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Unification Scenario of the Gravitational and Gauge Forces via the Higher Dimensional Soliton

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In a dynamically localized subspacetime via the solitonic solution in the higher dimensional space-time, the gravitational and gauge forces are shown to be unifiedly induced as its intrinsic and extrinsic curvature effects.

Some dynamical models are known to have solutions localized in the neighborhood of a subspace-time with lower dimensions.[1] Our (3+1)-space-time itself could be a dynamically localized one in a higher dimensional spacetime, as is studied by several authors.[2)~10)] In this paper, we show that, in the dynamically localized subspace-time, the gravitational and the gauge forces are unifiedly induced through its intrinsic and extrinsic curvature effects. This suggests a new unification scenario alternative to the ordinary Kaluza-Klein, supergravity or superstring theories.

We denote the action by $S=\int L(\Phi^{(i)}, \partial\Phi^{(i)})dX^{N+M+1}$, where $\Phi^{(i)}$ $(i=1, 2, \cdots)$ are the basic fields belonging to some representations of the Lorentz group $O(N+M, 1)$. We assume that the equation of motion $\delta S/\delta\Phi^{(i)}=0$ has the solitonic solution

$$\Phi^{(i)}(X^\lambda, X^{\underline{\lambda}})=\Phi_0{}^{(i)}(X^{\underline{\lambda}}), \qquad (\lambda=0, 1, \cdots, M, \underline{\lambda}=M+1, \cdots, M+N) \tag{1}$$

which is independent of X^λ and localized near the $(M+1)$-dimensional subspace-time at $X^{\underline{\lambda}}=0$.[11] The small fluctuation around the solution (1), $\varphi^{(i)}=\Phi^{(i)}-\Phi_0{}^{(i)}$ satisfies $\Delta^{(ij)}\varphi^{(j)}=0$ where $\Delta^{(ij)}=\delta S/\delta\Phi^{(i)}\delta\Phi^{(j)}|_{\Phi=\Phi_0}$. Let $M^{(ij)}$ be the operator obtained by restricting $\Delta^{(ij)}$ in the subspace $X^\lambda=$constant. Then, we expand $\varphi^{(i)}$ in terms of the eigenfunctions $\varphi_n{}^{(i)}(n=1, 2, \cdots)$ of $M^{(ij)}$,

$$\varphi^{(i)}=\sum_n \chi_n{}^{(i)}(X^\lambda)\varphi_n{}^{(i)}(X^{\underline{\lambda}}),$$

where $\chi_n{}^{(i)}$ should be taken so that $\Phi^{(i)}$ satisfies the equation of motion.

The eigenfunctions $\varphi_n{}^{(i)}(X^{\underline{\lambda}})$ involve the zero modes associated with the translation invariance of the action. It means that the 'soliton' is displaced from $X^{\underline{\lambda}}=0$, and the subspace-time gets curved. We denote its position by $\boldsymbol{y}(z^\lambda)$ with $M+1$ parameters z^λ, and the orthonormal local Lorentz frame by $\boldsymbol{n}_I(z^\lambda)$, where $\boldsymbol{n}_i$ are tangential and $\boldsymbol{n}_{\underline{i}}$ are normal to the subspace-time. These vectors define the vielbein $e_{k\lambda}=\boldsymbol{n}_k\boldsymbol{y}_{,\lambda}$ and the connection $\Gamma_{IJ\lambda}=\boldsymbol{n}_I\boldsymbol{n}_{J,\lambda}$ of the subspace-time, or equivalently[12]

$$\boldsymbol{y}_{,\lambda}=\boldsymbol{n}_k e^k{}_\lambda \quad \text{and} \quad \boldsymbol{n}_{J,\lambda}=\boldsymbol{n}_I\Gamma^I{}_{J\lambda}. \tag{2}$$

We denote the normal components of $\Gamma_{IJ\lambda}$ by $A_{\underline{ij}\lambda}=\Gamma_{\underline{ij}\lambda}$ and $B_{\underline{i}j\lambda}=\Gamma_{\underline{i}j\lambda}$. The $\Gamma_{ij\lambda}$, $A_{\underline{ij}\lambda}$ and $B_{\underline{i}\mu\nu}$ are respectively called the affine connection, the normal connection and the second fundamental quantity.[12] Among them, $\Gamma_{ij\lambda}$ is written in terms of $e_{k\lambda}$,

$$\Gamma_{ij\lambda}=\frac{1}{2}(e_{k[i,j]}e^{k}{}_{\lambda}-e_{i[j,\lambda]}+e_{j[i,\lambda]})\,. \tag{3}$$

The $A_{\underline{ij}\lambda}$ and $B_{\underline{i}\mu\nu}$ have the symmetry properties,

$$A_{\underline{ij}\lambda}=-A_{\underline{ji}\lambda}\,,\qquad B_{\underline{i}\mu\nu}=B_{\underline{i}\nu\mu}\,. \tag{4}$$

Since the whole space-time is flat, the equation of Gauss-Codazzi-Ricci reads[13]

$$\Gamma_{IJ[\mu,\nu]}+\Gamma_{KI[\mu}\Gamma^{K}{}_{J\nu]}=0\,. \tag{5}$$

In terms of the components,

$$\Gamma_{ij[\mu,\nu]}+\Gamma_{ki[\mu}\Gamma^{k}{}_{j\nu]}+B_{\underline{k}i[\mu}B^{\underline{k}}{}_{j\nu]}=0\,, \tag{5a}$$

$$B_{\underline{i}j[\mu,\nu]}+B_{\underline{i}k[\mu}\Gamma_{j}^{\;k}{}_{\nu]}+A_{\underline{ki}[\mu}B^{\underline{k}}{}_{j\nu]}=0\,, \tag{5b}$$

$$A_{\underline{ij}[\mu,\nu]}+A_{\underline{ik}[\mu}A_{\underline{j}}^{\;\underline{k}}{}_{\nu]}+B_{\underline{i}k[\mu}B_{\underline{j}}^{\;k}{}_{\nu]}=0\,. \tag{5c}$$

On the contrary, when $e_{k\lambda}$, $A_{\underline{ij}\lambda}$ and $B_{\underline{i}j\lambda}$ satisfying Eqs. (3)~(5) are given, the system of the $\boldsymbol{y}(z^{\lambda})$ and orthonormal $\boldsymbol{n}_{k}(z^{\lambda})$ satisfying Eq. (2) exists uniquely up to overall translations. This is because Eqs. (3)~(5) prove integrability of Eq. (2). Thus, the set of variables $e_{k\lambda}$, $A_{\underline{ij}\lambda}$ and $B_{\underline{i}j\lambda}$ with the constraints (3)~(5) can be taken as the collective modes which specify the subspace-time.[2)]

In order to treat more properly the $\Phi^{(i)}$ with zero modes excited, we transform the coordinate X^{A} to the curvilinear coordinate $z^{A}=(z^{\lambda},z^{\underline{\lambda}})$ by $\boldsymbol{X}=\boldsymbol{y}(z^{\lambda})+z^{\underline{\lambda}}\boldsymbol{n}_{\underline{\lambda}}(z^{\lambda})$, and the local Lorentz frame at z^{A} to $\boldsymbol{n}_{I}(z^{\lambda})$, independently of $z^{\underline{\lambda}}$. The vielbein and the connection in this system are respectively given by $E_{KA}=\boldsymbol{n}_{K}\boldsymbol{X}_{,A}$ and $\Gamma_{IJA}=\boldsymbol{n}_{I}\boldsymbol{n}_{J,A}$. The E_{KA} is written in terms of $e_{k\lambda}$, $B_{\underline{i}j\lambda}$, $A_{\underline{ij}\lambda}$ and $z^{\underline{\lambda}}$ as

$$\begin{bmatrix} E_{k\lambda} & E_{k\underline{\lambda}} \\ E_{\underline{k}\lambda} & E_{\underline{k\lambda}} \end{bmatrix}=\begin{bmatrix} e_{k\lambda}-z^{\underline{i}}B_{\underline{i}k\lambda} & 0 \\ -z^{\underline{i}}A_{\underline{ik}\lambda} & -\delta_{\underline{k\lambda}} \end{bmatrix}.$$

$\Gamma_{IJ\lambda}$ is equal to that of the subspace-time, and $\Gamma_{IJ\underline{\lambda}}=0$. The curvilinear coordinate z^{A} becomes singular at the point where $E=\det(E_{KA})=0$, since there adjacent normal spaces intersect. If it happens within the nonasymptotic region of the 'soliton', the description with z^{A} fails, and the quantities $e_{k\lambda}$, $A_{\underline{ij}\lambda}$ and $B_{\underline{i}j\lambda}$ lose their meaning. However, we are concerned with so low energies or small curvatures that the singularities are in the far asymptotic region.

The field $\Phi^{(i)}(X^{A})$ is transformed into $\tilde{\Phi}^{(i)}(z^{A})$ with $\Phi^{(i)}=U\tilde{\Phi}^{(i)}$ where U is the representation of the Lorentz transformation to the local Lorentz frame $\boldsymbol{n}_{I}$. The derivative is rewritten as $\boldsymbol{\partial}\Phi^{(i)}=U\boldsymbol{n}_{K}D^{K}\tilde{\Phi}^{(i)}$ with the covariant differentiation

$$D^{K}=E^{KA}\Big(\partial_{A}-\frac{1}{2}i\Gamma_{IJA}T^{IJ}\Big)\,,$$

where $E^{KA}=\boldsymbol{n}^{K}\boldsymbol{\partial}z^{A}$ is the inverse of E_{KA}. Explicitly in terms of $e_{k\lambda}$, $A_{\underline{ij}\lambda}$, $B_{\underline{i}j\lambda}$ and $z^{\underline{\lambda}}$,

$$D^{k}=H^{k}{}_{l}e^{l\lambda}\Big(\partial_{\lambda}-\frac{1}{2}i\Gamma_{ij\lambda}T^{ij}-\frac{1}{2}iA_{\underline{ij}\lambda}(T^{\underline{ij}}+2iz^{\underline{i}}\partial^{\underline{j}})-iB_{\underline{i}j\lambda}T^{\underline{i}j}\Big)\,,$$

$$D^{\underline{k}}=\partial^{\underline{k}}\,,$$

where $H=1+B+B^2+B^3+\cdots$ with $B^i{}_j=z^{\underline{k}}B_{\underline{k}j}{}^i$.

The action is rewritten as

$$S=\int EL(\tilde{\Phi}^{(i)}, D_K\tilde{\Phi}^{(i)})dz^{N+M+1}\,. \tag{6}$$

The equation of motion $\delta S/\delta\tilde{\Phi}^{(i)}=0$ has the solitonic solution

$$\tilde{\Phi}^{(i)}(z^\lambda, z^{\underline{\lambda}})=\Phi_0{}^{(i)}(z^{\underline{\lambda}})$$

for $e_{k\lambda}=\eta_{k\lambda}$, $A_{\underline{ij}\lambda}=0$, $B_{\underline{i}j\lambda}=0$. We expand the fluctuation $\tilde{\varphi}^{(i)}=\tilde{\Phi}^{(i)}-\Phi_0{}^{(i)}$ in terms of $\varphi_n{}^{(i)}$, the eigenfunctions of $M^{(ij)}$,

$$\tilde{\varphi}^{(i)}=\sum_n \tilde{\chi}_n{}^{(i)}(z^\lambda)\varphi_n{}^{(i)}(z^{\underline{\lambda}})\,, \tag{7}$$

where $\tilde{\chi}_n{}^{(i)}$ should be chosen so that $\tilde{\Phi}^{(i)}$ satisfies the equation of motion. In (7) we exclude the zero modes associated with the translation, by the constraint that $z^{\underline{\lambda}}=0$ is the center of the 'soliton'. The dynamical freedom is converted into that of $e_{k\lambda}$, $A_{\underline{ij}\lambda}$ and $B_{\underline{i}j\lambda}$. Now we substitute the $\tilde{\Phi}^{(i)}$ back into the action (6). Then, the action is invariant under the general coordinate transformations (GCT) and the local Lorentz transformations (LLT) of the $(M+1)$-subspace-time, since each piece of the arguments of the integrand is invariant or covariant under them. We assume that the 'soliton' solution (1) is invariant under the rotations of the normal space around the center of the 'soliton'. It is the global $O(N)$ symmetry. Then, the action (6) is invariant under the local $O(N)$ transformations, where the normal connection $A_{\underline{ij}\lambda}$ plays the role of the gauge field. Next we integrate the Lagrangian density over $z^{\underline{\lambda}}$ to get that in the $(M+1)$-subspace-time.

$$eL_{\text{sub}}(\tilde{\chi}_n{}^{(i)}, D_k\tilde{\chi}_n{}^{(i)}, e_{k\lambda}, A_{\underline{ij}\lambda}, B_{\underline{i}j\lambda})=\int EL(\tilde{\Phi}^{(i)}, D_k\tilde{\Phi}^{(i)})dz^N\,, \tag{8}$$

where

$$D_k=H_{kl}e^{l\lambda}\Big(\partial_\lambda-\frac{1}{2}i\Gamma_{ij\lambda}T^{ij}-\frac{1}{2}iA_{\underline{ij}\lambda}T^{\underline{ij}}-iB_{\underline{i}j\lambda}T^{\underline{i}j}\Big)\,, \tag{9}$$

and the integration region is a sphere which covers the whole non-asymptotic region.

Now we consider the quantum effects due to the loop diagrams with $\tilde{\chi}_n{}^{(i)}$-internal lines and with $e_{k\lambda}$-, $A_{\underline{ij}\lambda}$- and $B_{\underline{i}j\lambda}$-external lines. We argue that they are automatically convergent, and give rise to the kinetic terms of $e_{k\lambda}$, $A_{\underline{ij}\lambda}$ and $B_{\underline{i}j\lambda}$.[14] Suppose that $\tilde{\chi}_n{}^{(i)}$ has as high momentum as the inverse size of the 'soliton'. Then, it would easily be transferred to $e_{k\lambda}$, $A_{\underline{ij}\lambda}$ and $B_{\underline{i}j\lambda}$, and accordingly the subspace-time would be so strongly curved that the $E=0$ singularities might be met within the size of the 'soliton'. It means that the descriptions in terms of the curvilinear coordinate z^A fail, and the collective modes $e_{k\lambda}$, $A_{\underline{ij}\lambda}$ and $B_{\underline{i}j\lambda}$ no longer exist. Therefore, the fields $e_{k\lambda}$, $A_{\underline{ij}\lambda}$ and $B_{\underline{i}j\lambda}$ decouple from the fields with so high momenta. Thus the loop momenta are effectively cut off below the inverse size of the 'soliton'. As far as the fields $e_{k\lambda}$, $A_{\underline{ij}\lambda}$ and $B_{\underline{i}j\lambda}$ exist, the $E=0$ singularities reside outside the 'soliton', and the action (6) is invariant under the GCT, LLT and local $O(N)$ transformations. Accordingly, the quantum loop effects should reflect those symmetries. The dimensional analysis shows that its leading terms are

$$L_{\rm eff}=c_0+c_1R+aF_{\underline{ij}\mu\nu}F^{\underline{ij}\mu\nu}+bB_{\underline{ij}\lambda}B^{\underline{ij}\lambda}+\cdots\,, \tag{10}$$

where c_0, c_1, a and b are constants, and $R=e^{i\mu}e^{j\nu}R_{ij\mu\nu}$,

$$R_{ij\mu\nu}=\Gamma_{ij[\mu,\nu]}+\Gamma_{ki[\mu}\Gamma^{k}{}_{j\nu]}\,,$$

$$F_{\underline{ij}\mu\nu}=A_{\underline{ij}[\mu,\nu]}+A_{\underline{ki}[\mu}A^{\underline{k}}{}_{\underline{j}\nu]}\,.$$

The first term in (10) is the cosmological term, which is large in general. We need to keep it small by fine tuning of the original parameters.[15] At this point, the present model does not improve the situation of the ordinary quantum gravity theory, but not worse. The second and the third terms give the kinetic and self-interaction terms of $e_{k\lambda}$ and $A_{\underline{ij}\lambda}$. They are the Lagrangians of the ordinary gravitational and gauge theory. The kinetic term of $B_{\underline{ij}\lambda}$ is among the non-leading terms. The fourth term is characteristic of such embedded-space-time model, and would modify the ordinary gravitational theory. We need to extremize the action under the constraint of Eq. (5).

Now we turn to the effective theory at low energies. There the massive mode of $\tilde{\chi}_n{}^{(i)}$'s cannot be excited. The field $B_{\underline{ij}\lambda}$, though it is massive, can be active because of the constraints in Eq. (5). The terms which are linear in $z^{\underline{\lambda}}$ in the integrand of Eq. (8) vanish when integrated over $z^{\underline{\lambda}}$. The terms with the more factors of $z^{\underline{\lambda}}$ are suppressed by the more factors of the order of the size in $L_{\rm sub}$. Therefore, at low energies, $H=1$ in Eq. (9). Furthermore, if $\Phi^{(i)}$ is a scalar or a gauge field, $\tilde{\chi}_n{}^{(i)}$'s decouple from $B_{\underline{ij}\lambda}$, and if $\tilde{\chi}_n{}^{(i)}$ is a chiral zero-mode fermion, it decouples from $B_{\underline{ij}\lambda}$. Notice that, in many existing models of solitons, these conditions are satisfied, i.e., the basic fields are scalar, spinor or gauge fields, and the fermion zero modes are chiral. Then, D_k in Eq. (9) is reduced to the covariant derivative of the GCT, LLT and the local $O(N)$ transformations. Thus, the low energy effective theory in the subspace-time is a unified theory of the gravitation and gauge forces with the constraint (5). This constraint does not immediately contradict with the observations. For example, if N is sufficiently large, we can always arrange the $A_{\underline{ij}\lambda}$ and $B_{\underline{ij}\lambda}$ so as to reconcile with the given configurations of the gravitational, electroweak and gluonic field. The physical implications of the constraint (5) may be an interesting problem to be investigated in future. In the above the induced gauge group was $O(N)$. However, it can be subgroup of $O(N)$, if the solution (1) is invariant only under the subgroup rotations. To complete this scenario, we need in future to search for models which yield more realistic gauge symmetry such as $SU(3)\times SU(2)\times U(1)$, $SU(5)$, $SO(10)$, etc.

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1) For example, solitons, monopoles, instantons, skyrmions, vortices, domain walls, bags, QCD-strings, etc. For a review, see, e.g., R. Rajaraman, *Solitons and Instantons* (North-Holland, 1982), and references therein.

2) This model is different from that of Regge and Teitelboim where they started with the Einstein-

Hilbert action taking the position y as the dynamical variable instead of the metric $g_{\mu\nu}$, and showed that it results in difficulties. See T. Regge and C. Teitelboim, *Marcel Grossman Meeting on Relativity, 1975* (North Holland, 1977), p. 77.
S. Deser, F. Pirani and D. Robinson, Phys. Rev. **D14** (1976), 3301.
In our case, the subspace-time itself is dynamically induced. The action of gravity is not a priori assumed but derived [see Ref. 3)]. The dynamical variable is not y but $g_{\mu\nu}$.
3) K. Akama, *Gauge Theory and Gravitation, Proceedings of the International Symposium, Nara, Japan, 1982,* ed. K. Kikkawa, N. Nakanishi and H. Nariai (Springer-Verlag, 1983), p. 267.
4) V. Rubakov and M. Shaposhnikov, Phys. Lett. **125B** (1983), 136.
5) B. Holdom, Stanford Preprint ITP-744 (1983); Nucl. Phys. **B233** (1983), 413.
6) C. G. Callan, Jr. and J. A. Harvey, Nucl. Phys. **B250** (1985), 427.
7) M. Visser, Phys. Lett. **159B** (1985), 22.
E. J. Squires, Phys. Lett. **167B** (1986), 286.
8) J. Huges, J. Liu and J. Polchinski, Phys. Lett. **B180** (1986), 370.
9) A model where our spacetime is identified with a submanifold of a fibre bundle as higher dimensional space-time is studied by M. Chaichan, A. P. Demichev and N. F. Nelipa, Phys. Lett. **196B** (1986), 327.
10) For models of soliton in higher dimensions, see for example, C. Teitelboim, Phys. Lett. **B167** (1986), 69.
H. Date, K. Fujii and H. So, Kyoto Preprint, RIFP-679 (1986).
11) Our conventions of the suffices are as follows:
(upper-case letter)$=0, 1, \cdots, M+N$,
(lower-case letter)$=0, 1, \cdots, M$,
(underlined letter)$=M+1, \cdots, M+N$,
(Greek letter)=(general coordinate index),
(Latin letter)=(local Lorentz index).
The indices with [and] are antisymmetrized. The indices after comma indicate differentiations.
12) $e_k{}^\lambda$ is defined by $e_k{}^\lambda e_{l\lambda}=\eta_{kl}=\mathrm{diag}(1, -1, -1, \cdots, -1)$. $g_{\mu\nu}=e_{k\mu}e_{l\nu}\eta^{kl}$ and $g^{\mu\nu}=e_k{}^\mu e_l{}^\nu \eta^{kl}$. The Greek (Latin) indices are uppered and lowered by g's (η's), and mutually converted by e's. The underlined Greek and Latin indices are uppered, lowered and mutually converted by η's.
13) Importance of these quantities and equations is suggested in Ref. 5).
14) The idea that the quantum fluctuations of matter give rise to the kinetic terms of composite fields has a long history. For example, see Y. Nambu and G. Jona-Lasinio, Phys. Rev. **122** (1961), 345; J. D. Bjorken, Ann. of Phys. **24** (1963), 174; A. D. Sakharov, Doklady Akad. Nauk SSSR **177** (1967), 70; H. Terazawa, Y. Chikashige and K. Akama, Phys. Rev. **D15** (1977), 480; K. Akama, Y. Chikashige T. Matsuki and H. Terazawa, Prog. Theor. Phys. **60** (1978), 868.
15) See, for example, K. Akama, Phys. Rev. **D24** (1981), 3073; Phys. Lett. **140B** (1984), 197.