac{1}{4}\breve{\lambda}_2 \,\mathrm{Wi}^2 \psi_8^{(8)}\right)_{15}
\end{bmatrix} \tag{58}
$$

$$
\phi_1^{(i)} \equiv \begin{bmatrix}
\left(\dfrac{J_{2k}}{1+\left(2k+\frac{\mathrm{Wi}}{\mathrm{De}}\right)^2 \mathrm{De}^2}\right)_{i=1}, \left(\dfrac{-J_{2k-1}}{1+\left(2k-1-\frac{\mathrm{Wi}}{\mathrm{De}}\right)^2 \mathrm{De}^2}\right)_2, \\
\left(\dfrac{J_{2k-1}}{1+\left(2k-1+\frac{\mathrm{Wi}}{\mathrm{De}}\right)^2 \mathrm{De}^2}\right)_3, \left(\dfrac{J_{2k}}{1+\left(2k-\frac{\mathrm{Wi}}{\mathrm{De}}\right)^2 \mathrm{De}^2}\right)_4
\end{bmatrix} \tag{59}
$$

$$
\phi_2^{(i)} \equiv \begin{bmatrix}
\left(\dfrac{-\left(2k+\frac{\mathrm{Wi}}{\mathrm{De}}\right)\mathrm{De}\,J_{2k}}{1+\left(2k+\frac{\mathrm{Wi}}{\mathrm{De}}\right)^2 \mathrm{De}^2}\right)_{i=1}, \left(\dfrac{-\left(2k-1-\frac{\mathrm{Wi}}{\mathrm{De}}\right)\mathrm{De}\,J_{2k-1}}{1+\left(2k-1-\frac{\mathrm{Wi}}{\mathrm{De}}\right)^2 \mathrm{De}^2}\right)_2, \\
\left(\dfrac{-\left(2k-1+\frac{\mathrm{Wi}}{\mathrm{De}}\right)\mathrm{De}\,J_{2k-1}}{1+\left(2k-1+\frac{\mathrm{Wi}}{\mathrm{De}}\right)^2 \mathrm{De}^2}\right)_3, \left(\dfrac{\left(2k-\frac{\mathrm{Wi}}{\mathrm{De}}\right)\mathrm{De}\,J_{2k}}{1+\left(2k-\frac{\mathrm{Wi}}{\mathrm{De}}\right)^2 \mathrm{De}^2}\right)_4
\end{bmatrix} \tag{60}
$$

$$\phi_3^{(i)} \equiv \left[\begin{array}{l} \left(\dfrac{J_{2k}}{1+\left(2k+1-\frac{\mathrm{Wi}}{\mathrm{De}}\right)^2\mathrm{De}^2}\right)_{i=1}, \left(\dfrac{J_{2k}}{1+\left(2k+1+\frac{\mathrm{Wi}}{\mathrm{De}}\right)^2\mathrm{De}^2}\right)_{2}, \left(\dfrac{-J_{2k-1}}{1+\left(2k-2-\frac{\mathrm{Wi}}{\mathrm{De}}\right)^2\mathrm{De}^2}\right)_{3}, \\ \left(\dfrac{J_{2k-1}}{1+\left(2k-2+\frac{\mathrm{Wi}}{\mathrm{De}}\right)^2\mathrm{De}^2}\right)_{4}, \left(\dfrac{J_{2k}}{1+\left(2k-1-\frac{\mathrm{Wi}}{\mathrm{De}}\right)^2\mathrm{De}^2}\right)_{5}, \left(\dfrac{J_{2k}}{1+\left(2k-1+\frac{\mathrm{Wi}}{\mathrm{De}}\right)^2\mathrm{De}^2}\right)_{6}, \\ \left(\dfrac{-J_{2k-1}}{1+\left(2k-\frac{\mathrm{Wi}}{\mathrm{De}}\right)^2\mathrm{De}^2}\right)_{7}, \left(\dfrac{J_{2k-1}}{1+\left(2k+\frac{\mathrm{Wi}}{\mathrm{De}}\right)^2\mathrm{De}^2}\right)_{8} \end{array}\right] \tag{61}$$

$$\phi_4^{(i)} \equiv \left[\begin{array}{l} \left(\dfrac{\left(2k+1-\frac{\mathrm{Wi}}{\mathrm{De}}\right)\mathrm{De}\,J_{2k}}{1+\left(2k+1-\frac{\mathrm{Wi}}{\mathrm{De}}\right)^2\mathrm{De}^2}\right)_{i=1}, \left(\dfrac{-\left(2k+1+\frac{\mathrm{Wi}}{\mathrm{De}}\right)\mathrm{De}\,J_{2k}}{1+\left(2k+1+\frac{\mathrm{Wi}}{\mathrm{De}}\right)^2\mathrm{De}^2}\right)_{2}, \left(\dfrac{-\left(2k-2-\frac{\mathrm{Wi}}{\mathrm{De}}\right)\mathrm{De}\,J_{2k-1}}{1+\left(2k-2-\frac{\mathrm{Wi}}{\mathrm{De}}\right)^2\mathrm{De}^2}\right)_{3}, \\ \left(\dfrac{-\left(2k-2+\frac{\mathrm{Wi}}{\mathrm{De}}\right)\mathrm{De}\,J_{2k-1}}{1+\left(2k-2+\frac{\mathrm{Wi}}{\mathrm{De}}\right)^2\mathrm{De}^2}\right)_{4}, \left(\dfrac{\left(2k-1-\frac{\mathrm{Wi}}{\mathrm{De}}\right)\mathrm{De}\,J_{2k}}{1+\left(2k-1-\frac{\mathrm{Wi}}{\mathrm{De}}\right)^2\mathrm{De}^2}\right)_{5}, \left(\dfrac{-\left(2k-1+\frac{\mathrm{Wi}}{\mathrm{De}}\right)\mathrm{De}\,J_{2k}}{1+\left(2k-1+\frac{\mathrm{Wi}}{\mathrm{De}}\right)^2\mathrm{De}^2}\right)_{6}, \\ \left(\dfrac{-\left(2k-\frac{\mathrm{Wi}}{\mathrm{De}}\right)\mathrm{De}\,J_{2k-1}}{1+\left(2k-\frac{\mathrm{Wi}}{\mathrm{De}}\right)^2\mathrm{De}^2}\right)_{7}, \left(\dfrac{-\left(2k+\frac{\mathrm{Wi}}{\mathrm{De}}\right)\mathrm{De}\,J_{2k-1}}{1+\left(2k+\frac{\mathrm{Wi}}{\mathrm{De}}\right)^2\mathrm{De}^2}\right)_{8} \end{array}\right] \tag{62}$$

$$\phi_5^{(i)} \equiv \left[\begin{array}{l} \left(\dfrac{-\left(2k+1-\frac{\mathrm{Wi}}{\mathrm{De}}\right)\mathrm{De}\,J_{2k}}{1+\left(2k+1-\frac{\mathrm{Wi}}{\mathrm{De}}\right)^2\mathrm{De}^2}\right)_{i=1}, \left(\dfrac{-\left(2k+1+\frac{\mathrm{Wi}}{\mathrm{De}}\right)\mathrm{De}\,J_{2k}}{1+\left(2k+1+\frac{\mathrm{Wi}}{\mathrm{De}}\right)^2\mathrm{De}^2}\right)_{2}, \left(\dfrac{-\left(2k-2-\frac{\mathrm{Wi}}{\mathrm{De}}\right)\mathrm{De}\,J_{2k-1}}{1+\left(2k-2-\frac{\mathrm{Wi}}{\mathrm{De}}\right)^2\mathrm{De}^2}\right)_{3}, \\ \left(\dfrac{\left(2k-2+\frac{\mathrm{Wi}}{\mathrm{De}}\right)\mathrm{De}\,J_{2k-1}}{1+\left(2k-2+\frac{\mathrm{Wi}}{\mathrm{De}}\right)^2\mathrm{De}^2}\right)_{4}, \left(\dfrac{\left(2k-1-\frac{\mathrm{Wi}}{\mathrm{De}}\right)\mathrm{De}\,J_{2k}}{1+\left(2k-1-\frac{\mathrm{Wi}}{\mathrm{De}}\right)^2\mathrm{De}^2}\right)_{5}, \left(\dfrac{\left(2k-1+\frac{\mathrm{Wi}}{\mathrm{De}}\right)\mathrm{De}\,J_{2k}}{1+\left(2k-1+\frac{\mathrm{Wi}}{\mathrm{De}}\right)^2\mathrm{De}^2}\right)_{6}, \\ \left(\dfrac{\left(2k-\frac{\mathrm{Wi}}{\mathrm{De}}\right)\mathrm{De}\,J_{2k-1}}{1+\left(2k-\frac{\mathrm{Wi}}{\mathrm{De}}\right)^2\mathrm{De}^2}\right)_{7}, \left(\dfrac{-\left(2k+\frac{\mathrm{Wi}}{\mathrm{De}}\right)\mathrm{De}\,J_{2k-1}}{1+\left(2k+\frac{\mathrm{Wi}}{\mathrm{De}}\right)^2\mathrm{De}^2}\right)_{8} \end{array}\right] \tag{63}$$

$$\phi_6^{(i)} \equiv \left[\begin{array}{l} \left(\dfrac{J_{2k}}{1+\left(2k+1-\frac{\mathrm{Wi}}{\mathrm{De}}\right)^2\mathrm{De}^2}\right)_{i=1}, \left(\dfrac{-J_{2k}}{1+\left(2k+1+\frac{\mathrm{Wi}}{\mathrm{De}}\right)^2\mathrm{De}^2}\right)_{2}, \left(\dfrac{J_{2k-1}}{1+\left(2k-2-\frac{\mathrm{Wi}}{\mathrm{De}}\right)^2\mathrm{De}^2}\right)_{3}, \\ \left(\dfrac{J_{2k-1}}{1+\left(2k-2+\frac{\mathrm{Wi}}{\mathrm{De}}\right)^2\mathrm{De}^2}\right)_{4}, \left(\dfrac{-J_{2k}}{1+\left(2k-1-\frac{\mathrm{Wi}}{\mathrm{De}}\right)^2\mathrm{De}^2}\right)_{5}, \left(\dfrac{J_{2k}}{1+\left(2k-1+\frac{\mathrm{Wi}}{\mathrm{De}}\right)^2\mathrm{De}^2}\right)_{6}, \\ \left(\dfrac{-J_{2k-1}}{1+\left(2k-\frac{\mathrm{Wi}}{\mathrm{De}}\right)^2\mathrm{De}^2}\right)_{7}, \left(\dfrac{-J_{2k-1}}{1+\left(2k+\frac{\mathrm{Wi}}{\mathrm{De}}\right)^2\mathrm{De}^2}\right)_{8} \end{array}\right] \tag{64}$$

$$\phi_7^{(i)} \equiv \left[\begin{array}{l} \left(\frac{\left(2k+2-\frac{\mathrm{Wi}}{\mathrm{De}}\right)\mathrm{De}\,J_{2k}}{1+\left(2k+2-\frac{\mathrm{Wi}}{\mathrm{De}}\right)^2\mathrm{De}^2}\right)_{i=1},\left(\frac{-\left(2k+2+\frac{\mathrm{Wi}}{\mathrm{De}}\right)\mathrm{De}\,J_{2k}}{1+\left(2k+2+\frac{\mathrm{Wi}}{\mathrm{De}}\right)^2\mathrm{De}^2}\right)_2,\left(\frac{-\left(2k-3-\frac{\mathrm{Wi}}{\mathrm{De}}\right)\mathrm{De}\,J_{2k-1}}{1+\left(2k-3-\frac{\mathrm{Wi}}{\mathrm{De}}\right)^2\mathrm{De}^2}\right)_3, \\ \left(\frac{-\left(2k-3+\frac{\mathrm{Wi}}{\mathrm{De}}\right)\mathrm{De}\,J_{2k-1}}{1+\left(2k-3+\frac{\mathrm{Wi}}{\mathrm{De}}\right)^2\mathrm{De}^2}\right)_4,\left(\frac{\left(2k-2-\frac{\mathrm{Wi}}{\mathrm{De}}\right)\mathrm{De}\,J_{2k}}{1+\left(2k-2-\frac{\mathrm{Wi}}{\mathrm{De}}\right)^2\mathrm{De}^2}\right)_5,\left(\frac{-\left(2k-2+\frac{\mathrm{Wi}}{\mathrm{De}}\right)\mathrm{De}\,J_{2k}}{1+\left(2k-2+\frac{\mathrm{Wi}}{\mathrm{De}}\right)^2\mathrm{De}^2}\right)_6, \\ \left(\frac{-\left(2k+1-\frac{\mathrm{Wi}}{\mathrm{De}}\right)\mathrm{De}\,J_{2k-1}}{1+\left(2k+1-\frac{\mathrm{Wi}}{\mathrm{De}}\right)^2\mathrm{De}^2}\right)_7,\left(\frac{-\left(2k+1+\frac{\mathrm{Wi}}{\mathrm{De}}\right)\mathrm{De}\,J_{2k-1}}{1+\left(2k+1+\frac{\mathrm{Wi}}{\mathrm{De}}\right)^2\mathrm{De}^2}\right)_8 \end{array}\right] \tag{65}$$

$$\phi_8^{(i)} \equiv \left[\begin{array}{l} \left(\frac{J_{2k}}{1+\left(2k+2-\frac{\mathrm{Wi}}{\mathrm{De}}\right)^2\mathrm{De}^2}\right)_{i=1},\left(\frac{J_{2k}}{1+\left(2k+2+\frac{\mathrm{Wi}}{\mathrm{De}}\right)^2\mathrm{De}^2}\right)_2,\left(\frac{-J_{2k-1}}{1+\left(2k-3-\frac{\mathrm{Wi}}{\mathrm{De}}\right)^2\mathrm{De}^2}\right)_3, \\ \left(\frac{J_{2k-1}}{1+\left(2k-3+\frac{\mathrm{Wi}}{\mathrm{De}}\right)^2\mathrm{De}^2}\right)_4,\left(\frac{J_{2k}}{1+\left(2k-2-\frac{\mathrm{Wi}}{\mathrm{De}}\right)^2\mathrm{De}^2}\right)_5,\left(\frac{-J_{2k}}{1+\left(2k-2+\frac{\mathrm{Wi}}{\mathrm{De}}\right)^2\mathrm{De}^2}\right)_6, \\ \left(\frac{J_{2k-1}}{1+\left(2k+1-\frac{\mathrm{Wi}}{\mathrm{De}}\right)^2\mathrm{De}^2}\right)_7,\left(\frac{J_{2k-1}}{1+\left(2k+1+\frac{\mathrm{Wi}}{\mathrm{De}}\right)^2\mathrm{De}^2}\right)_8 \end{array}\right] \tag{66}$$

$$\psi_1^{(i)} \equiv \left[\begin{array}{l} \left(\frac{\left(2k+\frac{\mathrm{Wi}}{\mathrm{De}}\right)\mathrm{De}\,J_{2k}}{1+\left(2k+\frac{\mathrm{Wi}}{\mathrm{De}}\right)^2\mathrm{De}^2}\right)_{i=1},\left(\frac{-\left(2k-1-\frac{\mathrm{Wi}}{\mathrm{De}}\right)\mathrm{De}\,J_{2k-1}}{1+\left(2k-1-\frac{\mathrm{Wi}}{\mathrm{De}}\right)^2\mathrm{De}^2}\right)_2, \\ \left(\frac{\left(2k-1+\frac{\mathrm{Wi}}{\mathrm{De}}\right)\mathrm{De}\,J_{2k-1}}{1+\left(2k-1+\frac{\mathrm{Wi}}{\mathrm{De}}\right)^2\mathrm{De}^2}\right)_3,\left(\frac{\left(2k-\frac{\mathrm{Wi}}{\mathrm{De}}\right)\mathrm{De}\,J_{2k}}{1+\left(2k-\frac{\mathrm{Wi}}{\mathrm{De}}\right)^2\mathrm{De}^2}\right)_4 \end{array}\right] \tag{67}$$

$$\psi_2^{(i)} \equiv \left[\begin{array}{l} \left(\frac{J_{2k}}{1+\left(2k+\frac{\mathrm{Wi}}{\mathrm{De}}\right)^2\mathrm{De}^2}\right)_{i=1},\left(\frac{J_{2k-1}}{1+\left(2k-1-\frac{\mathrm{Wi}}{\mathrm{De}}\right)^2\mathrm{De}^2}\right)_2, \\ \left(\frac{J_{2k-1}}{1+\left(2k-1+\frac{\mathrm{Wi}}{\mathrm{De}}\right)^2\mathrm{De}^2}\right)_3,\left(\frac{-J_{2k}}{1+\left(2k-\frac{\mathrm{Wi}}{\mathrm{De}}\right)^2\mathrm{De}^2}\right)_4 \end{array}\right] \tag{68}$$

$$\psi_3^{(i)} \equiv \left[\begin{array}{l} \left(\frac{\left(2k+1-\frac{\mathrm{Wi}}{\mathrm{De}}\right)\mathrm{De}\,J_{2k}}{1+\left(2k+1-\frac{\mathrm{Wi}}{\mathrm{De}}\right)^2\mathrm{De}^2}\right)_{i=1},\left(\frac{\left(2k+1+\frac{\mathrm{Wi}}{\mathrm{De}}\right)\mathrm{De}\,J_{2k}}{1+\left(2k+1+\frac{\mathrm{Wi}}{\mathrm{De}}\right)^2\mathrm{De}^2}\right)_2,\left(\frac{-\left(2k-2-\frac{\mathrm{Wi}}{\mathrm{De}}\right)\mathrm{De}\,J_{2k-1}}{1+\left(2k-2-\frac{\mathrm{Wi}}{\mathrm{De}}\right)^2\mathrm{De}^2}\right)_3, \\ \left(\frac{\left(2k-2+\frac{\mathrm{Wi}}{\mathrm{De}}\right)\mathrm{De}\,J_{2k-1}}{1+\left(2k-2+\frac{\mathrm{Wi}}{\mathrm{De}}\right)^2\mathrm{De}^2}\right)_4,\left(\frac{\left(2k-1-\frac{\mathrm{Wi}}{\mathrm{De}}\right)\mathrm{De}\,J_{2k}}{1+\left(2k-1-\frac{\mathrm{Wi}}{\mathrm{De}}\right)^2\mathrm{De}^2}\right)_5,\left(\frac{\left(2k-1+\frac{\mathrm{Wi}}{\mathrm{De}}\right)\mathrm{De}\,J_{2k}}{1+\left(2k-1+\frac{\mathrm{Wi}}{\mathrm{De}}\right)^2\mathrm{De}^2}\right)_6, \\ \left(\frac{-\left(2k-\frac{\mathrm{Wi}}{\mathrm{De}}\right)\mathrm{De}\,J_{2k-1}}{1+\left(2k-\frac{\mathrm{Wi}}{\mathrm{De}}\right)^2\mathrm{De}^2}\right)_7,\left(\frac{\left(2k+\frac{\mathrm{Wi}}{\mathrm{De}}\right)\mathrm{De}\,J_{2k-1}}{1+\left(2k+\frac{\mathrm{Wi}}{\mathrm{De}}\right)^2\mathrm{De}^2}\right)_8 \end{array}\right] \tag{69}$$

$$\boldsymbol{\psi}_4^{(i)} \equiv \left[\begin{array}{l} \left(\frac{-J_{2k}}{1+\left(2k+1-\frac{\mathrm{Wi}}{\mathrm{De}}\right)^2\mathrm{De}^2}\right)_{i=1}, \left(\frac{J_{2k}}{1+\left(2k+1+\frac{\mathrm{Wi}}{\mathrm{De}}\right)^2\mathrm{De}^2}\right)_2, \left(\frac{J_{2k-1}}{1+\left(2k-2-\frac{\mathrm{Wi}}{\mathrm{De}}\right)^2\mathrm{De}^2}\right)_3, \\ \left(\frac{J_{2k-1}}{1+\left(2k-2+\frac{\mathrm{Wi}}{\mathrm{De}}\right)^2\mathrm{De}^2}\right)_4, \left(\frac{-J_{2k}}{1+\left(2k-1-\frac{\mathrm{Wi}}{\mathrm{De}}\right)^2\mathrm{De}^2}\right)_5, \left(\frac{J_{2k}}{1+\left(2k-1+\frac{\mathrm{Wi}}{\mathrm{De}}\right)^2\mathrm{De}^2}\right)_6, \\ \left(\frac{J_{2k-1}}{1+\left(2k-\frac{\mathrm{Wi}}{\mathrm{De}}\right)^2\mathrm{De}^2}\right)_7, \left(\frac{J_{2k-1}}{1+\left(2k+\frac{\mathrm{Wi}}{\mathrm{De}}\right)^2\mathrm{De}^2}\right)_8 \end{array}\right] \tag{70}$$

$$\boldsymbol{\psi}_5^{(i)} \equiv \left[\begin{array}{l} \left(\frac{J_{2k}}{1+\left(2k+1-\frac{\mathrm{Wi}}{\mathrm{De}}\right)^2\mathrm{De}^2}\right)_{i=1}, \left(\frac{J_{2k}}{1+\left(2k+1+\frac{\mathrm{Wi}}{\mathrm{De}}\right)^2\mathrm{De}^2}\right)_2, \left(\frac{J_{2k-1}}{1+\left(2k-2-\frac{\mathrm{Wi}}{\mathrm{De}}\right)^2\mathrm{De}^2}\right)_3, \\ \left(\frac{-J_{2k-1}}{1+\left(2k-2+\frac{\mathrm{Wi}}{\mathrm{De}}\right)^2\mathrm{De}^2}\right)_4, \left(\frac{-J_{2k}}{1+\left(2k-1-\frac{\mathrm{Wi}}{\mathrm{De}}\right)^2\mathrm{De}^2}\right)_5, \left(\frac{-J_{2k}}{1+\left(2k-1+\frac{\mathrm{Wi}}{\mathrm{De}}\right)^2\mathrm{De}^2}\right)_6, \\ \left(\frac{-J_{2k-1}}{1+\left(2k-\frac{\mathrm{Wi}}{\mathrm{De}}\right)^2\mathrm{De}^2}\right)_7, \left(\frac{J_{2k-1}}{1+\left(2k+\frac{\mathrm{Wi}}{\mathrm{De}}\right)^2\mathrm{De}^2}\right)_8 \end{array}\right] \tag{71}$$

$$\boldsymbol{\psi}_6^{(i)} \equiv \left[\begin{array}{l} \left(\frac{\left(2k+1-\frac{\mathrm{Wi}}{\mathrm{De}}\right)\mathrm{De}\,J_{2k}}{1+\left(2k+1-\frac{\mathrm{Wi}}{\mathrm{De}}\right)^2\mathrm{De}^2}\right)_{i=1}, \left(\frac{-\left(2k+1+\frac{\mathrm{Wi}}{\mathrm{De}}\right)\mathrm{De}\,J_{2k}}{1+\left(2k+1+\frac{\mathrm{Wi}}{\mathrm{De}}\right)^2\mathrm{De}^2}\right)_2, \left(\frac{\left(2k-2-\frac{\mathrm{Wi}}{\mathrm{De}}\right)\mathrm{De}\,J_{2k-1}}{1+\left(2k-2-\frac{\mathrm{Wi}}{\mathrm{De}}\right)^2\mathrm{De}^2}\right)_3, \\ \left(\frac{\left(2k-2+\frac{\mathrm{Wi}}{\mathrm{De}}\right)\mathrm{De}\,J_{2k-1}}{1+\left(2k-2+\frac{\mathrm{Wi}}{\mathrm{De}}\right)^2\mathrm{De}^2}\right)_4, \left(\frac{-\left(2k-1-\frac{\mathrm{Wi}}{\mathrm{De}}\right)\mathrm{De}\,J_{2k}}{1+\left(2k-1-\frac{\mathrm{Wi}}{\mathrm{De}}\right)^2\mathrm{De}^2}\right)_5, \left(\frac{\left(2k-1+\frac{\mathrm{Wi}}{\mathrm{De}}\right)\mathrm{De}\,J_{2k}}{1+\left(2k-1+\frac{\mathrm{Wi}}{\mathrm{De}}\right)^2\mathrm{De}^2}\right)_6, \\ \left(\frac{-\left(2k-\frac{\mathrm{Wi}}{\mathrm{De}}\right)\mathrm{De}\,J_{2k-1}}{1+\left(2k-\frac{\mathrm{Wi}}{\mathrm{De}}\right)^2\mathrm{De}^2}\right)_7, \left(\frac{-\left(2k+\frac{\mathrm{Wi}}{\mathrm{De}}\right)\mathrm{De}\,J_{2k-1}}{1+\left(2k+\frac{\mathrm{Wi}}{\mathrm{De}}\right)^2\mathrm{De}^2}\right)_8 \end{array}\right] \tag{72}$$

$$\boldsymbol{\psi}_7^{(i)} \equiv \left[\begin{array}{l} \left(\frac{-J_{2k}}{1+\left(2k+2-\frac{\mathrm{Wi}}{\mathrm{De}}\right)^2\mathrm{De}^2}\right)_{i=1}, \left(\frac{J_{2k}}{1+\left(2k+2+\frac{\mathrm{Wi}}{\mathrm{De}}\right)^2\mathrm{De}^2}\right)_2, \left(\frac{J_{2k-1}}{1+\left(2k-3-\frac{\mathrm{Wi}}{\mathrm{De}}\right)^2\mathrm{De}^2}\right)_3, \\ \left(\frac{J_{2k-1}}{1+\left(2k-3+\frac{\mathrm{Wi}}{\mathrm{De}}\right)^2\mathrm{De}^2}\right)_4, \left(\frac{-J_{2k}}{1+\left(2k-2-\frac{\mathrm{Wi}}{\mathrm{De}}\right)^2\mathrm{De}^2}\right)_5, \left(\frac{J_{2k}}{1+\left(2k-2+\frac{\mathrm{Wi}}{\mathrm{De}}\right)^2\mathrm{De}^2}\right)_6, \\ \left(\frac{J_{2k-1}}{1+\left(2k+1-\frac{\mathrm{Wi}}{\mathrm{De}}\right)^2\mathrm{De}^2}\right)_7, \left(\frac{J_{2k-1}}{1+\left(2k+1+\frac{\mathrm{Wi}}{\mathrm{De}}\right)^2\mathrm{De}^2}\right)_8 \end{array}\right] \tag{73}$$

$$\boldsymbol{\psi}_8^{(i)} \equiv \left[\begin{array}{l} \left(\frac{\left(2k+2-\frac{\mathrm{Wi}}{\mathrm{De}}\right)\mathrm{De}\,J_{2k}}{1+\left(2k+2-\frac{\mathrm{Wi}}{\mathrm{De}}\right)^2\mathrm{De}^2} \right)_{i=1}, \left(\frac{\left(2k+2+\frac{\mathrm{Wi}}{\mathrm{De}}\right)\mathrm{De}\,J_{2k}}{1+\left(2k+2+\frac{\mathrm{Wi}}{\mathrm{De}}\right)^2\mathrm{De}^2} \right)_{2}, \left(\frac{-\left(2k-3-\frac{\mathrm{Wi}}{\mathrm{De}}\right)\mathrm{De}\,J_{2k-1}}{1+\left(2k-3-\frac{\mathrm{Wi}}{\mathrm{De}}\right)^2\mathrm{De}^2} \right)_{3}, \\ \left(\frac{\left(2k-3+\frac{\mathrm{Wi}}{\mathrm{De}}\right)\mathrm{De}\,J_{2k-1}}{1+\left(2k-3+\frac{\mathrm{Wi}}{\mathrm{De}}\right)^2\mathrm{De}^2} \right)_{4}, \left(\frac{\left(2k-2-\frac{\mathrm{Wi}}{\mathrm{De}}\right)\mathrm{De}\,J_{2k}}{1+\left(2k-2-\frac{\mathrm{Wi}}{\mathrm{De}}\right)^2\mathrm{De}^2} \right)_{5}, \left(\frac{\left(2k-2+\frac{\mathrm{Wi}}{\mathrm{De}}\right)\mathrm{De}\,J_{2k}}{1+\left(2k-2+\frac{\mathrm{Wi}}{\mathrm{De}}\right)^2\mathrm{De}^2} \right)_{6}, \\ \left(\frac{-\left(2k+1-\frac{\mathrm{Wi}}{\mathrm{De}}\right)\mathrm{De}\,J_{2k-1}}{1+\left(2k+1-\frac{\mathrm{Wi}}{\mathrm{De}}\right)^2\mathrm{De}^2} \right)_{7}, \left(\frac{\left(2k+1+\frac{\mathrm{Wi}}{\mathrm{De}}\right)\mathrm{De}\,J_{2k-1}}{1+\left(2k+1+\frac{\mathrm{Wi}}{\mathrm{De}}\right)^2\mathrm{De}^2} \right)_{8} \end{array} \right] \tag{74}$$

Table I: Dimensional variables

Name	Unit	Symbol
Shear rate	t^{-1}	$\dot{\gamma}$
Shear rate, steady shear	t^{-1}	$\dot{\gamma}_s$
Shear rate, oscillatory amplitude	t^{-1}	$\dot{\gamma}^0$
Derivative, corotational, of a tensor **b**	t^{-1}	$\frac{\mathscr{D}\mathbf{b}}{\mathscr{D}t} \equiv \frac{D\mathbf{b}}{Dt} + \frac{1}{2}\{\boldsymbol{\omega}\cdot\mathbf{b} - \mathbf{b}\cdot\boldsymbol{\omega}\}$
Derivative, material, of a tensor **b**	t^{-1}	$\frac{D\mathbf{b}}{Dt} \equiv \frac{\partial\mathbf{b}}{\partial t} + (\mathbf{v}\cdot\nabla\mathbf{b})$
Angular frequency	t^{-1}	ω
Velocity profile	M/L	$\mathbf{v}$
Cartesian coordinates	L	x, y, z
Extra stress tensor	M/Lt^2	$\boldsymbol{\tau}$
Extra stress tensor, *k*th spectrum	M/Lt^2	$\boldsymbol{\tau}_k$
Rate-of-strain tensor	t^{-1}	$\dot{\boldsymbol{\gamma}} \equiv \nabla\mathbf{v} + (\nabla\mathbf{v})^\dagger$
Zero-shear viscosity	M/Lt	η_0
Relaxation time	t	λ_1
Retardation time	t	λ_2
Oldroyd 8-constant parameters	t	$\mu_0, \mu_1, \mu_2, \nu_1, \nu_2$
Extra stress tensor, *ij*-component	M/Lt^2	τ_{ij}
First normal stress difference	M/Lt^2	$N_1 \equiv \tau_{xx} - \tau_{yy}$
Second normal stress difference	M/Lt^2	$N_2 \equiv \tau_{yy} - \tau_{zz}$
Extra stress tensor, *ij*-component, *k*th spectrum	M/Lt^2	$\tau_{ij,k}$
Zero-shear viscosity, *k*th spectrum	M/Lt	η_k
Relaxation time, *k*th spectrum	t	$\lambda_{1,k}$
Retardation time, *k*th spectrum	t	$\lambda_{2,k}$
Oldroyd 8-constant parameters, *k*th spectrum	t	$\mu_{0,k}, \mu_{1,k}, \mu_{2,k}, \nu_{1,k}, \nu_{2,k}$
Vorticity tensor	t^{-1}	$\boldsymbol{\omega} \equiv \nabla\mathbf{v} - (\nabla\mathbf{v})^\dagger$

Legend: $M \equiv$ mass; $L \equiv$ length; $t \equiv$ time; $T \equiv$ temperature

Table II: Dimensionless variables and groups

Name	Symbol
Direc delta function [Eq. (6)]	δ
vth order Bessel function of first kind of argument Wi/De	$J_v \equiv J_v(\mathrm{Wi}/\mathrm{De})$
Extra stress tensor, ij-component [Eq. (7)]	$\breve{\tau}_{ij}$
Time [Eq. (5)]	$\tau \equiv \omega t$
Deborah number [Eq. (4)]	De
Weissenberg number [Eq. (4)]	Wi
Retardation time [Eq. (7)]	$\breve{\lambda}_2 \equiv \lambda_2/\lambda_1$
Oldroyd 8-constant parameters [Eq. (7)]	$\breve{\mu}_0 \equiv \mu_0/\lambda_1, \breve{\mu}_1 \equiv \mu_1/\lambda_1, \breve{\mu}_2 \equiv \mu_2/\lambda_1$ $\breve{v}_1 \equiv v_1/\lambda_1, \breve{v}_2 \equiv v_2/\lambda_1$
Shear stress, Fourierized parameters [Eqs. (21)–(24)]	$\alpha_m, \beta_m, \gamma_m, \delta_m$
Shear stress, Fourierized parameters, transient [Eqs. (26)–(29)]	$\alpha_{m,0}, \beta_{m,0}, \gamma_{m,0}, \delta_{m,0}$
Integral defined in Eqs. (10)–(17) and their limits	$I_1, \ldots, I_8$ and $I_{1,0} \equiv \lim_{\tau\to 0} I_1, \ldots, I_{8,0} \equiv \lim_{\tau\to 0} I_8$
Parameters defined in Eqs. (55)–(58)	$C_1^{(2j)}, C_2^{(2j)}, S_1^{(2j)}, S_2^{(2j)}$
Parameters defined in Eqs. (59)–(74)	$\phi_1^{(i)}, \ldots, \phi_8^{(i)}, \psi_1^{(i)}, \ldots, \psi_8^{(i)}$
Spriggs exponent	α

Table III: Summary of fitted model parameters for all three donors.

Multimode CJ fluid		
Donor 1	Donor 2	Donor 3
$\eta_0 = 0.0315\,\mathrm{Pa\cdot s}, \alpha = 2.05$ $\lambda_1 = 2.95\,\mathrm{s}, \lambda_2 = 0.285\,\mathrm{s}$	$\eta_0 = 0.0495\,\mathrm{Pa\cdot s}, \alpha = 2.25$ $\lambda_1 = 2.65\,\mathrm{s}, \lambda_2 = 0.185\,\mathrm{s}$	$\eta_0 = 0.0455\,\mathrm{Pa\cdot s}, \alpha = 2.35$ $\lambda_1 = 2.25\,\mathrm{s}, \lambda_2 = 0.160\,\mathrm{s}$
Multimode O8 framework		
Donor 1	Donor 2	Donor 3
$\eta_0 = 0.0315\,\mathrm{Pa\cdot s}, \alpha = 2.05,$ $\lambda_1 = 2.95\,\mathrm{s}, \lambda_2 = 0.285\,\mathrm{s},$ $\mu_0 = -0.500\,\mathrm{s}, \mu_2 = -0.120\,\mathrm{s},$ $\mu_1 = \nu_1 = \nu_2 = 0\,\mathrm{s}$	$\eta_0 = 0.0495\,\mathrm{Pa\cdot s}, \alpha = 2.25,$ $\lambda_1 = 2.65\,\mathrm{s}, \lambda_2 = 0.185\,\mathrm{s},$ $\mu_0 = -0.500\,\mathrm{s}, \mu_2 = -0.100\,\mathrm{s},$ $\mu_1 = \nu_1 = \nu_2 = 0\,\mathrm{s}$	$\eta_0 = 0.0455\,\mathrm{Pa\cdot s}, \alpha = 2.35,$ $\lambda_1 = 2.25\,\mathrm{s}, \lambda_2 = 0.160\,\mathrm{s},$ $\mu_0 = \mu_2 = -0.200\,\mathrm{s},$ $\mu_1 = \nu_1 = \nu_2 = 0\,\mathrm{s}$

Table IV: Residual sum of squares (RSS) for fitted curves from the multimode corotational Jeffreys (CJ) fluid and the multimode Oldroyd 8-constant (O8) framework.

Experiment	RSS (Donor 1, Donor 2, Donor 3)	
	Multimode CJ	Multimode O8
Steady shear	0.0108,0.0213,0.0091	0.0100,0.0195,0.0082
$\omega = 10\,\text{rad}/s$ and $\dot{\gamma}^0 = 10\,\text{s}^{-1}$	0.0118, 0.0212, 0.0100	0.00802, 0.00676, 0.00265
$\omega = 10\,\text{rad}/s$ and $\dot{\gamma}^0 = 100\,\text{s}^{-1}$	0.223,0.336, 0.244	0.134,0.214,0.176
$\omega = 1\,\text{rad}/s$ and $\dot{\gamma}^0 = 5\,\text{s}^{-1}$	0.00285,0.0808, 0.0615	0.00231, 0.0828, 0.0636

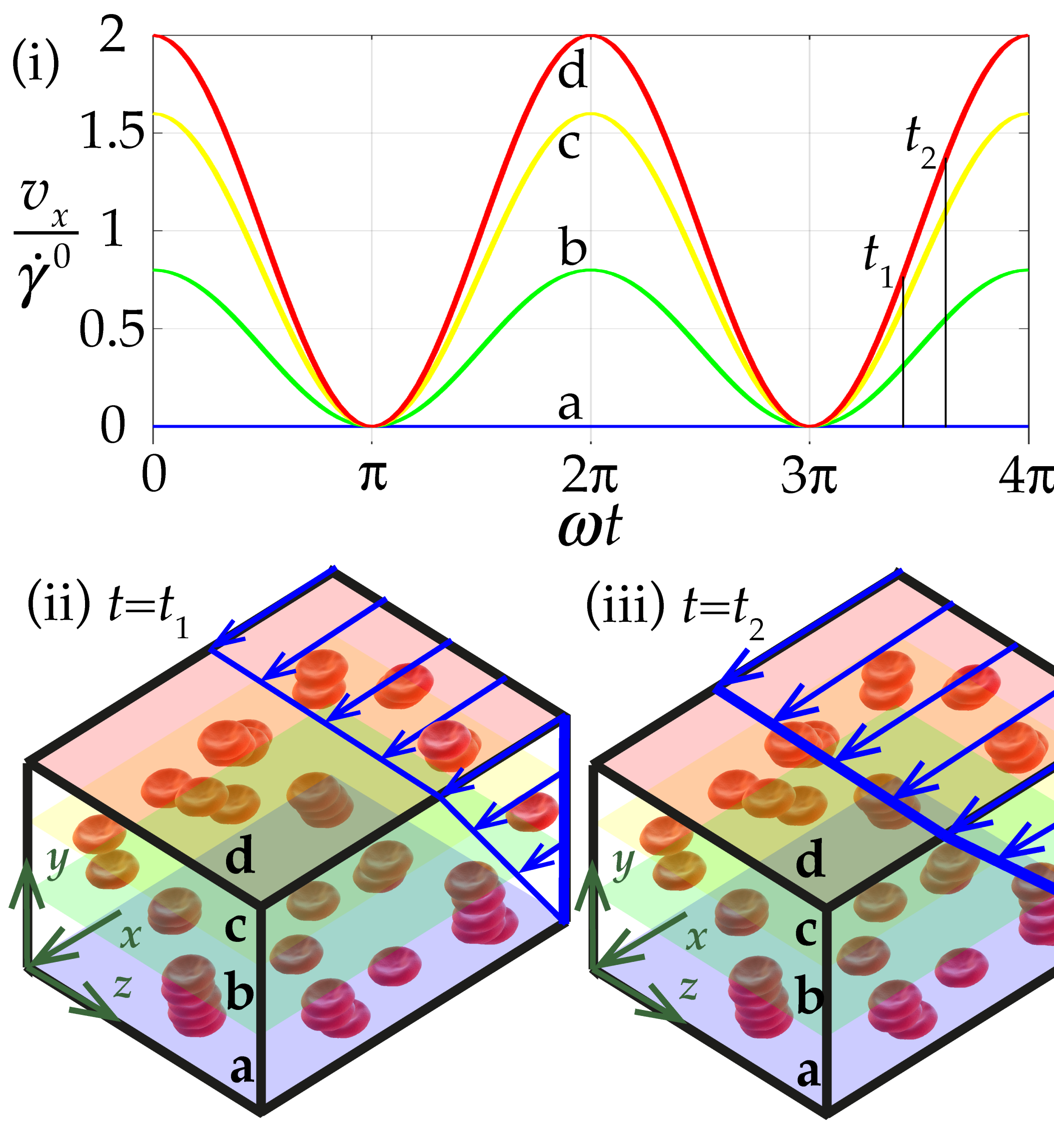

Figure 1: (i) *ud*LAOS velocity profile showing $v_x/\dot{\gamma}^0 \geq 0$ indicating no backflow. Curves a, b, c and d shows velocity profiles of corresponding planes illustrated in (ii) and (iii). Fluid elements in (iii), at $t = t_2$, have twice the speed of those in (ii).

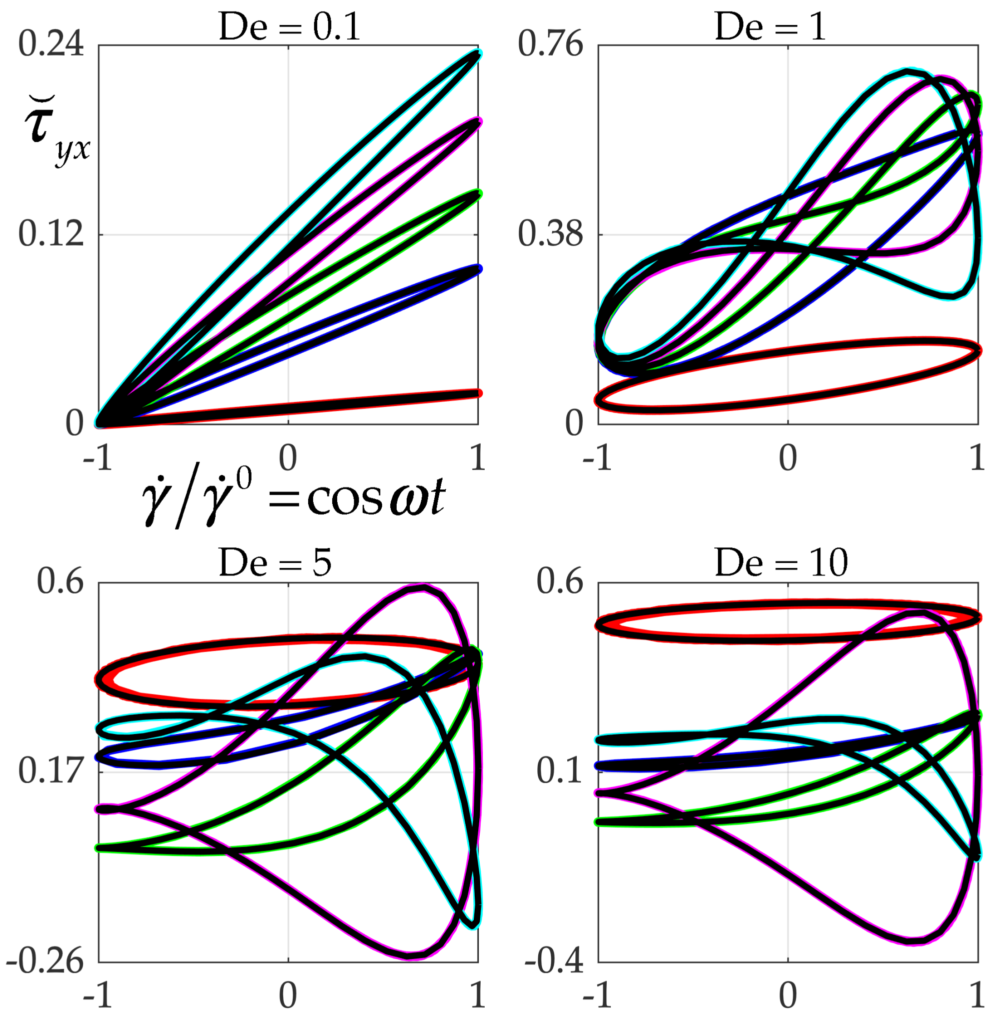

Figure 2: Alternant response of the shear stress [Eq. (20) with Eqs. (21)–(24) (in **black**)] and finite difference calculations of Eq. (8) (in **color**). Loops of dimensionless shear stress, $\breve{\tau}_{yx} \equiv \lambda_1 \tau_{yx} / \eta_0$, *versus* dimensionless shear rate, $\dot{\gamma}/\dot{\gamma}^0 = \cos\omega t$ for $\text{Wi/De} = 1/10$, $1/2$, $3/4$, 1, $5/4$ and $\lambda_2/\lambda_1 = 0$ (the special case of the corotational Maxwell fluid). Four subfigures are for $\text{De} = 0.1, 1, 5, 10$.

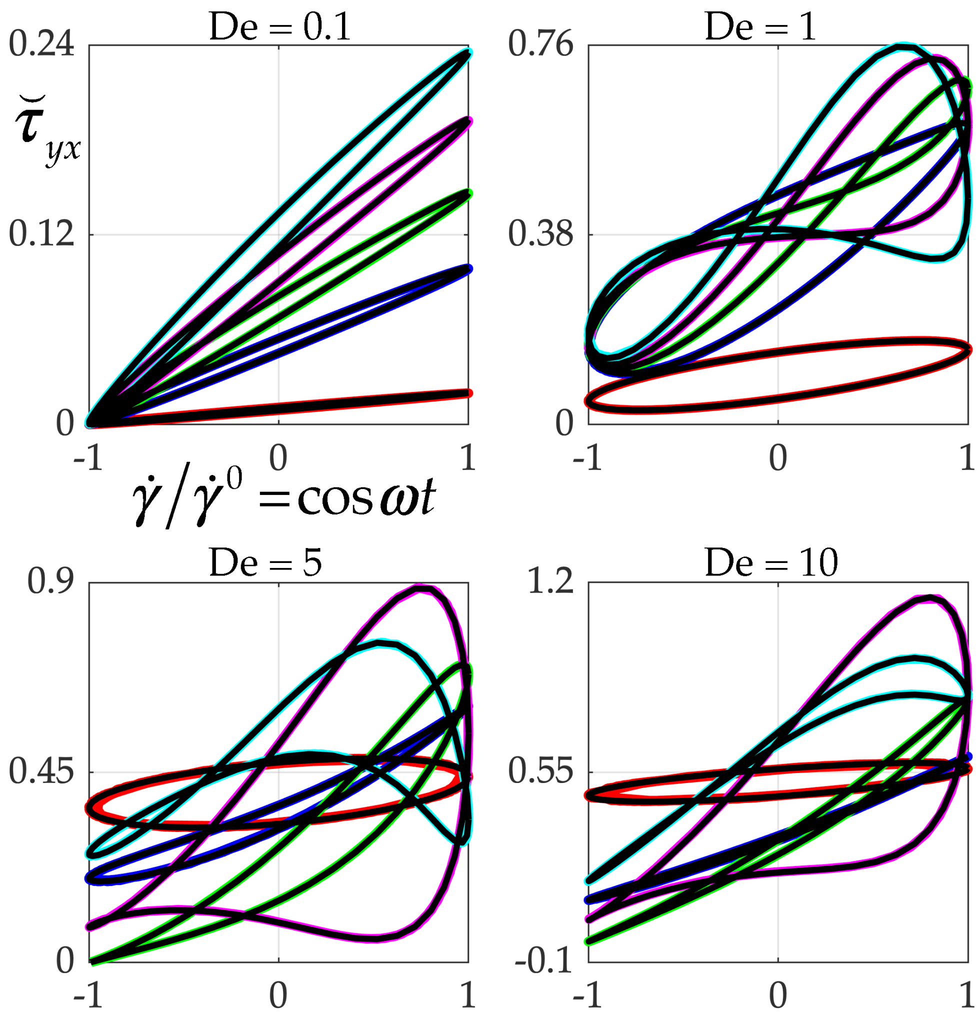

Figure 3: Alternant response of the shear stress [Eq. (20) with Eqs. (21)–(24) (**black**)] *versus* finite difference calculations of Eq. (8) (in **color**). Loops of dimensionless shear stress, $\breve{\tau}_{yx} \equiv \lambda_1 \tau_{yx}/\eta_0$, *versus* dimensionless shear rate, $\dot{\gamma}/\dot{\gamma}^0 = \cos\omega t$ for $\mathrm{Wi/De} = 1/10$, $1/2$, $3/4$, 1, $5/4$ and $\lambda_2/\lambda_1 = 1/27$. Four subfigures are for $\mathrm{De} = 0.1, 1, 5, 10$.

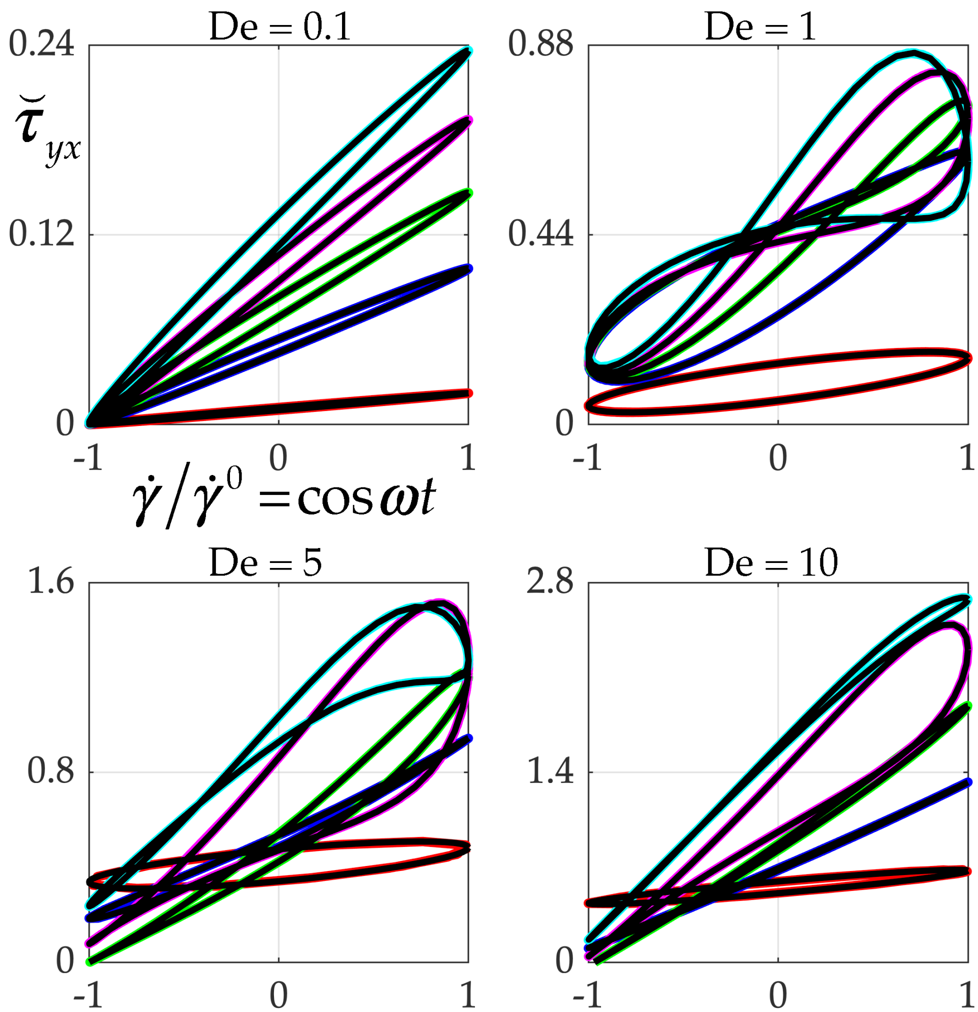

Figure 4: Alternant response of the shear stress [Eq. (20) with Eqs. (21)–(24) (**black**)] *versus* finite difference calculations of Eq. (8) (in **color**). Loops of dimensionless shear stress, $\breve{\tau}_{yx} \equiv \lambda_1 \tau_{yx}/\eta_0$, *versus* dimensionless shear rate, $\dot{\gamma}/\dot{\gamma}^0 = \cos\omega t$ for $\mathrm{Wi/De} = 1/10$, $1/2$, $3/4$, 1, $5/4$ and $\lambda_2/\lambda_1 = 1/9$. Four subfigures are for $\mathrm{De} = 0.1, 1, 5, 10$.

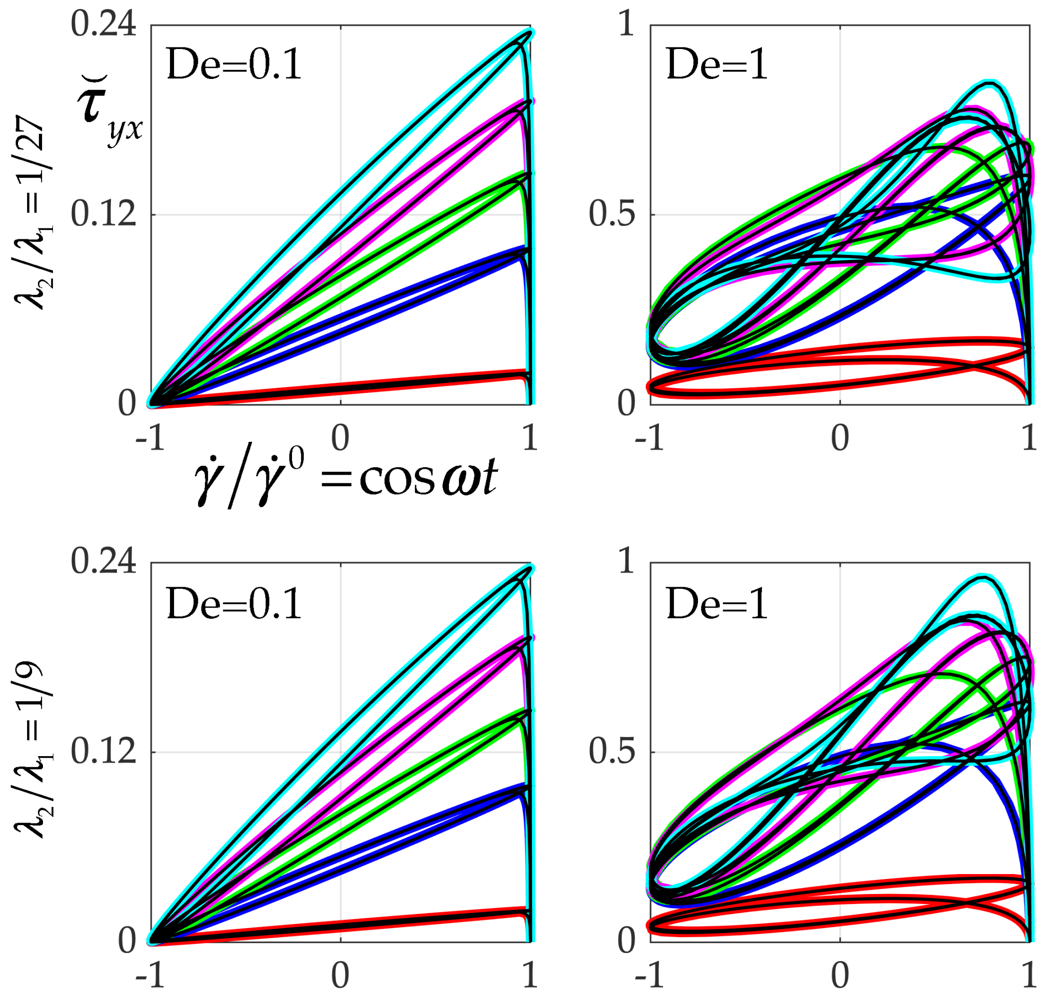

Figure 5: Startup shear stress response [Eqs. (20) and (25) with Eqs. (21)–(24) and Eqs. (26)–(29) (in **black**)] and finite difference calculations of Eq. (8) (in **color**). Loops of dimensionless shear stress, $\breve{\tau}_{yx} \equiv \lambda_1 \tau_{yx}/\eta_0$, *versus* dimensionless shear rate, $\dot{\gamma}/\dot{\gamma}^0 = \cos\omega t$ for $\mathrm{Wi/De} = 1/10, 1/2, 3/4, 1, 5/4$, and $\mathrm{De} = 0.1, 1$ and for $\lambda_2/\lambda_1 = 1/27$ (top row) and $\lambda_2/\lambda_1 = 1/9$ (bottom row).

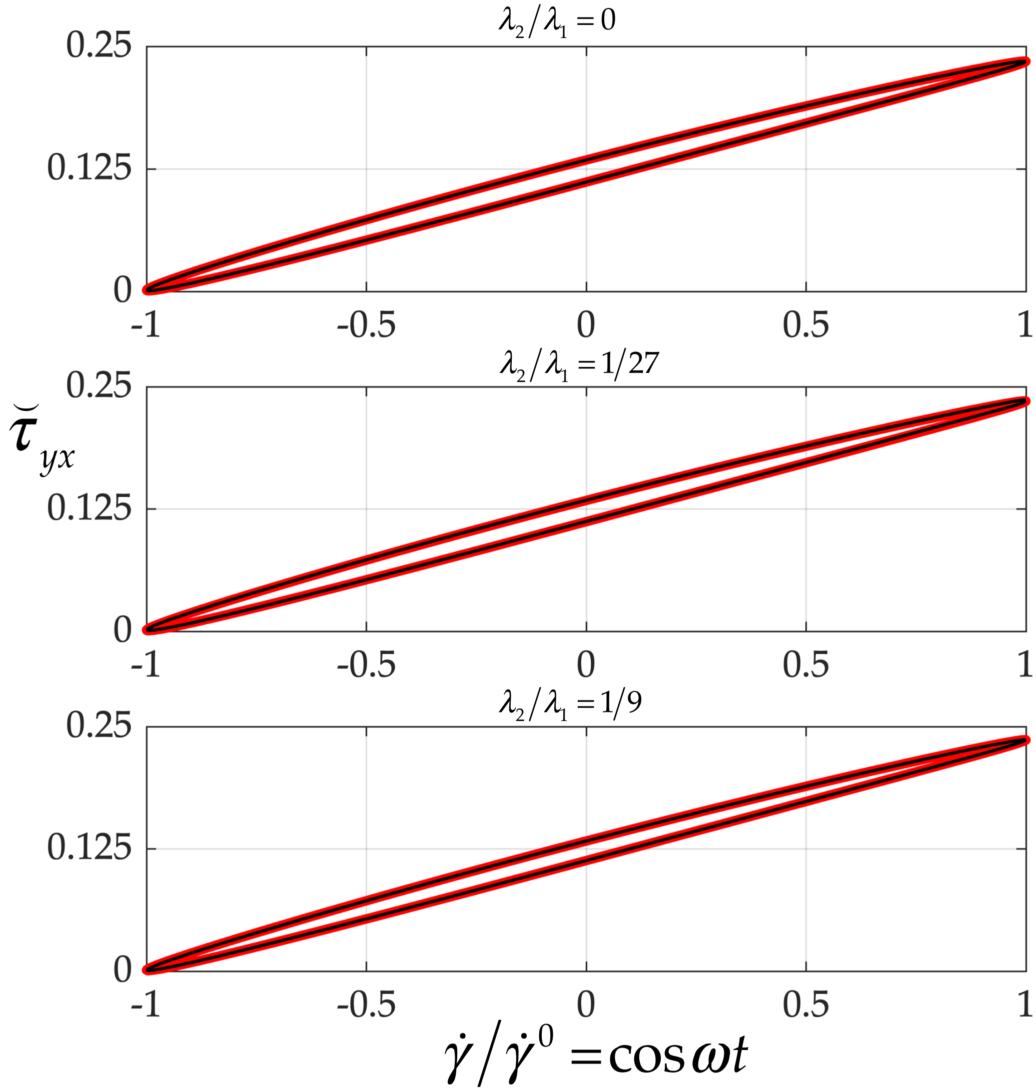

Figure 6: Alternant response of the exact solution [Eq. (20) with Eqs. (21)–(24) (**black**)] *versus* finite difference calculations of Eq. (8) (**red**). Loops of dimensionless shear stress, $\breve{\tau}_{yx} \equiv \lambda_1 \tau_{yx}/\eta_0$, *versus* dimensionless shear rate, $\dot{\gamma}/\dot{\gamma}^0 = \cos\omega t$ at $\mathrm{De} \to 0$. Three subfigures are for $\lambda_2/\lambda_1 = 0, 1/27, 1/9$.

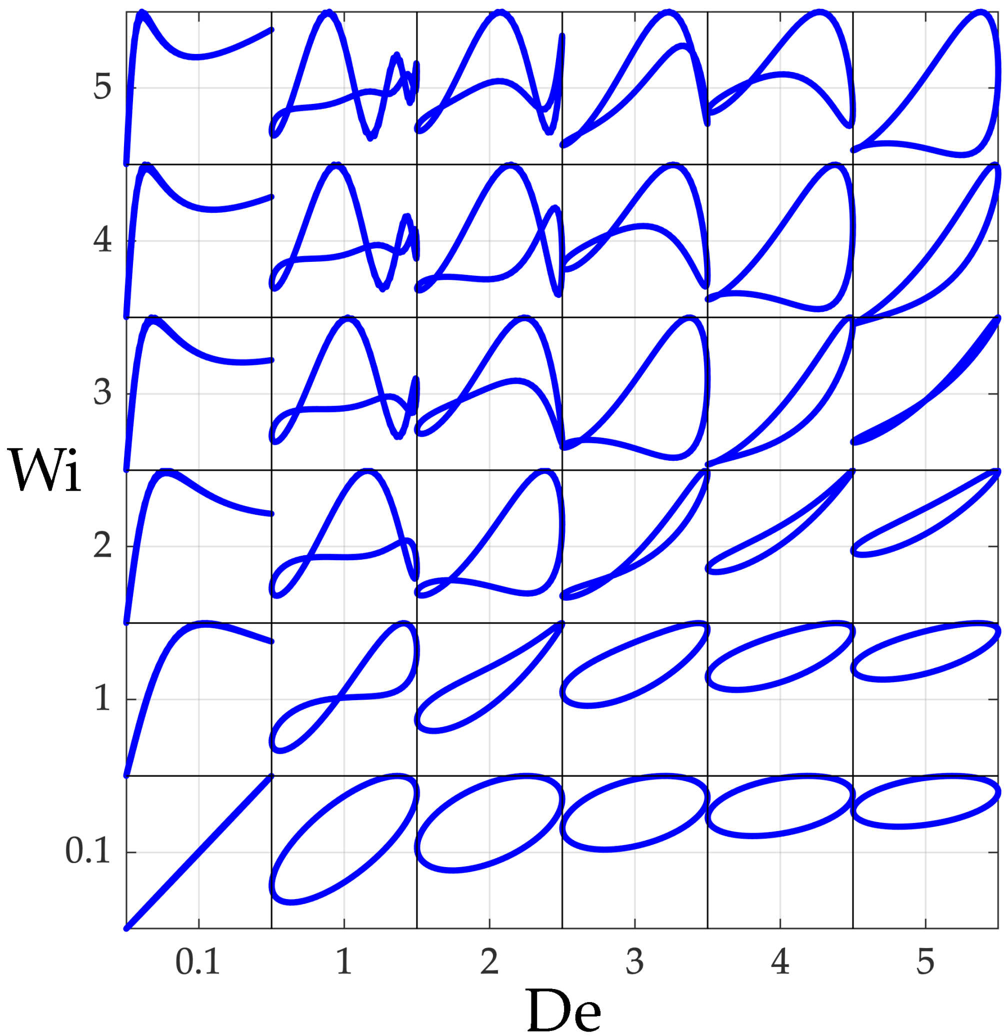

Figure 7: Ewoldt grid of the alternant response [Eq. (20) with Eqs. (21)–(24)] for $\lambda_2/\lambda_1 = 1/9$.

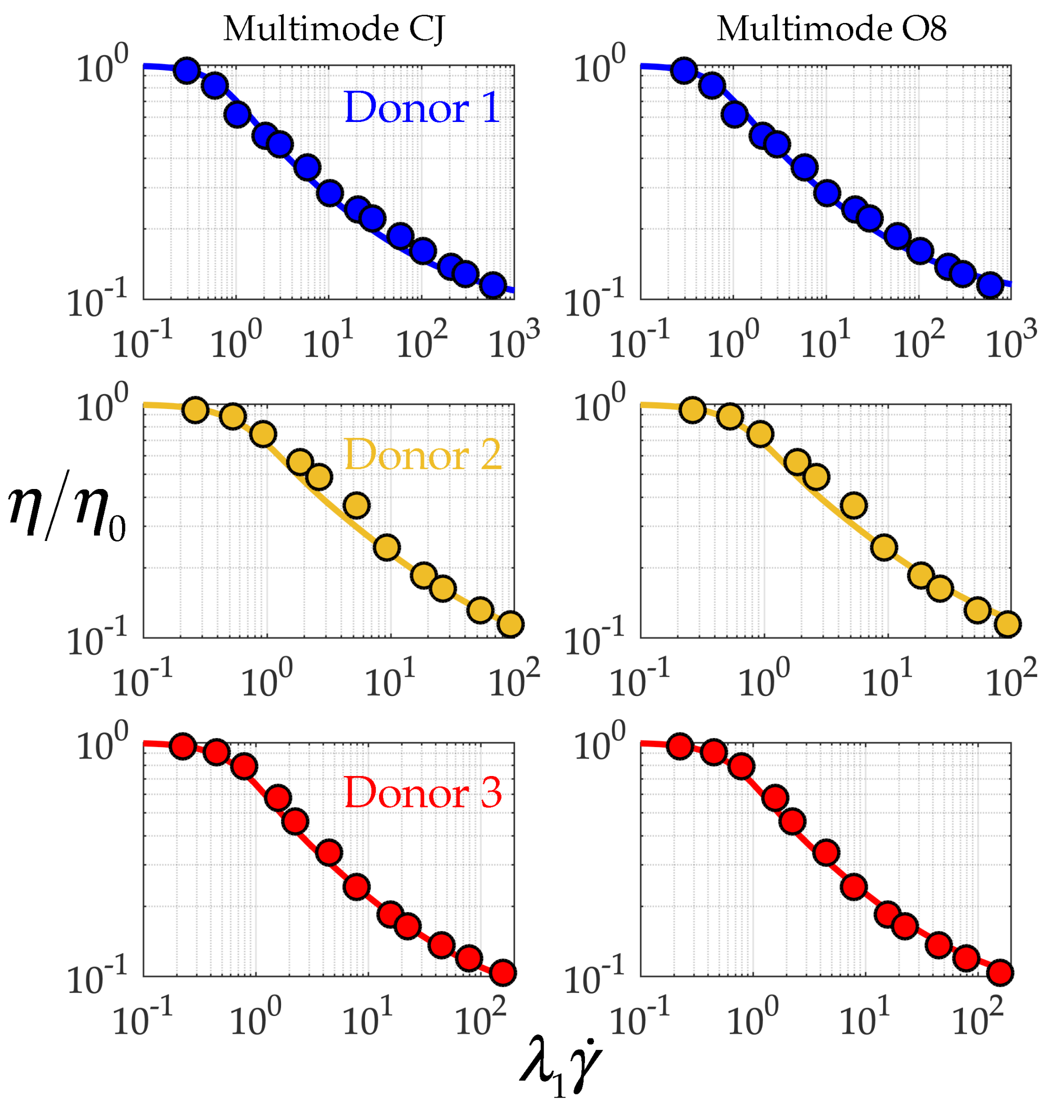

Figure 8: Measured (**circles**) and fitted (**lines**) η/η_0 *versus* $\lambda_1\dot{\gamma}$. Left column is from the multimode CJ [Eq. (36)]. Right column is finite difference calculation for the multimode O8 [Eq. (35)]. The fitted model parameters for both models are summarized in Table III and the RSS are summarized in Table IV.

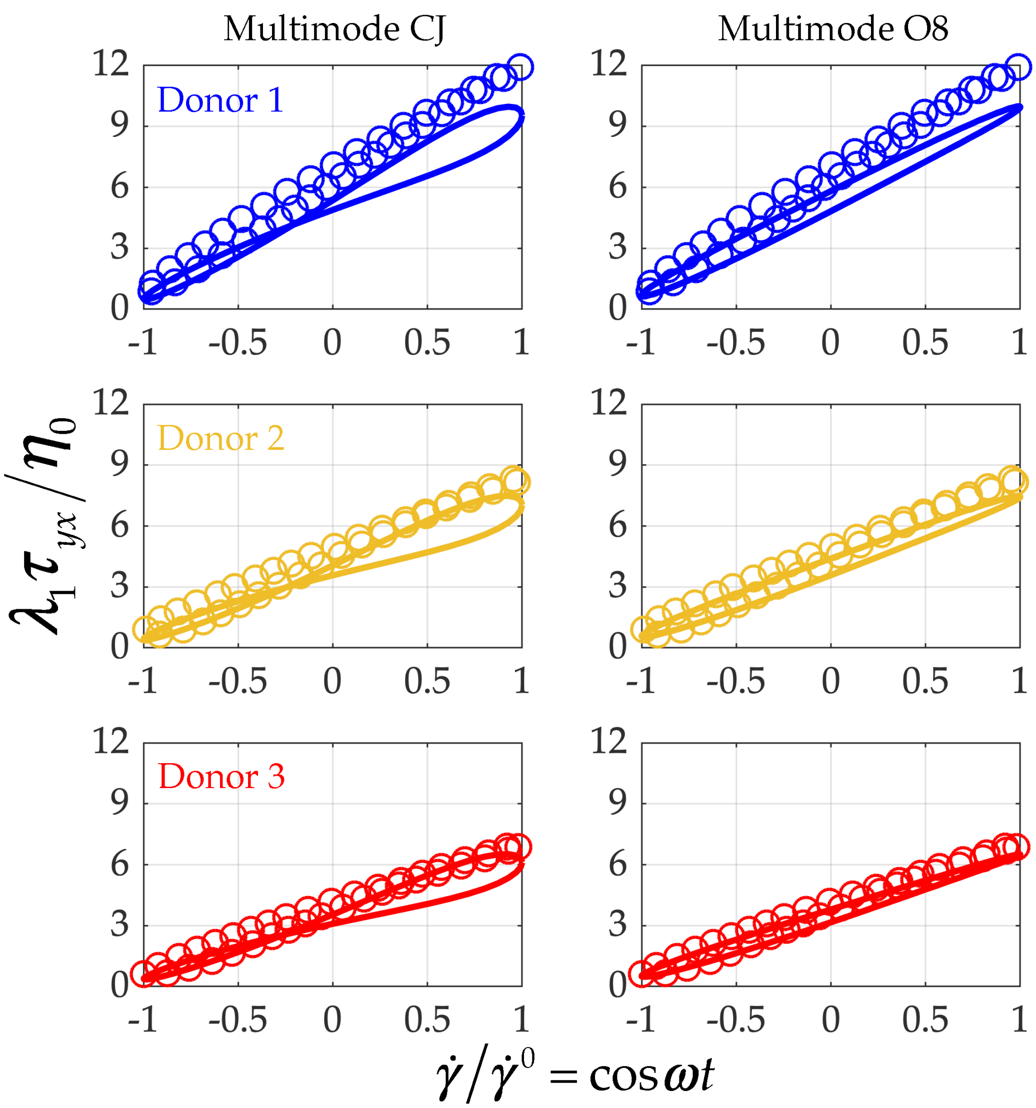

Figure 9: Measured (**circles**) and predicted (**lines**) $\lambda_1 \tau_{yx} / \eta_0$ *versus* $\dot{\gamma}/\dot{\gamma}^0$ for $\omega = 10\,\mathrm{rad/s}$ and $\dot{\gamma}^0 = 10\,\mathrm{s}^{-1}$. Using the best-fit parameters from steady shear experiment, left column is from the multimode CJ [Eq. (37)], right column is finite difference calculation of the multimode O8 [Eq. (38) with Eq. (33)] using the fitted model parameters in Table III. RSS are summarized in Table IV.

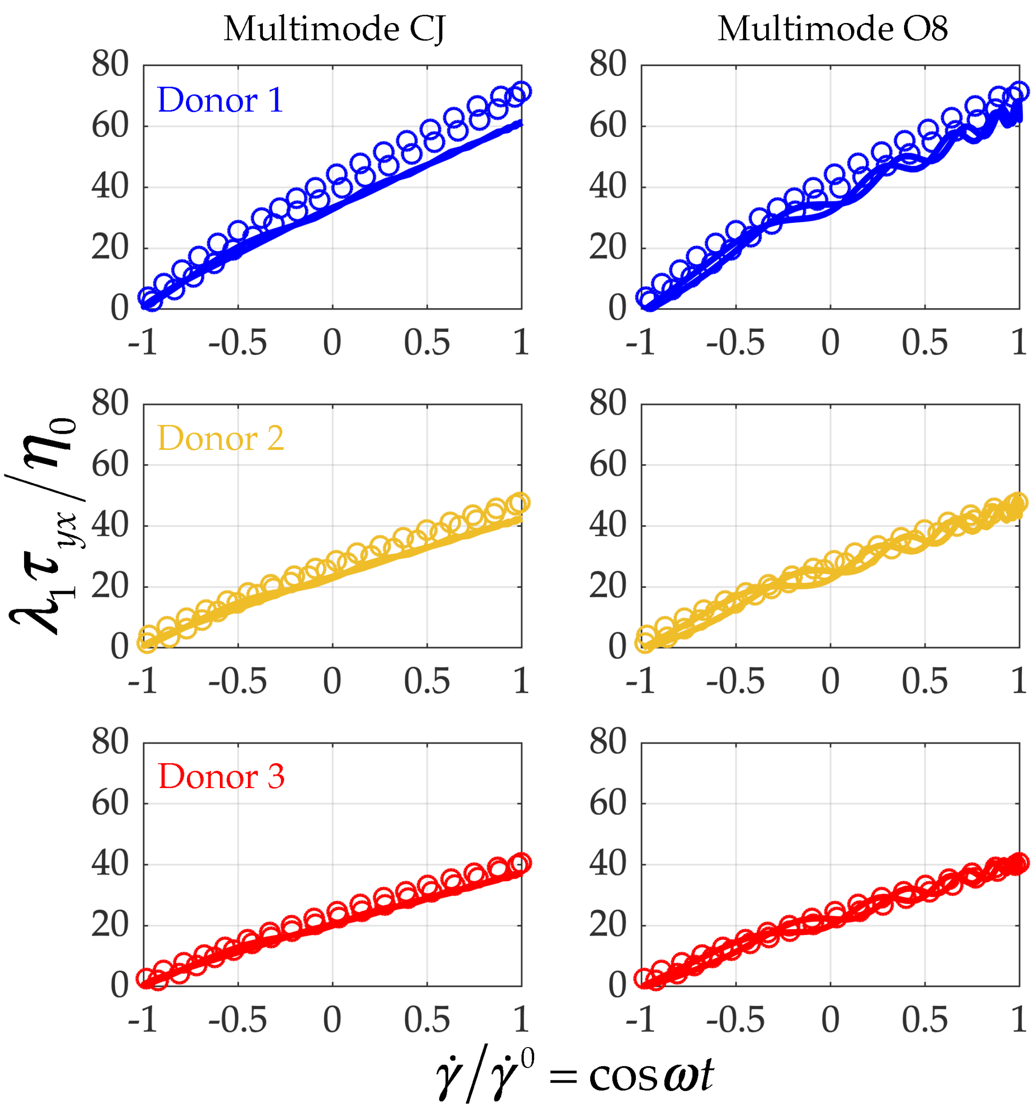

Figure 10: Measured (**circles**) and predicted (**lines**) $\lambda_1 \tau_{yx}/\eta_0$ *versus* $\dot{\gamma}/\dot{\gamma}^0$ for $\omega = 10\,\text{rad/s}$ and $\dot{\gamma}^0 = 100\,\text{s}^{-1}$. Using the best-fit parameters from steady shear experiment, left column is from the multimode CJ [Eq. (37)], right column is finite difference calculation of the multimode O8 [Eq. (38) with Eq. (33)] using the fitted model parameters in Table III. RSS are summarized in Table IV.

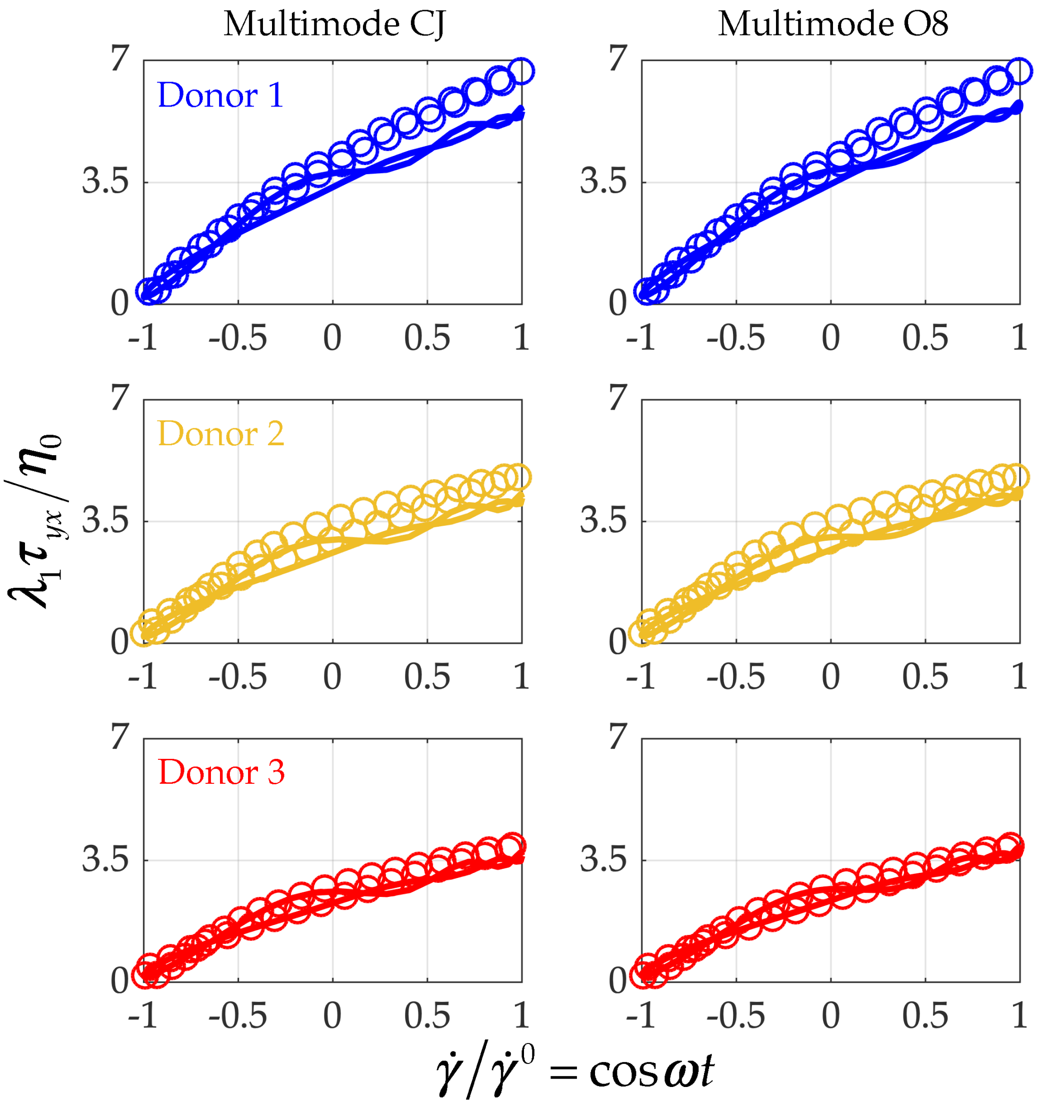

Figure 11: Measured (**circles**) and predicted (**lines**) $\lambda_1\tau_{yx}/\eta_0$ *versus* $\dot{\gamma}/\dot{\gamma}^0$ for $\omega = 1\,\text{rad/s}$ and $\dot{\gamma}^0 = 5\,\text{s}^{-1}$. Using the best-fit parameters from steady shear experiment, left column is from the multimode CJ [Eq. (37)], right column is finite difference calculation of the multimode O8 [Eq. (38) with Eq. (33)] using the fitted model parameters in Table III. RSS are summarized in Table IV.

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SUPPLEMENTARY MATERIAL TO: UNIDIRECTIONAL LARGE-AMPLITUDE OSCILLATORY SHEAR FLOW OF HUMAN BLOOD

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CONTENTS

This supplements [1,2] with the proof of the stress governing equation and our main results. Section I of the Supplementary Material provides the detailed derivation of the stress governing equation for both the Oldroyd 8-constant framework, Eq. (7), and its special case, the corotational Jeffreys fluid, Eq. (8). Section II of the Supplementary Material provides the detailed derivation of our main results, for the alternant part, Eq. (20) [with Eqs. (21)–(24)], and the transient part, Eq. (25) [with Eqs. (26)–(29)].

I DERIVATION OF THE STRESS GOVERNING EQUATION

The shear rate expression for *ud*LAOS is given by:

$$\dot{\gamma} = \dot{\gamma}^0 (1+\cos\omega t) \tag{1}$$

and the Oldroyd 8-constant framework is given by:

$$\begin{aligned} &\boldsymbol{\tau} + \lambda_1 \frac{\mathscr{D}\boldsymbol{\tau}}{\mathscr{D}t} + \tfrac{1}{2}\mu_0 (\operatorname{tr}\boldsymbol{\tau})\dot{\boldsymbol{\gamma}} - \tfrac{1}{2}\mu_1 \{\boldsymbol{\tau}\cdot\dot{\boldsymbol{\gamma}} + \dot{\boldsymbol{\gamma}}\cdot\boldsymbol{\tau}\} + \tfrac{1}{2}\nu_1 (\boldsymbol{\tau}:\dot{\boldsymbol{\gamma}})\boldsymbol{\delta} \\ &\quad = -\eta_0 \left(\dot{\boldsymbol{\gamma}} + \lambda_2 \frac{\mathscr{D}\dot{\boldsymbol{\gamma}}}{\mathscr{D}t} - \mu_2 \{\dot{\boldsymbol{\gamma}}\cdot\dot{\boldsymbol{\gamma}}\} + \tfrac{1}{2}\nu_2 (\dot{\boldsymbol{\gamma}}:\dot{\boldsymbol{\gamma}})\boldsymbol{\delta} \right) \end{aligned} \tag{2}$$

To derive the stress governing equation for this framework under *ud*LAOS, we need the following ingredients:

$$\dot{\boldsymbol{\gamma}} \equiv \nabla\mathbf{v} + (\nabla\mathbf{v})^\dagger = \dot{\gamma}^0 (1+\cos\omega t) \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{3}$$

$$\boldsymbol{\omega} \equiv \nabla\mathbf{v} - (\nabla\mathbf{v})^\dagger = \dot{\gamma}^0 (1+\cos\omega t) \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{4}$$

$$\dot{\gamma}_{yx} = \dot{\gamma}^0 (1+\cos\omega t) \tag{5}$$

$$\boldsymbol{\tau} = \begin{bmatrix} \tau_{xx} & \tau_{yx} & 0 \\ \tau_{yx} & \tau_{yy} & 0 \\ 0 & 0 & \tau_{zz} \end{bmatrix} \tag{6}$$

$$\begin{aligned} \frac{\mathscr{D}\boldsymbol{\tau}}{\mathscr{D}t} &\equiv \frac{D\boldsymbol{\tau}}{Dt} + \frac{1}{2}\{\boldsymbol{\omega}\cdot\boldsymbol{\tau} - \boldsymbol{\tau}\cdot\boldsymbol{\omega}\} = \frac{\partial\boldsymbol{\tau}}{\partial t} + \frac{1}{2}\{\boldsymbol{\omega}\cdot\boldsymbol{\tau} - \boldsymbol{\tau}\cdot\boldsymbol{\omega}\} \\ &= \begin{bmatrix} \dot{\tau}_{xx} & \dot{\tau}_{yx} & 0 \\ \dot{\tau}_{yx} & \dot{\tau}_{yy} & 0 \\ 0 & 0 & \dot{\tau}_{zz} \end{bmatrix} + \frac{1}{2}\dot{\gamma}^0 (1+\cos\omega t) \begin{bmatrix} -2\tau_{yx} & \tau_{xx}-\tau_{yy} & 0 \\ \tau_{xx}-\tau_{yy} & 2\tau_{yx} & 0 \\ 0 & 0 & 0 \end{bmatrix} \end{aligned} \tag{7}$$

$$(\mathrm{tr}\,\boldsymbol{\tau})\dot{\boldsymbol{\gamma}}=\left(\tau_{xx}+\tau_{yy}+\tau_{zz}\right)\dot{\gamma}^0\left(1+\cos\omega t\right)\begin{bmatrix}0&1&0\\1&0&0\\0&0&0\end{bmatrix} \tag{8}$$

$$\left\{\boldsymbol{\tau}\cdot\dot{\boldsymbol{\gamma}}+\dot{\boldsymbol{\gamma}}\cdot\boldsymbol{\tau}\right\}=\dot{\gamma}^0\left(1+\cos\omega t\right)\begin{bmatrix}2\tau_{yx}&\tau_{yy}+\tau_{xx}&0\\\tau_{yy}+\tau_{xx}&2\tau_{yx}&0\\0&0&0\end{bmatrix} \tag{9}$$

$$\left(\boldsymbol{\tau}:\dot{\boldsymbol{\gamma}}\right)\boldsymbol{\delta}=2\dot{\gamma}^0\left(1+\cos\omega t\right)\tau_{yx}\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix} \tag{10}$$

$$\begin{aligned}\frac{\mathscr{D}\dot{\boldsymbol{\gamma}}}{\mathscr{D}t}&\equiv\frac{D\dot{\boldsymbol{\gamma}}}{Dt}+\frac{1}{2}\left\{\boldsymbol{\omega}\cdot\dot{\boldsymbol{\gamma}}-\dot{\boldsymbol{\gamma}}\cdot\boldsymbol{\omega}\right\}=\frac{\partial\dot{\boldsymbol{\gamma}}}{\partial t}+\frac{1}{2}\left\{\boldsymbol{\omega}\cdot\dot{\boldsymbol{\gamma}}-\dot{\boldsymbol{\gamma}}\cdot\boldsymbol{\omega}\right\}\\&=-\dot{\gamma}^0\omega\sin\omega t\begin{bmatrix}0&1&0\\1&0&0\\0&0&0\end{bmatrix}+\left[\dot{\gamma}^0\left(1+\cos\omega t\right)\right]^2\begin{bmatrix}-1&0&0\\0&1&0\\0&0&0\end{bmatrix}\end{aligned} \tag{11}$$

$$\left\{\dot{\boldsymbol{\gamma}}\cdot\dot{\boldsymbol{\gamma}}\right\}=\left[\dot{\gamma}^0\left(1+\cos\omega t\right)\right]^2\begin{bmatrix}1&0&0\\0&1&0\\0&0&0\end{bmatrix} \tag{12}$$

$$\left(\dot{\boldsymbol{\gamma}}:\dot{\boldsymbol{\gamma}}\right)\boldsymbol{\delta}=2\left[\dot{\gamma}^0\left(1+\cos\omega t\right)\right]^2\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix} \tag{13}$$

Substituting Eqs. (3) through (13) into Eq. (2), we find:

$$\begin{aligned}&\begin{bmatrix}\tau_{xx}&\tau_{yx}&0\\\tau_{yx}&\tau_{yy}&0\\0&0&\tau_{zz}\end{bmatrix}+\lambda_1\begin{bmatrix}\dot{\tau}_{xx}&\dot{\tau}_{yx}&0\\\dot{\tau}_{yx}&\dot{\tau}_{yy}&0\\0&0&\dot{\tau}_{zz}\end{bmatrix}+\frac{\lambda_1}{2}\dot{\gamma}_{yx}\begin{bmatrix}-2\tau_{yx}&\tau_{xx}-\tau_{yy}&0\\\tau_{xx}-\tau_{yy}&2\tau_{yx}&0\\0&0&0\end{bmatrix}+\frac{\mu_0}{2}\left(\tau_{xx}+\tau_{yy}+\tau_{zz}\right)\dot{\gamma}_{yx}\begin{bmatrix}0&1&0\\1&0&0\\0&0&0\end{bmatrix}\\&-\frac{\mu_1}{2}\dot{\gamma}_{yx}\begin{bmatrix}2\tau_{yx}&\tau_{yy}+\tau_{xx}&0\\\tau_{yy}+\tau_{xx}&2\tau_{yx}&0\\0&0&0\end{bmatrix}+\nu_1\dot{\gamma}_{yx}\tau_{yx}\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}\\&=-\eta_0\dot{\gamma}_{yx}\begin{bmatrix}0&1&0\\1&0&0\\0&0&0\end{bmatrix}+\eta_0\lambda_2\dot{\gamma}^0\omega\sin\omega t\begin{bmatrix}0&1&0\\1&0&0\\0&0&0\end{bmatrix}-\eta_0\lambda_2\dot{\gamma}_{yx}^2\begin{bmatrix}-1&0&0\\0&1&0\\0&0&0\end{bmatrix}\\&\quad+\eta_0\mu_2\dot{\gamma}_{yx}^2\begin{bmatrix}1&0&0\\0&1&0\\0&0&0\end{bmatrix}-\eta_0\nu_2\dot{\gamma}_{yx}^2\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}\end{aligned} \tag{14}$$

from which we extract:

$$\lambda_1\dot{\tau}_{xx}+\tau_{xx}-\left[\lambda_1+\mu_1-\nu_1\right]\dot{\gamma}^0\left(1+\cos\omega t\right)\tau_{yx}=\eta_0\left(\lambda_2+\mu_2-\nu_2\right)\left[\dot{\gamma}^0\left(1+\cos\omega t\right)\right]^2 \tag{15}$$

$$\lambda_1\dot{\tau}_{yy}+\tau_{yy}+\left(\lambda_1-\mu_1+\nu_1\right)\dot{\gamma}^0\left(1+\cos\omega t\right)\tau_{yx}=-\eta_0\left(\lambda_2-\mu_2+\nu_2\right)\left[\dot{\gamma}^0\left(1+\cos\omega t\right)\right]^2 \tag{16}$$

$$\lambda_1\dot{\tau}_{zz}=-\tau_{zz}-\nu_1\dot{\gamma}^0\left(1+\cos\omega t\right)\tau_{yx}-\eta_0\nu_2\left[\dot{\gamma}^0\left(1+\cos\omega t\right)\right]^2 \tag{17}$$

$$\begin{aligned}\lambda_1\dot{\tau}_{yx}=&-\tfrac{1}{2}\left[\lambda_1-\mu_1+\mu_0\right]\dot{\gamma}^0\left(1+\cos\omega t\right)N_1-\left[\mu_0-\mu_1\right]\dot{\gamma}^0\left(1+\cos\omega t\right)N_2\\&-\left[\tfrac{3}{2}\mu_0-\mu_1\right]\dot{\gamma}^0\left(1+\cos\omega t\right)\tau_{zz}-\tau_{yx}-\eta_0\dot{\gamma}^0\left(1+\cos\omega t\right)+\eta_0\lambda_2\dot{\gamma}^0\omega\sin\omega t\end{aligned} \tag{18}$$

Eqs. (15) through (18) can be nondimensionalized to:

$$\begin{aligned}\frac{d\breve{\tau}_{yx}}{d\tau}=&-\frac{1}{\mathrm{De}}\breve{\tau}_{yx}-\tfrac{1}{2}\left[1+\breve{\mu}_0-\breve{\mu}_1\right]\frac{\mathrm{Wi}}{\mathrm{De}}\left(1+\cos\tau\right)\breve{N}_1-\left[\breve{\mu}_0-\breve{\mu}_1\right]\frac{\mathrm{Wi}}{\mathrm{De}}\left(1+\cos\tau\right)\breve{N}_2\\&-\left[\tfrac{3}{2}\breve{\mu}_0-\breve{\mu}_1\right]\frac{\mathrm{Wi}}{\mathrm{De}}\left(1+\cos\tau\right)\breve{\tau}_{zz}-\frac{\mathrm{Wi}}{\mathrm{De}}\left(1+\cos\tau\right)+\breve{\lambda}_2\,\mathrm{Wi}\sin\tau\end{aligned} \tag{19}$$

$$\frac{d\breve{N}_1}{d\tau}=2\frac{\mathrm{Wi}}{\mathrm{De}}\left(1+\cos\tau\right)\breve{\tau}_{yx}-\frac{1}{\mathrm{De}}\breve{N}_1+2\breve{\lambda}_2\frac{\mathrm{Wi}^2}{\mathrm{De}}\left(1+\cos\tau\right)^2 \tag{20}$$

$$\frac{d\breve{N}_2}{d\tau}=-\left(1-\breve{\mu}_1\right)\frac{\mathrm{Wi}}{\mathrm{De}}\left(1+\cos\tau\right)\breve{\tau}_{yx}-\frac{1}{\mathrm{De}}\breve{N}_2-\left(\breve{\lambda}_2-\breve{\mu}_2\right)\frac{\mathrm{Wi}^2}{\mathrm{De}}\left(1+\cos\tau\right)^2 \tag{21}$$

$$\frac{d\breve{\tau}_{zz}}{d\tau}=-\breve{\nu}_1\frac{\mathrm{Wi}}{\mathrm{De}}\left(1+\cos\tau\right)\breve{\tau}_{yx}-\frac{1}{\mathrm{De}}\breve{\tau}_{zz}-\breve{\nu}_2\frac{\mathrm{Wi}^2}{\mathrm{De}}\left(1+\cos\tau\right)^2 \tag{22}$$

which can be rewritten into matrix form as in Eq. (7) of the main manuscript [1,2]. For the special case of the corotational Jeffreys fluid, we set $\mu_0=\mu_1=\mu_2=\nu_1=\nu_2=0$ to find:

$$\frac{d\breve{\tau}_{yx}}{d\tau}=-\frac{1}{\mathrm{De}}\breve{\tau}_{yx}-\frac{1}{2}\frac{\mathrm{Wi}}{\mathrm{De}}\left(1+\cos\tau\right)\breve{N}_1-\frac{\mathrm{Wi}}{\mathrm{De}}\left(1+\cos\tau\right)+\breve{\lambda}_2\,\mathrm{Wi}\sin\tau \tag{23}$$

$$\frac{d\breve{N}_1}{d\tau}=2\frac{\mathrm{Wi}}{\mathrm{De}}\left(1+\cos\tau\right)\breve{\tau}_{yx}-\frac{1}{\mathrm{De}}\breve{N}_1+2\breve{\lambda}_2\frac{\mathrm{Wi}^2}{\mathrm{De}}\left(1+\cos\tau\right)^2 \tag{24}$$

$$\frac{d\breve{N}_2}{d\tau}=-\frac{\mathrm{Wi}}{\mathrm{De}}\left(1+\cos\tau\right)\breve{\tau}_{yx}-\frac{1}{\mathrm{De}}\breve{N}_2-\breve{\lambda}_2\frac{\mathrm{Wi}^2}{\mathrm{De}}\left(1+\cos\tau\right)^2 \tag{25}$$

$$\frac{d\breve{\tau}_{zz}}{d\tau}+\frac{1}{\mathrm{De}}\breve{\tau}_{zz}=0 \tag{26}$$

By inspection of Eqs. (24) and (25), we find $\breve{N}_1=-2\breve{N}_2$, as it should [6], and shear stress response is independent of $\breve{\tau}_{zz}$. Therefore, the relevant governing equation for stress responses are:

$$\frac{d\breve{\tau}_{yx}}{d\tau}=-\frac{1}{\mathrm{De}}\breve{\tau}_{yx}-\frac{1}{2}\frac{\mathrm{Wi}}{\mathrm{De}}\left(1+\cos\tau\right)\breve{N}_1-\frac{\mathrm{Wi}}{\mathrm{De}}\left(1+\cos\tau\right)+\breve{\lambda}_2\,\mathrm{Wi}\sin\tau \tag{27}$$

$$\frac{d\breve{N}_1}{d\tau}=2\frac{\mathrm{Wi}}{\mathrm{De}}\left(1+\cos\tau\right)\breve{\tau}_{yx}-\frac{1}{\mathrm{De}}\breve{N}_1+2\breve{\lambda}_2\frac{\mathrm{Wi}^2}{\mathrm{De}}\left(1+\cos\tau\right)^2 \tag{28}$$

which can be rewritten into matrix form as in Eq. (8) of the main manuscript [1,2]. Table I of [1,2] summarizes all dimensional variables and symbols, whereas Table II of [1,2] gathers all dimensionless ones.

II EXACT SOLUTIONS

We derive the exact shear stress expression from the corotational Jeffreys fluid from the governing equations for stresses, given by Eqs. (27) and (28). We first derive the homogeneous part in Subsection IIa, and then the particular part in Subsection IIb.

a Homogeneous Part

The homogeneous parts in Eqs. (23) and (24) are [3]:

$$\frac{d\breve{\tau}_{yx,h}}{d\tau}=-\frac{1}{\mathrm{De}}\breve{\tau}_{yx,h}-\frac{1}{2}\frac{\mathrm{Wi}}{\mathrm{De}}\left(1+\cos\tau\right)\breve{N}_{1,h} \tag{29}$$

$$\frac{d\breve{N}_{1,h}}{d\tau}=2\frac{\mathrm{Wi}}{\mathrm{De}}\left(1+\cos\tau\right)\breve{\tau}_{yx,h}-\frac{1}{\mathrm{De}}\breve{N}_{1,h} \tag{30}$$

Solving these by following the method of [3] as described in Section **4.** of [6], we find:

$$\breve{\tau}_{yx,h}=\frac{1}{2}e^{\frac{-\tau}{\mathrm{De}}}\left[ie^{\frac{\pi}{2\mathrm{De}}}\left(C_1-C_2\right)\cos\left(\frac{\mathrm{Wi}}{\mathrm{De}}\left(\tau+\sin\tau\right)\right)-e^{\frac{\pi}{2\mathrm{De}}}\left(C_1+C_2\right)\sin\left(\frac{\mathrm{Wi}}{\mathrm{De}}\left(\tau+\sin\tau\right)\right)\right] \tag{31}$$

$$\breve{N}_{1,h}=e^{\frac{-\tau}{\mathrm{De}}}\left[e^{\frac{\pi}{2\mathrm{De}}}\left(C_1+C_2\right)\cos\left(\frac{\mathrm{Wi}}{\mathrm{De}}\left(\tau+\sin\tau\right)\right)+ie^{\frac{\pi}{2\mathrm{De}}}\left(C_1-C_2\right)\sin\left(\frac{\mathrm{Wi}}{\mathrm{De}}\left(\tau+\sin\tau\right)\right)\right] \tag{32}$$

For simplicity, we group the integration constants:

$$C_3\equiv ie^{\frac{\pi}{2\mathrm{De}}}\left(C_1-C_2\right);\quad C_4\equiv e^{\frac{\pi}{2\mathrm{De}}}\left(C_1+C_2\right) \tag{33}$$

Substituting Eq. (33) into Eqs. (31) and (32), we find:

$$\breve{\tau}_{yx,h}=\frac{1}{2}e^{\frac{-\tau}{\mathrm{De}}}\left[C_3\cos\left(\frac{\mathrm{Wi}}{\mathrm{De}}\left(\tau+\sin\tau\right)\right)-C_4\sin\left(\frac{\mathrm{Wi}}{\mathrm{De}}\left(\tau+\sin\tau\right)\right)\right] \tag{34}$$

$$\breve{N}_{1,h}=e^{\frac{-\tau}{\mathrm{De}}}\left[C_3\sin\left(\frac{\mathrm{Wi}}{\mathrm{De}}\left(\tau+\sin\tau\right)\right)+C_4\cos\left(\frac{\mathrm{Wi}}{\mathrm{De}}\left(\tau+\sin\tau\right)\right)\right] \tag{35}$$

Rewriting Eqs. (34) and (35) into matrix form, we find:

$$\begin{bmatrix}\breve{\tau}_{yx,h}\\ \breve{N}_{1,h}\end{bmatrix}=\mathbf{\Phi}\begin{bmatrix}C_3\\ C_4\end{bmatrix} \tag{36}$$

in which the fundamental matrix is given by:

$$\mathbf{\Phi}\equiv\begin{bmatrix}\frac{1}{2}\exp\left(\frac{-\tau}{\mathrm{De}}\right)\cos\left(\frac{\mathrm{Wi}}{\mathrm{De}}\left(\tau+\sin\tau\right)\right) & -\frac{1}{2}\exp\left(\frac{-\tau}{\mathrm{De}}\right)\sin\left(\frac{\mathrm{Wi}}{\mathrm{De}}\left(\tau+\sin\tau\right)\right)\\ \exp\left(\frac{-\tau}{\mathrm{De}}\right)\sin\left(\frac{\mathrm{Wi}}{\mathrm{De}}\left(\tau+\sin\tau\right)\right) & \exp\left(\frac{-\tau}{\mathrm{De}}\right)\cos\left(\frac{\mathrm{Wi}}{\mathrm{De}}\left(\tau+\sin\tau\right)\right)\end{bmatrix} \tag{37}$$

which we will use to calculate the particular part of the stress responses. Eq. (36) is the homogeneous response of *ud*LAOS from the corotational Jeffreys.

b Particular Part

The particular responses can be calculated from (see Eq. (10) on p. 711 of [4]):

$$\begin{bmatrix} \breve{\tau}_{yx,p} \\ \breve{N}_{1,p} \end{bmatrix} = \Phi(\tau) \int \Phi^{-1}(\tau') \begin{bmatrix} -\frac{\mathrm{Wi}}{\mathrm{De}}(1+\cos\tau') + \breve{\lambda}_2 \,\mathrm{Wi} \sin\tau' \\ 2\breve{\lambda}_2 \frac{\mathrm{Wi}^2}{\mathrm{De}}(1+\cos\tau')^2 \end{bmatrix} d\tau' \tag{38}$$

To evaluate this, we must invert Eq. (37):

$$\Phi^{-1} = \begin{bmatrix} 2\exp\left(\frac{\tau}{\mathrm{De}}\right)\cos\left(\frac{\mathrm{Wi}}{\mathrm{De}}(\tau+\sin\tau)\right) & \exp\left(\frac{\tau}{\mathrm{De}}\right)\sin\left(\frac{\mathrm{Wi}}{\mathrm{De}}(\tau+\sin\tau)\right) \\ -2\exp\left(\frac{\tau}{\mathrm{De}}\right)\sin\left(\frac{\mathrm{Wi}}{\mathrm{De}}(\tau+\sin\tau)\right) & \exp\left(\frac{\tau}{\mathrm{De}}\right)\cos\left(\frac{\mathrm{Wi}}{\mathrm{De}}(\tau+\sin\tau)\right) \end{bmatrix} \tag{39}$$

Substituting this into Eq. (38) and considering only the integrand, we find:

$$\int \Phi^{-1} \begin{bmatrix} -\frac{\mathrm{Wi}}{\mathrm{De}}(1+\cos\tau') + \breve{\lambda}_2 \,\mathrm{Wi} \sin\tau' \\ 2\breve{\lambda}_2 \frac{\mathrm{Wi}^2}{\mathrm{De}}(1+\cos\tau')^2 \end{bmatrix} d\tau' = 2 \begin{bmatrix} -\frac{\mathrm{Wi}}{\mathrm{De}} I_1 - \frac{\mathrm{Wi}}{\mathrm{De}} I_3 + \breve{\lambda}_2 \,\mathrm{Wi}\, I_5 + \breve{\lambda}_2 \frac{\mathrm{Wi}^2}{\mathrm{De}}\left(\frac{3}{2} I_2 + 2I_4 + \frac{1}{2} I_7\right) \\ \frac{\mathrm{Wi}}{\mathrm{De}} I_2 + \frac{\mathrm{Wi}}{\mathrm{De}} I_4 - \breve{\lambda}_2 \,\mathrm{Wi}\, I_6 + \breve{\lambda}_2 \frac{\mathrm{Wi}^2}{\mathrm{De}}\left(\frac{3}{2} I_1 + 2I_3 + \frac{1}{2} I_8\right) \end{bmatrix} \tag{40}$$

where:

$$I_1 \equiv \int e^{\frac{\tau'}{\mathrm{De}}} \cos\left(\frac{\mathrm{Wi}}{\mathrm{De}}(\tau'+\sin\tau')\right) d\tau' \tag{41}$$

$$I_2 \equiv \int e^{\frac{\tau'}{\mathrm{De}}} \sin\left(\frac{\mathrm{Wi}}{\mathrm{De}}(\tau'+\sin\tau')\right) d\tau' \tag{42}$$

$$I_3 \equiv \int e^{\frac{\tau'}{\mathrm{De}}} \cos\tau' \cos\left(\frac{\mathrm{Wi}}{\mathrm{De}}(\tau'+\sin\tau')\right) d\tau' \tag{43}$$

$$I_4 \equiv \int e^{\frac{\tau'}{\mathrm{De}}} \cos\tau' \sin\left(\frac{\mathrm{Wi}}{\mathrm{De}}(\tau'+\sin\tau')\right) d\tau' \tag{44}$$

$$I_5 \equiv \int e^{\frac{\tau'}{\mathrm{De}}} \sin\tau' \cos\left(\frac{\mathrm{Wi}}{\mathrm{De}}(\tau'+\sin\tau')\right) d\tau' \tag{45}$$

$$I_6 \equiv \int e^{\frac{\tau'}{\mathrm{De}}} \sin\tau' \sin\left(\frac{\mathrm{Wi}}{\mathrm{De}}(\tau'+\sin\tau')\right) d\tau' \tag{46}$$

$$I_7 \equiv \int e^{\frac{\tau'}{\mathrm{De}}} \cos 2\tau' \sin\left(\frac{\mathrm{Wi}}{\mathrm{De}}(\tau'+\sin\tau')\right) d\tau' \tag{47}$$

$$I_8 \equiv \int e^{\frac{\tau'}{\mathrm{De}}} \cos 2\tau' \cos\left(\frac{\mathrm{Wi}}{\mathrm{De}}(\tau'+\sin\tau')\right) d\tau' \tag{48}$$

To evaluate the constant of integration, C_3 and C_4, we substitute Eqs. (37) and (40) into Eq. (38) to get:

$$\begin{bmatrix} \breve{\tau}_{yx,p} \\ \breve{N}_{1,p} \end{bmatrix} = e^{\frac{-\tau}{\mathrm{De}}} \begin{bmatrix} \left(-\frac{\mathrm{Wi}}{\mathrm{De}} I_1 - \frac{\mathrm{Wi}}{\mathrm{De}} I_3 + \breve{\lambda}_2 \,\mathrm{Wi}\, I_5 + \breve{\lambda}_2 \frac{\mathrm{Wi}^2}{\mathrm{De}}\left(\frac{3}{2} I_2 + 2I_4 + \frac{1}{2} I_7\right)\right)\cos\frac{\mathrm{Wi}}{\mathrm{De}}(\tau+\sin\tau) \\ +\left(-\frac{\mathrm{Wi}}{\mathrm{De}} I_2 - \frac{\mathrm{Wi}}{\mathrm{De}} I_4 + \breve{\lambda}_2 \,\mathrm{Wi}\, I_6 - \breve{\lambda}_2 \frac{\mathrm{Wi}^2}{\mathrm{De}}\left(\frac{3}{2} I_1 + 2I_3 + \frac{1}{2} I_8\right)\right)\sin\frac{\mathrm{Wi}}{\mathrm{De}}(\tau+\sin\tau) \\ 2\left(-\frac{\mathrm{Wi}}{\mathrm{De}} I_1 - \frac{\mathrm{Wi}}{\mathrm{De}} I_3 + \breve{\lambda}_2 \,\mathrm{Wi}\, I_5 + \breve{\lambda}_2 \frac{\mathrm{Wi}^2}{\mathrm{De}}\left(\frac{3}{2} I_2 + 2I_4 + \frac{1}{2} I_7\right)\right)\sin\frac{\mathrm{Wi}}{\mathrm{De}}(\tau+\sin\tau) \\ +2\left(\frac{\mathrm{Wi}}{\mathrm{De}} I_2 + \frac{\mathrm{Wi}}{\mathrm{De}} I_4 - \breve{\lambda}_2 \,\mathrm{Wi}\, I_6 + \breve{\lambda}_2 \frac{\mathrm{Wi}^2}{\mathrm{De}}\left(\frac{3}{2} I_1 + 2I_3 + \frac{1}{2} I_8\right)\right)\cos\frac{\mathrm{Wi}}{\mathrm{De}}(\tau+\sin\tau) \end{bmatrix} \tag{49}$$

Combining Eqs. (36) and (49) gives the complete compact solution:

$$\begin{bmatrix} \breve{\tau}_{yx} \\ \breve{N}_1 \end{bmatrix} = \begin{bmatrix} \frac{C_3}{2} e^{\frac{-\tau}{\mathrm{De}}} \cos\frac{\mathrm{Wi}}{\mathrm{De}}(\tau+\sin\tau) - \frac{C_4}{2} e^{\frac{-\tau}{\mathrm{De}}} \sin\frac{\mathrm{Wi}}{\mathrm{De}}(\tau+\sin\tau) \\ C_3 e^{\frac{-\tau}{\mathrm{De}}} \sin\frac{\mathrm{Wi}}{\mathrm{De}}(\tau+\sin\tau) + C_4 e^{\frac{-\tau}{\mathrm{De}}} \cos\frac{\mathrm{Wi}}{\mathrm{De}}(\tau+\sin\tau) \end{bmatrix}$$

$$+e^{\frac{-\tau}{\mathrm{De}}} \begin{bmatrix} \left(-\frac{\mathrm{Wi}}{\mathrm{De}} I_1 - \frac{\mathrm{Wi}}{\mathrm{De}} I_3 + \breve{\lambda}_2 \mathrm{Wi}\, I_5 + \breve{\lambda}_2 \frac{\mathrm{Wi}^2}{\mathrm{De}}\left(\frac{3}{2} I_2 + 2I_4 + \frac{1}{2} I_7\right)\right) \cos\frac{\mathrm{Wi}}{\mathrm{De}}(\tau+\sin\tau) \\ \quad + \left(-\frac{\mathrm{Wi}}{\mathrm{De}} I_2 - \frac{\mathrm{Wi}}{\mathrm{De}} I_4 + \breve{\lambda}_2 \mathrm{Wi}\, I_6 - \breve{\lambda}_2 \frac{\mathrm{Wi}^2}{\mathrm{De}}\left(\frac{3}{2} I_1 + 2I_3 + \frac{1}{2} I_8\right)\right) \sin\frac{\mathrm{Wi}}{\mathrm{De}}(\tau+\sin\tau) \\ 2\left(-\frac{\mathrm{Wi}}{\mathrm{De}} I_1 - \frac{\mathrm{Wi}}{\mathrm{De}} I_3 + \breve{\lambda}_2 \mathrm{Wi}\, I_5 + \breve{\lambda}_2 \frac{\mathrm{Wi}^2}{\mathrm{De}}\left(\frac{3}{2} I_2 + 2I_4 + \frac{1}{2} I_7\right)\right) \sin\frac{\mathrm{Wi}}{\mathrm{De}}(\tau+\sin\tau) \\ \quad + 2\left(\frac{\mathrm{Wi}}{\mathrm{De}} I_2 + \frac{\mathrm{Wi}}{\mathrm{De}} I_4 - \breve{\lambda}_2 \mathrm{Wi}\, I_6 + \breve{\lambda}_2 \frac{\mathrm{Wi}^2}{\mathrm{De}}\left(\frac{3}{2} I_1 + 2I_3 + \frac{1}{2} I_8\right)\right) \cos\frac{\mathrm{Wi}}{\mathrm{De}}(\tau+\sin\tau) \end{bmatrix} \quad (50)$$

Subjecting this to the stress-free initial conditions:

$$\begin{bmatrix} \breve{\tau}_{yx}(0) \\ \breve{N}_1(0) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad (51)$$

we find:

$$\begin{bmatrix} C_3 \\ C_4 \end{bmatrix} = \begin{bmatrix} -2\left(-\frac{\mathrm{Wi}}{\mathrm{De}} I_{1,0} - \frac{\mathrm{Wi}}{\mathrm{De}} I_{3,0} + \breve{\lambda}_2 \mathrm{Wi}\, I_{5,0} + \breve{\lambda}_2 \frac{\mathrm{Wi}^2}{\mathrm{De}}\left(\frac{3}{2} I_{2,0} + 2I_{4,0} + \frac{1}{2} I_{7,0}\right)\right) \\ -2\left(\frac{\mathrm{Wi}}{\mathrm{De}} I_{2,0} + \frac{\mathrm{Wi}}{\mathrm{De}} I_{4,0} - \breve{\lambda}_2 \mathrm{Wi}\, I_{6,0} + \breve{\lambda}_2 \frac{\mathrm{Wi}^2}{\mathrm{De}}\left(\frac{3}{2} I_{1,0} + 2I_{3,0} + \frac{1}{2} I_{8,0}\right)\right) \end{bmatrix} \quad (52)$$

where $I_{n,0} \equiv I_n\big|_{\tau=0}$.

Substituting Eq. (52) into the shear stress component of Eq. (50):

$$\breve{\tau}_{yx} = \begin{bmatrix} \begin{pmatrix} -\frac{\mathrm{Wi}}{\mathrm{De}}\left(I_1 - I_{1,0}\right) - \frac{\mathrm{Wi}}{\mathrm{De}}\left(I_3 - I_{3,0}\right) + \breve{\lambda}_2 \mathrm{Wi}\left(I_5 - I_{5,0}\right) \\ + \breve{\lambda}_2 \frac{\mathrm{Wi}^2}{\mathrm{De}}\left(\frac{3}{2}\left(I_2 - I_{2,0}\right) + 2\left(I_4 - I_{4,0}\right) + \frac{1}{2}\left(I_7 - I_{7,0}\right)\right) \end{pmatrix} e^{\frac{-\tau}{\mathrm{De}}} \cos\frac{\mathrm{Wi}}{\mathrm{De}}(\tau+\sin\tau) \\ + \begin{pmatrix} -\frac{\mathrm{Wi}}{\mathrm{De}}\left(I_2 - I_{2,0}\right) - \frac{\mathrm{Wi}}{\mathrm{De}}\left(I_4 - I_{4,0}\right) + \breve{\lambda}_2 \mathrm{Wi}\left(I_6 - I_{6,0}\right) \\ - \breve{\lambda}_2 \frac{\mathrm{Wi}^2}{\mathrm{De}}\left(\frac{3}{2}\left(I_1 - I_{1,0}\right) + 2\left(I_3 - I_{3,0}\right) + \frac{1}{2}\left(I_8 - I_{8,0}\right)\right) \end{pmatrix} e^{\frac{-\tau}{\mathrm{De}}} \sin\frac{\mathrm{Wi}}{\mathrm{De}}(\tau+\sin\tau) \end{bmatrix} \quad (53)$$

where the alternant part can be extracted as follow:

$$\begin{aligned} \breve{\tau}_{yx} &= \left(-\frac{\mathrm{Wi}}{\mathrm{De}} I_1 - \frac{\mathrm{Wi}}{\mathrm{De}} I_3 + \breve{\lambda}_2 \mathrm{Wi}\, I_5 + \breve{\lambda}_2 \frac{\mathrm{Wi}^2}{\mathrm{De}}\left(\frac{3}{2} I_2 + 2I_4 + \frac{1}{2} I_7\right)\right) e^{\frac{-\tau}{\mathrm{De}}} \cos\frac{\mathrm{Wi}}{\mathrm{De}}(\tau+\sin\tau) \\ &\quad + \left(-\frac{\mathrm{Wi}}{\mathrm{De}} I_2 - \frac{\mathrm{Wi}}{\mathrm{De}} I_4 + \breve{\lambda}_2 \mathrm{Wi}\, I_6 - \breve{\lambda}_2 \frac{\mathrm{Wi}^2}{\mathrm{De}}\left(\frac{3}{2} I_1 + 2I_3 + \frac{1}{2} I_8\right)\right) e^{\frac{-\tau}{\mathrm{De}}} \sin\frac{\mathrm{Wi}}{\mathrm{De}}(\tau+\sin\tau) \end{aligned} \quad (54)$$

and the transient part can be extracted as follow:

$$\breve{\tau}_{yx,0} = -e^{\frac{-\tau}{\mathrm{De}}} \begin{bmatrix} \left(-\frac{\mathrm{Wi}}{\mathrm{De}} I_{1,0} - \frac{\mathrm{Wi}}{\mathrm{De}} I_{3,0} + \breve{\lambda}_2 \mathrm{Wi}\, I_{5,0} + \breve{\lambda}_2 \frac{\mathrm{Wi}^2}{\mathrm{De}}\left(\frac{3}{2} I_{2,0} + 2I_{4,0} + \frac{1}{2} I_{7,0}\right)\right) \cos\frac{\mathrm{Wi}}{\mathrm{De}}(\tau+\sin\tau) \\ + \left(-\frac{\mathrm{Wi}}{\mathrm{De}} I_{2,0} - \frac{\mathrm{Wi}}{\mathrm{De}} I_{4,0} + \breve{\lambda}_2 \mathrm{Wi}\, I_{6,0} - \breve{\lambda}_2 \frac{\mathrm{Wi}^2}{\mathrm{De}}\left(\frac{3}{2} I_{1,0} + 2I_{3,0} + \frac{1}{2} I_{8,0}\right)\right) \sin\frac{\mathrm{Wi}}{\mathrm{De}}(\tau+\sin\tau) \end{bmatrix} \quad (55)$$

Eq. (53) is the *compact form* of the shear stress response in *ud*LAOS.

We next evaluate the integrals $I_1 - I_8$, defined in Eqs. (41) through (48). For these, we need the following trigonometric identities (Eqs. **4.3.16** and **4.3.17** of [5] and Eqs. (117) and (120) of [6]):

$$\cos\left(\tfrac{\mathrm{Wi}}{\mathrm{De}}\tau+\tfrac{\mathrm{Wi}}{\mathrm{De}}\sin\tau\right)=\cos\tfrac{\mathrm{Wi}}{\mathrm{De}}\tau\cos\left(\tfrac{\mathrm{Wi}}{\mathrm{De}}\sin\tau\right)-\sin\tfrac{\mathrm{Wi}}{\mathrm{De}}\tau\sin\left(\tfrac{\mathrm{Wi}}{\mathrm{De}}\sin\tau\right) \tag{56}$$

$$\sin\left(\tfrac{\mathrm{Wi}}{\mathrm{De}}\tau+\tfrac{\mathrm{Wi}}{\mathrm{De}}\sin\tau\right)=\sin\tfrac{\mathrm{Wi}}{\mathrm{De}}\tau\cos\left(\tfrac{\mathrm{Wi}}{\mathrm{De}}\sin\tau\right)+\cos\tfrac{\mathrm{Wi}}{\mathrm{De}}\tau\sin\left(\tfrac{\mathrm{Wi}}{\mathrm{De}}\sin\tau\right) \tag{57}$$

$$\cos\left(\tfrac{\mathrm{Wi}}{\mathrm{De}}\sin\tau\right)=J_0+2\sum_{k=1}^{\infty}J_{2k}\cos 2k\tau \tag{58}$$

$$\sin\left(\tfrac{\mathrm{Wi}}{\mathrm{De}}\sin\tau\right)=2\sum_{k=1}^{\infty}J_{2k-1}\sin\left(2k-1\right)\tau \tag{59}$$

i. I_1

Substituting Eq. (56) into Eq. (41) yields:

$$I_1=\int e^{\frac{\tau'}{\mathrm{De}}}\cos\tfrac{\mathrm{Wi}}{\mathrm{De}}\tau'\cos\left(\tfrac{\mathrm{Wi}}{\mathrm{De}}\sin\tau'\right)d\tau'-\int e^{\frac{\tau'}{\mathrm{De}}}\sin\tfrac{\mathrm{Wi}}{\mathrm{De}}\tau'\sin\left(\tfrac{\mathrm{Wi}}{\mathrm{De}}\sin\tau'\right)d\tau' \tag{60}$$

and then Eqs. (58) and (59) into the result, we find:

$$I_1=J_0I_{1,1}+2\sum_{k=1}^{\infty}\left(J_{2k}I_{1,2}-J_{2k-1}I_{1,3}\right) \tag{61}$$

$$I_{1,1}\equiv\int e^{\frac{\tau'}{\mathrm{De}}}\cos\tfrac{\mathrm{Wi}}{\mathrm{De}}\tau'\,d\tau' \tag{62}$$

$$I_{1,2}\equiv\int e^{\frac{\tau'}{\mathrm{De}}}\cos\tfrac{\mathrm{Wi}}{\mathrm{De}}\tau'\cos 2k\tau\,d\tau' \tag{63}$$

$$I_{1,3}\equiv\int e^{\frac{\tau'}{\mathrm{De}}}\sin\tfrac{\mathrm{Wi}}{\mathrm{De}}\tau'\sin\left(2k-1\right)\tau'\,d\tau' \tag{64}$$

Evaluating these, and then substituting the results back into Eq. (61) yields:

$$\begin{aligned}I_1&=\mathrm{De}\,e^{\frac{\tau}{\mathrm{De}}}\left[\frac{J_0\cos\frac{\mathrm{Wi}}{\mathrm{De}}\tau}{1+\mathrm{Wi}^2}+\frac{\mathrm{Wi}\,J_0\sin\frac{\mathrm{Wi}}{\mathrm{De}}\tau}{1+\mathrm{Wi}^2}\right]\\
&+\mathrm{De}\,e^{\frac{\tau}{\mathrm{De}}}\sum_{k=1}^{\infty}\left[\begin{array}{l}\frac{J_{2k}\cos\left(2k-\frac{\mathrm{Wi}}{\mathrm{De}}\right)\tau}{1+\left(2k-\frac{\mathrm{Wi}}{\mathrm{De}}\right)^2\mathrm{De}^2}+\frac{J_{2k}\cos\left(2k+\frac{\mathrm{Wi}}{\mathrm{De}}\right)\tau}{1+\left(2k+\frac{\mathrm{Wi}}{\mathrm{De}}\right)^2\mathrm{De}^2}-\frac{J_{2k-1}\cos\left(2k-1-\frac{\mathrm{Wi}}{\mathrm{De}}\right)\tau}{1+\left(2k-1-\frac{\mathrm{Wi}}{\mathrm{De}}\right)^2\mathrm{De}^2}+\frac{J_{2k-1}\cos\left(2k-1+\frac{\mathrm{Wi}}{\mathrm{De}}\right)\tau}{1+\left(2k-1+\frac{\mathrm{Wi}}{\mathrm{De}}\right)^2\mathrm{De}^2}\\
+\frac{\left(2k-\frac{\mathrm{Wi}}{\mathrm{De}}\right)\mathrm{De}\,J_{2k}\sin\left(2k-\frac{\mathrm{Wi}}{\mathrm{De}}\right)\tau}{1+\left(2k-\frac{\mathrm{Wi}}{\mathrm{De}}\right)^2\mathrm{De}^2}+\frac{\left(2k+\frac{\mathrm{Wi}}{\mathrm{De}}\right)\mathrm{De}\,J_{2k}\sin\left(2k+\frac{\mathrm{Wi}}{\mathrm{De}}\right)\tau}{1+\left(2k+\frac{\mathrm{Wi}}{\mathrm{De}}\right)^2\mathrm{De}^2}\\
-\frac{\left(2k-1-\frac{\mathrm{Wi}}{\mathrm{De}}\right)\mathrm{De}\,J_{2k-1}\sin\left(2k-1-\frac{\mathrm{Wi}}{\mathrm{De}}\right)\tau}{1+\left(2k-1-\frac{\mathrm{Wi}}{\mathrm{De}}\right)^2\mathrm{De}^2}+\frac{\left(2k-1+\frac{\mathrm{Wi}}{\mathrm{De}}\right)\mathrm{De}\,J_{2k-1}\sin\left(2k-1+\frac{\mathrm{Wi}}{\mathrm{De}}\right)\tau}{1+\left(2k-1+\frac{\mathrm{Wi}}{\mathrm{De}}\right)^2\mathrm{De}^2}\end{array}\right]\\
&-\left[\frac{J_0\,\mathrm{De}}{1+\mathrm{Wi}^2}+\mathrm{De}\sum_{k=1}^{\infty}\left[\begin{array}{l}\frac{J_{2k}}{1+\left(2k-\frac{\mathrm{Wi}}{\mathrm{De}}\right)^2\mathrm{De}^2}+\frac{J_{2k}}{1+\left(2k+\frac{\mathrm{Wi}}{\mathrm{De}}\right)^2\mathrm{De}^2}\\
-\frac{J_{2k-1}}{1+\left(2k-1-\frac{\mathrm{Wi}}{\mathrm{De}}\right)^2\mathrm{De}^2}+\frac{J_{2k-1}}{1+\left(2k-1+\frac{\mathrm{Wi}}{\mathrm{De}}\right)^2\mathrm{De}^2}\end{array}\right]\right]\end{aligned} \tag{65}$$

Subtracting the transient part, $I_{1,0}\equiv\lim_{\tau\to0}I_1$, gives:

$$I_1 - I_{1,0} = \mathrm{De}\, e^{\frac{\tau}{\mathrm{De}}} \left[\begin{array}{l} \sum_{k=0}^{\infty} \left[\phi_1^{(1)} \cos\left(2k + \frac{\mathrm{Wi}}{\mathrm{De}}\right)\tau + \psi_1^{(1)} \sin\left(2k + \frac{\mathrm{Wi}}{\mathrm{De}}\right)\tau \right] \\ + \sum_{k=1}^{\infty} \left[\begin{array}{l} \phi_1^{(2)} \cos\left(2k - 1 - \frac{\mathrm{Wi}}{\mathrm{De}}\right)\tau + \phi_1^{(3)} \cos\left(2k - 1 + \frac{\mathrm{Wi}}{\mathrm{De}}\right)\tau + \phi_1^{(4)} \cos\left(2k - \frac{\mathrm{Wi}}{\mathrm{De}}\right)\tau \\ + \psi_1^{(2)} \sin\left(2k - 1 - \frac{\mathrm{Wi}}{\mathrm{De}}\right)\tau + \psi_1^{(3)} \sin\left(2k - 1 + \frac{\mathrm{Wi}}{\mathrm{De}}\right)\tau + \psi_1^{(4)} \sin\left(2k - \frac{\mathrm{Wi}}{\mathrm{De}}\right)\tau \end{array} \right] \end{array} \right] - \mathrm{De}\left[\sum_{k=0}^{\infty} \phi_1^{(1)} + \sum_{k=1}^{\infty} \left(\phi_1^{(2)} + \phi_1^{(3)} + \phi_1^{(4)} \right) \right] \tag{66}$$

where:

$$\phi_1^{(i)} \equiv \left[\begin{array}{l} \left(\frac{J_{2k}}{1 + \left(2k + \frac{\mathrm{Wi}}{\mathrm{De}}\right)^2 \mathrm{De}^2} \right)_{i=1}, \left(\frac{-J_{2k-1}}{1 + \left(2k - 1 - \frac{\mathrm{Wi}}{\mathrm{De}}\right)^2 \mathrm{De}^2} \right)_2, \\ \left(\frac{J_{2k-1}}{1 + \left(2k - 1 + \frac{\mathrm{Wi}}{\mathrm{De}}\right)^2 \mathrm{De}^2} \right)_3, \left(\frac{J_{2k}}{1 + \left(2k - \frac{\mathrm{Wi}}{\mathrm{De}}\right)^2 \mathrm{De}^2} \right)_4 \end{array} \right] \tag{67}$$

$$\psi_1^{(i)} \equiv \left[\begin{array}{l} \left(\frac{\left(2k + \frac{\mathrm{Wi}}{\mathrm{De}}\right)\mathrm{De}\, J_{2k}}{1 + \left(2k + \frac{\mathrm{Wi}}{\mathrm{De}}\right)^2 \mathrm{De}^2} \right)_{i=1}, \left(\frac{-\left(2k - 1 - \frac{\mathrm{Wi}}{\mathrm{De}}\right)\mathrm{De}\, J_{2k-1}}{1 + \left(2k - 1 - \frac{\mathrm{Wi}}{\mathrm{De}}\right)^2 \mathrm{De}^2} \right)_2, \\ \left(\frac{\left(2k - 1 + \frac{\mathrm{Wi}}{\mathrm{De}}\right)\mathrm{De}\, J_{2k-1}}{1 + \left(2k - 1 + \frac{\mathrm{Wi}}{\mathrm{De}}\right)^2 \mathrm{De}^2} \right)_3, \left(\frac{\left(2k - \frac{\mathrm{Wi}}{\mathrm{De}}\right)\mathrm{De}\, J_{2k}}{1 + \left(2k - \frac{\mathrm{Wi}}{\mathrm{De}}\right)^2 \mathrm{De}^2} \right)_4 \end{array} \right] \tag{68}$$

ii. I_2

Substituting Eq. (57) into Eq. (42) yields:

$$I_2 = \int e^{\frac{\tau'}{\mathrm{De}}} \sin \tfrac{\mathrm{Wi}}{\mathrm{De}} \tau' \cos\left(\tfrac{\mathrm{Wi}}{\mathrm{De}} \sin \tau'\right) d\tau' + \int e^{\frac{\tau'}{\mathrm{De}}} \cos \tfrac{\mathrm{Wi}}{\mathrm{De}} \tau' \sin\left(\tfrac{\mathrm{Wi}}{\mathrm{De}} \sin \tau'\right) d\tau' \tag{69}$$

and then Eqs. (58) and (59) into the result, we find:

$$I_2 = J_0 I_{2,1} + 2\sum_{k=1}^{\infty} \left(J_{2k} I_{2,2} + J_{2k-1} I_{2,3} \right) \tag{70}$$

$$I_{2,1} \equiv \int e^{\frac{\tau'}{\mathrm{De}}} \sin \tfrac{\mathrm{Wi}}{\mathrm{De}} \tau' \, d\tau' \tag{71}$$

$$I_{2,2} \equiv \int e^{\frac{\tau'}{\mathrm{De}}} \sin \tfrac{\mathrm{Wi}}{\mathrm{De}} \tau' \cos 2k\tau \, d\tau' \tag{72}$$

$$I_{2,3} \equiv \int e^{\frac{\tau'}{\mathrm{De}}} \cos \tfrac{\mathrm{Wi}}{\mathrm{De}} \tau' \sin\left(2k - 1\right)\tau \, d\tau' \tag{73}$$

Evaluating these, and then substituting the results back into Eq. (70) yields:

$$I_2 = \mathrm{De}\, e^{\frac{\tau}{\mathrm{De}}} \left[\begin{array}{l} \left(\dfrac{-\mathrm{Wi}\, J_0 \cos \frac{\mathrm{Wi}}{\mathrm{De}}\tau}{1+\mathrm{Wi}^2} + \dfrac{J_0 \sin \frac{\mathrm{Wi}}{\mathrm{De}}\tau}{1+\mathrm{Wi}^2} \right) \\ +\sum_{k=1}^{\infty} \left[\begin{array}{l} \dfrac{\left(2k-\frac{\mathrm{Wi}}{\mathrm{De}}\right)\mathrm{De}\, J_{2k} \cos\left(2k-\frac{\mathrm{Wi}}{\mathrm{De}}\right)\tau}{1+\left(2k-\frac{\mathrm{Wi}}{\mathrm{De}}\right)^2 \mathrm{De}^2} - \dfrac{\left(2k+\frac{\mathrm{Wi}}{\mathrm{De}}\right)\mathrm{De}\, J_{2k} \cos\left(2k+\frac{\mathrm{Wi}}{\mathrm{De}}\right)\tau}{1+\left(2k+\frac{\mathrm{Wi}}{\mathrm{De}}\right)^2 \mathrm{De}^2} \\ -\dfrac{\left(2k-1-\frac{\mathrm{Wi}}{\mathrm{De}}\right)\mathrm{De}\, J_{2k-1} \cos\left(2k-1-\frac{\mathrm{Wi}}{\mathrm{De}}\right)\tau}{1+\left(2k-1-\frac{\mathrm{Wi}}{\mathrm{De}}\right)^2 \mathrm{De}^2} - \dfrac{\left(2k-1+\frac{\mathrm{Wi}}{\mathrm{De}}\right)\mathrm{De}\, J_{2k-1} \cos\left(2k-1+\frac{\mathrm{Wi}}{\mathrm{De}}\right)\tau}{1+\left(2k-1+\frac{\mathrm{Wi}}{\mathrm{De}}\right)^2 \mathrm{De}^2} \\ -\dfrac{J_{2k} \sin\left(2k-\frac{\mathrm{Wi}}{\mathrm{De}}\right)\tau}{1+\left(2k-\frac{\mathrm{Wi}}{\mathrm{De}}\right)^2 \mathrm{De}^2} + \dfrac{J_{2k} \sin\left(2k+\frac{\mathrm{Wi}}{\mathrm{De}}\right)\tau}{1+\left(2k+\frac{\mathrm{Wi}}{\mathrm{De}}\right)^2 \mathrm{De}^2} \\ +\dfrac{J_{2k-1} \sin\left(2k-1-\frac{\mathrm{Wi}}{\mathrm{De}}\right)\tau}{1+\left(2k-1-\frac{\mathrm{Wi}}{\mathrm{De}}\right)^2 \mathrm{De}^2} + \dfrac{J_{2k-1} \sin\left(2k-1+\frac{\mathrm{Wi}}{\mathrm{De}}\right)\tau}{1+\left(2k-1+\frac{\mathrm{Wi}}{\mathrm{De}}\right)^2 \mathrm{De}^2} \end{array} \right] \end{array} \right]$$

$$+\frac{\mathrm{Wi}\,\mathrm{De}\, J_0}{1+\mathrm{Wi}^2} + \sum_{k=1}^{\infty} \left[\begin{array}{l} \dfrac{-\left(2k-\frac{\mathrm{Wi}}{\mathrm{De}}\right)\mathrm{De}^2 J_{2k}}{1+\left(2k-\frac{\mathrm{Wi}}{\mathrm{De}}\right)^2 \mathrm{De}^2} + \dfrac{\left(2k+\frac{\mathrm{Wi}}{\mathrm{De}}\right)\mathrm{De}^2 J_{2k}}{1+\left(2k+\frac{\mathrm{Wi}}{\mathrm{De}}\right)^2 \mathrm{De}^2} \\ +\dfrac{\left(2k-1-\frac{\mathrm{Wi}}{\mathrm{De}}\right)\mathrm{De}^2 J_{2k-1}}{1+\left(2k-1-\frac{\mathrm{Wi}}{\mathrm{De}}\right)^2 \mathrm{De}^2} + \dfrac{\left(2k-1+\frac{\mathrm{Wi}}{\mathrm{De}}\right)\mathrm{De}^2 J_{2k-1}}{1+\left(2k-1+\frac{\mathrm{Wi}}{\mathrm{De}}\right)^2 \mathrm{De}^2} \end{array} \right] \tag{74}$$

Subtracting the transient part, $I_{2,0} \equiv \lim_{\tau\to 0} I_2$, gives:

$$I_2 - I_{2,0} = \mathrm{De}\, e^{\frac{\tau}{\mathrm{De}}} \left[\begin{array}{l} \sum_{k=0}^{\infty} \left[\phi_2^{(1)} \cos\left(2k+\frac{\mathrm{Wi}}{\mathrm{De}}\right)\tau + \psi_2^{(1)} \sin\left(2k+\frac{\mathrm{Wi}}{\mathrm{De}}\right)\tau \right] \\ +\sum_{k=1}^{\infty} \left[\begin{array}{l} \phi_2^{(2)} \cos\left(2k-1-\frac{\mathrm{Wi}}{\mathrm{De}}\right)\tau + \phi_2^{(3)} \cos\left(2k-1+\frac{\mathrm{Wi}}{\mathrm{De}}\right)\tau + \phi_2^{(4)} \cos\left(2k-\frac{\mathrm{Wi}}{\mathrm{De}}\right)\tau \\ +\psi_2^{(2)} \sin\left(2k-1-\frac{\mathrm{Wi}}{\mathrm{De}}\right)\tau + \psi_2^{(3)} \sin\left(2k-1+\frac{\mathrm{Wi}}{\mathrm{De}}\right)\tau + \psi_2^{(4)} \sin\left(2k-\frac{\mathrm{Wi}}{\mathrm{De}}\right)\tau \end{array} \right] \end{array} \right]$$
$$-\mathrm{De}\left[\sum_{k=0}^{\infty} \phi_2^{(1)} + \sum_{k=1}^{\infty} \left(\phi_2^{(2)} + \phi_2^{(3)} + \phi_2^{(4)} \right) \right] \tag{75}$$

where:

$$\phi_2^{(i)} \equiv \left[\begin{array}{l} \left(\dfrac{-\left(2k+\frac{\mathrm{Wi}}{\mathrm{De}}\right)\mathrm{De}\, J_{2k}}{1+\left(2k+\frac{\mathrm{Wi}}{\mathrm{De}}\right)^2 \mathrm{De}^2} \right)_{i=1}, \left(\dfrac{-\left(2k-1-\frac{\mathrm{Wi}}{\mathrm{De}}\right)\mathrm{De}\, J_{2k-1}}{1+\left(2k-1-\frac{\mathrm{Wi}}{\mathrm{De}}\right)^2 \mathrm{De}^2} \right)_2, \\ \left(\dfrac{-\left(2k-1+\frac{\mathrm{Wi}}{\mathrm{De}}\right)\mathrm{De}\, J_{2k-1}}{1+\left(2k-1+\frac{\mathrm{Wi}}{\mathrm{De}}\right)^2 \mathrm{De}^2} \right)_3, \left(\dfrac{\left(2k-\frac{\mathrm{Wi}}{\mathrm{De}}\right)\mathrm{De}\, J_{2k}}{1+\left(2k-\frac{\mathrm{Wi}}{\mathrm{De}}\right)^2 \mathrm{De}^2} \right)_4 \end{array} \right] \tag{76}$$

$$\psi_2^{(i)} \equiv \left[\begin{array}{l} \left(\dfrac{J_{2k}}{1+\left(2k+\frac{\mathrm{Wi}}{\mathrm{De}}\right)^2 \mathrm{De}^2} \right)_{i=1}, \left(\dfrac{J_{2k-1}}{1+\left(2k-1-\frac{\mathrm{Wi}}{\mathrm{De}}\right)^2 \mathrm{De}^2} \right)_2, \\ \left(\dfrac{J_{2k-1}}{1+\left(2k-1+\frac{\mathrm{Wi}}{\mathrm{De}}\right)^2 \mathrm{De}^2} \right)_3, \left(\dfrac{-J_{2k}}{1+\left(2k-\frac{\mathrm{Wi}}{\mathrm{De}}\right)^2 \mathrm{De}^2} \right)_4 \end{array} \right] \tag{77}$$

iii. I_3

Substituting Eq. (56), into Eq. (43) yields:

$$I_3 = \int e^{\frac{\tau'}{\mathrm{De}}} \cos\tau' \cos\tfrac{\mathrm{Wi}}{\mathrm{De}}\tau' \cos\left(\tfrac{\mathrm{Wi}}{\mathrm{De}}\sin\tau'\right) d\tau' - \int e^{\frac{\tau'}{\mathrm{De}}} \cos\tau' \sin\tfrac{\mathrm{Wi}}{\mathrm{De}}\tau' \sin\left(\tfrac{\mathrm{Wi}}{\mathrm{De}}\sin\tau'\right) d\tau' \tag{78}$$

and then Eqs. (58) and (59) into the result, we find:

$$I_3 = J_0 I_{3,1} + 2\sum_{k=1}^{\infty}\left(J_{2k} I_{3,2} - J_{2k-1} I_{3,3}\right) \tag{79}$$

$$I_{3,1} \equiv \int e^{\frac{\tau'}{\mathrm{De}}} \cos\tau' \cos\tfrac{\mathrm{Wi}}{\mathrm{De}}\tau' \, d\tau' \tag{80}$$

$$I_{3,2} \equiv \int e^{\frac{\tau'}{\mathrm{De}}} \cos\tau' \cos\tfrac{\mathrm{Wi}}{\mathrm{De}}\tau' \cos 2k\tau' \, d\tau' \tag{81}$$

$$I_{3,3} \equiv \int e^{\frac{\tau'}{\mathrm{De}}} \cos\tau' \sin\tfrac{\mathrm{Wi}}{\mathrm{De}}\tau' \sin\left(2k-1\right)\tau' \, d\tau' \tag{82}$$

Evaluating these, and then substituting the results back into Eq. (79) yields:

$$\begin{aligned}
I_3 = \frac{\mathrm{De}\, e^{\frac{\tau}{\mathrm{De}}}}{2} \Bigg[& \frac{J_0\cos\left(1-\tfrac{\mathrm{Wi}}{\mathrm{De}}\right)\tau}{1+\left(1-\tfrac{\mathrm{Wi}}{\mathrm{De}}\right)^2\mathrm{De}^2} + \frac{J_0\cos\left(1+\tfrac{\mathrm{Wi}}{\mathrm{De}}\right)\tau}{1+\left(1+\tfrac{\mathrm{Wi}}{\mathrm{De}}\right)^2\mathrm{De}^2} + \frac{\left(1-\tfrac{\mathrm{Wi}}{\mathrm{De}}\right)\mathrm{De}\, J_0\sin\left(1-\tfrac{\mathrm{Wi}}{\mathrm{De}}\right)\tau}{1+\left(1-\tfrac{\mathrm{Wi}}{\mathrm{De}}\right)^2\mathrm{De}^2} + \frac{\left(1+\tfrac{\mathrm{Wi}}{\mathrm{De}}\right)\mathrm{De}\, J_0\sin\left(1+\tfrac{\mathrm{Wi}}{\mathrm{De}}\right)\tau}{1+\left(1+\tfrac{\mathrm{Wi}}{\mathrm{De}}\right)^2\mathrm{De}^2} \\
& \sum_{k=1}^{\infty}\Bigg[-\frac{J_{2k-1}\cos\left(2k-\tfrac{\mathrm{Wi}}{\mathrm{De}}\right)\tau}{1+\left(2k-\tfrac{\mathrm{Wi}}{\mathrm{De}}\right)^2\mathrm{De}^2} + \frac{J_{2k-1}\cos\left(2k+\tfrac{\mathrm{Wi}}{\mathrm{De}}\right)\tau}{1+\left(2k+\tfrac{\mathrm{Wi}}{\mathrm{De}}\right)^2\mathrm{De}^2} - \frac{J_{2k-1}\cos\left(2k-2-\tfrac{\mathrm{Wi}}{\mathrm{De}}\right)\tau}{1+\left(2k-2-\tfrac{\mathrm{Wi}}{\mathrm{De}}\right)^2\mathrm{De}^2} + \frac{J_{2k-1}\cos\left(2k-2+\tfrac{\mathrm{Wi}}{\mathrm{De}}\right)\tau}{1+\left(2k-2+\tfrac{\mathrm{Wi}}{\mathrm{De}}\right)^2\mathrm{De}^2} + \frac{J_{2k}\cos\left(2k-1-\tfrac{\mathrm{Wi}}{\mathrm{De}}\right)\tau}{1+\left(2k-1-\tfrac{\mathrm{Wi}}{\mathrm{De}}\right)^2\mathrm{De}^2} \\
& + \frac{J_{2k}\cos\left(2k-1+\tfrac{\mathrm{Wi}}{\mathrm{De}}\right)\tau}{1+\left(2k-1+\tfrac{\mathrm{Wi}}{\mathrm{De}}\right)^2\mathrm{De}^2} + \frac{J_{2k}\cos\left(2k+1-\tfrac{\mathrm{Wi}}{\mathrm{De}}\right)\tau}{1+\left(2k+1-\tfrac{\mathrm{Wi}}{\mathrm{De}}\right)^2\mathrm{De}^2} + \frac{J_{2k}\cos\left(2k+1+\tfrac{\mathrm{Wi}}{\mathrm{De}}\right)\tau}{1+\left(2k+1+\tfrac{\mathrm{Wi}}{\mathrm{De}}\right)^2\mathrm{De}^2} - \frac{\left(2k-\tfrac{\mathrm{Wi}}{\mathrm{De}}\right)\mathrm{De}\, J_{2k-1}\sin\left(2k-\tfrac{\mathrm{Wi}}{\mathrm{De}}\right)\tau}{1+\left(2k-\tfrac{\mathrm{Wi}}{\mathrm{De}}\right)^2\mathrm{De}^2} \\
& + \frac{\left(2k+\tfrac{\mathrm{Wi}}{\mathrm{De}}\right)\mathrm{De}\, J_{2k-1}\sin\left(2k+\tfrac{\mathrm{Wi}}{\mathrm{De}}\right)\tau}{1+\left(2k+\tfrac{\mathrm{Wi}}{\mathrm{De}}\right)^2\mathrm{De}^2} - \frac{\left(2k-2-\tfrac{\mathrm{Wi}}{\mathrm{De}}\right)\mathrm{De}\, J_{2k-1}\sin\left(2k-2-\tfrac{\mathrm{Wi}}{\mathrm{De}}\right)\tau}{1+\left(2k-2-\tfrac{\mathrm{Wi}}{\mathrm{De}}\right)^2\mathrm{De}^2} + \frac{\left(2k-2+\tfrac{\mathrm{Wi}}{\mathrm{De}}\right)\mathrm{De}\, J_{2k-1}\sin\left(2k-2+\tfrac{\mathrm{Wi}}{\mathrm{De}}\right)\tau}{1+\left(2k-2+\tfrac{\mathrm{Wi}}{\mathrm{De}}\right)^2\mathrm{De}^2} \\
& + \frac{\left(2k-1-\tfrac{\mathrm{Wi}}{\mathrm{De}}\right)\mathrm{De}\, J_{2k}\sin\left(2k-1-\tfrac{\mathrm{Wi}}{\mathrm{De}}\right)\tau}{1+\left(2k-1-\tfrac{\mathrm{Wi}}{\mathrm{De}}\right)^2\mathrm{De}^2} + \frac{\left(2k-1+\tfrac{\mathrm{Wi}}{\mathrm{De}}\right)\mathrm{De}\, J_{2k}\sin\left(2k-1+\tfrac{\mathrm{Wi}}{\mathrm{De}}\right)\tau}{1+\left(2k-1+\tfrac{\mathrm{Wi}}{\mathrm{De}}\right)^2\mathrm{De}^2} + \frac{\left(2k+1-\tfrac{\mathrm{Wi}}{\mathrm{De}}\right)\mathrm{De}\, J_{2k}\sin\left(2k+1-\tfrac{\mathrm{Wi}}{\mathrm{De}}\right)\tau}{1+\left(2k+1-\tfrac{\mathrm{Wi}}{\mathrm{De}}\right)^2\mathrm{De}^2} \\
& + \frac{\left(2k+1+\tfrac{\mathrm{Wi}}{\mathrm{De}}\right)\mathrm{De}\, J_{2k}\sin\left(2k+1+\tfrac{\mathrm{Wi}}{\mathrm{De}}\right)\tau}{1+\left(2k+1+\tfrac{\mathrm{Wi}}{\mathrm{De}}\right)^2\mathrm{De}^2} \Bigg]\Bigg] \\
- \frac{\mathrm{De}}{2}\Bigg[& \frac{J_0}{1+\left(1-\tfrac{\mathrm{Wi}}{\mathrm{De}}\right)^2\mathrm{De}^2} + \frac{J_0}{1+\left(1+\tfrac{\mathrm{Wi}}{\mathrm{De}}\right)^2\mathrm{De}^2} \\
& + \sum_{k=1}^{\infty}\Bigg[\frac{J_{2k}}{1+\left(2k-1-\tfrac{\mathrm{Wi}}{\mathrm{De}}\right)^2\mathrm{De}^2} + \frac{J_{2k}}{1+\left(2k-1+\tfrac{\mathrm{Wi}}{\mathrm{De}}\right)^2\mathrm{De}^2} + \frac{J_{2k}}{1+\left(2k+1-\tfrac{\mathrm{Wi}}{\mathrm{De}}\right)^2\mathrm{De}^2} + \frac{J_{2k}}{1+\left(2k+1+\tfrac{\mathrm{Wi}}{\mathrm{De}}\right)^2\mathrm{De}^2} \\
& - \frac{J_{2k-1}}{1+\left(2k-\tfrac{\mathrm{Wi}}{\mathrm{De}}\right)^2\mathrm{De}^2} + \frac{J_{2k-1}}{1+\left(2k+\tfrac{\mathrm{Wi}}{\mathrm{De}}\right)^2\mathrm{De}^2} - \frac{J_{2k-1}}{1+\left(2k-2-\tfrac{\mathrm{Wi}}{\mathrm{De}}\right)^2\mathrm{De}^2} + \frac{J_{2k-1}}{1+\left(2k-2+\tfrac{\mathrm{Wi}}{\mathrm{De}}\right)^2\mathrm{De}^2} \Bigg]\Bigg]
\end{aligned} \tag{83}$$

Subtracting the transient part, $I_{3,0} \equiv \lim_{\tau\to 0} I_3$, from Eq. (83) gives:

$$\begin{aligned}
I_3 - I_{3,0} = \frac{\mathrm{De}\, e^{\frac{\tau}{\mathrm{De}}}}{2}\Bigg[& \sum_{k=0}^{\infty}\left[\begin{aligned} & \phi_3^{(1)}\cos\left(2k+1-\tfrac{\mathrm{Wi}}{\mathrm{De}}\right)\tau + \phi_3^{(2)}\cos\left(2k+1+\tfrac{\mathrm{Wi}}{\mathrm{De}}\right)\tau \\ & + \psi_3^{(1)}\sin\left(2k+1-\tfrac{\mathrm{Wi}}{\mathrm{De}}\right)\tau + \psi_3^{(2)}\sin\left(2k+1+\tfrac{\mathrm{Wi}}{\mathrm{De}}\right)\tau \end{aligned}\right] \\
& + \sum_{k=1}^{\infty}\left[\begin{aligned} & \phi_3^{(3)}\cos\left(2k-2-\tfrac{\mathrm{Wi}}{\mathrm{De}}\right)\tau + \phi_3^{(4)}\cos\left(2k-2+\tfrac{\mathrm{Wi}}{\mathrm{De}}\right)\tau + \phi_3^{(5)}\cos\left(2k-1-\tfrac{\mathrm{Wi}}{\mathrm{De}}\right)\tau \\ & + \phi_3^{(6)}\cos\left(2k-1+\tfrac{\mathrm{Wi}}{\mathrm{De}}\right)\tau + \phi_3^{(7)}\cos\left(2k-\tfrac{\mathrm{Wi}}{\mathrm{De}}\right)\tau + \phi_3^{(8)}\cos\left(2k+\tfrac{\mathrm{Wi}}{\mathrm{De}}\right)\tau \\ & + \psi_3^{(3)}\sin\left(2k-2-\tfrac{\mathrm{Wi}}{\mathrm{De}}\right)\tau + \psi_3^{(4)}\sin\left(2k-2+\tfrac{\mathrm{Wi}}{\mathrm{De}}\right)\tau + \psi_3^{(5)}\sin\left(2k-1-\tfrac{\mathrm{Wi}}{\mathrm{De}}\right)\tau \\ & + \psi_3^{(6)}\sin\left(2k-1+\tfrac{\mathrm{Wi}}{\mathrm{De}}\right)\tau + \psi_3^{(7)}\sin\left(2k-\tfrac{\mathrm{Wi}}{\mathrm{De}}\right)\tau + \psi_3^{(8)}\sin\left(2k+\tfrac{\mathrm{Wi}}{\mathrm{De}}\right)\tau \end{aligned}\right]\Bigg] \\
& - \frac{\mathrm{De}}{2}\left[\sum_{k=0}^{\infty}\left(\phi_3^{(1)}+\phi_3^{(2)}\right) + \sum_{k=1}^{\infty}\left(\phi_3^{(3)}+\phi_3^{(4)}+\phi_3^{(5)}+\phi_3^{(6)}+\phi_3^{(7)}+\phi_3^{(8)}\right)\right]
\end{aligned} \tag{84}$$

where:

$$\phi_3^{(i)} \equiv \begin{bmatrix} \left(\frac{J_{2k}}{1+\left(2k+1-\frac{\mathrm{Wi}}{\mathrm{De}}\right)^2\mathrm{De}^2}\right)_{i=1}, \left(\frac{J_{2k}}{1+\left(2k+1+\frac{\mathrm{Wi}}{\mathrm{De}}\right)^2\mathrm{De}^2}\right)_2, \left(\frac{-J_{2k-1}}{1+\left(2k-2-\frac{\mathrm{Wi}}{\mathrm{De}}\right)^2\mathrm{De}^2}\right)_3, \\ \left(\frac{J_{2k-1}}{1+\left(2k-2+\frac{\mathrm{Wi}}{\mathrm{De}}\right)^2\mathrm{De}^2}\right)_4, \left(\frac{J_{2k}}{1+\left(2k-1-\frac{\mathrm{Wi}}{\mathrm{De}}\right)^2\mathrm{De}^2}\right)_5, \left(\frac{J_{2k}}{1+\left(2k-1+\frac{\mathrm{Wi}}{\mathrm{De}}\right)^2\mathrm{De}^2}\right)_6, \\ \left(\frac{-J_{2k-1}}{1+\left(2k-\frac{\mathrm{Wi}}{\mathrm{De}}\right)^2\mathrm{De}^2}\right)_7, \left(\frac{J_{2k-1}}{1+\left(2k+\frac{\mathrm{Wi}}{\mathrm{De}}\right)^2\mathrm{De}^2}\right)_8 \end{bmatrix} \tag{85}$$

$$\psi_3^{(i)} \equiv \begin{bmatrix} \left(\frac{\left(2k+1-\frac{\mathrm{Wi}}{\mathrm{De}}\right)\mathrm{De}\,J_{2k}}{1+\left(2k+1-\frac{\mathrm{Wi}}{\mathrm{De}}\right)^2\mathrm{De}^2}\right)_{i=1}, \left(\frac{\left(2k+1+\frac{\mathrm{Wi}}{\mathrm{De}}\right)\mathrm{De}\,J_{2k}}{1+\left(2k+1+\frac{\mathrm{Wi}}{\mathrm{De}}\right)^2\mathrm{De}^2}\right)_2, \left(\frac{-\left(2k-2-\frac{\mathrm{Wi}}{\mathrm{De}}\right)\mathrm{De}\,J_{2k-1}}{1+\left(2k-2-\frac{\mathrm{Wi}}{\mathrm{De}}\right)^2\mathrm{De}^2}\right)_3, \\ \left(\frac{\left(2k-2+\frac{\mathrm{Wi}}{\mathrm{De}}\right)\mathrm{De}\,J_{2k-1}}{1+\left(2k-2+\frac{\mathrm{Wi}}{\mathrm{De}}\right)^2\mathrm{De}^2}\right)_4, \left(\frac{\left(2k-1-\frac{\mathrm{Wi}}{\mathrm{De}}\right)\mathrm{De}\,J_{2k}}{1+\left(2k-1-\frac{\mathrm{Wi}}{\mathrm{De}}\right)^2\mathrm{De}^2}\right)_5, \left(\frac{\left(2k-1+\frac{\mathrm{Wi}}{\mathrm{De}}\right)\mathrm{De}\,J_{2k}}{1+\left(2k-1+\frac{\mathrm{Wi}}{\mathrm{De}}\right)^2\mathrm{De}^2}\right)_6, \\ \left(\frac{-\left(2k-\frac{\mathrm{Wi}}{\mathrm{De}}\right)\mathrm{De}\,J_{2k-1}}{1+\left(2k-\frac{\mathrm{Wi}}{\mathrm{De}}\right)^2\mathrm{De}^2}\right)_7, \left(\frac{\left(2k+\frac{\mathrm{Wi}}{\mathrm{De}}\right)\mathrm{De}\,J_{2k-1}}{1+\left(2k+\frac{\mathrm{Wi}}{\mathrm{De}}\right)^2\mathrm{De}^2}\right)_8 \end{bmatrix} \tag{86}$$

iv. I_4

Substituting Eq. (57) into Eq. (44) yields:

$$I_4 = \int e^{\frac{\tau'}{\mathrm{De}}}\cos\tau'\sin\tfrac{\mathrm{Wi}}{\mathrm{De}}\tau'\cos\left(\tfrac{\mathrm{Wi}}{\mathrm{De}}\sin\tau'\right)d\tau' + \int e^{\frac{\tau'}{\mathrm{De}}}\cos\tau'\cos\tfrac{\mathrm{Wi}}{\mathrm{De}}\tau'\sin\left(\tfrac{\mathrm{Wi}}{\mathrm{De}}\sin\tau'\right)d\tau' \tag{87}$$

and then Eqs. (58) and (59) into the result, we find:

$$I_4 = J_0 I_{4,1} + 2\sum_{k=1}^{\infty}\left(J_{2k}I_{4,2} + J_{2k-1}I_{4,3}\right) \tag{88}$$

$$I_{4,1} \equiv \int e^{\frac{\tau'}{\mathrm{De}}}\cos\tau'\sin\tfrac{\mathrm{Wi}}{\mathrm{De}}\tau'\,d\tau' \tag{89}$$

$$I_{4,2} \equiv \int e^{\frac{\tau'}{\mathrm{De}}}\cos\tau'\sin\tfrac{\mathrm{Wi}}{\mathrm{De}}\tau'\cos 2k\tau\,d\tau' \tag{90}$$

$$I_{4,3} \equiv \int e^{\frac{\tau'}{\mathrm{De}}}\cos\tau'\cos\tfrac{\mathrm{Wi}}{\mathrm{De}}\tau'\sin\left(2k-1\right)\tau\,d\tau' \tag{91}$$

Evaluating these, and then substituting back into Eq. (88) yields:

$$I_4 = \frac{\mathrm{De}\,e^{\frac{\tau}{\mathrm{De}}}}{2}\left[\begin{array}{l} \frac{\left(1-\frac{\mathrm{Wi}}{\mathrm{De}}\right)\mathrm{De}\,J_0\cos\left(1-\frac{\mathrm{Wi}}{\mathrm{De}}\right)\tau}{1+\left(1-\frac{\mathrm{Wi}}{\mathrm{De}}\right)^2\mathrm{De}^2} - \frac{\left(1+\frac{\mathrm{Wi}}{\mathrm{De}}\right)\mathrm{De}\,J_0\cos\left(1+\frac{\mathrm{Wi}}{\mathrm{De}}\right)\tau}{1+\left(1+\frac{\mathrm{Wi}}{\mathrm{De}}\right)^2\mathrm{De}^2} - \frac{J_0\sin\left(1-\frac{\mathrm{Wi}}{\mathrm{De}}\right)\tau}{1+\left(1-\frac{\mathrm{Wi}}{\mathrm{De}}\right)^2\mathrm{De}^2} + \frac{J_0\sin\left(1+\frac{\mathrm{Wi}}{\mathrm{De}}\right)\tau}{1+\left(1+\frac{\mathrm{Wi}}{\mathrm{De}}\right)^2\mathrm{De}^2} \\ +\sum_{k=1}^{\infty}\left[\begin{array}{l} \frac{\left(2k-1-\frac{\mathrm{Wi}}{\mathrm{De}}\right)\mathrm{De}\,J_{2k}\cos\left(2k-1-\frac{\mathrm{Wi}}{\mathrm{De}}\right)\tau}{1+\left(2k-1-\frac{\mathrm{Wi}}{\mathrm{De}}\right)^2\mathrm{De}^2} - \frac{\left(2k-1+\frac{\mathrm{Wi}}{\mathrm{De}}\right)\mathrm{De}\,J_{2k}\cos\left(2k-1+\frac{\mathrm{Wi}}{\mathrm{De}}\right)\tau}{1+\left(2k-1+\frac{\mathrm{Wi}}{\mathrm{De}}\right)^2\mathrm{De}^2} \\ +\frac{\left(2k+1-\frac{\mathrm{Wi}}{\mathrm{De}}\right)\mathrm{De}\,J_{2k}\cos\left(2k+1-\frac{\mathrm{Wi}}{\mathrm{De}}\right)\tau}{1+\left(2k+1-\frac{\mathrm{Wi}}{\mathrm{De}}\right)^2\mathrm{De}^2} - \frac{\left(2k+1+\frac{\mathrm{Wi}}{\mathrm{De}}\right)\mathrm{De}\,J_{2k}\cos\left(2k+1+\frac{\mathrm{Wi}}{\mathrm{De}}\right)\tau}{1+\left(2k+1+\frac{\mathrm{Wi}}{\mathrm{De}}\right)^2\mathrm{De}^2} \\ -\frac{J_{2k}\sin\left(2k-1-\frac{\mathrm{Wi}}{\mathrm{De}}\right)\tau}{1+\left(2k-1-\frac{\mathrm{Wi}}{\mathrm{De}}\right)^2\mathrm{De}^2} + \frac{J_{2k}\sin\left(2k-1+\frac{\mathrm{Wi}}{\mathrm{De}}\right)\tau}{1+\left(2k-1+\frac{\mathrm{Wi}}{\mathrm{De}}\right)^2\mathrm{De}^2} - \frac{J_{2k}\sin\left(2k+1-\frac{\mathrm{Wi}}{\mathrm{De}}\right)\tau}{1+\left(2k+1-\frac{\mathrm{Wi}}{\mathrm{De}}\right)^2\mathrm{De}^2} \\ +\frac{J_{2k}\sin\left(2k+1+\frac{\mathrm{Wi}}{\mathrm{De}}\right)\tau}{1+\left(2k+1+\frac{\mathrm{Wi}}{\mathrm{De}}\right)^2\mathrm{De}^2} - \frac{\left(2k-\frac{\mathrm{Wi}}{\mathrm{De}}\right)\mathrm{De}\,J_{2k-1}\cos\left(2k-\frac{\mathrm{Wi}}{\mathrm{De}}\right)\tau}{1+\left(2k-\frac{\mathrm{Wi}}{\mathrm{De}}\right)^2\mathrm{De}^2} \\ -\frac{\left(2k+\frac{\mathrm{Wi}}{\mathrm{De}}\right)\mathrm{De}\,J_{2k-1}\cos\left(2k+\frac{\mathrm{Wi}}{\mathrm{De}}\right)\tau}{1+\left(2k+\frac{\mathrm{Wi}}{\mathrm{De}}\right)^2\mathrm{De}^2} - \frac{\left(2k-2-\frac{\mathrm{Wi}}{\mathrm{De}}\right)\mathrm{De}\,J_{2k-1}\cos\left(2k-2-\frac{\mathrm{Wi}}{\mathrm{De}}\right)\tau}{1+\left(2k-2-\frac{\mathrm{Wi}}{\mathrm{De}}\right)^2\mathrm{De}^2} \\ -\frac{\left(2k-2+\frac{\mathrm{Wi}}{\mathrm{De}}\right)\mathrm{De}\,J_{2k-1}\cos\left(2k-2+\frac{\mathrm{Wi}}{\mathrm{De}}\right)\tau}{1+\left(2k-2+\frac{\mathrm{Wi}}{\mathrm{De}}\right)^2\mathrm{De}^2} + \frac{J_{2k-1}\sin\left(2k-\frac{\mathrm{Wi}}{\mathrm{De}}\right)\tau}{1+\left(2k-\frac{\mathrm{Wi}}{\mathrm{De}}\right)^2\mathrm{De}^2} + \frac{J_{2k-1}\sin\left(2k+\frac{\mathrm{Wi}}{\mathrm{De}}\right)\tau}{1+\left(2k+\frac{\mathrm{Wi}}{\mathrm{De}}\right)^2\mathrm{De}^2} \\ +\frac{J_{2k-1}\sin\left(2k-2-\frac{\mathrm{Wi}}{\mathrm{De}}\right)\tau}{1+\left(2k-2-\frac{\mathrm{Wi}}{\mathrm{De}}\right)^2\mathrm{De}^2} + \frac{J_{2k-1}\sin\left(2k-2+\frac{\mathrm{Wi}}{\mathrm{De}}\right)\tau}{1+\left(2k-2+\frac{\mathrm{Wi}}{\mathrm{De}}\right)^2\mathrm{De}^2} \end{array}\right] \end{array}\right] \qquad (92)$$

$$-\frac{\left(1-\frac{\mathrm{Wi}}{\mathrm{De}}\right)\mathrm{De}^2 J_0}{2\left(1+\left(1-\frac{\mathrm{Wi}}{\mathrm{De}}\right)^2\mathrm{De}^2\right)} + \frac{\left(1+\frac{\mathrm{Wi}}{\mathrm{De}}\right)\mathrm{De}^2 J_0}{2\left(1+\left(1+\frac{\mathrm{Wi}}{\mathrm{De}}\right)^2\mathrm{De}^2\right)}$$

$$+\sum_{k=1}^{\infty}\left[\begin{array}{l} -\frac{\left(2k-1-\frac{\mathrm{Wi}}{\mathrm{De}}\right)\mathrm{De}^2 J_{2k}}{2\left(1+\left(2k-1-\frac{\mathrm{Wi}}{\mathrm{De}}\right)^2\mathrm{De}^2\right)} + \frac{\left(2k-1+\frac{\mathrm{Wi}}{\mathrm{De}}\right)\mathrm{De}^2 J_{2k}}{2\left(1+\left(2k-1+\frac{\mathrm{Wi}}{\mathrm{De}}\right)^2\mathrm{De}^2\right)} - \frac{\left(2k+1-\frac{\mathrm{Wi}}{\mathrm{De}}\right)\mathrm{De}^2 J_{2k}}{2\left(1+\left(2k+1-\frac{\mathrm{Wi}}{\mathrm{De}}\right)^2\mathrm{De}^2\right)} + \frac{\left(2k+1+\frac{\mathrm{Wi}}{\mathrm{De}}\right)\mathrm{De}^2 J_{2k}}{2\left(1+\left(2k+1+\frac{\mathrm{Wi}}{\mathrm{De}}\right)^2\mathrm{De}^2\right)} \\ +\frac{\left(2k-\frac{\mathrm{Wi}}{\mathrm{De}}\right)\mathrm{De}^2 J_{2k-1}}{2\left(1+\left(2k-\frac{\mathrm{Wi}}{\mathrm{De}}\right)^2\mathrm{De}^2\right)} + \frac{\left(2k+\frac{\mathrm{Wi}}{\mathrm{De}}\right)\mathrm{De}^2 J_{2k-1}}{2\left(1+\left(2k+\frac{\mathrm{Wi}}{\mathrm{De}}\right)^2\mathrm{De}^2\right)} + \frac{\left(2k-2-\frac{\mathrm{Wi}}{\mathrm{De}}\right)\mathrm{De}^2 J_{2k-1}}{2\left(1+\left(2k-2-\frac{\mathrm{Wi}}{\mathrm{De}}\right)^2\mathrm{De}^2\right)} + \frac{\left(2k-2+\frac{\mathrm{Wi}}{\mathrm{De}}\right)\mathrm{De}^2 J_{2k-1}}{2\left(1+\left(2k-2+\frac{\mathrm{Wi}}{\mathrm{De}}\right)^2\mathrm{De}^2\right)} \end{array}\right]$$

Subtracting the transient part, $I_{4,0} \equiv \lim_{\tau\to 0} I_4$, from Eq. (92) gives:

$$I_4 - I_{4,0} = \frac{\mathrm{De}\,e^{\frac{\tau}{\mathrm{De}}}}{2}\left[\begin{array}{l} \sum_{k=0}^{\infty}\left[\begin{array}{l} \phi_4^{(1)}\cos\left(2k+1-\frac{\mathrm{Wi}}{\mathrm{De}}\right)\tau + \phi_4^{(2)}\cos\left(2k+1+\frac{\mathrm{Wi}}{\mathrm{De}}\right)\tau \\ +\psi_4^{(1)}\sin\left(2k+1-\frac{\mathrm{Wi}}{\mathrm{De}}\right)\tau + \psi_4^{(2)}\sin\left(2k+1+\frac{\mathrm{Wi}}{\mathrm{De}}\right)\tau \end{array}\right] \\ +\sum_{k=1}^{\infty}\left[\begin{array}{l} \phi_4^{(3)}\cos\left(2k-2-\frac{\mathrm{Wi}}{\mathrm{De}}\right)\tau + \phi_4^{(4)}\cos\left(2k-2+\frac{\mathrm{Wi}}{\mathrm{De}}\right)\tau + \phi_4^{(5)}\cos\left(2k-1-\frac{\mathrm{Wi}}{\mathrm{De}}\right)\tau \\ +\phi_4^{(6)}\cos\left(2k-1+\frac{\mathrm{Wi}}{\mathrm{De}}\right)\tau + \phi_4^{(7)}\cos\left(2k-\frac{\mathrm{Wi}}{\mathrm{De}}\right)\tau + \phi_4^{(8)}\cos\left(2k+\frac{\mathrm{Wi}}{\mathrm{De}}\right)\tau \\ +\psi_4^{(3)}\sin\left(2k-2-\frac{\mathrm{Wi}}{\mathrm{De}}\right)\tau + \psi_4^{(4)}\sin\left(2k-2+\frac{\mathrm{Wi}}{\mathrm{De}}\right)\tau + \psi_4^{(5)}\sin\left(2k-1-\frac{\mathrm{Wi}}{\mathrm{De}}\right)\tau \\ +\psi_4^{(6)}\sin\left(2k-1+\frac{\mathrm{Wi}}{\mathrm{De}}\right)\tau + \psi_4^{(7)}\sin\left(2k-\frac{\mathrm{Wi}}{\mathrm{De}}\right)\tau + \psi_4^{(8)}\sin\left(2k+\frac{\mathrm{Wi}}{\mathrm{De}}\right)\tau \end{array}\right] \end{array}\right] \qquad (93)$$

$$-\frac{\mathrm{De}}{2}\left[\sum_{k=0}^{\infty}\left(\phi_4^{(1)}+\phi_4^{(2)}\right) + \sum_{k=1}^{\infty}\left(\phi_4^{(3)}+\phi_4^{(4)}+\phi_4^{(5)}+\phi_4^{(6)}+\phi_4^{(7)}+\phi_4^{(8)}\right)\right]$$

where:

$$\phi_4^{(i)} \equiv \left[\begin{array}{l} \left(\frac{\left(2k+1-\frac{\mathrm{Wi}}{\mathrm{De}}\right)\mathrm{De}\,J_{2k}}{1+\left(2k+1-\frac{\mathrm{Wi}}{\mathrm{De}}\right)^2\mathrm{De}^2} \right)_{i=1}, \left(\frac{-\left(2k+1+\frac{\mathrm{Wi}}{\mathrm{De}}\right)\mathrm{De}\,J_{2k}}{1+\left(2k+1+\frac{\mathrm{Wi}}{\mathrm{De}}\right)^2\mathrm{De}^2} \right)_2, \left(\frac{-\left(2k-2-\frac{\mathrm{Wi}}{\mathrm{De}}\right)\mathrm{De}\,J_{2k-1}}{1+\left(2k-2-\frac{\mathrm{Wi}}{\mathrm{De}}\right)^2\mathrm{De}^2} \right)_3, \\ \left(\frac{-\left(2k-2+\frac{\mathrm{Wi}}{\mathrm{De}}\right)\mathrm{De}\,J_{2k-1}}{1+\left(2k-2+\frac{\mathrm{Wi}}{\mathrm{De}}\right)^2\mathrm{De}^2} \right)_4, \left(\frac{\left(2k-1-\frac{\mathrm{Wi}}{\mathrm{De}}\right)\mathrm{De}\,J_{2k}}{1+\left(2k-1-\frac{\mathrm{Wi}}{\mathrm{De}}\right)^2\mathrm{De}^2} \right)_5, \left(\frac{-\left(2k-1+\frac{\mathrm{Wi}}{\mathrm{De}}\right)\mathrm{De}\,J_{2k}}{1+\left(2k-1+\frac{\mathrm{Wi}}{\mathrm{De}}\right)^2\mathrm{De}^2} \right)_6, \\ \left(\frac{-\left(2k-\frac{\mathrm{Wi}}{\mathrm{De}}\right)\mathrm{De}\,J_{2k-1}}{1+\left(2k-\frac{\mathrm{Wi}}{\mathrm{De}}\right)^2\mathrm{De}^2} \right)_7, \left(\frac{-\left(2k+\frac{\mathrm{Wi}}{\mathrm{De}}\right)\mathrm{De}\,J_{2k-1}}{1+\left(2k+\frac{\mathrm{Wi}}{\mathrm{De}}\right)^2\mathrm{De}^2} \right)_8 \end{array} \right] \tag{94}$$

$$\psi_4^{(i)} \equiv \left[\begin{array}{l} \left(\frac{-J_{2k}}{1+\left(2k+1-\frac{\mathrm{Wi}}{\mathrm{De}}\right)^2\mathrm{De}^2} \right)_{i=1}, \left(\frac{J_{2k}}{1+\left(2k+1+\frac{\mathrm{Wi}}{\mathrm{De}}\right)^2\mathrm{De}^2} \right)_2, \left(\frac{J_{2k-1}}{1+\left(2k-2-\frac{\mathrm{Wi}}{\mathrm{De}}\right)^2\mathrm{De}^2} \right)_3, \\ \left(\frac{J_{2k-1}}{1+\left(2k-2+\frac{\mathrm{Wi}}{\mathrm{De}}\right)^2\mathrm{De}^2} \right)_4, \left(\frac{-J_{2k}}{1+\left(2k-1-\frac{\mathrm{Wi}}{\mathrm{De}}\right)^2\mathrm{De}^2} \right)_5, \left(\frac{J_{2k}}{1+\left(2k-1+\frac{\mathrm{Wi}}{\mathrm{De}}\right)^2\mathrm{De}^2} \right)_6, \\ \left(\frac{J_{2k-1}}{1+\left(2k-\frac{\mathrm{Wi}}{\mathrm{De}}\right)^2\mathrm{De}^2} \right)_7, \left(\frac{J_{2k-1}}{1+\left(2k+\frac{\mathrm{Wi}}{\mathrm{De}}\right)^2\mathrm{De}^2} \right)_8 \end{array} \right] \tag{95}$$

v. I_5

Substituting Eq. (56) into Eq. (45) yields:

$$I_5 = \int e^{\frac{\tau'}{\mathrm{De}}} \sin\tau' \cos\tfrac{\mathrm{Wi}}{\mathrm{De}}\tau' \cos\left(\tfrac{\mathrm{Wi}}{\mathrm{De}}\sin\tau'\right)d\tau' - \int e^{\frac{\tau'}{\mathrm{De}}} \sin\tau' \sin\tfrac{\mathrm{Wi}}{\mathrm{De}}\tau' \sin\left(\tfrac{\mathrm{Wi}}{\mathrm{De}}\sin\tau'\right)d\tau' \tag{96}$$

and then Eqs. (58) and (59) into the result, we find:

$$I_5 = J_0 I_{5,1} + 2\sum_{k=1}^{\infty}\left(J_{2k}I_{5,2} - J_{2k-1}I_{5,3}\right) \tag{97}$$

$$I_{5,1} \equiv \int e^{\frac{\tau'}{\mathrm{De}}} \sin\tau' \cos\tfrac{\mathrm{Wi}}{\mathrm{De}}\tau'\, d\tau' \tag{98}$$

$$I_{5,2} \equiv \int e^{\frac{\tau'}{\mathrm{De}}} \sin\tau' \cos\tfrac{\mathrm{Wi}}{\mathrm{De}}\tau' \cos 2k\tau'\, d\tau' \tag{99}$$

$$I_{5,3} \equiv \int e^{\frac{\tau'}{\mathrm{De}}} \sin\tau' \sin\tfrac{\mathrm{Wi}}{\mathrm{De}}\tau' \sin\left(2k-1\right)\tau'\, d\tau' \tag{100}$$

Evaluating these, and then substituting back into Eq. (97) yields:

$$
\begin{aligned}
I_5 = \frac{\mathrm{De}\,e^{\frac{\tau}{\mathrm{De}}}}{2}\Bigg[& -\frac{\left(1-\tfrac{\mathrm{Wi}}{\mathrm{De}}\right)\mathrm{De}\,J_0\cos\left(1-\tfrac{\mathrm{Wi}}{\mathrm{De}}\right)\tau}{1+\left(1-\tfrac{\mathrm{Wi}}{\mathrm{De}}\right)^2\mathrm{De}^2}-\frac{\left(1+\tfrac{\mathrm{Wi}}{\mathrm{De}}\right)\mathrm{De}\,J_0\cos\left(1+\tfrac{\mathrm{Wi}}{\mathrm{De}}\right)\tau}{1+\left(1+\tfrac{\mathrm{Wi}}{\mathrm{De}}\right)^2\mathrm{De}^2}+\frac{J_0\sin\left(1-\tfrac{\mathrm{Wi}}{\mathrm{De}}\right)\tau}{1+\left(1-\tfrac{\mathrm{Wi}}{\mathrm{De}}\right)^2\mathrm{De}^2}+\frac{J_0\sin\left(1+\tfrac{\mathrm{Wi}}{\mathrm{De}}\right)\tau}{1+\left(1+\tfrac{\mathrm{Wi}}{\mathrm{De}}\right)^2\mathrm{De}^2} \\
& +\sum_{k=1}^{\infty}\Bigg[\frac{\left(2k-1-\tfrac{\mathrm{Wi}}{\mathrm{De}}\right)\mathrm{De}\,J_{2k}\cos\left(2k-1-\tfrac{\mathrm{Wi}}{\mathrm{De}}\right)\tau}{1+\left(2k-1-\tfrac{\mathrm{Wi}}{\mathrm{De}}\right)^2\mathrm{De}^2}+\frac{\left(2k-1+\tfrac{\mathrm{Wi}}{\mathrm{De}}\right)\mathrm{De}\,J_{2k}\cos\left(2k-1+\tfrac{\mathrm{Wi}}{\mathrm{De}}\right)\tau}{1+\left(2k-1+\tfrac{\mathrm{Wi}}{\mathrm{De}}\right)^2\mathrm{De}^2} \\
& -\frac{\left(2k+1-\tfrac{\mathrm{Wi}}{\mathrm{De}}\right)\mathrm{De}\,J_{2k}\cos\left(2k+1-\tfrac{\mathrm{Wi}}{\mathrm{De}}\right)\tau}{1+\left(2k+1-\tfrac{\mathrm{Wi}}{\mathrm{De}}\right)^2\mathrm{De}^2}-\frac{\left(2k+1+\tfrac{\mathrm{Wi}}{\mathrm{De}}\right)\mathrm{De}\,J_{2k}\cos\left(2k+1+\tfrac{\mathrm{Wi}}{\mathrm{De}}\right)\tau}{1+\left(2k+1+\tfrac{\mathrm{Wi}}{\mathrm{De}}\right)^2\mathrm{De}^2} \\
& -\frac{J_{2k}\sin\left(2k-1-\tfrac{\mathrm{Wi}}{\mathrm{De}}\right)\tau}{1+\left(2k-1-\tfrac{\mathrm{Wi}}{\mathrm{De}}\right)^2\mathrm{De}^2}-\frac{J_{2k}\sin\left(2k-1+\tfrac{\mathrm{Wi}}{\mathrm{De}}\right)\tau}{1+\left(2k-1+\tfrac{\mathrm{Wi}}{\mathrm{De}}\right)^2\mathrm{De}^2}+\frac{J_{2k}\sin\left(2k+1-\tfrac{\mathrm{Wi}}{\mathrm{De}}\right)\tau}{1+\left(2k+1-\tfrac{\mathrm{Wi}}{\mathrm{De}}\right)^2\mathrm{De}^2} \\
& +\frac{J_{2k}\sin\left(2k+1+\tfrac{\mathrm{Wi}}{\mathrm{De}}\right)\tau}{1+\left(2k+1+\tfrac{\mathrm{Wi}}{\mathrm{De}}\right)^2\mathrm{De}^2}+\frac{\left(2k-\tfrac{\mathrm{Wi}}{\mathrm{De}}\right)\mathrm{De}\,J_{2k-1}\cos\left(2k-\tfrac{\mathrm{Wi}}{\mathrm{De}}\right)\tau}{1+\left(2k-\tfrac{\mathrm{Wi}}{\mathrm{De}}\right)^2\mathrm{De}^2} \\
& -\frac{\left(2k+\tfrac{\mathrm{Wi}}{\mathrm{De}}\right)\mathrm{De}\,J_{2k-1}\cos\left(2k+\tfrac{\mathrm{Wi}}{\mathrm{De}}\right)\tau}{1+\left(2k+\tfrac{\mathrm{Wi}}{\mathrm{De}}\right)^2\mathrm{De}^2}-\frac{\left(2k-2-\tfrac{\mathrm{Wi}}{\mathrm{De}}\right)\mathrm{De}\,J_{2k-1}\cos\left(2k-2-\tfrac{\mathrm{Wi}}{\mathrm{De}}\right)\tau}{1+\left(2k-2-\tfrac{\mathrm{Wi}}{\mathrm{De}}\right)^2\mathrm{De}^2} \\
& +\frac{\left(2k-2+\tfrac{\mathrm{Wi}}{\mathrm{De}}\right)\mathrm{De}\,J_{2k-1}\cos\left(2k-2+\tfrac{\mathrm{Wi}}{\mathrm{De}}\right)\tau}{1+\left(2k-2+\tfrac{\mathrm{Wi}}{\mathrm{De}}\right)^2\mathrm{De}^2}-\frac{J_{2k-1}\sin\left(2k-\tfrac{\mathrm{Wi}}{\mathrm{De}}\right)\tau}{1+\left(2k-\tfrac{\mathrm{Wi}}{\mathrm{De}}\right)^2\mathrm{De}^2}+\frac{J_{2k-1}\sin\left(2k+\tfrac{\mathrm{Wi}}{\mathrm{De}}\right)\tau}{1+\left(2k+\tfrac{\mathrm{Wi}}{\mathrm{De}}\right)^2\mathrm{De}^2} \\
& +\frac{J_{2k-1}\sin\left(2k-2-\tfrac{\mathrm{Wi}}{\mathrm{De}}\right)\tau}{1+\left(2k-2-\tfrac{\mathrm{Wi}}{\mathrm{De}}\right)^2\mathrm{De}^2}-\frac{J_{2k-1}\sin\left(2k-2+\tfrac{\mathrm{Wi}}{\mathrm{De}}\right)\tau}{1+\left(2k-2+\tfrac{\mathrm{Wi}}{\mathrm{De}}\right)^2\mathrm{De}^2}\Bigg]\Bigg] \\
+\frac{\mathrm{De}^2}{2}\Bigg[& \frac{\left(1-\tfrac{\mathrm{Wi}}{\mathrm{De}}\right)J_0}{1+\left(1-\tfrac{\mathrm{Wi}}{\mathrm{De}}\right)^2\mathrm{De}^2}+\frac{\left(1+\tfrac{\mathrm{Wi}}{\mathrm{De}}\right)J_0}{1+\left(1+\tfrac{\mathrm{Wi}}{\mathrm{De}}\right)^2\mathrm{De}^2} \\
& +\sum_{k=1}^{\infty}\Bigg\{-\frac{\left(2k-1-\tfrac{\mathrm{Wi}}{\mathrm{De}}\right)J_{2k}}{1+\left(2k-1-\tfrac{\mathrm{Wi}}{\mathrm{De}}\right)^2\mathrm{De}^2}-\frac{\left(2k-1+\tfrac{\mathrm{Wi}}{\mathrm{De}}\right)J_{2k}}{1+\left(2k-1+\tfrac{\mathrm{Wi}}{\mathrm{De}}\right)^2\mathrm{De}^2}+\frac{\left(2k+1-\tfrac{\mathrm{Wi}}{\mathrm{De}}\right)J_{2k}}{1+\left(2k+1-\tfrac{\mathrm{Wi}}{\mathrm{De}}\right)^2\mathrm{De}^2}+\frac{\left(2k+1+\tfrac{\mathrm{Wi}}{\mathrm{De}}\right)J_{2k}}{1+\left(2k+1+\tfrac{\mathrm{Wi}}{\mathrm{De}}\right)^2\mathrm{De}^2} \\
& -\frac{\left(2k-\tfrac{\mathrm{Wi}}{\mathrm{De}}\right)J_{2k-1}}{1+\left(2k-\tfrac{\mathrm{Wi}}{\mathrm{De}}\right)^2\mathrm{De}^2}+\frac{\left(2k+\tfrac{\mathrm{Wi}}{\mathrm{De}}\right)J_{2k-1}}{1+\left(2k+\tfrac{\mathrm{Wi}}{\mathrm{De}}\right)^2\mathrm{De}^2}+\frac{\left(2k-2-\tfrac{\mathrm{Wi}}{\mathrm{De}}\right)J_{2k-1}}{1+\left(2k-2-\tfrac{\mathrm{Wi}}{\mathrm{De}}\right)^2\mathrm{De}^2}-\frac{\left(2k-2+\tfrac{\mathrm{Wi}}{\mathrm{De}}\right)J_{2k-1}}{1+\left(2k-2+\tfrac{\mathrm{Wi}}{\mathrm{De}}\right)^2\mathrm{De}^2}\Bigg\}\Bigg]
\end{aligned}
\tag{101}
$$

Subtracting the transient part, $I_{5,0}\equiv\lim_{\tau\to 0}I_5$, from Eq. (101) gives:

$$
\begin{aligned}
I_5-I_{5,0}=\frac{\mathrm{De}\,e^{\frac{\tau}{\mathrm{De}}}}{2}\Bigg[& \sum_{k=0}^{\infty}\left[\begin{aligned}&\phi_5^{(1)}\cos\left(2k+1-\tfrac{\mathrm{Wi}}{\mathrm{De}}\right)\tau+\phi_5^{(2)}\cos\left(2k+1+\tfrac{\mathrm{Wi}}{\mathrm{De}}\right)\tau\\&+\psi_5^{(1)}\sin\left(2k+1-\tfrac{\mathrm{Wi}}{\mathrm{De}}\right)\tau+\psi_5^{(2)}\sin\left(2k+1+\tfrac{\mathrm{Wi}}{\mathrm{De}}\right)\tau\end{aligned}\right] \\
& +\sum_{k=1}^{\infty}\left[\begin{aligned}&+\phi_5^{(3)}\cos\left(2k-2-\tfrac{\mathrm{Wi}}{\mathrm{De}}\right)\tau+\phi_5^{(4)}\cos\left(2k-2+\tfrac{\mathrm{Wi}}{\mathrm{De}}\right)\tau+\phi_5^{(5)}\cos\left(2k-1-\tfrac{\mathrm{Wi}}{\mathrm{De}}\right)\tau\\&+\phi_5^{(6)}\cos\left(2k-1+\tfrac{\mathrm{Wi}}{\mathrm{De}}\right)\tau+\phi_5^{(7)}\cos\left(2k-\tfrac{\mathrm{Wi}}{\mathrm{De}}\right)\tau+\phi_5^{(8)}\cos\left(2k+\tfrac{\mathrm{Wi}}{\mathrm{De}}\right)\tau\\&+\psi_5^{(3)}\sin\left(2k-2-\tfrac{\mathrm{Wi}}{\mathrm{De}}\right)\tau+\psi_5^{(4)}\sin\left(2k-2+\tfrac{\mathrm{Wi}}{\mathrm{De}}\right)\tau+\psi_5^{(5)}\sin\left(2k-1-\tfrac{\mathrm{Wi}}{\mathrm{De}}\right)\tau\\&+\psi_5^{(6)}\sin\left(2k-1+\tfrac{\mathrm{Wi}}{\mathrm{De}}\right)\tau+\psi_5^{(7)}\sin\left(2k-\tfrac{\mathrm{Wi}}{\mathrm{De}}\right)\tau+\psi_5^{(8)}\sin\left(2k+\tfrac{\mathrm{Wi}}{\mathrm{De}}\right)\tau\end{aligned}\right]\Bigg] \\
& -\frac{\mathrm{De}}{2}\left[\sum_{k=0}^{\infty}\left(\phi_5^{(1)}+\phi_5^{(2)}\right)+\sum_{k=1}^{\infty}\left(\phi_5^{(3)}+\phi_5^{(4)}+\phi_5^{(5)}+\phi_5^{(6)}+\phi_5^{(7)}+\phi_5^{(8)}\right)\right]
\end{aligned}
\tag{102}
$$

where:

$$\phi_5^{(i)} \equiv \begin{bmatrix} \left(\frac{-\left(2k+1-\frac{\mathrm{Wi}}{\mathrm{De}}\right)\mathrm{De}\,J_{2k}}{1+\left(2k+1-\frac{\mathrm{Wi}}{\mathrm{De}}\right)^2\mathrm{De}^2}\right)_{i=1}, \left(\frac{-\left(2k+1+\frac{\mathrm{Wi}}{\mathrm{De}}\right)\mathrm{De}\,J_{2k}}{1+\left(2k+1+\frac{\mathrm{Wi}}{\mathrm{De}}\right)^2\mathrm{De}^2}\right)_2, \left(\frac{-\left(2k-2-\frac{\mathrm{Wi}}{\mathrm{De}}\right)\mathrm{De}\,J_{2k-1}}{1+\left(2k-2-\frac{\mathrm{Wi}}{\mathrm{De}}\right)^2\mathrm{De}^2}\right)_3, \\ \left(\frac{\left(2k-2+\frac{\mathrm{Wi}}{\mathrm{De}}\right)\mathrm{De}\,J_{2k-1}}{1+\left(2k-2+\frac{\mathrm{Wi}}{\mathrm{De}}\right)^2\mathrm{De}^2}\right)_4, \left(\frac{\left(2k-1-\frac{\mathrm{Wi}}{\mathrm{De}}\right)\mathrm{De}\,J_{2k}}{1+\left(2k-1-\frac{\mathrm{Wi}}{\mathrm{De}}\right)^2\mathrm{De}^2}\right)_5, \left(\frac{\left(2k-1+\frac{\mathrm{Wi}}{\mathrm{De}}\right)\mathrm{De}\,J_{2k}}{1+\left(2k-1+\frac{\mathrm{Wi}}{\mathrm{De}}\right)^2\mathrm{De}^2}\right)_6, \\ \left(\frac{\left(2k-\frac{\mathrm{Wi}}{\mathrm{De}}\right)\mathrm{De}\,J_{2k-1}}{1+\left(2k-\frac{\mathrm{Wi}}{\mathrm{De}}\right)^2\mathrm{De}^2}\right)_7, \left(\frac{-\left(2k+\frac{\mathrm{Wi}}{\mathrm{De}}\right)\mathrm{De}\,J_{2k-1}}{1+\left(2k+\frac{\mathrm{Wi}}{\mathrm{De}}\right)^2\mathrm{De}^2}\right)_8 \end{bmatrix} \tag{103}$$

$$\psi_5^{(i)} \equiv \begin{bmatrix} \left(\frac{J_{2k}}{1+\left(2k+1-\frac{\mathrm{Wi}}{\mathrm{De}}\right)^2\mathrm{De}^2}\right)_{i=1}, \left(\frac{J_{2k}}{1+\left(2k+1+\frac{\mathrm{Wi}}{\mathrm{De}}\right)^2\mathrm{De}^2}\right)_2, \left(\frac{J_{2k-1}}{1+\left(2k-2-\frac{\mathrm{Wi}}{\mathrm{De}}\right)^2\mathrm{De}^2}\right)_3, \\ \left(\frac{-J_{2k-1}}{1+\left(2k-2+\frac{\mathrm{Wi}}{\mathrm{De}}\right)^2\mathrm{De}^2}\right)_4, \left(\frac{-J_{2k}}{1+\left(2k-1-\frac{\mathrm{Wi}}{\mathrm{De}}\right)^2\mathrm{De}^2}\right)_5, \left(\frac{-J_{2k}}{1+\left(2k-1+\frac{\mathrm{Wi}}{\mathrm{De}}\right)^2\mathrm{De}^2}\right)_6, \\ \left(\frac{-J_{2k-1}}{1+\left(2k-\frac{\mathrm{Wi}}{\mathrm{De}}\right)^2\mathrm{De}^2}\right)_7, \left(\frac{J_{2k-1}}{1+\left(2k+\frac{\mathrm{Wi}}{\mathrm{De}}\right)^2\mathrm{De}^2}\right)_8 \end{bmatrix} \tag{104}$$

vi. I_6

Substituting Eq. (57) into Eq. (46) gives:

$$I_6 = \int e^{\frac{\tau'}{\mathrm{De}}} \sin\tau' \sin\tfrac{\mathrm{Wi}}{\mathrm{De}}\tau' \cos\left(\tfrac{\mathrm{Wi}}{\mathrm{De}}\sin\tau'\right)d\tau' + \int e^{\frac{\tau'}{\mathrm{De}}} \sin\tau' \cos\tfrac{\mathrm{Wi}}{\mathrm{De}}\tau' \sin\left(\tfrac{\mathrm{Wi}}{\mathrm{De}}\sin\tau'\right)d\tau' \tag{105}$$

and then Eqs. (58) and (59) into the result, we find:

$$I_6 = J_0 I_{6,1} + 2\sum_{k=1}^{\infty}\left(J_{2k} I_{6,2} + J_{2k-1} I_{6,3}\right) \tag{106}$$

$$I_{6,1} \equiv \int e^{\frac{\tau'}{\mathrm{De}}} \sin\tau' \sin\tfrac{\mathrm{Wi}}{\mathrm{De}}\tau'\, d\tau' \tag{107}$$

$$I_{6,2} \equiv \int e^{\frac{\tau'}{\mathrm{De}}} \sin\tau' \sin\tfrac{\mathrm{Wi}}{\mathrm{De}}\tau' \cos 2k\tau'\, d\tau' \tag{108}$$

$$I_{6,3} \equiv \int e^{\frac{\tau'}{\mathrm{De}}} \sin\tau' \cos\tfrac{\mathrm{Wi}}{\mathrm{De}}\tau' \sin\left(2k-1\right)\tau'\, d\tau' \tag{109}$$

Evaluating these, and then substituting back into Eq. (106) yields:

$$
I_6=\frac{\mathrm{De}\,e^{\frac{\tau}{\mathrm{De}}}}{2}\left[\begin{array}{l}
\frac{J_0\cos\left(1-\frac{\mathrm{Wi}}{\mathrm{De}}\right)\tau}{1+\left(1-\frac{\mathrm{Wi}}{\mathrm{De}}\right)^2\mathrm{De}^2}-\frac{J_0\cos\left(1+\frac{\mathrm{Wi}}{\mathrm{De}}\right)\tau}{1+\left(1+\frac{\mathrm{Wi}}{\mathrm{De}}\right)^2\mathrm{De}^2}+\frac{\left(1-\frac{\mathrm{Wi}}{\mathrm{De}}\right)\mathrm{De}J_0\sin\left(1-\frac{\mathrm{Wi}}{\mathrm{De}}\right)\tau}{1+\left(1-\frac{\mathrm{Wi}}{\mathrm{De}}\right)^2\mathrm{De}^2}-\frac{\left(1+\frac{\mathrm{Wi}}{\mathrm{De}}\right)\mathrm{De}J_0\sin\left(1+\frac{\mathrm{Wi}}{\mathrm{De}}\right)\tau}{1+\left(1+\frac{\mathrm{Wi}}{\mathrm{De}}\right)^2\mathrm{De}^2}\\
+\sum_{k=1}^{\infty}\left[\begin{array}{l}
-\frac{J_{2k}\cos\left(2k-1-\frac{\mathrm{Wi}}{\mathrm{De}}\right)\tau}{1+\left(2k-1-\frac{\mathrm{Wi}}{\mathrm{De}}\right)^2\mathrm{De}^2}+\frac{J_{2k}\cos\left(2k-1+\frac{\mathrm{Wi}}{\mathrm{De}}\right)\tau}{1+\left(2k-1+\frac{\mathrm{Wi}}{\mathrm{De}}\right)^2\mathrm{De}^2}+\frac{J_{2k}\cos\left(2k+1-\frac{\mathrm{Wi}}{\mathrm{De}}\right)\tau}{1+\left(2k+1-\frac{\mathrm{Wi}}{\mathrm{De}}\right)^2\mathrm{De}^2}-\frac{J_{2k}\cos\left(2k+1+\frac{\mathrm{Wi}}{\mathrm{De}}\right)\tau}{1+\left(2k+1+\frac{\mathrm{Wi}}{\mathrm{De}}\right)^2\mathrm{De}^2}\\
-\frac{\left(2k-1-\frac{\mathrm{Wi}}{\mathrm{De}}\right)\mathrm{De}J_{2k}\sin\left(2k-1-\frac{\mathrm{Wi}}{\mathrm{De}}\right)\tau}{1+\left(2k-1-\frac{\mathrm{Wi}}{\mathrm{De}}\right)^2\mathrm{De}^2}+\frac{\left(2k-1+\frac{\mathrm{Wi}}{\mathrm{De}}\right)\mathrm{De}J_{2k}\sin\left(2k-1+\frac{\mathrm{Wi}}{\mathrm{De}}\right)\tau}{1+\left(2k-1+\frac{\mathrm{Wi}}{\mathrm{De}}\right)^2\mathrm{De}^2}\\
+\frac{\left(2k+1-\frac{\mathrm{Wi}}{\mathrm{De}}\right)\mathrm{De}J_{2k}\sin\left(2k+1-\frac{\mathrm{Wi}}{\mathrm{De}}\right)\tau}{1+\left(2k+1-\frac{\mathrm{Wi}}{\mathrm{De}}\right)^2\mathrm{De}^2}-\frac{\left(2k+1+\frac{\mathrm{Wi}}{\mathrm{De}}\right)\mathrm{De}J_{2k}\sin\left(2k+1+\frac{\mathrm{Wi}}{\mathrm{De}}\right)\tau}{1+\left(2k+1+\frac{\mathrm{Wi}}{\mathrm{De}}\right)^2\mathrm{De}^2}-\frac{J_{2k-1}\cos\left(2k-\frac{\mathrm{Wi}}{\mathrm{De}}\right)\tau}{1+\left(2k-\frac{\mathrm{Wi}}{\mathrm{De}}\right)^2\mathrm{De}^2}\\
-\frac{J_{2k-1}\cos\left(2k+\frac{\mathrm{Wi}}{\mathrm{De}}\right)\tau}{1+\left(2k+\frac{\mathrm{Wi}}{\mathrm{De}}\right)^2\mathrm{De}^2}+\frac{J_{2k-1}\cos\left(2k-2-\frac{\mathrm{Wi}}{\mathrm{De}}\right)\tau}{1+\left(2k-2-\frac{\mathrm{Wi}}{\mathrm{De}}\right)^2\mathrm{De}^2}+\frac{J_{2k-1}\cos\left(2k-2+\frac{\mathrm{Wi}}{\mathrm{De}}\right)\tau}{1+\left(2k-2+\frac{\mathrm{Wi}}{\mathrm{De}}\right)^2\mathrm{De}^2}-\frac{\left(2k-\frac{\mathrm{Wi}}{\mathrm{De}}\right)\mathrm{De}J_{2k-1}\sin\left(2k-\frac{\mathrm{Wi}}{\mathrm{De}}\right)\tau}{1+\left(2k-\frac{\mathrm{Wi}}{\mathrm{De}}\right)^2\mathrm{De}^2}\\
-\frac{\left(2k+\frac{\mathrm{Wi}}{\mathrm{De}}\right)\mathrm{De}J_{2k-1}\sin\left(2k+\frac{\mathrm{Wi}}{\mathrm{De}}\right)\tau}{1+\left(2k+\frac{\mathrm{Wi}}{\mathrm{De}}\right)^2\mathrm{De}^2}+\frac{\left(2k-2-\frac{\mathrm{Wi}}{\mathrm{De}}\right)\mathrm{De}J_{2k-1}\sin\left(2k-2-\frac{\mathrm{Wi}}{\mathrm{De}}\right)\tau}{1+\left(2k-2-\frac{\mathrm{Wi}}{\mathrm{De}}\right)^2\mathrm{De}^2}+\frac{\left(2k-2+\frac{\mathrm{Wi}}{\mathrm{De}}\right)\mathrm{De}J_{2k-1}\sin\left(2k-2+\frac{\mathrm{Wi}}{\mathrm{De}}\right)\tau}{1+\left(2k-2+\frac{\mathrm{Wi}}{\mathrm{De}}\right)^2\mathrm{De}^2}
\end{array}\right]
\end{array}\right]
$$

$$
+\frac{\mathrm{De}}{2}\left[\begin{array}{l}
-\frac{J_0}{1+\left(1-\frac{\mathrm{Wi}}{\mathrm{De}}\right)^2\mathrm{De}^2}+\frac{J_0}{1+\left(1+\frac{\mathrm{Wi}}{\mathrm{De}}\right)^2\mathrm{De}^2}\\
+\sum_{k=1}^{\infty}\left[\begin{array}{l}
\frac{J_{2k}}{1+\left(2k-1-\frac{\mathrm{Wi}}{\mathrm{De}}\right)^2\mathrm{De}^2}-\frac{J_{2k}}{1+\left(2k-1+\frac{\mathrm{Wi}}{\mathrm{De}}\right)^2\mathrm{De}^2}-\frac{J_{2k}}{1+\left(2k+1-\frac{\mathrm{Wi}}{\mathrm{De}}\right)^2\mathrm{De}^2}+\frac{J_{2k}}{1+\left(2k+1+\frac{\mathrm{Wi}}{\mathrm{De}}\right)^2\mathrm{De}^2}\\
+\frac{J_{2k-1}}{1+\left(2k-\frac{\mathrm{Wi}}{\mathrm{De}}\right)^2\mathrm{De}^2}+\frac{J_{2k-1}}{1+\left(2k+\frac{\mathrm{Wi}}{\mathrm{De}}\right)^2\mathrm{De}^2}-\frac{J_{2k-1}}{1+\left(2k-2-\frac{\mathrm{Wi}}{\mathrm{De}}\right)^2\mathrm{De}^2}-\frac{J_{2k-1}}{1+\left(2k-2+\frac{\mathrm{Wi}}{\mathrm{De}}\right)^2\mathrm{De}^2}
\end{array}\right]
\end{array}\right] \tag{110}
$$

Subtracting the transient part, $I_{6,0}\equiv\lim_{\tau\to 0}I_6$, from Eq. (110) gives:

$$
I_6-I_{6,0}=\frac{\mathrm{De}\,e^{\frac{\tau}{\mathrm{De}}}}{2}\left[\begin{array}{l}
\sum_{k=0}^{\infty}\left[\begin{array}{l}
\phi_6^{(1)}\cos\left(2k+1-\frac{\mathrm{Wi}}{\mathrm{De}}\right)\tau+\phi_6^{(2)}\cos\left(2k+1+\frac{\mathrm{Wi}}{\mathrm{De}}\right)\tau\\
+\psi_6^{(1)}\sin\left(2k+1-\frac{\mathrm{Wi}}{\mathrm{De}}\right)\tau+\psi_6^{(2)}\sin\left(2k+1+\frac{\mathrm{Wi}}{\mathrm{De}}\right)\tau
\end{array}\right]\\
+\sum_{k=1}^{\infty}\left[\begin{array}{l}
\phi_6^{(3)}\cos\left(2k-2-\frac{\mathrm{Wi}}{\mathrm{De}}\right)\tau+\phi_6^{(4)}\cos\left(2k-2+\frac{\mathrm{Wi}}{\mathrm{De}}\right)\tau+\phi_6^{(5)}\cos\left(2k-1-\frac{\mathrm{Wi}}{\mathrm{De}}\right)\tau\\
+\phi_6^{(6)}\cos\left(2k-1+\frac{\mathrm{Wi}}{\mathrm{De}}\right)\tau+\phi_6^{(7)}\cos\left(2k-\frac{\mathrm{Wi}}{\mathrm{De}}\right)\tau+\phi_6^{(8)}\cos\left(2k+\frac{\mathrm{Wi}}{\mathrm{De}}\right)\tau\\
+\psi_6^{(3)}\sin\left(2k-2-\frac{\mathrm{Wi}}{\mathrm{De}}\right)\tau+\psi_6^{(4)}\sin\left(2k-2+\frac{\mathrm{Wi}}{\mathrm{De}}\right)\tau+\psi_6^{(5)}\sin\left(2k-1-\frac{\mathrm{Wi}}{\mathrm{De}}\right)\tau\\
+\psi_6^{(6)}\sin\left(2k-1+\frac{\mathrm{Wi}}{\mathrm{De}}\right)\tau+\psi_6^{(7)}\sin\left(2k-\frac{\mathrm{Wi}}{\mathrm{De}}\right)\tau+\psi_6^{(8)}\sin\left(2k+\frac{\mathrm{Wi}}{\mathrm{De}}\right)\tau
\end{array}\right]
\end{array}\right]
$$

$$
-\frac{\mathrm{De}}{2}\left[\sum_{k=0}^{\infty}\left(\phi_6^{(1)}+\phi_6^{(2)}\right)+\sum_{k=1}^{\infty}\left(\phi_6^{(3)}+\phi_6^{(4)}+\phi_6^{(5)}+\phi_6^{(6)}+\phi_6^{(7)}+\phi_6^{(8)}\right)\right] \tag{111}
$$

where:

$$
\phi_6^{(i)}=\left[\begin{array}{l}
\left(\frac{J_{2k}}{1+\left(2k+1-\frac{\mathrm{Wi}}{\mathrm{De}}\right)^2\mathrm{De}^2}\right)_{i=1},\left(\frac{-J_{2k}}{1+\left(2k+1+\frac{\mathrm{Wi}}{\mathrm{De}}\right)^2\mathrm{De}^2}\right)_2,\left(\frac{J_{2k-1}}{1+\left(2k-2-\frac{\mathrm{Wi}}{\mathrm{De}}\right)^2\mathrm{De}^2}\right)_3,\\
\left(\frac{J_{2k-1}}{1+\left(2k-2+\frac{\mathrm{Wi}}{\mathrm{De}}\right)^2\mathrm{De}^2}\right)_4,\left(\frac{-J_{2k}}{1+\left(2k-1-\frac{\mathrm{Wi}}{\mathrm{De}}\right)^2\mathrm{De}^2}\right)_5,\left(\frac{J_{2k}}{1+\left(2k-1+\frac{\mathrm{Wi}}{\mathrm{De}}\right)^2\mathrm{De}^2}\right)_6,\\
\left(\frac{-J_{2k-1}}{1+\left(2k-\frac{\mathrm{Wi}}{\mathrm{De}}\right)^2\mathrm{De}^2}\right)_7,\left(\frac{-J_{2k-1}}{1+\left(2k+\frac{\mathrm{Wi}}{\mathrm{De}}\right)^2\mathrm{De}^2}\right)_8
\end{array}\right] \tag{112}
$$

$$\psi_6^{(i)} = \begin{bmatrix} \left(\frac{\left(2k+1-\tfrac{\mathrm{Wi}}{\mathrm{De}}\right)\mathrm{De}\,J_{2k}}{1+\left(2k+1-\tfrac{\mathrm{Wi}}{\mathrm{De}}\right)^2\mathrm{De}^2}\right)_{i=1}, \left(\frac{-\left(2k+1+\tfrac{\mathrm{Wi}}{\mathrm{De}}\right)\mathrm{De}\,J_{2k}}{1+\left(2k+1+\tfrac{\mathrm{Wi}}{\mathrm{De}}\right)^2\mathrm{De}^2}\right)_2, \left(\frac{\left(2k-2-\tfrac{\mathrm{Wi}}{\mathrm{De}}\right)\mathrm{De}\,J_{2k-1}}{1+\left(2k-2-\tfrac{\mathrm{Wi}}{\mathrm{De}}\right)^2\mathrm{De}^2}\right)_3, \\ \left(\frac{\left(2k-2+\tfrac{\mathrm{Wi}}{\mathrm{De}}\right)\mathrm{De}\,J_{2k-1}}{1+\left(2k-2+\tfrac{\mathrm{Wi}}{\mathrm{De}}\right)^2\mathrm{De}^2}\right)_4, \left(\frac{-\left(2k-1-\tfrac{\mathrm{Wi}}{\mathrm{De}}\right)\mathrm{De}\,J_{2k}}{1+\left(2k-1-\tfrac{\mathrm{Wi}}{\mathrm{De}}\right)^2\mathrm{De}^2}\right)_5, \left(\frac{\left(2k-1+\tfrac{\mathrm{Wi}}{\mathrm{De}}\right)\mathrm{De}\,J_{2k}}{1+\left(2k-1+\tfrac{\mathrm{Wi}}{\mathrm{De}}\right)^2\mathrm{De}^2}\right)_6, \\ \left(\frac{-\left(2k-\tfrac{\mathrm{Wi}}{\mathrm{De}}\right)\mathrm{De}\,J_{2k-1}}{1+\left(2k-\tfrac{\mathrm{Wi}}{\mathrm{De}}\right)^2\mathrm{De}^2}\right)_7, \left(\frac{-\left(2k+\tfrac{\mathrm{Wi}}{\mathrm{De}}\right)\mathrm{De}\,J_{2k-1}}{1+\left(2k+\tfrac{\mathrm{Wi}}{\mathrm{De}}\right)^2\mathrm{De}^2}\right)_8 \end{bmatrix} \tag{113}$$

vii. I_7

Substituting Eq. (57) into Eq. (47) gives:

$$I_7 = \int e^{\frac{\tau'}{\mathrm{De}}}\cos 2\tau' \sin\tfrac{\mathrm{Wi}}{\mathrm{De}}\tau' \cos\left(\tfrac{\mathrm{Wi}}{\mathrm{De}}\sin\tau'\right)d\tau' + \int e^{\frac{\tau'}{\mathrm{De}}}\cos 2\tau' \cos\tfrac{\mathrm{Wi}}{\mathrm{De}}\tau' \sin\left(\tfrac{\mathrm{Wi}}{\mathrm{De}}\sin\tau'\right)d\tau' \tag{114}$$

and then Eqs. (58) and (59) into the result, we find:

$$I_7 = J_0 I_{7,1} + 2\sum_{k=1}^{\infty}\left(J_{2k}I_{7,2} + J_{2k-1}I_{7,3}\right) \tag{115}$$

$$I_{7,1} \equiv \int e^{\frac{\tau'}{\mathrm{De}}}\cos 2\tau' \sin\tfrac{\mathrm{Wi}}{\mathrm{De}}\tau'\,d\tau' \tag{116}$$

$$I_{7,2} \equiv \int e^{\frac{\tau'}{\mathrm{De}}}\cos 2\tau' \sin\tfrac{\mathrm{Wi}}{\mathrm{De}}\tau' \cos 2k\tau'\,d\tau' \tag{117}$$

$$I_{7,3} \equiv \int e^{\frac{\tau'}{\mathrm{De}}}\cos 2\tau' \cos\tfrac{\mathrm{Wi}}{\mathrm{De}}\tau' \sin(2k-1)\tau'\,d\tau' \tag{118}$$

Evaluating these, and then substituting into Eq. (115) yields:

$$\begin{aligned} I_7 = \frac{\mathrm{De}\,e^{\frac{\tau}{\mathrm{De}}}}{2} \Bigg[& \frac{\left(2-\tfrac{\mathrm{Wi}}{\mathrm{De}}\right)\mathrm{De}\,J_0\cos\left(2-\tfrac{\mathrm{Wi}}{\mathrm{De}}\right)\tau}{1+\left(2-\tfrac{\mathrm{Wi}}{\mathrm{De}}\right)^2\mathrm{De}^2} - \frac{\left(2+\tfrac{\mathrm{Wi}}{\mathrm{De}}\right)\mathrm{De}\,J_0\cos\left(2+\tfrac{\mathrm{Wi}}{\mathrm{De}}\right)\tau}{1+\left(2+\tfrac{\mathrm{Wi}}{\mathrm{De}}\right)^2\mathrm{De}^2} - \frac{J_0\sin\left(2-\tfrac{\mathrm{Wi}}{\mathrm{De}}\right)\tau}{1+\left(2-\tfrac{\mathrm{Wi}}{\mathrm{De}}\right)^2\mathrm{De}^2} + \frac{J_0\sin\left(2+\tfrac{\mathrm{Wi}}{\mathrm{De}}\right)\tau}{1+\left(2+\tfrac{\mathrm{Wi}}{\mathrm{De}}\right)^2\mathrm{De}^2} \\ & + \sum_{k=1}^{\infty}\Bigg[\frac{\left(2k-2-\tfrac{\mathrm{Wi}}{\mathrm{De}}\right)\mathrm{De}\,J_{2k}\cos\left(2k-2-\tfrac{\mathrm{Wi}}{\mathrm{De}}\right)\tau}{1+\left(2k-2-\tfrac{\mathrm{Wi}}{\mathrm{De}}\right)^2\mathrm{De}^2} - \frac{\left(2k-2+\tfrac{\mathrm{Wi}}{\mathrm{De}}\right)\mathrm{De}\,J_{2k}\cos\left(2k-2+\tfrac{\mathrm{Wi}}{\mathrm{De}}\right)\tau}{1+\left(2k-2+\tfrac{\mathrm{Wi}}{\mathrm{De}}\right)^2\mathrm{De}^2} + \frac{\left(2k+2-\tfrac{\mathrm{Wi}}{\mathrm{De}}\right)\mathrm{De}\,J_{2k}\cos\left(2k+2-\tfrac{\mathrm{Wi}}{\mathrm{De}}\right)\tau}{1+\left(2k+2-\tfrac{\mathrm{Wi}}{\mathrm{De}}\right)^2\mathrm{De}^2} \\ & - \frac{\left(2k+2+\tfrac{\mathrm{Wi}}{\mathrm{De}}\right)\mathrm{De}\,J_{2k}\cos\left(2k+2+\tfrac{\mathrm{Wi}}{\mathrm{De}}\right)\tau}{1+\left(2k+2+\tfrac{\mathrm{Wi}}{\mathrm{De}}\right)^2\mathrm{De}^2} - \frac{J_{2k}\sin\left(2k-2-\tfrac{\mathrm{Wi}}{\mathrm{De}}\right)\tau}{1+\left(2k-2-\tfrac{\mathrm{Wi}}{\mathrm{De}}\right)^2\mathrm{De}^2} + \frac{J_{2k}\sin\left(2k-2+\tfrac{\mathrm{Wi}}{\mathrm{De}}\right)\tau}{1+\left(2k-2+\tfrac{\mathrm{Wi}}{\mathrm{De}}\right)^2\mathrm{De}^2} - \frac{J_{2k}\sin\left(2k+2-\tfrac{\mathrm{Wi}}{\mathrm{De}}\right)\tau}{1+\left(2k+2-\tfrac{\mathrm{Wi}}{\mathrm{De}}\right)^2\mathrm{De}^2} \\ & + \frac{J_{2k}\sin\left(2k+2+\tfrac{\mathrm{Wi}}{\mathrm{De}}\right)\tau}{1+\left(2k+2+\tfrac{\mathrm{Wi}}{\mathrm{De}}\right)^2\mathrm{De}^2} - \frac{\left(2k-3-\tfrac{\mathrm{Wi}}{\mathrm{De}}\right)\mathrm{De}\,J_{2k-1}\cos\left(2k-3-\tfrac{\mathrm{Wi}}{\mathrm{De}}\right)\tau}{1+\left(2k-3-\tfrac{\mathrm{Wi}}{\mathrm{De}}\right)^2\mathrm{De}^2} - \frac{\left(2k-3+\tfrac{\mathrm{Wi}}{\mathrm{De}}\right)\mathrm{De}\,J_{2k-1}\cos\left(2k-3+\tfrac{\mathrm{Wi}}{\mathrm{De}}\right)\tau}{1+\left(2k-3+\tfrac{\mathrm{Wi}}{\mathrm{De}}\right)^2\mathrm{De}^2} \\ & - \frac{\left(2k+1-\tfrac{\mathrm{Wi}}{\mathrm{De}}\right)\mathrm{De}\,J_{2k-1}\cos\left(2k+1-\tfrac{\mathrm{Wi}}{\mathrm{De}}\right)\tau}{1+\left(2k+1-\tfrac{\mathrm{Wi}}{\mathrm{De}}\right)^2\mathrm{De}^2} - \frac{\left(2k+1+\tfrac{\mathrm{Wi}}{\mathrm{De}}\right)\mathrm{De}\,J_{2k-1}\cos\left(2k+1+\tfrac{\mathrm{Wi}}{\mathrm{De}}\right)\tau}{1+\left(2k+1+\tfrac{\mathrm{Wi}}{\mathrm{De}}\right)^2\mathrm{De}^2} + \frac{J_{2k-1}\sin\left(2k-3-\tfrac{\mathrm{Wi}}{\mathrm{De}}\right)\tau}{1+\left(2k-3-\tfrac{\mathrm{Wi}}{\mathrm{De}}\right)^2\mathrm{De}^2} \\ & + \frac{J_{2k-1}\sin\left(2k-3+\tfrac{\mathrm{Wi}}{\mathrm{De}}\right)\tau}{1+\left(2k-3+\tfrac{\mathrm{Wi}}{\mathrm{De}}\right)^2\mathrm{De}^2} + \frac{J_{2k-1}\sin\left(2k+1-\tfrac{\mathrm{Wi}}{\mathrm{De}}\right)\tau}{1+\left(2k+1-\tfrac{\mathrm{Wi}}{\mathrm{De}}\right)^2\mathrm{De}^2} + \frac{J_{2k-1}\sin\left(2k+1+\tfrac{\mathrm{Wi}}{\mathrm{De}}\right)\tau}{1+\left(2k+1+\tfrac{\mathrm{Wi}}{\mathrm{De}}\right)^2\mathrm{De}^2} \Bigg]\Bigg] \\ + \frac{\mathrm{De}}{2}\Bigg[& -\frac{\left(2-\tfrac{\mathrm{Wi}}{\mathrm{De}}\right)J_0}{1+\left(2-\tfrac{\mathrm{Wi}}{\mathrm{De}}\right)^2\mathrm{De}^2} + \frac{\left(2+\tfrac{\mathrm{Wi}}{\mathrm{De}}\right)J_0}{1+\left(2+\tfrac{\mathrm{Wi}}{\mathrm{De}}\right)^2\mathrm{De}^2} \\ & + \sum_{k=1}^{\infty}\Bigg[\frac{-\left(2k-2-\tfrac{\mathrm{Wi}}{\mathrm{De}}\right)J_{2k}}{1+\left(2k-2-\tfrac{\mathrm{Wi}}{\mathrm{De}}\right)^2\mathrm{De}^2} + \frac{\left(2k-2+\tfrac{\mathrm{Wi}}{\mathrm{De}}\right)J_{2k}}{1+\left(2k-2+\tfrac{\mathrm{Wi}}{\mathrm{De}}\right)^2\mathrm{De}^2} - \frac{\left(2k+2-\tfrac{\mathrm{Wi}}{\mathrm{De}}\right)J_{2k}}{1+\left(2k+2-\tfrac{\mathrm{Wi}}{\mathrm{De}}\right)^2\mathrm{De}^2} + \frac{\left(2k+2+\tfrac{\mathrm{Wi}}{\mathrm{De}}\right)J_{2k}}{1+\left(2k+2+\tfrac{\mathrm{Wi}}{\mathrm{De}}\right)^2\mathrm{De}^2} \\ & + \frac{\left(2k-3-\tfrac{\mathrm{Wi}}{\mathrm{De}}\right)J_{2k-1}}{1+\left(2k-3-\tfrac{\mathrm{Wi}}{\mathrm{De}}\right)^2\mathrm{De}^2} + \frac{\left(2k-3+\tfrac{\mathrm{Wi}}{\mathrm{De}}\right)J_{2k-1}}{1+\left(2k-3+\tfrac{\mathrm{Wi}}{\mathrm{De}}\right)^2\mathrm{De}^2} + \frac{\left(2k+1-\tfrac{\mathrm{Wi}}{\mathrm{De}}\right)J_{2k-1}}{1+\left(2k+1-\tfrac{\mathrm{Wi}}{\mathrm{De}}\right)^2\mathrm{De}^2} + \frac{\left(2k+1+\tfrac{\mathrm{Wi}}{\mathrm{De}}\right)J_{2k-1}}{1+\left(2k+1+\tfrac{\mathrm{Wi}}{\mathrm{De}}\right)^2\mathrm{De}^2} \Bigg]\Bigg] \end{aligned} \tag{119}$$

Subtracting the transient part, $I_{7,0} \equiv \lim_{\tau\to 0} I_7$, from Eq. (119) gives:

$$I_7 - I_{7,0} = \frac{\mathrm{De}\, e^{\frac{\tau}{\mathrm{De}}}}{2} \begin{bmatrix} \sum_{k=0}^{\infty} \begin{bmatrix} \phi_7^{(1)} \cos\left(2k+2-\frac{\mathrm{Wi}}{\mathrm{De}}\right)\tau + \phi_7^{(2)} \cos\left(2k+2+\frac{\mathrm{Wi}}{\mathrm{De}}\right)\tau \\ +\psi_7^{(1)} \sin\left(2k+2-\frac{\mathrm{Wi}}{\mathrm{De}}\right)\tau + \psi_7^{(2)} \sin\left(2k+2+\frac{\mathrm{Wi}}{\mathrm{De}}\right)\tau \end{bmatrix} \\ + \sum_{k=1}^{\infty} \begin{bmatrix} \phi_7^{(3)} \cos\left(2k-3-\frac{\mathrm{Wi}}{\mathrm{De}}\right)\tau + \phi_7^{(4)} \cos\left(2k-3+\frac{\mathrm{Wi}}{\mathrm{De}}\right)\tau + \phi_7^{(5)} \cos\left(2k-2-\frac{\mathrm{Wi}}{\mathrm{De}}\right)\tau \\ +\phi_7^{(6)} \cos\left(2k-2+\frac{\mathrm{Wi}}{\mathrm{De}}\right)\tau + \phi_7^{(7)} \cos\left(2k+1-\frac{\mathrm{Wi}}{\mathrm{De}}\right)\tau + \phi_7^{(8)} \cos\left(2k+1+\frac{\mathrm{Wi}}{\mathrm{De}}\right)\tau \\ +\psi_7^{(3)} \sin\left(2k-3-\frac{\mathrm{Wi}}{\mathrm{De}}\right)\tau + \psi_7^{(4)} \sin\left(2k-3+\frac{\mathrm{Wi}}{\mathrm{De}}\right)\tau + \psi_7^{(5)} \sin\left(2k-2-\frac{\mathrm{Wi}}{\mathrm{De}}\right)\tau \\ +\psi_7^{(6)} \sin\left(2k-2+\frac{\mathrm{Wi}}{\mathrm{De}}\right)\tau + \psi_7^{(7)} \sin\left(2k+1-\frac{\mathrm{Wi}}{\mathrm{De}}\right)\tau + \psi_7^{(8)} \sin\left(2k+1+\frac{\mathrm{Wi}}{\mathrm{De}}\right)\tau \end{bmatrix} \end{bmatrix} - \frac{\mathrm{De}}{2}\left[\sum_{k=0}^{\infty}\left(\phi_7^{(1)}+\phi_7^{(2)}\right) + \sum_{k=1}^{\infty}\left(\phi_7^{(3)}+\phi_7^{(4)}+\phi_7^{(5)}+\phi_7^{(6)}+\phi_7^{(7)}+\phi_7^{(8)}\right)\right] \quad (120)$$

where:

$$\phi_7^{(i)} \equiv \begin{bmatrix} \left(\frac{\left(2k+2-\frac{\mathrm{Wi}}{\mathrm{De}}\right)\mathrm{De}\, J_{2k}}{1+\left(2k+2-\frac{\mathrm{Wi}}{\mathrm{De}}\right)^2 \mathrm{De}^2}\right)_{i=1}, \left(\frac{-\left(2k+2+\frac{\mathrm{Wi}}{\mathrm{De}}\right)\mathrm{De}\, J_{2k}}{1+\left(2k+2+\frac{\mathrm{Wi}}{\mathrm{De}}\right)^2 \mathrm{De}^2}\right)_2, \left(\frac{-\left(2k-3-\frac{\mathrm{Wi}}{\mathrm{De}}\right)\mathrm{De}\, J_{2k-1}}{1+\left(2k-3-\frac{\mathrm{Wi}}{\mathrm{De}}\right)^2 \mathrm{De}^2}\right)_3, \\ \left(\frac{-\left(2k-3+\frac{\mathrm{Wi}}{\mathrm{De}}\right)\mathrm{De}\, J_{2k-1}}{1+\left(2k-3+\frac{\mathrm{Wi}}{\mathrm{De}}\right)^2 \mathrm{De}^2}\right)_4, \left(\frac{\left(2k-2-\frac{\mathrm{Wi}}{\mathrm{De}}\right)\mathrm{De}\, J_{2k}}{1+\left(2k-2-\frac{\mathrm{Wi}}{\mathrm{De}}\right)^2 \mathrm{De}^2}\right)_5, \left(\frac{-\left(2k-2+\frac{\mathrm{Wi}}{\mathrm{De}}\right)\mathrm{De}\, J_{2k}}{1+\left(2k-2+\frac{\mathrm{Wi}}{\mathrm{De}}\right)^2 \mathrm{De}^2}\right)_6, \\ \left(\frac{-\left(2k+1-\frac{\mathrm{Wi}}{\mathrm{De}}\right)\mathrm{De}\, J_{2k-1}}{1+\left(2k+1-\frac{\mathrm{Wi}}{\mathrm{De}}\right)^2 \mathrm{De}^2}\right)_7, \left(\frac{-\left(2k+1+\frac{\mathrm{Wi}}{\mathrm{De}}\right)\mathrm{De}\, J_{2k-1}}{1+\left(2k+1+\frac{\mathrm{Wi}}{\mathrm{De}}\right)^2 \mathrm{De}^2}\right)_8 \end{bmatrix} \quad (121)$$

$$\psi_7^{(i)} = \begin{bmatrix} \left(-\frac{J_{2k}}{1+\left(2k+2-\frac{\mathrm{Wi}}{\mathrm{De}}\right)^2 \mathrm{De}^2}\right)_{i=1}, \left(\frac{J_{2k}}{1+\left(2k+2+\frac{\mathrm{Wi}}{\mathrm{De}}\right)^2 \mathrm{De}^2}\right)_2, \left(\frac{J_{2k-1}}{1+\left(2k-3-\frac{\mathrm{Wi}}{\mathrm{De}}\right)^2 \mathrm{De}^2}\right)_3, \\ \left(\frac{J_{2k-1}}{1+\left(2k-3+\frac{\mathrm{Wi}}{\mathrm{De}}\right)^2 \mathrm{De}^2}\right)_4, \left(-\frac{J_{2k}}{1+\left(2k-2-\frac{\mathrm{Wi}}{\mathrm{De}}\right)^2 \mathrm{De}^2}\right)_5, \left(\frac{J_{2k}}{1+\left(2k-2+\frac{\mathrm{Wi}}{\mathrm{De}}\right)^2 \mathrm{De}^2}\right)_6, \\ \left(\frac{J_{2k-1}}{1+\left(2k+1-\frac{\mathrm{Wi}}{\mathrm{De}}\right)^2 \mathrm{De}^2}\right)_7, \left(\frac{J_{2k-1}}{1+\left(2k+1+\frac{\mathrm{Wi}}{\mathrm{De}}\right)^2 \mathrm{De}^2}\right)_8 \end{bmatrix} \quad (122)$$

viii. I_8

Substituting Eq. (56) into Eq. (48) yields:

$$I_8 = \int e^{\frac{\tau'}{\mathrm{De}}} \cos 2\tau' \cos \tfrac{\mathrm{Wi}}{\mathrm{De}}\tau' \cos\left(\tfrac{\mathrm{Wi}}{\mathrm{De}} \sin\tau'\right) d\tau' - \int e^{\frac{\tau'}{\mathrm{De}}} \cos 2\tau' \sin \tfrac{\mathrm{Wi}}{\mathrm{De}}\tau' \sin\left(\tfrac{\mathrm{Wi}}{\mathrm{De}} \sin\tau'\right) d\tau' \quad (123)$$

and then Eqs. (58) and (59) into the result, we find:

$$I_8 = J_0 I_{8,1} + 2\sum_{k=1}^{\infty}\left(J_{2k} I_{8,2} - J_{2k-1} I_{8,3}\right) \quad (124)$$

$$I_{8,1} \equiv \int e^{\frac{\tau'}{\mathrm{De}}} \cos 2\tau' \cos \tfrac{\mathrm{Wi}}{\mathrm{De}}\tau'\, d\tau' \quad (125)$$

$$I_{8,2} \equiv \int e^{\frac{\tau'}{\mathrm{De}}} \cos 2\tau' \cos \tfrac{\mathrm{Wi}}{\mathrm{De}}\tau' \cos 2k\tau'\, d\tau' \quad (126)$$

$$I_{8,3} \equiv \int e^{\frac{\tau'}{\mathrm{De}}} \cos 2\tau' \sin \tfrac{\mathrm{Wi}}{\mathrm{De}} \tau' \sin\left(2k-1\right)\tau' \, d\tau' \tag{127}$$

Evaluating these, and then substituting into Eq. (124) yields:

$$I_8 = \frac{\mathrm{De}\, e^{\frac{\tau}{\mathrm{De}}}}{2} \left[\begin{array}{l} \frac{J_0 \cos\left(2-\frac{\mathrm{Wi}}{\mathrm{De}}\right)\tau}{1+\left(2-\frac{\mathrm{Wi}}{\mathrm{De}}\right)^2 \mathrm{De}^2} + \frac{J_0 \cos\left(2+\frac{\mathrm{Wi}}{\mathrm{De}}\right)\tau}{1+\left(2+\frac{\mathrm{Wi}}{\mathrm{De}}\right)^2 \mathrm{De}^2} + \frac{\left(2-\frac{\mathrm{Wi}}{\mathrm{De}}\right)\mathrm{De}\, J_0 \sin\left(2-\frac{\mathrm{Wi}}{\mathrm{De}}\right)\tau}{1+\left(2-\frac{\mathrm{Wi}}{\mathrm{De}}\right)^2 \mathrm{De}^2} + \frac{\left(2+\frac{\mathrm{Wi}}{\mathrm{De}}\right)\mathrm{De}\, J_0 \sin\left(2+\frac{\mathrm{Wi}}{\mathrm{De}}\right)\tau}{1+\left(2+\frac{\mathrm{Wi}}{\mathrm{De}}\right)^2 \mathrm{De}^2} \\ + \sum_{k=1}^{\infty} \left[\begin{array}{l} -\frac{J_{2k-1} \cos\left(2k-3-\frac{\mathrm{Wi}}{\mathrm{De}}\right)\tau}{1+\left(2k-3-\frac{\mathrm{Wi}}{\mathrm{De}}\right)^2 \mathrm{De}^2} + \frac{J_{2k-1} \cos\left(2k-3+\frac{\mathrm{Wi}}{\mathrm{De}}\right)\tau}{1+\left(2k-3+\frac{\mathrm{Wi}}{\mathrm{De}}\right)^2 \mathrm{De}^2} + \frac{J_{2k} \cos\left(2k-2-\frac{\mathrm{Wi}}{\mathrm{De}}\right)\tau}{1+\left(2k-2-\frac{\mathrm{Wi}}{\mathrm{De}}\right)^2 \mathrm{De}^2} + \frac{J_{2k} \cos\left(2k-2+\frac{\mathrm{Wi}}{\mathrm{De}}\right)\tau}{1+\left(2k-2+\frac{\mathrm{Wi}}{\mathrm{De}}\right)^2 \mathrm{De}^2} \\ -\frac{J_{2k-1} \cos\left(2k+1-\frac{\mathrm{Wi}}{\mathrm{De}}\right)\tau}{1+\left(2k+1-\frac{\mathrm{Wi}}{\mathrm{De}}\right)^2 \mathrm{De}^2} + \frac{J_{2k-1} \cos\left(2k+1+\frac{\mathrm{Wi}}{\mathrm{De}}\right)\tau}{1+\left(2k+1+\frac{\mathrm{Wi}}{\mathrm{De}}\right)^2 \mathrm{De}^2} + \frac{J_{2k} \cos\left(2k+2-\frac{\mathrm{Wi}}{\mathrm{De}}\right)\tau}{1+\left(2k+2-\frac{\mathrm{Wi}}{\mathrm{De}}\right)^2 \mathrm{De}^2} + \frac{J_{2k} \cos\left(2k+2+\frac{\mathrm{Wi}}{\mathrm{De}}\right)\tau}{1+\left(2k+2+\frac{\mathrm{Wi}}{\mathrm{De}}\right)^2 \mathrm{De}^2} \\ -\frac{\left(2k-3-\frac{\mathrm{Wi}}{\mathrm{De}}\right)\mathrm{De}\, J_{2k-1} \sin\left(2k-3-\frac{\mathrm{Wi}}{\mathrm{De}}\right)\tau}{1+\left(2k-3-\frac{\mathrm{Wi}}{\mathrm{De}}\right)^2 \mathrm{De}^2} + \frac{\left(2k-3+\frac{\mathrm{Wi}}{\mathrm{De}}\right)\mathrm{De}\, J_{2k-1} \sin\left(2k-3+\frac{\mathrm{Wi}}{\mathrm{De}}\right)\tau}{1+\left(2k-3+\frac{\mathrm{Wi}}{\mathrm{De}}\right)^2 \mathrm{De}^2} \\ + \frac{\left(2k-2-\frac{\mathrm{Wi}}{\mathrm{De}}\right)\mathrm{De}\, J_{2k} \sin\left(2k-2-\frac{\mathrm{Wi}}{\mathrm{De}}\right)\tau}{1+\left(2k-2-\frac{\mathrm{Wi}}{\mathrm{De}}\right)^2 \mathrm{De}^2} + \frac{\left(2k-2+\frac{\mathrm{Wi}}{\mathrm{De}}\right)\mathrm{De}\, J_{2k} \sin\left(2k-2+\frac{\mathrm{Wi}}{\mathrm{De}}\right)\tau}{1+\left(2k-2+\frac{\mathrm{Wi}}{\mathrm{De}}\right)^2 \mathrm{De}^2} \\ -\frac{\left(2k+1-\frac{\mathrm{Wi}}{\mathrm{De}}\right)\mathrm{De}\, J_{2k-1} \sin\left(2k+1-\frac{\mathrm{Wi}}{\mathrm{De}}\right)\tau}{1+\left(2k+1-\frac{\mathrm{Wi}}{\mathrm{De}}\right)^2 \mathrm{De}^2} + \frac{\left(2k+1+\frac{\mathrm{Wi}}{\mathrm{De}}\right)\mathrm{De}\, J_{2k-1} \sin\left(2k+1+\frac{\mathrm{Wi}}{\mathrm{De}}\right)\tau}{1+\left(2k+1+\frac{\mathrm{Wi}}{\mathrm{De}}\right)^2 \mathrm{De}^2} \\ + \frac{\left(2k+2-\frac{\mathrm{Wi}}{\mathrm{De}}\right)\mathrm{De}\, J_{2k} \sin\left(2k+2-\frac{\mathrm{Wi}}{\mathrm{De}}\right)\tau}{1+\left(2k+2-\frac{\mathrm{Wi}}{\mathrm{De}}\right)^2 \mathrm{De}^2} + \frac{\left(2k+2+\frac{\mathrm{Wi}}{\mathrm{De}}\right)\mathrm{De}\, J_{2k} \sin\left(2k+2+\frac{\mathrm{Wi}}{\mathrm{De}}\right)\tau}{1+\left(2k+2+\frac{\mathrm{Wi}}{\mathrm{De}}\right)^2 \mathrm{De}^2} \end{array} \right] \end{array} \right] + \frac{\mathrm{De}}{2} \left[\begin{array}{l} -\frac{J_0}{1+\left(2-\frac{\mathrm{Wi}}{\mathrm{De}}\right)^2 \mathrm{De}^2} - \frac{J_0}{1+\left(2+\frac{\mathrm{Wi}}{\mathrm{De}}\right)^2 \mathrm{De}^2} \\ + \sum_{k=1}^{\infty} \left[\begin{array}{l} -\frac{J_{2k}}{1+\left(2k-2-\frac{\mathrm{Wi}}{\mathrm{De}}\right)^2 \mathrm{De}^2} - \frac{J_{2k}}{1+\left(2k-2+\frac{\mathrm{Wi}}{\mathrm{De}}\right)^2 \mathrm{De}^2} - \frac{J_{2k}}{1+\left(2k+2-\frac{\mathrm{Wi}}{\mathrm{De}}\right)^2 \mathrm{De}^2} - \frac{J_{2k}}{1+\left(2k+2+\frac{\mathrm{Wi}}{\mathrm{De}}\right)^2 \mathrm{De}^2} \\ + \frac{J_{2k-1}}{1+\left(2k-3-\frac{\mathrm{Wi}}{\mathrm{De}}\right)^2 \mathrm{De}^2} - \frac{J_{2k-1}}{1+\left(2k-3+\frac{\mathrm{Wi}}{\mathrm{De}}\right)^2 \mathrm{De}^2} + \frac{J_{2k-1}}{1+\left(2k+1-\frac{\mathrm{Wi}}{\mathrm{De}}\right)^2 \mathrm{De}^2} - \frac{J_{2k-1}}{1+\left(2k+1+\frac{\mathrm{Wi}}{\mathrm{De}}\right)^2 \mathrm{De}^2} \end{array} \right] \end{array} \right] \tag{128}$$

Subtracting the transient part, $I_{8,0} \equiv \lim_{\tau\to 0} I_8$, from Eq. (128) gives:

$$I_8 - I_{8,0} = \frac{\mathrm{De}\, e^{\frac{\tau}{\mathrm{De}}}}{2} \left[\begin{array}{l} \sum_{k=0}^{\infty} \left[\begin{array}{l} \phi_8^{(1)} \cos\left(2k+2-\frac{\mathrm{Wi}}{\mathrm{De}}\right)\tau + \phi_8^{(2)} \cos\left(2k+2+\frac{\mathrm{Wi}}{\mathrm{De}}\right)\tau \\ +\psi_8^{(1)} \sin\left(2k+2-\frac{\mathrm{Wi}}{\mathrm{De}}\right)\tau + \psi_8^{(2)} \sin\left(2k+2+\frac{\mathrm{Wi}}{\mathrm{De}}\right)\tau \end{array} \right] \\ + \sum_{k=1}^{\infty} \left[\begin{array}{l} \phi_8^{(3)} \cos\left(2k-3-\frac{\mathrm{Wi}}{\mathrm{De}}\right)\tau + \phi_8^{(4)} \cos\left(2k-3+\frac{\mathrm{Wi}}{\mathrm{De}}\right)\tau + \phi_8^{(5)} \cos\left(2k-2-\frac{\mathrm{Wi}}{\mathrm{De}}\right)\tau \\ +\phi_8^{(6)} \cos\left(2k-2+\frac{\mathrm{Wi}}{\mathrm{De}}\right)\tau + \phi_8^{(7)} \cos\left(2k+1-\frac{\mathrm{Wi}}{\mathrm{De}}\right)\tau + \phi_8^{(8)} \cos\left(2k+1+\frac{\mathrm{Wi}}{\mathrm{De}}\right)\tau \\ +\psi_8^{(3)} \sin\left(2k-3-\frac{\mathrm{Wi}}{\mathrm{De}}\right)\tau + \psi_8^{(4)} \sin\left(2k-3+\frac{\mathrm{Wi}}{\mathrm{De}}\right)\tau + \psi_8^{(5)} \sin\left(2k-2-\frac{\mathrm{Wi}}{\mathrm{De}}\right)\tau \\ +\psi_8^{(6)} \sin\left(2k-2+\frac{\mathrm{Wi}}{\mathrm{De}}\right)\tau + \psi_8^{(7)} \sin\left(2k+1-\frac{\mathrm{Wi}}{\mathrm{De}}\right)\tau + \psi_8^{(8)} \sin\left(2k+1+\frac{\mathrm{Wi}}{\mathrm{De}}\right)\tau \end{array} \right] \end{array} \right] - \frac{\mathrm{De}}{2} \left[\sum_{k=0}^{\infty} \left(\phi_8^{(1)} + \phi_8^{(2)}\right) + \sum_{k=1}^{\infty} \left(\phi_8^{(3)} + \phi_8^{(4)} + \phi_8^{(5)} + \phi_8^{(6)} + \phi_8^{(7)} + \phi_8^{(8)}\right) \right] \tag{129}$$

where:

$$\phi_8^{(i)} \equiv \left[\begin{array}{l} \left(\frac{J_{2k}}{1+\left(2k+2-\frac{\mathrm{Wi}}{\mathrm{De}}\right)^2 \mathrm{De}^2} \right)_{i=1}, \left(\frac{J_{2k}}{1+\left(2k+2+\frac{\mathrm{Wi}}{\mathrm{De}}\right)^2 \mathrm{De}^2} \right)_2, \left(\frac{-J_{2k-1}}{1+\left(2k-3-\frac{\mathrm{Wi}}{\mathrm{De}}\right)^2 \mathrm{De}^2} \right)_3, \\ \left(\frac{J_{2k-1}}{1+\left(2k-3+\frac{\mathrm{Wi}}{\mathrm{De}}\right)^2 \mathrm{De}^2} \right)_4, \left(\frac{J_{2k}}{1+\left(2k-2-\frac{\mathrm{Wi}}{\mathrm{De}}\right)^2 \mathrm{De}^2} \right)_5, \left(\frac{J_{2k}}{1+\left(2k-2+\frac{\mathrm{Wi}}{\mathrm{De}}\right)^2 \mathrm{De}^2} \right)_6, \\ \left(\frac{-J_{2k-1}}{1+\left(2k+1-\frac{\mathrm{Wi}}{\mathrm{De}}\right)^2 \mathrm{De}^2} \right)_7, \left(\frac{J_{2k-1}}{1+\left(2k+1+\frac{\mathrm{Wi}}{\mathrm{De}}\right)^2 \mathrm{De}^2} \right)_8 \end{array} \right] \tag{130}$$

$$\psi_8^{(i)} \equiv \begin{bmatrix} \left(\frac{\left(2k+2-\frac{\mathrm{Wi}}{\mathrm{De}}\right)\mathrm{De}\,J_{2k}}{1+\left(2k+2-\frac{\mathrm{Wi}}{\mathrm{De}}\right)^2\mathrm{De}^2}\right)_{i=1}, \left(\frac{\left(2k+2+\frac{\mathrm{Wi}}{\mathrm{De}}\right)\mathrm{De}\,J_{2k}}{1+\left(2k+2+\frac{\mathrm{Wi}}{\mathrm{De}}\right)^2\mathrm{De}^2}\right)_2, \left(\frac{-\left(2k-3-\frac{\mathrm{Wi}}{\mathrm{De}}\right)\mathrm{De}\,J_{2k-1}}{1+\left(2k-3-\frac{\mathrm{Wi}}{\mathrm{De}}\right)^2\mathrm{De}^2}\right)_3, \\ \left(\frac{\left(2k-3+\frac{\mathrm{Wi}}{\mathrm{De}}\right)\mathrm{De}\,J_{2k-1}}{1+\left(2k-3+\frac{\mathrm{Wi}}{\mathrm{De}}\right)^2\mathrm{De}^2}\right)_4, \left(\frac{\left(2k-2-\frac{\mathrm{Wi}}{\mathrm{De}}\right)\mathrm{De}\,J_{2k}}{1+\left(2k-2-\frac{\mathrm{Wi}}{\mathrm{De}}\right)^2\mathrm{De}^2}\right)_5, \left(\frac{\left(2k-2+\frac{\mathrm{Wi}}{\mathrm{De}}\right)\mathrm{De}\,J_{2k}}{1+\left(2k-2+\frac{\mathrm{Wi}}{\mathrm{De}}\right)^2\mathrm{De}^2}\right)_6, \\ \left(\frac{-\left(2k+1-\frac{\mathrm{Wi}}{\mathrm{De}}\right)\mathrm{De}\,J_{2k-1}}{1+\left(2k+1-\frac{\mathrm{Wi}}{\mathrm{De}}\right)^2\mathrm{De}^2}\right)_7, \left(\frac{\left(2k+1+\frac{\mathrm{Wi}}{\mathrm{De}}\right)\mathrm{De}\,J_{2k-1}}{1+\left(2k+1+\frac{\mathrm{Wi}}{\mathrm{De}}\right)^2\mathrm{De}^2}\right)_8 \end{bmatrix} \tag{131}$$

c Fourier Series for Shear Stress

This subsection details how to rewrite our exact solution for shear stress from the corotational Jeffreys fluid into the Fourier series. Subsection I.a.i concerns the alternant part of our exact solution, whereas Subsection I.a.ii concerns the transient part. The complete solution, arrived at by adding the transient part to the alternance, explains the *ud*LAOS shear stress response before and after the alternant state has reach.

i. Alternant Part

Substituting $I_1 - I_8$, given in Eqs. (66), (75), (84), (93), (102), (111), (120) and (129), into Eq. (53), and then rearranging, we find:

$$\breve{\tau}_{yx} = \breve{\tau}_{yx,1}\cos\tfrac{\mathrm{Wi}}{\mathrm{De}}\left(\tau+\sin\tau\right)+\breve{\tau}_{yx,2}\sin\tfrac{\mathrm{Wi}}{\mathrm{De}}\left(\tau+\sin\tau\right) \tag{132}$$

where:

$$\begin{aligned}\breve{\tau}_{yx,1} \equiv \sum_{k=0}^{\infty} &\begin{bmatrix} C_1^{(1)}\cos\left(2k+\frac{\mathrm{Wi}}{\mathrm{De}}\right)\tau + C_1^{(2)}\cos\left(2k+1-\frac{\mathrm{Wi}}{\mathrm{De}}\right)\tau + C_1^{(3)}\cos\left(2k+1+\frac{\mathrm{Wi}}{\mathrm{De}}\right)\tau \\ +C_1^{(4)}\cos\left(2k+2-\frac{\mathrm{Wi}}{\mathrm{De}}\right)\tau + C_1^{(5)}\cos\left(2k+2+\frac{\mathrm{Wi}}{\mathrm{De}}\right)\tau \\ +S_1^{(1)}\sin\left(2k+\frac{\mathrm{Wi}}{\mathrm{De}}\right)\tau + S_1^{(2)}\sin\left(2k+1-\frac{\mathrm{Wi}}{\mathrm{De}}\right)\tau + S_1^{(3)}\sin\left(2k+1+\frac{\mathrm{Wi}}{\mathrm{De}}\right)\tau \\ +S_1^{(4)}\sin\left(2k+2-\frac{\mathrm{Wi}}{\mathrm{De}}\right)\tau + S_1^{(5)}\sin\left(2k+2+\frac{\mathrm{Wi}}{\mathrm{De}}\right)\tau \end{bmatrix} \\ +\sum_{k=1}^{\infty} &\begin{bmatrix} +C_1^{(6)}\cos\left(2k-3-\frac{\mathrm{Wi}}{\mathrm{De}}\right)\tau + C_1^{(7)}\cos\left(2k-3+\frac{\mathrm{Wi}}{\mathrm{De}}\right)\tau + C_1^{(8)}\cos\left(2k-2-\frac{\mathrm{Wi}}{\mathrm{De}}\right)\tau \\ +C_1^{(9)}\cos\left(2k-2+\frac{\mathrm{Wi}}{\mathrm{De}}\right)\tau + C_1^{(10)}\cos\left(2k-1-\frac{\mathrm{Wi}}{\mathrm{De}}\right)\tau + C_1^{(11)}\cos\left(2k-1+\frac{\mathrm{Wi}}{\mathrm{De}}\right)\tau \\ +C_1^{(12)}\cos\left(2k-\frac{\mathrm{Wi}}{\mathrm{De}}\right)\tau + C_1^{(13)}\cos\left(2k+\frac{\mathrm{Wi}}{\mathrm{De}}\right)\tau + C_1^{(14)}\cos\left(2k+1-\frac{\mathrm{Wi}}{\mathrm{De}}\right)\tau + C_1^{(15)}\cos\left(2k+1+\frac{\mathrm{Wi}}{\mathrm{De}}\right)\tau \\ +S_1^{(6)}\sin\left(2k-3-\frac{\mathrm{Wi}}{\mathrm{De}}\right)\tau + S_1^{(7)}\sin\left(2k-3+\frac{\mathrm{Wi}}{\mathrm{De}}\right)\tau + S_1^{(8)}\sin\left(2k-2-\frac{\mathrm{Wi}}{\mathrm{De}}\right)\tau \\ +S_1^{(9)}\sin\left(2k-2+\frac{\mathrm{Wi}}{\mathrm{De}}\right)\tau + S_1^{(10)}\sin\left(2k-1-\frac{\mathrm{Wi}}{\mathrm{De}}\right)\tau + S_1^{(11)}\sin\left(2k-1+\frac{\mathrm{Wi}}{\mathrm{De}}\right)\tau \\ +S_1^{(12)}\sin\left(2k-\frac{\mathrm{Wi}}{\mathrm{De}}\right)\tau + S_1^{(13)}\sin\left(2k+\frac{\mathrm{Wi}}{\mathrm{De}}\right)\tau + S_1^{(14)}\sin\left(2k+1-\frac{\mathrm{Wi}}{\mathrm{De}}\right)\tau + S_1^{(15)}\sin\left(2k+1+\frac{\mathrm{Wi}}{\mathrm{De}}\right)\tau \end{bmatrix}\end{aligned} \tag{133}$$

$$\breve{\tau}_{yx,2} \equiv \sum_{k=0}^{\infty} \left[\begin{array}{l} C_2^{(1)} \cos\left(2k + \frac{\mathrm{Wi}}{\mathrm{De}}\right)\tau + C_2^{(2)} \cos\left(2k+1-\frac{\mathrm{Wi}}{\mathrm{De}}\right)\tau + C_2^{(3)} \cos\left(2k+1+\frac{\mathrm{Wi}}{\mathrm{De}}\right)\tau \\ +C_2^{(4)} \cos\left(2k+2-\frac{\mathrm{Wi}}{\mathrm{De}}\right)\tau + C_2^{(5)} \cos\left(2k+2+\frac{\mathrm{Wi}}{\mathrm{De}}\right)\tau \\ +S_2^{(1)} \sin\left(2k + \frac{\mathrm{Wi}}{\mathrm{De}}\right)\tau + S_2^{(2)} \sin\left(2k+1-\frac{\mathrm{Wi}}{\mathrm{De}}\right)\tau + S_2^{(3)} \sin\left(2k+1+\frac{\mathrm{Wi}}{\mathrm{De}}\right)\tau \\ +S_2^{(4)} \sin\left(2k+2-\frac{\mathrm{Wi}}{\mathrm{De}}\right)\tau + S_2^{(5)} \sin\left(2k+2+\frac{\mathrm{Wi}}{\mathrm{De}}\right)\tau \end{array} \right]$$

$$+\sum_{k=1}^{\infty} \left[\begin{array}{l} +C_2^{(6)} \cos\left(2k-3-\frac{\mathrm{Wi}}{\mathrm{De}}\right)\tau + C_2^{(7)} \cos\left(2k-3+\frac{\mathrm{Wi}}{\mathrm{De}}\right)\tau + C_2^{(8)} \cos\left(2k-2-\frac{\mathrm{Wi}}{\mathrm{De}}\right)\tau \\ +C_2^{(9)} \cos\left(2k-2+\frac{\mathrm{Wi}}{\mathrm{De}}\right)\tau + C_2^{(10)} \cos\left(2k-1-\frac{\mathrm{Wi}}{\mathrm{De}}\right)\tau + C_2^{(11)} \cos\left(2k-1+\frac{\mathrm{Wi}}{\mathrm{De}}\right)\tau \\ +C_2^{(12)} \cos\left(2k-\frac{\mathrm{Wi}}{\mathrm{De}}\right)\tau + C_2^{(13)} \cos\left(2k+\frac{\mathrm{Wi}}{\mathrm{De}}\right)\tau + C_2^{(14)} \cos\left(2k+1-\frac{\mathrm{Wi}}{\mathrm{De}}\right)\tau + C_2^{(15)} \cos\left(2k+1+\frac{\mathrm{Wi}}{\mathrm{De}}\right)\tau \\ +S_2^{(6)} \sin\left(2k-3-\frac{\mathrm{Wi}}{\mathrm{De}}\right)\tau + S_2^{(7)} \sin\left(2k-3+\frac{\mathrm{Wi}}{\mathrm{De}}\right)\tau + S_2^{(8)} \sin\left(2k-2-\frac{\mathrm{Wi}}{\mathrm{De}}\right)\tau \\ +S_2^{(9)} \sin\left(2k-2+\frac{\mathrm{Wi}}{\mathrm{De}}\right)\tau + S_2^{(10)} \sin\left(2k-1-\frac{\mathrm{Wi}}{\mathrm{De}}\right)\tau + S_2^{(11)} \sin\left(2k-1+\frac{\mathrm{Wi}}{\mathrm{De}}\right)\tau \\ +S_2^{(12)} \sin\left(2k-\frac{\mathrm{Wi}}{\mathrm{De}}\right)\tau + S_2^{(13)} \sin\left(2k+\frac{\mathrm{Wi}}{\mathrm{De}}\right)\tau + S_2^{(14)} \sin\left(2k+1-\frac{\mathrm{Wi}}{\mathrm{De}}\right)\tau + S_2^{(15)} \sin\left(2k+1+\frac{\mathrm{Wi}}{\mathrm{De}}\right)\tau \end{array} \right] \quad (134)$$

where:

$$C_1^{(i)} \equiv \left[\begin{array}{l} \left(-\mathrm{Wi}\phi_1^{(1)} + \frac{3}{2}\breve{\lambda}_2 \mathrm{Wi}^2 \phi_2^{(1)}\right)_{i=1}, \left(-\frac{1}{2}\mathrm{Wi}\phi_3^{(1)} + \breve{\lambda}_2 \mathrm{Wi}^2 \phi_4^{(1)} + \frac{1}{2}\breve{\lambda}_2 \mathrm{De}\,\mathrm{Wi}\phi_5^{(1)}\right)_2 \\ \left(-\frac{1}{2}\mathrm{Wi}\phi_3^{(2)} + \breve{\lambda}_2 \mathrm{Wi}^2 \phi_4^{(2)} + \frac{1}{2}\breve{\lambda}_2 \mathrm{De}\,\mathrm{Wi}\phi_5^{(2)}\right)_3, \left(\frac{1}{4}\breve{\lambda}_2 \mathrm{Wi}^2 \phi_7^{(1)}\right)_4, \left(\frac{1}{4}\breve{\lambda}_2 \mathrm{Wi}^2 \phi_7^{(2)}\right)_5, \\ \left(\frac{1}{4}\breve{\lambda}_2 \mathrm{Wi}^2 \phi_7^{(3)}\right)_6, \left(\frac{1}{4}\breve{\lambda}_2 \mathrm{Wi}^2 \phi_7^{(4)}\right)_7, \left(-\frac{1}{2}\mathrm{Wi}\phi_3^{(3)} + \breve{\lambda}_2 \mathrm{Wi}^2 \phi_4^{(3)} + \frac{1}{2}\breve{\lambda}_2 \mathrm{De}\,\mathrm{Wi}\phi_5^{(3)} + \frac{1}{4}\breve{\lambda}_2 \mathrm{Wi}^2 \phi_7^{(5)}\right)_8 \\ \left(-\frac{1}{2}\mathrm{Wi}\phi_3^{(4)} + \breve{\lambda}_2 \mathrm{Wi}^2 \phi_4^{(4)} + \frac{1}{2}\breve{\lambda}_2 \mathrm{De}\,\mathrm{Wi}\phi_5^{(4)} + \frac{1}{4}\breve{\lambda}_2 \mathrm{Wi}^2 \phi_7^{(6)}\right)_9 \\ \left(-\mathrm{Wi}\phi_1^{(2)} + \frac{3}{2}\breve{\lambda}_2 \mathrm{Wi}^2 \phi_2^{(2)} - \frac{1}{2}\mathrm{Wi}\phi_3^{(5)} + \breve{\lambda}_2 \mathrm{Wi}^2 \phi_4^{(5)} + \frac{1}{2}\breve{\lambda}_2 \mathrm{De}\,\mathrm{Wi}\phi_5^{(5)}\right)_{10} \\ \left(-\mathrm{Wi}\phi_1^{(3)} + \frac{3}{2}\breve{\lambda}_2 \mathrm{Wi}^2 \phi_2^{(3)} - \frac{1}{2}\mathrm{Wi}\phi_3^{(6)} + \breve{\lambda}_2 \mathrm{Wi}^2 \phi_4^{(6)} + \frac{1}{2}\breve{\lambda}_2 \mathrm{De}\,\mathrm{Wi}\phi_5^{(6)}\right)_{11} \\ \left(-\mathrm{Wi}\phi_1^{(4)} + \frac{3}{2}\breve{\lambda}_2 \mathrm{Wi}^2 \phi_2^{(4)} - \frac{1}{2}\mathrm{Wi}\phi_3^{(7)} + \breve{\lambda}_2 \mathrm{Wi}^2 \phi_4^{(7)} + \frac{1}{2}\breve{\lambda}_2 \mathrm{De}\,\mathrm{Wi}\phi_5^{(7)}\right)_{12} \\ \left(-\frac{1}{2}\mathrm{Wi}\phi_3^{(8)} + \breve{\lambda}_2 \mathrm{Wi}^2 \phi_4^{(8)} + \frac{1}{2}\breve{\lambda}_2 \mathrm{De}\,\mathrm{Wi}\phi_5^{(8)}\right)_{13}, \left(\frac{1}{4}\breve{\lambda}_2 \mathrm{Wi}^2 \phi_7^{(7)}\right)_{14}, \left(\frac{1}{4}\breve{\lambda}_2 \mathrm{Wi}^2 \phi_7^{(8)}\right)_{15} \end{array} \right] \quad (135)$$

$$S_1^{(i)} \equiv \left[\begin{array}{l} \left(-\mathrm{Wi}\psi_1^{(1)} + \frac{3}{2}\breve{\lambda}_2 \mathrm{Wi}^2 \psi_2^{(1)}\right)_{i=1}, \left(-\frac{1}{2}\mathrm{Wi}\psi_3^{(1)} + \breve{\lambda}_2 \mathrm{Wi}^2 \psi_4^{(1)} + \frac{1}{2}\breve{\lambda}_2 \mathrm{De}\,\mathrm{Wi}\psi_5^{(1)}\right)_2, \\ \left(-\frac{1}{2}\mathrm{Wi}\psi_3^{(2)} + \breve{\lambda}_2 \mathrm{Wi}^2 \psi_4^{(2)} + \frac{1}{2}\breve{\lambda}_2 \mathrm{De}\,\mathrm{Wi}\psi_5^{(2)}\right)_3, \left(\frac{1}{4}\breve{\lambda}_2 \mathrm{Wi}^2 \psi_7^{(1)}\right)_4, \left(\frac{1}{4}\breve{\lambda}_2 \mathrm{Wi}^2 \psi_7^{(2)}\right)_5, \\ \left(\frac{1}{4}\breve{\lambda}_2 \mathrm{Wi}^2 \psi_7^{(3)}\right)_6, \left(\frac{1}{4}\breve{\lambda}_2 \mathrm{Wi}^2 \psi_7^{(4)}\right)_7, \left(-\frac{1}{2}\mathrm{Wi}\psi_3^{(3)} + \breve{\lambda}_2 \mathrm{Wi}^2 \psi_4^{(3)} + \frac{1}{2}\breve{\lambda}_2 \mathrm{De}\,\mathrm{Wi}\psi_5^{(3)} + \frac{1}{4}\breve{\lambda}_2 \mathrm{Wi}^2 \psi_7^{(5)}\right)_8, \\ \left(-\frac{1}{2}\mathrm{Wi}\psi_3^{(4)} + \breve{\lambda}_2 \mathrm{Wi}^2 \psi_4^{(4)} + \frac{1}{2}\breve{\lambda}_2 \mathrm{De}\,\mathrm{Wi}\psi_5^{(4)} + \frac{1}{4}\breve{\lambda}_2 \mathrm{Wi}^2 \psi_7^{(6)}\right)_9, \\ \left(-\mathrm{Wi}\psi_1^{(2)} + \frac{3}{2}\breve{\lambda}_2 \mathrm{Wi}^2 \psi_2^{(2)} - \frac{1}{2}\mathrm{Wi}\psi_3^{(5)} + \breve{\lambda}_2 \mathrm{Wi}^2 \psi_4^{(5)} + \frac{1}{2}\breve{\lambda}_2 \mathrm{De}\,\mathrm{Wi}\psi_5^{(5)}\right)_{10}, \\ \left(-\mathrm{Wi}\psi_1^{(3)} + \frac{3}{2}\breve{\lambda}_2 \mathrm{Wi}^2 \psi_2^{(3)} - \frac{1}{2}\mathrm{Wi}\psi_3^{(6)} + \breve{\lambda}_2 \mathrm{Wi}^2 \psi_4^{(6)} + \frac{1}{2}\breve{\lambda}_2 \mathrm{De}\,\mathrm{Wi}\psi_5^{(6)}\right)_{11}, \\ \left(-\mathrm{Wi}\psi_1^{(4)} + \frac{3}{2}\breve{\lambda}_2 \mathrm{Wi}^2 \psi_2^{(4)} - \frac{1}{2}\mathrm{Wi}\psi_3^{(7)} + \breve{\lambda}_2 \mathrm{Wi}^2 \psi_4^{(7)} + \frac{1}{2}\breve{\lambda}_2 \mathrm{De}\,\mathrm{Wi}\psi_5^{(7)}\right)_{12}, \\ \left(-\frac{1}{2}\mathrm{Wi}\psi_3^{(8)} + \breve{\lambda}_2 \mathrm{Wi}^2 \psi_4^{(8)} + \frac{1}{2}\breve{\lambda}_2 \mathrm{De}\,\mathrm{Wi}\psi_5^{(8)}\right)_{13}, \left(\frac{1}{4}\breve{\lambda}_2 \mathrm{Wi}^2 \psi_7^{(7)}\right)_{14}, \left(\frac{1}{4}\breve{\lambda}_2 \mathrm{Wi}^2 \psi_7^{(8)}\right)_{15} \end{array} \right] \quad (136)$$

$$C_2^{(i)} \equiv \begin{bmatrix} \left(-\frac{3}{2}\breve{\lambda}_2\,\mathrm{Wi}^2\phi_1^{(1)} - \mathrm{Wi}\phi_2^{(1)}\right)_{i=1}, \left(-\breve{\lambda}_2\,\mathrm{Wi}^2\phi_3^{(1)} - \frac{1}{2}\mathrm{Wi}\phi_4^{(1)} + \frac{1}{2}\breve{\lambda}_2\,\mathrm{De}\,\mathrm{Wi}\phi_6^{(1)}\right)_2, \\ \left(-\breve{\lambda}_2\,\mathrm{Wi}^2\phi_3^{(2)} - \frac{1}{2}\mathrm{Wi}\phi_4^{(2)} + \frac{1}{2}\breve{\lambda}_2\,\mathrm{De}\,\mathrm{Wi}\phi_6^{(2)}\right)_3, \left(-\frac{1}{4}\breve{\lambda}_2\,\mathrm{Wi}^2\phi_8^{(1)}\right)_4, \left(-\frac{1}{4}\breve{\lambda}_2\,\mathrm{Wi}^2\phi_8^{(2)}\right)_5, \\ \left(-\frac{1}{4}\breve{\lambda}_2\,\mathrm{Wi}^2\phi_8^{(3)}\right)_6, \left(-\frac{1}{4}\breve{\lambda}_2\,\mathrm{Wi}^2\phi_8^{(4)}\right)_7, \left(-\breve{\lambda}_2\,\mathrm{Wi}^2\phi_3^{(3)} - \frac{1}{2}\mathrm{Wi}\phi_4^{(3)} + \frac{1}{2}\breve{\lambda}_2\,\mathrm{De}\,\mathrm{Wi}\phi_6^{(3)} - \frac{1}{4}\breve{\lambda}_2\,\mathrm{Wi}^2\phi_8^{(5)}\right)_8, \\ \left(-\breve{\lambda}_2\,\mathrm{Wi}^2\phi_3^{(4)} - \frac{1}{2}\mathrm{Wi}\phi_4^{(4)} + \frac{1}{2}\breve{\lambda}_2\,\mathrm{De}\,\mathrm{Wi}\phi_6^{(4)} - \frac{1}{4}\breve{\lambda}_2\,\mathrm{Wi}^2\phi_8^{(6)}\right)_9, \\ \left(-\frac{3}{2}\breve{\lambda}_2\,\mathrm{Wi}^2\phi_1^{(2)} - \mathrm{Wi}\phi_2^{(2)} - \breve{\lambda}_2\,\mathrm{Wi}^2\phi_3^{(5)} - \frac{1}{2}\mathrm{Wi}\phi_4^{(5)} + \frac{1}{2}\breve{\lambda}_2\,\mathrm{De}\,\mathrm{Wi}\phi_6^{(5)}\right)_{10}, \\ \left(-\frac{3}{2}\breve{\lambda}_2\,\mathrm{Wi}^2\phi_1^{(3)} - \mathrm{Wi}\phi_2^{(3)} - \breve{\lambda}_2\,\mathrm{Wi}^2\phi_3^{(6)} - \frac{1}{2}\mathrm{Wi}\phi_4^{(6)} + \frac{1}{2}\breve{\lambda}_2\,\mathrm{De}\,\mathrm{Wi}\phi_6^{(6)}\right)_{11}, \\ \left(-\frac{3}{2}\breve{\lambda}_2\,\mathrm{Wi}^2\phi_1^{(4)} - \mathrm{Wi}\phi_2^{(4)} - \breve{\lambda}_2\,\mathrm{Wi}^2\phi_3^{(7)} - \frac{1}{2}\mathrm{Wi}\phi_4^{(7)} + \frac{1}{2}\breve{\lambda}_2\,\mathrm{De}\,\mathrm{Wi}\phi_6^{(7)}\right)_{12}, \\ \left(-\breve{\lambda}_2\,\mathrm{Wi}^2\phi_3^{(8)} - \frac{1}{2}\mathrm{Wi}\phi_4^{(8)} + \frac{1}{2}\breve{\lambda}_2\,\mathrm{De}\,\mathrm{Wi}\phi_6^{(8)}\right)_{13}, \left(-\frac{1}{4}\breve{\lambda}_2\,\mathrm{Wi}^2\phi_8^{(7)}\right)_{14}, \left(-\frac{1}{4}\breve{\lambda}_2\,\mathrm{Wi}^2\phi_8^{(8)}\right)_{15} \end{bmatrix} \tag{137}$$

$$S_2^{(i)} \equiv \begin{bmatrix} \left(-\frac{3}{2}\breve{\lambda}_2\,\mathrm{Wi}^2\psi_1^{(1)} - \mathrm{Wi}\psi_2^{(1)}\right)_{i=1}, \left(-\breve{\lambda}_2\,\mathrm{Wi}^2\psi_3^{(1)} - \frac{1}{2}\mathrm{Wi}\psi_4^{(1)} + \frac{1}{2}\breve{\lambda}_2\,\mathrm{De}\,\mathrm{Wi}\psi_6^{(1)}\right)_2, \\ \left(-\breve{\lambda}_2\,\mathrm{Wi}^2\psi_3^{(2)} - \frac{1}{2}\mathrm{Wi}\psi_4^{(2)} + \frac{1}{2}\breve{\lambda}_2\,\mathrm{De}\,\mathrm{Wi}\psi_6^{(2)}\right)_3, \left(-\frac{1}{4}\breve{\lambda}_2\,\mathrm{Wi}^2\psi_8^{(1)}\right)_4, \left(-\frac{1}{4}\breve{\lambda}_2\,\mathrm{Wi}^2\psi_8^{(2)}\right)_5, \\ \left(-\frac{1}{4}\breve{\lambda}_2\,\mathrm{Wi}^2\psi_8^{(3)}\right)_6, \left(-\frac{1}{4}\breve{\lambda}_2\,\mathrm{Wi}^2\psi_8^{(4)}\right)_7, \left(-\breve{\lambda}_2\,\mathrm{Wi}^2\psi_3^{(3)} - \frac{1}{2}\mathrm{Wi}\psi_4^{(3)} + \frac{1}{2}\breve{\lambda}_2\,\mathrm{De}\,\mathrm{Wi}\psi_6^{(3)} - \frac{1}{4}\breve{\lambda}_2\,\mathrm{Wi}^2\psi_8^{(5)}\right)_8, \\ \left(-\breve{\lambda}_2\,\mathrm{Wi}^2\psi_3^{(4)} - \frac{1}{2}\mathrm{Wi}\psi_4^{(4)} + \frac{1}{2}\breve{\lambda}_2\,\mathrm{De}\,\mathrm{Wi}\psi_6^{(4)} - \frac{1}{4}\breve{\lambda}_2\,\mathrm{Wi}^2\psi_8^{(6)}\right)_9, \\ \left(-\frac{3}{2}\breve{\lambda}_2\,\mathrm{Wi}^2\psi_1^{(2)} - \mathrm{Wi}\psi_2^{(2)} - \breve{\lambda}_2\,\mathrm{Wi}^2\psi_3^{(5)} - \frac{1}{2}\mathrm{Wi}\psi_4^{(5)} + \frac{1}{2}\breve{\lambda}_2\,\mathrm{De}\,\mathrm{Wi}\psi_6^{(5)}\right)_{10}, \\ \left(-\frac{3}{2}\breve{\lambda}_2\,\mathrm{Wi}^2\psi_1^{(3)} - \mathrm{Wi}\psi_2^{(3)} - \breve{\lambda}_2\,\mathrm{Wi}^2\psi_3^{(6)} - \frac{1}{2}\mathrm{Wi}\psi_4^{(6)} + \frac{1}{2}\breve{\lambda}_2\,\mathrm{De}\,\mathrm{Wi}\psi_6^{(6)}\right)_{11}, \\ \left(-\frac{3}{2}\breve{\lambda}_2\,\mathrm{Wi}^2\psi_1^{(4)} - \mathrm{Wi}\psi_2^{(4)} - \breve{\lambda}_2\,\mathrm{Wi}^2\psi_3^{(7)} - \frac{1}{2}\mathrm{Wi}\psi_4^{(7)} + \frac{1}{2}\breve{\lambda}_2\,\mathrm{De}\,\mathrm{Wi}\psi_6^{(7)}\right)_{12}, \\ \left(-\breve{\lambda}_2\,\mathrm{Wi}^2\psi_3^{(8)} - \frac{1}{2}\mathrm{Wi}\psi_4^{(8)} + \frac{1}{2}\breve{\lambda}_2\,\mathrm{De}\,\mathrm{Wi}\psi_6^{(8)}\right)_{13}, \left(-\frac{1}{4}\breve{\lambda}_2\,\mathrm{Wi}^2\psi_8^{(7)}\right)_{14}, \left(-\frac{1}{4}\breve{\lambda}_2\,\mathrm{Wi}^2\psi_8^{(8)}\right)_{15} \end{bmatrix} \tag{138}$$

in which $\phi_1^{(i)}-\phi_8^{(i)}$ and $\psi_1^{(i)}-\psi_8^{(i)}$ are defined at the end of Subsections I.a.i through I.a.viii.

To further process, we need the following trigonometric identities, arrived at by substituting Eqs. (58) and (59) into Eqs. (56) and (57):

$$\begin{aligned} \cos\left(\tfrac{\mathrm{Wi}}{\mathrm{De}}\left(\tau+\sin\tau\right)\right) &= J_0\cos\tfrac{\mathrm{Wi}}{\mathrm{De}}\tau + 2\sum_{k=1}^{\infty}J_{2k}\cos\tfrac{\mathrm{Wi}}{\mathrm{De}}\tau\cos 2k\tau - 2\sum_{k=1}^{\infty}J_{2k-1}\sin\tfrac{\mathrm{Wi}}{\mathrm{De}}\tau\sin\left(2k-1\right)\tau \\ &= \sum_{k=0}^{\infty}J_{2k}\cos\left(2k+\tfrac{\mathrm{Wi}}{\mathrm{De}}\right)\tau + \sum_{k=1}^{\infty}J_{2k}\cos\left(2k-\tfrac{\mathrm{Wi}}{\mathrm{De}}\right)\tau \\ &\quad -\sum_{k=1}^{\infty}J_{2k-1}\cos\left(2k-1-\tfrac{\mathrm{Wi}}{\mathrm{De}}\right)\tau + \sum_{k=1}^{\infty}J_{2k-1}\cos\left(2k-1+\tfrac{\mathrm{Wi}}{\mathrm{De}}\right)\tau \end{aligned} \tag{139}$$

$$\begin{aligned} \sin\left(\tfrac{\mathrm{Wi}}{\mathrm{De}}\left(\tau+\sin\tau\right)\right) &= \sum_{k=0}^{\infty}J_{2k}\sin\left(2k+\tfrac{\mathrm{Wi}}{\mathrm{De}}\right)\tau - \sum_{k=1}^{\infty}J_{2k}\sin\left(2k-\tfrac{\mathrm{Wi}}{\mathrm{De}}\right)\tau \\ &\quad +\sum_{k=1}^{\infty}J_{2k-1}\sin\left(2k-1-\tfrac{\mathrm{Wi}}{\mathrm{De}}\right)\tau + \sum_{k=1}^{\infty}J_{2k-1}\sin\left(2k-1+\tfrac{\mathrm{Wi}}{\mathrm{De}}\right)\tau \end{aligned} \tag{140}$$

In conjunction with Eqs. (139) and (140), substituting Eqs. (133) and (134) into Eq. (132), we find:

$$\breve{\tau}_{yx} = \alpha_0 + \sum_{m=1}^{\infty}\left[\alpha_m + \beta_m + \gamma_m + \delta_m\right] \tag{141}$$

where:

$$\begin{aligned}
\alpha_m \equiv \frac{J_{2m}}{2}\sum_{k=0}^{\infty} & \left[\begin{array}{l}
\displaystyle\sum_{j=1}^{2}\left[\begin{array}{l}
\left(C_1^{(2j)} - S_2^{(2j)}\right)\cos\left(2k+2m+j\right)\tau + \left(C_1^{(2j)} + S_2^{(2j)}\right)\cos\left(2k-2m+j-2\tfrac{\mathrm{Wi}}{\mathrm{De}}\right)\tau \\
+\left(C_2^{(2j)} + S_1^{(2j)}\right)\sin\left(2k+2m+j\right)\tau + \left(-C_2^{(2j)} + S_1^{(2j)}\right)\sin\left(2k-2m+j-2\tfrac{\mathrm{Wi}}{\mathrm{De}}\right)\tau
\end{array}\right] \\
+\displaystyle\sum_{j=1}^{3}\left[\begin{array}{l}
\left(C_1^{(2j-1)} + S_2^{(2j-1)}\right)\cos\left(2k-2m+j-1\right)\tau + \left(C_1^{(2j-1)} - S_2^{(2j-1)}\right)\cos\left(2k+2m+j-1+2\tfrac{\mathrm{Wi}}{\mathrm{De}}\right)\tau \\
+\left(-C_2^{(2j-1)} + S_1^{(2j-1)}\right)\sin\left(2k-2m+j-1\right)\tau + \left(C_2^{(2j-1)} + S_1^{(2j-1)}\right)\sin\left(2k+2m+j-1+2\tfrac{\mathrm{Wi}}{\mathrm{De}}\right)\tau
\end{array}\right]
\end{array}\right] \\
+\frac{J_{2m}}{2}\sum_{k=1}^{\infty} & \left[\begin{array}{l}
\displaystyle\sum_{j=3}^{7}\left[\begin{array}{l}
\left(C_1^{(2j)} - S_2^{(2j)}\right)\cos\left(2k+2m+j-6\right)\tau + \left(C_1^{(2j)} + S_2^{(2j)}\right)\cos\left(2k-2m+j-6-2\tfrac{\mathrm{Wi}}{\mathrm{De}}\right)\tau \\
+\left(C_2^{(2j)} + S_1^{(2j)}\right)\sin\left(2k+2m+j-6\right)\tau + \left(-C_2^{(2j)} + S_1^{(2j)}\right)\sin\left(2k-2m+j-6-2\tfrac{\mathrm{Wi}}{\mathrm{De}}\right)\tau
\end{array}\right] \\
+\displaystyle\sum_{j=4}^{8}\left[\begin{array}{l}
\left(C_1^{(2j-1)} + S_2^{(2j-1)}\right)\cos\left(2k-2m+j-7\right)\tau + \left(C_1^{(2j-1)} - S_2^{(2j-1)}\right)\cos\left(2k+2m+j-7+2\tfrac{\mathrm{Wi}}{\mathrm{De}}\right)\tau \\
+\left(-C_2^{(2j-1)} + S_1^{(2j-1)}\right)\sin\left(2k-2m+j-7\right)\tau + \left(C_2^{(2j-1)} + S_1^{(2j-1)}\right)\sin\left(2k+2m+j-7+2\tfrac{\mathrm{Wi}}{\mathrm{De}}\right)\tau
\end{array}\right]
\end{array}\right]
\end{aligned} \tag{142}$$

$$\begin{aligned}
\beta_m \equiv \frac{J_{2m}}{2}\sum_{k=0}^{\infty} & \left[\begin{array}{l}
\displaystyle\sum_{j=1}^{2}\left[\begin{array}{l}
\left(C_1^{(2j)} - S_2^{(2j)}\right)\cos\left(2k-2m+j\right)\tau + \left(C_1^{(2j)} + S_2^{(2j)}\right)\cos\left(2k+2m+j-2\tfrac{\mathrm{Wi}}{\mathrm{De}}\right)\tau \\
+\left(C_2^{(2j)} + S_1^{(2j)}\right)\sin\left(2k-2m+j\right)\tau + \left(-C_2^{(2j)} + S_1^{(2j)}\right)\sin\left(2k+2m+j-2\tfrac{\mathrm{Wi}}{\mathrm{De}}\right)\tau
\end{array}\right] \\
+\displaystyle\sum_{j=1}^{3}\left[\begin{array}{l}
\left(C_1^{(2j-1)} + S_2^{(2j-1)}\right)\cos\left(2k+2m+j-1\right)\tau + \left(C_1^{(2j-1)} - S_2^{(2j-1)}\right)\cos\left(2k-2m+j-1+2\tfrac{\mathrm{Wi}}{\mathrm{De}}\right)\tau \\
+\left(-C_2^{(2j-1)} + S_1^{(2j-1)}\right)\sin\left(2k+2m+j-1\right)\tau + \left(C_2^{(2j-1)} + S_1^{(2j-1)}\right)\sin\left(2k-2m+j-1+2\tfrac{\mathrm{Wi}}{\mathrm{De}}\right)\tau
\end{array}\right]
\end{array}\right] \\
+\frac{J_{2m}}{2}\sum_{k=1}^{\infty} & \left[\begin{array}{l}
\displaystyle\sum_{j=3}^{7}\left[\begin{array}{l}
\left(C_1^{(2j)} - S_2^{(2j)}\right)\cos\left(2k-2m+j-6\right)\tau + \left(C_1^{(2j)} + S_2^{(2j)}\right)\cos\left(2k+2m+j-6-2\tfrac{\mathrm{Wi}}{\mathrm{De}}\right)\tau \\
+\left(C_2^{(2j)} + S_1^{(2j)}\right)\sin\left(2k-2m+j-6\right)\tau + \left(-C_2^{(2j)} + S_1^{(2j)}\right)\sin\left(2k+2m+j-6-2\tfrac{\mathrm{Wi}}{\mathrm{De}}\right)\tau
\end{array}\right] \\
+\displaystyle\sum_{j=4}^{8}\left[\begin{array}{l}
\left(C_1^{(2j-1)} + S_2^{(2j-1)}\right)\cos\left(2k+2m+j-7\right)\tau + \left(C_1^{(2j-1)} - S_2^{(2j-1)}\right)\cos\left(2k-2m+j-7+2\tfrac{\mathrm{Wi}}{\mathrm{De}}\right)\tau \\
+\left(-C_2^{(2j-1)} + S_1^{(2j-1)}\right)\sin\left(2k+2m+j-7\right)\tau + \left(C_2^{(2j-1)} + S_1^{(2j-1)}\right)\sin\left(2k-2m+j-7+2\tfrac{\mathrm{Wi}}{\mathrm{De}}\right)\tau
\end{array}\right]
\end{array}\right]
\end{aligned} \tag{143}$$

$$\begin{aligned}
\gamma_m \equiv \frac{J_{2m-1}}{2}\sum_{k=0}^{\infty} & \left[\begin{array}{l}
\displaystyle\sum_{j=1}^{2}\left[\begin{array}{l}
\left(-C_1^{(2j)} + S_2^{(2j)}\right)\cos\left(2k-2m+j+1\right)\tau + \left(-C_1^{(2j)} - S_2^{(2j)}\right)\cos\left(2k+2m+j-1-2\tfrac{\mathrm{Wi}}{\mathrm{De}}\right)\tau \\
+\left(-C_2^{(2j)} - S_1^{(2j)}\right)\sin\left(2k-2m+j+1\right)\tau + \left(C_2^{(2j)} - S_1^{(2j)}\right)\sin\left(2k+2m+j-1-2\tfrac{\mathrm{Wi}}{\mathrm{De}}\right)\tau
\end{array}\right] \\
+\displaystyle\sum_{j=1}^{3}\left[\begin{array}{l}
\left(-C_1^{(2j-1)} - S_2^{(2j-1)}\right)\cos\left(2k+2m+j-2\right)\tau + \left(-C_1^{(2j-1)} + S_2^{(2j-1)}\right)\cos\left(2k-2m+j+2\tfrac{\mathrm{Wi}}{\mathrm{De}}\right)\tau \\
+\left(C_2^{(2j-1)} - S_1^{(2j-1)}\right)\sin\left(2k+2m+j-2\right)\tau + \left(-C_2^{(2j-1)} - S_1^{(2j-1)}\right)\sin\left(2k-2m+j+2\tfrac{\mathrm{Wi}}{\mathrm{De}}\right)\tau
\end{array}\right]
\end{array}\right] \\
+\frac{J_{2m-1}}{2}\sum_{k=1}^{\infty} & \left[\begin{array}{l}
\displaystyle\sum_{j=3}^{7}\left[\begin{array}{l}
\left(-C_1^{(2j)} + S_2^{(2j)}\right)\cos\left(2k-2m+j-5\right)\tau + \left(-C_1^{(2j)} - S_2^{(2j)}\right)\cos\left(2k+2m+j-7-2\tfrac{\mathrm{Wi}}{\mathrm{De}}\right)\tau \\
+\left(-C_2^{(2j)} - S_1^{(2j)}\right)\sin\left(2k-2m+j-5\right)\tau + \left(C_2^{(2j)} - S_1^{(2j)}\right)\sin\left(2k+2m+j-7-2\tfrac{\mathrm{Wi}}{\mathrm{De}}\right)\tau
\end{array}\right] \\
+\displaystyle\sum_{j=4}^{8}\left[\begin{array}{l}
\left(-C_1^{(2j-1)} - S_2^{(2j-1)}\right)\cos\left(2k+2m+j-8\right)\tau + \left(-C_1^{(2j-1)} + S_2^{(2j-1)}\right)\cos\left(2k-2m+j-6+2\tfrac{\mathrm{Wi}}{\mathrm{De}}\right)\tau \\
+\left(C_2^{(2j-1)} - S_1^{(2j-1)}\right)\sin\left(2k+2m+j-8\right)\tau + \left(-C_2^{(2j-1)} - S_1^{(2j-1)}\right)\sin\left(2k-2m+j-6+2\tfrac{\mathrm{Wi}}{\mathrm{De}}\right)\tau
\end{array}\right]
\end{array}\right]
\end{aligned} \tag{144}$$

$$\delta_m \equiv \frac{J_{2m-1}}{2}\sum_{k=0}^{\infty}\left[\begin{aligned}&\sum_{j=1}^{2}\left[\begin{aligned}&\left(C_1^{(2j)}-S_2^{(2j)}\right)\cos\left(2k+2m+j-1\right)\tau+\left(C_1^{(2j)}+S_2^{(2j)}\right)\cos\left(2k-2m+j+1-2\tfrac{\mathrm{Wi}}{\mathrm{De}}\right)\tau\\&+\left(C_2^{(2j)}+S_1^{(2j)}\right)\sin\left(2k+2m+j-1\right)\tau+\left(-C_2^{(2j)}+S_1^{(2j)}\right)\sin\left(2k-2m+j+1-2\tfrac{\mathrm{Wi}}{\mathrm{De}}\right)\tau\end{aligned}\right]\\&+\sum_{j=1}^{3}\left[\begin{aligned}&\left(C_1^{(2j-1)}+S_2^{(2j-1)}\right)\cos\left(2k-2m+j\right)\tau+\left(C_1^{(2j-1)}-S_2^{(2j-1)}\right)\cos\left(2k+2m+j-2+2\tfrac{\mathrm{Wi}}{\mathrm{De}}\right)\tau\\&+\left(-C_2^{(2j-1)}+S_1^{(2j-1)}\right)\sin\left(2k-2m+j\right)\tau+\left(C_2^{(2j-1)}+S_1^{(2j-1)}\right)\sin\left(2k+2m+j-2+2\tfrac{\mathrm{Wi}}{\mathrm{De}}\right)\tau\end{aligned}\right]\end{aligned}\right]$$
$$+\frac{J_{2m-1}}{2}\sum_{k=1}^{\infty}\left[\begin{aligned}&\sum_{j=3}^{7}\left[\begin{aligned}&\left(C_1^{(2j)}-S_2^{(2j)}\right)\cos\left(2k+2m+j-7\right)\tau+\left(C_1^{(2j)}+S_2^{(2j)}\right)\cos\left(2k-2m+j-5-2\tfrac{\mathrm{Wi}}{\mathrm{De}}\right)\tau\\&+\left(C_2^{(2j)}+S_1^{(2j)}\right)\sin\left(2k+2m+j-7\right)\tau+\left(-C_2^{(2j)}+S_1^{(2j)}\right)\sin\left(2k-2m+j-5-2\tfrac{\mathrm{Wi}}{\mathrm{De}}\right)\tau\end{aligned}\right]\\&+\sum_{j=4}^{8}\left[\begin{aligned}&\left(C_1^{(2j-1)}+S_2^{(2j-1)}\right)\cos\left(2k-2m+j-6\right)\tau+\left(C_1^{(2j-1)}-S_2^{(2j-1)}\right)\cos\left(2k+2m+j-8+2\tfrac{\mathrm{Wi}}{\mathrm{De}}\right)\tau\\&+\left(-C_2^{(2j-1)}+S_1^{(2j-1)}\right)\sin\left(2k-2m+j-6\right)\tau+\left(C_2^{(2j-1)}+S_1^{(2j-1)}\right)\sin\left(2k+2m+j-8+2\tfrac{\mathrm{Wi}}{\mathrm{De}}\right)\tau\end{aligned}\right]\end{aligned}\right] \quad (145)$$

where $C_1^{(i)}, C_2^{(i)}, S_1^{(i)}$ and $S_2^{(i)}$ are defined in Eqs. (135)–(138). Eq. (141) [with Eqs. (142)–(145)] (*i.e.* Eq. (20) [with Eqs. (21)–(24)] of the main manuscript [1,2]) is the main result of this section.

ii. Transient Part

Substituting Eqs. (139), (140) and $I_{1,0} - I_{8,0}$, given in Eqs. (66), (75), (84), (93), (102), (111), (120) and (129), into Eq. (55), and then rearranging, we find:

$$\breve{\tau}_{yx} = -e^{\frac{-\tau}{\mathrm{De}}}\left(\breve{\tau}^{1}_{yx,0} + \breve{\tau}^{2}_{yx,0}\right) \quad (146)$$

in which:

$$\breve{\tau}^{1}_{yx,0} = \breve{\tau}^{1,1}_{yx,0} + \breve{\tau}^{1,2}_{yx,0} + \breve{\tau}^{1,3}_{yx,0} + \breve{\tau}^{1,4}_{yx,0} \quad (147)$$

where:

$$\breve{\tau}^{1,1}_{yx,0} \equiv \sum_{m=0}^{\infty} J_{2m}\left(\sum_{k=0}^{\infty}\sum_{i=1}^{5} C_1^{(i)} + \sum_{k=1}^{\infty}\sum_{i=6}^{15} C_1^{(i)}\right)\cos\left(2m+\tfrac{\mathrm{Wi}}{\mathrm{De}}\right)\tau \quad (148)$$

$$\breve{\tau}^{1,2}_{yx,0} \equiv \sum_{m=1}^{\infty} J_{2m}\left(\sum_{k=0}^{\infty}\sum_{i=1}^{5} C_1^{(i)} + \sum_{k=1}^{\infty}\sum_{i=6}^{15} C_1^{(i)}\right)\cos\left(2m-\tfrac{\mathrm{Wi}}{\mathrm{De}}\right)\tau \quad (149)$$

$$\breve{\tau}^{1,3}_{yx,0} \equiv -\sum_{m=1}^{\infty} J_{2m-1}\left(\sum_{k=0}^{\infty}\sum_{i=1}^{5} C_1^{(i)} + \sum_{k=1}^{\infty}\sum_{i=6}^{15} C_1^{(i)}\right)\cos\left(2m-1-\tfrac{\mathrm{Wi}}{\mathrm{De}}\right)\tau \quad (150)$$

$$\breve{\tau}^{1,4}_{yx,0} = \sum_{m=1}^{\infty} J_{2m-1}\left(\sum_{k=0}^{\infty}\sum_{i=1}^{5} C_1^{(i)} + \sum_{k=1}^{\infty}\sum_{i=6}^{15} C_1^{(i)}\right)\cos\left(2m-1+\tfrac{\mathrm{Wi}}{\mathrm{De}}\right)\tau \quad (151)$$

and:

$$\breve{\tau}^{2}_{yx,0} = \breve{\tau}^{2,1}_{yx,0} + \breve{\tau}^{2,2}_{yx,0} + \breve{\tau}^{2,3}_{yx,0} + \breve{\tau}^{2,4}_{yx,0} \quad (152)$$

where:

$$\breve{\tau}^{2,1}_{yx,0} \equiv \sum_{m=0}^{\infty} J_{2m}\left(\sum_{k=0}^{\infty}\sum_{i=1}^{5} C_2^{(i)} + \sum_{k=1}^{\infty}\sum_{i=6}^{15} C_2^{(i)}\right)\sin\left(2m+\tfrac{\mathrm{Wi}}{\mathrm{De}}\right)\tau \quad (153)$$

$$\breve{\tau}^{2,2}_{yx,0} \equiv -\sum_{m=1}^{\infty} J_{2m}\left(\sum_{k=0}^{\infty}\sum_{i=1}^{5} C_2^{(i)} + \sum_{k=1}^{\infty}\sum_{i=6}^{15} C_2^{(i)}\right)\sin\left(2m-\tfrac{\mathrm{Wi}}{\mathrm{De}}\right)\tau \quad (154)$$

$$\breve{\tau}_{yx,0}^{2,3} \equiv \sum_{m=1}^{\infty} J_{2m-1}\left(\sum_{k=0}^{\infty}\sum_{i=1}^{5} C_2^{(i)} + \sum_{k=1}^{\infty}\sum_{i=6}^{15} C_2^{(i)}\right)\sin\left(2m-1-\tfrac{\mathrm{Wi}}{\mathrm{De}}\right)\tau \tag{155}$$

$$\breve{\tau}_{yx,0}^{2,4} \equiv \sum_{m=1}^{\infty} J_{2m-1}\left(\sum_{k=0}^{\infty}\sum_{i=1}^{5} C_2^{(i)} + \sum_{k=1}^{\infty}\sum_{i=6}^{15} C_2^{(i)}\right)\sin\left(2m-1+\tfrac{\mathrm{Wi}}{\mathrm{De}}\right)\tau \tag{156}$$

Substituting Eqs. (147) and (152) [with Eqs. (148)–(151) and Eqs. (153)–(156)] into Eq. (146), we find:

$$\breve{\tau}_{yx,0} = -e^{\frac{-\tau}{\mathrm{De}}}\left(\alpha_{0,0} + \sum_{m=1}^{\infty}\left[\alpha_{m,0} + \beta_{m,0} + \gamma_{m,0} + \delta_{m,0}\right]\right) \tag{157}$$

where:

$$\begin{aligned}\alpha_{m,0} \equiv{}& J_{2m}\left(\sum_{k=0}^{\infty}\sum_{i=1}^{5} C_1^{(i)} + \sum_{k=1}^{\infty}\sum_{i=6}^{15} C_1^{(i)}\right)\cos\left(2m+\tfrac{\mathrm{Wi}}{\mathrm{De}}\right)\tau \\ &+ J_{2m}\left(\sum_{k=0}^{\infty}\sum_{i=1}^{5} C_2^{(i)} + \sum_{k=1}^{\infty}\sum_{i=6}^{15} C_2^{(i)}\right)\sin\left(2m+\tfrac{\mathrm{Wi}}{\mathrm{De}}\right)\tau\end{aligned} \tag{158}$$

$$\begin{aligned}\beta_{m,0} \equiv{}& J_{2m}\left(\sum_{k=0}^{\infty}\sum_{i=1}^{5} C_1^{(i)} + \sum_{k=1}^{\infty}\sum_{i=6}^{15} C_1^{(i)}\right)\cos\left(2m-\tfrac{\mathrm{Wi}}{\mathrm{De}}\right)\tau \\ &- J_{2m}\left(\sum_{k=0}^{\infty}\sum_{i=1}^{5} C_2^{(i)} + \sum_{k=1}^{\infty}\sum_{i=6}^{15} C_2^{(i)}\right)\sin\left(2m-\tfrac{\mathrm{Wi}}{\mathrm{De}}\right)\tau\end{aligned} \tag{159}$$

$$\begin{aligned}\gamma_{m,0} \equiv{}& -J_{2m-1}\left(\sum_{k=0}^{\infty}\sum_{i=1}^{5} C_1^{(i)} + \sum_{k=1}^{\infty}\sum_{i=6}^{15} C_1^{(i)}\right)\cos\left(2m-1-\tfrac{\mathrm{Wi}}{\mathrm{De}}\right)\tau \\ &+ J_{2m-1}\left(\sum_{k=0}^{\infty}\sum_{i=1}^{5} C_2^{(i)} + \sum_{k=1}^{\infty}\sum_{i=6}^{15} C_2^{(i)}\right)\sin\left(2m-1-\tfrac{\mathrm{Wi}}{\mathrm{De}}\right)\tau\end{aligned} \tag{160}$$

$$\begin{aligned}\delta_{m,0} \equiv{}& J_{2m-1}\left(\sum_{k=0}^{\infty}\sum_{i=1}^{5} C_1^{(i)} + \sum_{k=1}^{\infty}\sum_{i=6}^{15} C_1^{(i)}\right)\cos\left(2m-1+\tfrac{\mathrm{Wi}}{\mathrm{De}}\right)\tau \\ &+ J_{2m-1}\left(\sum_{k=0}^{\infty}\sum_{i=1}^{5} C_2^{(i)} + \sum_{k=1}^{\infty}\sum_{i=6}^{15} C_2^{(i)}\right)\sin\left(2m-1+\tfrac{\mathrm{Wi}}{\mathrm{De}}\right)\tau\end{aligned} \tag{161}$$

in which $C_1^{(i)}$ and $C_2^{(i)}$ are defined in Eqs. (135) and (137). Eq. (157) [with Eqs. (158)–(161)] (*i.e.* Eq. (25) [with Eqs. (26)–(29)] of the main manuscript [1,2]) is the main result of this section.

III REFERENCES

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