Progress of Theoretical Physics, Vol. 107, No. 1, January 2002

Unified Description of q-Deformed Harmonic Oscillators

Ikuo S. Sogami*) and Kouzou Koizumi**)

Department of Physics, Kyoto Sangyo University, Kyoto 603-8555, Japan

(Received September 14, 2001)

It is shown that a wide class of q-deformed harmonic oscillators, including those of the Macfarlane type and Dubna type, can be described in a unified way. The Hamiltonian of the oscillator is assumed to be given by a q-deformed anti-commutator of the q-deformed ladder operators. By solving q-difference equations, explicit coordinate representations of ladder operators and wave functions are derived, and unified parametric representations are found for q-Hermite functions and related formulas for oscillators of the Macfarlane and Dubna types. In addition to the well-known solutions with globally periodic structure, it is found that there exist an infinite number of solutions with globally aperiodic structure.

§1. Introduction

A prototype of a q-deformed harmonic oscillator was found in the model investigated by Macfarlane in a "parametric coordinate" representation.[1] In his original theory, the Hilbert space is composed of wave functions related to the Rogers-Szegö polynomials on the unit circle.[2-4] The q-deformed oscillators with a real coordinate on the infinite interval were explored in succeeding models.[5,6] Independently, the Dubna group found a different kind of q-deformed harmonic oscillator in their study of a one-dimensional relativistic system.[7-13] In their approach also, a rather "abstract coordinate" was used for the oscillator.

In this article, we investigate these two seemingly different types of q-deformed harmonic oscillators, those of the Macfarlane type and the Dubna type, in a single real coordinate representation and find a wide class of new oscillators as their descendants. As a basic postulate, ladder operators of the oscillator are assumed to be composed of products of functions that include separately the position operator and the momentum operator.[7,10,11] We call such functions "part-functions". In particular, a kind of q-deformed derivative consisting of a difference of exponential functions is chosen for the part-function of the momentum. Other part-functions are obtained by solving the q-difference equations deduced from the condition that the ladder operators satisfy a q-deformed commutation relation.

One of the part-functions of the ladder operator is obtained as an infinite sum of Gaussian functions with arbitrary coefficients. It is the freedom in this arbitrariness that enables us to determine the global structure of the q-deformed oscillator system. Connection of the ladder operators and the wave functions realized by properly adjusting the coefficients results in the periodic global structure. This was recognized first in oscillators of the Macfarlane type.[5,6] In addition to such globally periodic

*) E-mail: sogami@cc.kyoto-su.ac.jp

**) E-mail: kkoizumi@cc.kyoto-su.ac.jp

solution, there exist infinitely many new solutions with aperiodic global structure.

The Hamiltonian, which is assumed to be given by a q-deformed anticommutator of the ladder operators, has the same eigenvalue spectrum (expressed by a definite function of the deformation parameter) for all q-deformed oscillators studied in this article. Derivation of its eigenfunctions results in different q-deformed Hermite functions [5), 10), 14)] for the oscillators of the Macfarlane and Dubna types.

With this constructive approach, we are able to clarify the similarities and differences of q-deformed oscillators of the Macfarlane and Dubna types in a unified way and, as an unexpected outcome, to disclose explicitly the existence of infinite families of new oscillator systems with different global structures.

§2. Algebra of q-deformed harmonic oscillator

We investigate a system consisting of a q-deformed harmonic oscillator with deformation parameter q and the Hamiltonian

$$\hat{H}_q = \frac{1}{2}\{\hat{A}, \hat{A}^\dagger\}_q = \frac{1}{2}(q\hat{A}\hat{A}^\dagger + q^{-1}\hat{A}^\dagger\hat{A}), \tag{2·1}$$

where $\hat{A}^\dagger$ and $\hat{A}$ are, respectively, the raising and lowering operators satisfying the q-deformed commutation relation (q-mutator)

$$[\hat{A}, \hat{A}^\dagger]_q = q\hat{A}\hat{A}^\dagger - q^{-1}\hat{A}^\dagger\hat{A} = 1. \tag{2·2}$$

As in the case of an ordinary (non-deformed) harmonic oscillator system, the ground state of the Hamiltonian $\hat{H}_q$ is defined by the conditions

$$\hat{A}|0\rangle = 0, \quad \langle 0|0\rangle = 1, \tag{2·3}$$

and excited states are generated from it by applying the raising operator as

$$|n\rangle = N_n(\hat{A}^\dagger)^n|0\rangle. \tag{2·4}$$

To confirm that the states $|n\rangle$ so constructed are eigenstates of the Hamiltonian and to obtain the eigenvalue, it is sufficient to use the q-deformed commutation relation

$$[\hat{A}, \hat{H}_q]_q = \frac{1}{2}(q + q^{-1})\hat{A}, \tag{2·5}$$

which leads readily to the recursion formula

$$q\left(E_n - \frac{1}{2}\frac{q+q^{-1}}{q-q^{-1}}\right) = q^{-1}\left(E_{n-1} - \frac{1}{2}\frac{q+q^{-1}}{q-q^{-1}}\right). \tag{2·6}$$

Using this relation and noting $E_0 = \frac{1}{2}$, we find the general formula for the energy eigenvalue spectrum of the q-deformed harmonic oscillator as follows:

$$E_n(q) = \frac{1}{2} + \frac{1-q^{-2n}}{q^2-1}. \tag{2·7}$$

The equal spacing law for eigenvalues of the ordinary harmonic oscillator becomes, in the present case, the geometrical progression

$$E_{n+1} - E_n = q^{-2}\left(E_n - E_{n-1}\right), \qquad n = 1, 2, \cdots \tag{2·8}$$

with constant ratio q^{-2} for the difference between adjacent eigenvalues. Note that this spectrum can be derived using an algebraic procedure only. Therefore, its form is common to all q-deformed oscillators specified in later sections and is independent of the details of their representations. Normalization of the eigenvectors results in the recursion formula

$$\left(\frac{N_n}{N_{n+1}}\right)^2 = \langle n|\hat{A}\hat{A}^\dagger|n\rangle = \frac{1}{2q}\left(2E_n + 1\right), \tag{2·9}$$

from which the normalization constant N_n in Eq. (2·4) is derived as

$$N_n = \prod_{m=1}^{n}\left(\frac{q - q^{-1}}{1 - q^{-2m}}\right)^{\frac{1}{2}}. \tag{2·10}$$

§3. *q*-deformed ladder operators in the *x*-representation

Following the basic postulate made in the Introduction, the part-function of the ladder operators that includes the momentum operator $\hat{p}$ is assumed to take the form

$$D(\hat{p}) = i\frac{\exp(s\hat{p}) - \exp(t\hat{p})}{s - t}, \tag{3·1}$$

where s and t are real parameters. Note that this part-function is a kind of q-deformed derivative which has the limit

$$\lim_{s\to 0}\lim_{t\to 0} D(\hat{p}) = \lim_{t\to 0}\lim_{s\to 0} D(\hat{p}) = i\hat{p}. \tag{3·2}$$

Other part-functions depending on the position operator $\hat{x}$ are introduced below at the stage when an explicit coordinate representation is chosen and determined by solving q-difference equations. It turns out to be appropriate to define the deformation parameter q as

$$q = \exp\Big(s^2 + t^2 + 3st\Big) \tag{3·3}$$

in terms of the real parameters s and t.

For an arbitrary state vector $|\psi\rangle$, the ladder operators $\hat{A}$ and $\hat{A}^\dagger$ consisting of separable part-functions have x-representations defined by

$$\langle x|\hat{A}|\psi\rangle = A(x)\psi(x), \quad \langle x|\hat{A}^\dagger|\psi\rangle = A^\dagger(x)\psi(x), \tag{3·4}$$

where $\psi(x) = \langle x|\psi\rangle$. The lowering and raising operators in the x-representation, $A(x)$ and $A(x)^\dagger$, are postulated to have the separable forms

$$A(x) = \frac{f(x)}{g(x)}\exp[-ih(x)]\, D\Big(\frac{1}{i}\frac{d}{dx}\Big)\,\frac{1}{f(x)g(x)} \tag{3·5}$$

and

$$A^\dagger(x) = -\frac{1}{f(x)g(x)} D\left(\frac{1}{i}\frac{d}{dx}\right)\frac{f(x)}{g(x)}\exp[ih(x)], \tag{3·6}$$

where $f(x)$, $g(x)^2$ and $h(x)$ are functions that take real values for $x \in \boldsymbol{R}$. It is essential to assume that the part-functions $f(x)$, $g(x)$ and $h(x)$ are continued analytically into the complex x plane. For brevity, dependence of the part-functions on the q-parameter is not explicitly expressed. These functions are determined by the conditions that the ladder operators satisfy the q-mutator in Eq. (2·2) and reduce to those of the ordinary (non-deformed) harmonic oscillator in the limit $q \to 1$. By definition, the function $f(x)$ has an intrinsic uncertainty of an arbitrary multiplicative constant, and the sign of the function $g(x)$ also is indeterminate. As will be clarified in §§5 and 6 and in the Appendix, the function $g(x)$ has additional intrinsic freedoms of uncertainty in q-deformed oscillator systems of the Dubna type.

As a necessary condition that the basic q-mutator in Eq. (2·2) includes a constant term, the parameters s and t must satisfy one of the following conditions:

$$s \neq 0 \text{ and } t = 0 \quad (s = 0 \text{ and } t \neq 0) \tag{3·7}$$

or

$$s + t = 0. \tag{3·8}$$

Without loss of generality, we choose the parameter s the "basic" one. Namely, the parameter t is eliminated by choosing the condition $t = 0$ in Eq. (3·7) and by setting $t = -s$ in Eq. (3·8). As shown in subsequent sections, systems subject to the former and latter conditions are identified generically as q-oscillators of the Macfarlane type and the Dubna type, respectively.

The x-representation of the Hamiltonian $\hat{H}_q$, H_q, is constructed as follows:

$$\langle x|\hat{H}_q|\psi\rangle = H_q\psi(x) = \frac{1}{2}\left[qA(x)A^\dagger(x) + q^{-1}A^\dagger(x)A(x)\right]\psi(x). \tag{3·9}$$

The ground-state eigenfunction $\psi_0(x)$ of the Hamiltonian H_q which is annihilated by the lowering operator (i.e., $A(x)\psi_0(x) = \langle x|\hat{A}|0\rangle = 0$) is determined to be

$$\psi_0(x) = \langle x|0\rangle = K_0(I)f(x)g(x), \tag{3·10}$$

where the normalization constant $K_0(I)$ is given by

$$K_0(I) = \left[\int_I f(x)^2|g(x)|^2dx\right]^{-\frac{1}{2}}. \tag{3·11}$$

For the wave functions $\psi(x)$ and $\phi(x)$, the inner product is defined by

$$\langle\psi|\phi\rangle = \int_I dx\,\langle\psi|x\rangle\langle x|\phi\rangle = \int_I dx\,\psi^*(x)\phi(x), \tag{3·12}$$

where I is the domain of integration which depends on global structure of the q-deformed oscillators specified in §6. For later use, it is necessary to prove that the operators $A(x)$ and $A^\dagger(x)$ are mutually adjoint on the Hilbert space generated by eigenstates of the Hamiltonian H_q.

For the following analysis, it is convenient to introduce the function

$$F(x) = \left[\frac{f(x+is)}{f(x)}\right]^2. \tag{3·13}$$

Note that, owing to the analyticity of the function $f(x)$, the function $F(x)$ must satisfy the condition $\lim_{s\to 0} F(x) = 1$.

§4. q-deformed harmonic oscillators of the Macfarlane type ($q = e^{s^2}$)

Let us first investigate q-deformed harmonic oscillators of the Macfarlane type. In this case ($t = 0$), where the deformation parameter is given by $q = e^{s^2} \geq 1$, there exist upper as well as lower bounds on the energy spectrum for every definite q except for $q \neq 1$. In fact, we have

$$\frac{1}{2} \leq E_n < \frac{1}{2} + \frac{1}{q^2 - 1}. \tag{4·1}$$

In the limit $s \to \infty$, all the eigenvalues accumulate at $\frac{1}{2}$. For the right-hand side of the basic q-mutator to be 1, the functions $f(x)$, $g(x)$ and $h(x)$ must satisfy the following three relations:

$$\frac{q - q^{-1}}{s^2 g(x)^4} = 1, \tag{4·2}$$

$$\begin{aligned}&\left[\frac{f(x-is)}{f(x)g(x)^2} + \frac{f(x)}{f(x-is)g(x-is)^2}\right] \\ &\quad = q^{-2}\left[\frac{f(x)}{f(x-is)g(x)^2} + \frac{f(x-is)}{f(x)g(x-is)^2}\right]\exp\{i[h(x) - h(x-is)]\}\end{aligned} \tag{4·3}$$

and

$$f(x)^2 f(x-2is)^2 = q^{-2} f(x-is)^4 \exp\{i[h(x) - h(x-2is)]\}. \tag{4·4}$$

Taking the sign ambiguity of the function $g(x)$ into consideration, we choose the solution of Eq. (4·2) as

$$g(x) = \left(\frac{q - q^{-1}}{s^2}\right)^{\frac{1}{4}}. \tag{4·5}$$

The fact that $g(x)$ is a constant simplifies Eq. (4·3) into the form

$$\exp\{i[h(x) - h(x-is)]\} = q^2, \tag{4·6}$$

which results in

$$h(x) - h(x-is) = -2is^2 + 2l\pi, \tag{4·7}$$

with arbitrary integer l. This difference equation has the general solution

$$h(x) = -2sx - i\frac{2l\pi}{s}x + \sum_{n=-\infty}^{\infty} a_n \exp\left(\frac{2n\pi}{s}x\right), \tag{4·8}$$

where the coefficients a_n are arbitrary. Note that, as a consequence of the difference equation, a periodic function of x of period is appears here as a power series in the scaled variable x/s in the solution $h(x)$. Owing to the condition that the function $h(x)$ must be real for $x \in \boldsymbol{R}$, the integer l must be 0. Further, for the function $h(x)$ to be definite for $x \in \boldsymbol{R}$ in the limit $s \to 0$, it must be the case that $a_n = 0$ for $n \neq 0$. As a result, we obtain

$$h(x) = -2sx + a_0. \tag{4·9}$$

In terms of the function $F(x)$ in Eq. (3·13), the relation (4·4) can be expressed by

$$F(x + is) = q^2 F(x), \tag{4·10}$$

which has the general solution

$$F(x) = \left[\sum_{n=-\infty}^{\infty} b_n \exp\left(\frac{2n\pi}{s}x\right)\right] \exp\left(-2isx\right). \tag{4·11}$$

Here, a periodic function of period is appears as a multiplicative uncertainty. Owing to the necessary condition $\lim_{s\to 0} F(x) = 1$, the coefficients in the periodic function are severely restricted to satisfy $b_0 \neq 0$ and $b_n = 0$ $(n \neq 0)$. Consequently, we obtain

$$F(x) \equiv \left[\frac{f(x+is)}{f(x)}\right]^2 = q\exp\left(-2isx\right) = \exp(s^2 - 2isx). \tag{4·12}$$

The coefficient $b_0 = q$ is specified from the definite form of the function $f(x)$ derived below. Then, solving Eq. (4·12) for $f(x)$, we find

$$\begin{aligned} f(x) &= \sum_{m=-\infty}^{\infty} c_m \exp\left(-\frac{s^2}{8} - \frac{2m^2\pi^2}{s^2}\right) \exp\left(\frac{2m\pi}{s}x\right) \exp\left[-\frac{1}{2}\left(x - i\frac{1}{2}s\right)^2 - i\frac{1}{2}sx\right] \\ &= \sum_{m=-\infty}^{\infty} c_m \exp\left[-\frac{1}{2}\left(x - \frac{2m\pi}{s}\right)^2\right], \end{aligned} \tag{4·13}$$

which is real for $x \in \boldsymbol{R}$. The coefficients c_m here must be chosen so as to make $f(x)$ square integrable, i.e.,

$$\int_{-\infty}^{\infty} f(x)^2 dx = \sqrt{\pi} \sum_{n,m} c_n c_m \exp\left[-\frac{(m-n)^2\pi^2}{s^2}\right] < \infty. \tag{4·14}$$

In the derivation of the part-function in Eq. (4·13), it is implicitly assumed that the central Gaussian component with $m = 0$ in the sum that survives in the limit $s \to 0$ is symmetric with respect to the origin, $x = 0$.

In this way, all part-functions of the ladder operators $A(x)$ and $A^\dagger(x)$ in Eqs. (3·5) and (3·6) have been obtained. In the case of $h(x) = -2sx$, where $a_0 = 0$ in Eq. (4·9), we find the solution of the Macfarlane type that was investigated by Shabanov.[5)]

In order to derive the eigenfunction of the q-Hamiltonian in Eq. (2·1), we utilize the relation among the ladder operators

$$A^{\dagger 2} = i\left(q - q^{-1}\right)^{-\frac{1}{2}} \exp[ih(x)] \left\{q^{-1}\exp(2isx) - 1\right\} A^\dagger \\ - q^{-1}\exp\left\{2i[h(x) + sx]\right\}\left(q^{-2}A^\dagger A + q^{-1}\right), \tag{4·15}$$

which leads readily to the recursion formula

$$\psi_{n+1}(x) = -i\left[1 - q^{-2(n+1)}\right]^{-\frac{1}{2}} \exp[ih(x)]\left[q^{-1}\exp(2isx) - 1\right]\psi_n(x) \\ - \exp\{2i[h(x) + sx]\}q^{-1}\left[\frac{1 - q^{-2n}}{1 - q^{-2(n+1)}}\right]^{\frac{1}{2}}\psi_{n-1}(x). \tag{4·16}$$

To extract the q-deformed Hermite functions from the eigenfunctions, which reduce properly to the Hermite polynomial in the limit $s \to 0$, we set

$$\psi_n(x) = K_0\, s^n\, f(x)g(x)\exp\left\{in[h(x) + sx]\right\} \\ \times \prod_{m=0}^{n-1}\left[q(1 - q^{-2(m+1)})\right]^{-\frac{1}{2}} H_n(x; q^{-1}), \tag{4·17}$$

provided that $\prod_{m=0}^{-1} 1/\sqrt{q(1 - q^{-2(m+1)})} = 1$. Then, Eq. (4·16) gives rise to the recursion formula of the q-Hermite function $H_n(x; q)$ as

$$H_{n+1}(x; q^{-1}) = \frac{i}{s}\left[q^{\frac{1}{2}}\exp(-isx) - q^{-\frac{1}{2}}\exp(isx)\right]H_n(x; q^{-1}) \\ -\frac{1}{s^2}(1 - q^{-2n})H_{n-1}(x; q^{-1}). \tag{4·18}$$

It is straightforward to prove that the relation $N_n\,\psi_{n+1} = N_{n+1}A^\dagger\psi_n$ is equivalent to the second recursion formula of the q-Hermite function as follows:

$$is\left[q^{-\frac{1}{2}}\exp(isx) + q^{\frac{1}{2}}\exp(-isx)\right]H_{n+1}(x; q^{-1}) \\ = q^{-n}\left[q^{-1}\exp(2isx)H_n(x - is; q^{-1}) - q\exp(-2isx)H_n(x + is; q^{-1})\right]. \tag{4·19}$$

With these formulas and the conditions $H_0(x; q^{-1}) = 1$ and $H_{-1}(x; q^{-1}) = 0$, we find that the q-Hermite function has the power series representation

$$H_n(x; q^{-1}) = \left(\frac{i}{s}\right)^n \sum_{m=0}^{n}(-1)^m q^{-\frac{(2m-n)}{2}}\left[\begin{array}{c} n \\ m \end{array}\right]_{q^{-1}} \exp\left[i(2m - n)sx\right], \tag{4·20}$$

where the q-binomial coefficient is defined by

$$\left[\begin{array}{c} n \\ m \end{array}\right]_z = \frac{\prod_{k=0}^{n-1}(1-z^{2(k+1)})}{\prod_{k=0}^{n-m-1}(1-z^{2(k+1)})\prod_{k=0}^{m-1}(1-z^{2(k+1)})}, \tag{4·21}$$

provided that $\prod_{k=0}^{-1}(1-z^{2(k+1)})=1$. This function $H_n(x;q)$ is the q-Hermite function of the Macfarlane type. [5)]

§5. q-deformed harmonic oscillators of the Dubna type ($q = e^{-s^2}$)

In a q-deformed harmonic oscillator of the Dubna type, no upper bound exists in the eigenvalue spectrum in Eq. (2·7), since $q = e^{-s^2} \leq 1$. In sharp contrast to the case of an ordinary harmonic oscillator, where the energy eigenvalue increases with equal spacing, the spacing between adjacent eigenvalues increases in powers of $q^{-2} = \exp(2s^2)$: $E_{n+1} - E_n = \exp[2(n+1)s^2]$.

For the q-deformed commutation relation in Eq. (2·2) to hold, the functions $f(x)$, $g(x)$ and $h(x)$ must satisfy the relations

$$\left[\frac{f(x+2is)}{f(x+is)}\right]^2 = q^{-2}\left[\frac{f(x+is)}{f(x)}\right]^2 \exp\{i[h(x)-h(x+2is)]\}, \tag{5·1}$$

$$\left[\frac{f(x)}{f(x-is)}\right]^2 = q^{-2}\left[\frac{f(x-is)}{f(x-2is)}\right]^2 \exp\{i[h(x)-h(x-2is)]\} \tag{5·2}$$

and

$$\begin{aligned} q&\left[\frac{f(x)^2 f(x+is)^{-2}}{g(x+is)^2} + \frac{f(x)^2 f(x-is)^{-2}}{g(x-is)^2}\right] \\ &-q^{-1}\left[\frac{f(x)^{-2} f(x+is)^2}{g(x+is)^2} + \frac{f(x)^{-2} f(x-is)^2}{g(x-is)^2}\right] = -4s^2 g(x)^2. \end{aligned} \tag{5·3}$$

The two relations in Eqs. (5·1) and (5·2) require that the functions $h(x)$ and $f(x)$, which are analytic in the complex x plane, satisfy the difference equations

$$h(x) - h(x+2is) = l\pi \tag{5·4}$$

with $l \in \boldsymbol{Z}$, and

$$F(x+is) = q^{-2}\exp(il\pi)F(x). \tag{5·5}$$

General solutions of these equations are given by

$$h(x) = i\frac{l\pi}{2s}x + \sum_{n=-\infty}^{\infty} a_n \exp\left(\frac{n\pi}{s}x\right) \tag{5·6}$$

and

$$F(x) \equiv \left[\frac{f(x+is)}{f(x)}\right]^2 = \sum_{n=-\infty}^{\infty} b_n \exp\left(\frac{2n\pi}{s}x + \frac{l\pi}{s}x - 2isx\right). \qquad (5\cdot7)$$

The conditions that $h(x)$ is real for $x \in \boldsymbol{R}$ and has a definite limit as $s \to 0$ specify the function $h(x)$ to be

$$h(x) = a_0. \qquad (5\cdot8)$$

Just as in the previous section, the function $F(x)$ in the present case $[q = \exp(-s^2)]$ is uniquely determined by the condition $\lim_{s\to 0} F(x) = 1$ as

$$F(x) \equiv \left[\frac{f(x+is)}{f(x)}\right]^2 = q^{-1}\exp(-2isx) = \exp(s^2 - 2isx). \qquad (5\cdot9)$$

This is identical to the function $F(x)$ in Eq. (4·12), which was obtained in the case $q = \exp(s^2)$. Therefore, we find that in the cases of q-deformed oscillators of both the Macfarlane and Dubna types, the part-function $f(x)$ takes the same form, which is given in Eq. (4·13).

The remaining equation (5·3) is expressed by

$$F(x)\left[q^3 g(x+is)^2 - q^{-1}g(x-is)^2\right] - F(x)^{-1}\left[q^{-3}g(x+is)^2 - qg(x-is)^2\right]$$
$$= -4s^2 g(x)^2 g(x-is)^2 g(x+is)^2 \qquad (5\cdot10)$$

in terms of the function $F(x)$ in Eq. (3·13). In the Appendix, we solve this nonlinear difference equation for $g(x)^2$, which must be real for $x \in \boldsymbol{R}$, and obtain

$$g^{\kappa,\lambda}_{\mu,\nu}(x)^2 = G^{\kappa,\lambda}_{\mu,\nu}(x)\left(\frac{q^{-1}-q}{s^2}\right)^{\frac{1}{2}} \cos sx, \qquad (5\cdot11)$$

with

$$G^{\kappa,\lambda}_{\mu,\nu}(x) = \tanh^{\kappa}\left[\frac{(2\mu+1)\pi}{2s}x\right]\coth^{\lambda}\left[\frac{(2\nu+1)\pi}{2s}x\right], \qquad (5\cdot12)$$

where κ and λ are arbitrary real numbers, and μ and ν are arbitrary integers. It is essential to recognize that the factors $G^{\kappa,\lambda}_{\mu,\nu}(x)$ satisfy the relations

$$G^{\kappa,\lambda}_{\mu,\nu}(x)\exp\left(is\frac{d}{dx}\right)G^{\kappa,\lambda}_{\mu,\nu}(x) = \exp\left(is\frac{d}{dx}\right). \qquad (5\cdot13)$$

Owing to these relations, the factors $G^{\kappa,\lambda}_{\mu,\nu}(x)$ have no influence on the structure of the ladder operators $A(x)$ and $A^\dagger(x)$. Therefore, it is possible to interpret that these factors are redundant. Then, the infinite number of solutions in Eq. (5·11) can be reduced to the simplest choice,

$$g(x) = \left(\frac{q^{-1}-q}{s^2}\right)^{\frac{1}{4}}\sqrt{\cos sx}. \qquad (5\cdot14)$$

Note that here again the phase of the part-function $g(x)$ is irrelevant, since only the product of this part-function appears in the ladder operators. With the part-functions $f(x)$, $g(x)$ and $h(x)$ thus obtained, the x dependence of the ladder operators is locally and naturally fixed.

In the previous section, Eq. (4·15) relating the operators $A^{\dagger 2}$, $A^\dagger A$ and $A^\dagger$ enabled us to derive the recursion formula for the eigenfunction of the Hamiltonian. In the case of q-deformed oscillators of the Dubna type, there exists no such relation. Following Mir-Kasimov, [10),11)] we introduce here a new operator as

$$T = \frac{1}{g(x)} \cosh\left(is\frac{d}{dx}\right)\frac{1}{g(x)}, \tag{5·15}$$

which also is not influenced by the factor $G^{\kappa,\lambda}_{\mu,\nu}(x)$, due to the relation (5·13). Among the operator T and the ladder operators, there exists the bilinear relation

$$T^2 = s^2 q^{-1}\left(A^\dagger A + \frac{q}{1-q^2}\right). \tag{5·16}$$

Therefore, the operator T^2 and the Hamiltonian H_q have the common eigenfunction $\psi_n(x)$ satisfying

$$T^2\psi_n(x) = s^2\frac{q^{-2n}}{1-q^2}\,\psi_n(x). \tag{5·17}$$

As the square-root of T^2, the operator T satisfies

$$T\psi_n(x) = \pm s\left(\frac{q^{-2n}}{1-q^2}\right)^{\frac{1}{2}}\psi_n(x), \tag{5·18}$$

since $\psi_n(x)$ is naturally assumed to be non-degenerate. Without loss of generality, the positive eigenvalue in Eq. (5·18) can be taken in the following argument. It is straightforward to prove that there exists the linear relation

$$T = \frac{s}{2\sin sx}\left[\frac{1}{\sqrt{q}}\exp[ih(x)]A + \sqrt{q}\exp\left[-ih(x)\right]A^\dagger\right] \tag{5·19}$$

among T and the ladder operators. Applying this relation to the eigenfunction $\psi_n(x)$, we find the recursion formula

$$\begin{aligned}\psi_{n+1}(x) = {}& 2\left[\frac{1}{1-q^{2(n+1)}}\right]^{\frac{1}{2}}\sin sx\,\exp\left[ih(x)\right]\,\psi_n(x)\\ &-\left\{\frac{(1-q^{2n})^2}{\left[1-q^{2n}\right]\left[1-q^{2(n+1)}\right]}\right\}^{\frac{1}{2}}\exp[2ih(x)]\,\psi_{n-1}(x).\end{aligned} \tag{5·20}$$

In parallel with Eq. (4·17) for the q-deformed oscillator of the Macfarlane type, let us define the q-deformed Hermite function $H_n(x;q)$ of the Dubna type by

$$\psi_n(x) = K_0 s^n f(x) g(x)\exp[inh(x)]\prod_{m=0}^{n-1}\left[1-q^{2(m+1)}\right]^{-\frac{1}{2}}H_n(x;q) \tag{5·21}$$

provided that $\prod_{m=0}^{-1} 1/\sqrt{1-q^{2(m+1)}} = 1$. Then, the relation (5·20) leads to the recursion formula

$$H_{n+1}(x;q) = \frac{2}{s}\sin sx\, H_n(x;q) - \frac{1}{s^2}\left(1-q^{2n}\right)H_{n-1}(x;q) \qquad (5{\cdot}22)$$

for the q-Hermite function. In analogy to Eq. (4·19), we find the second recursion formula

$$\begin{aligned} &2is\cos sxH_{n+1}(x;q) \\ &\quad = q^{-n}\left[\exp(2isx)H_n(x-is;q) - \exp(-2isx)H_n(x+is;q)\right]. \end{aligned} \qquad (5{\cdot}23)$$

From these formulas, we obtain the power series representation[10),14)]

$$H_n(x;q) = \left(\frac{i}{s}\right)^n \sum_{m=0}^{n} (-1)^m \begin{bmatrix} n \\ m \end{bmatrix}_q \exp\left[i(2m-n)sx\right]. \qquad (5{\cdot}24)$$

Note that the q-Hermite functions in Eqs. (4·20) and (5·24) are periodic functions of the same period $2\pi/s$.

The simplest choice of $h(x)$ for which the q-Hermite function reduces to the ordinary Hermite function in the limit $q \to 1$ is $h(x) = 0$. In this case, the eigenfunctions $\psi_n(x)$ are the same as those of the Kasimov solutions,[10)] up to factors of the cosine function related to a measure function that was introduced in the inner product in Kasimov's formalism in order to cancel singularities of the ladder operators.

It should be emphasized that the inner product is defined by Eq. (3·12) for all types of oscillators in the present unified theory. As shown at the end of the next section, singularities of the ladder operators are cancelled by zeros of the eigenfunctions without recourse to a measure function.

§6. Global structure of the operators and state vectors of *q*-deformed oscillator systems

In the preceding two sections, the part-functions $f(x)$, $g(x)$ and $h(x)$ of the ladder operators were derived locally from the condition that the q-mutator holds locally for each value of the coordinate x. With these components, we determine here the global structure of the ladder operators and the eigenfunctions of the Hamiltonian operator.

In oscillator systems of both the Macfarlane and Dubna types, the function $f(x)$ is represented as an arbitrary superposition of Gaussian functions centered at the positions $x = \frac{2\pi}{s}\times$(integers). The freedom existing in the way of choosing the superposition allows different types of the global structure of the ladder operators and the state vectors. More precisely, different global structures of the system are realized with different choices of the coefficients c_m in Eq. (4·13). We distinguish two kinds of global structure as follows:

- **Aperiodic structure**

The domain of the oscillator coordinate x is taken to be the infinite interval $I_\infty = (-\infty, \infty)$. The part-function $f(x)$ is defined over the whole interval I_∞ by the superposition

$$f(x) = \sum_{m=-\infty}^{\infty} c_m(s) f_m(x) \tag{6·1}$$

of the Gaussian functions

$$f_m(x) = \exp\left[-\frac{1}{2}\left(x - \frac{2m\pi}{s}\right)^2\right], \tag{6·2}$$

with the coefficients $c_m(s)$ that make the ground-state wave function $\psi_0(x)$ in Eq. (3·10) square-integrable. For the definite choice of the function $f(x)$, the ladder operators are defined in Eqs. (3·5) and (3·6), and the eigenvalue problem for the Hamiltonian consisting of those ladder operators is solved. There are systems with finite and infinite numbers of Gaussian factors. Note that the choice $c_m = c_{m'}$ for all m, m' is forbidden. Therefore, any choice of $f(x)$ of this kind over the whole interval I_∞ cannot allow for the periodic nature of the function $g(x)$ in Eqs. (4·5) and (5·14) and the q-Hermite functions in Eqs. (4·20) and (5·24) to be preserved.

- **Periodic structure**

  In this case, the domain of the oscillator coordinate I_∞ is taken to be the infinite sum of the finite interval as

$$I_\infty = \bigcup_{m=-\infty}^{\infty} I_m \tag{6·3}$$

where

$$I_m = \left[\frac{(2m-1)\pi}{s}, \frac{(2m+1)\pi}{s}\right]. \tag{6·4}$$

In the defining equations (3·5) and (3·6), the ladder operators $A(x)$ and $A^\dagger(x)$ are constructed independently on each interval I_m with the part-function $f_m(x)$ in Eq. (6·2). All operators and functions including the part-function $f(x)$ must be defined in this way. Then, all of them are smoothly connected over the whole interval I_∞. The resulting q-deformed oscillator system is periodic, in conformity with the property of the part-functions $g(x)$ and the q-Hermite functions.

The q-deformed harmonic oscillators of both the Macfarlane and Dubna types can have these global structures. Therefore, in addition to the periodic solution that have already been studied intensively,[5),6)] there exists an infinite variety of aperiodic solutions for the q-deformed oscillators of both types.

All eigenfunctions $\psi_n(x)$ in Eqs. (4·17) and (5·21) are proportional to the part-function $g(x)$. This is essential in an oscillator system of the Dubna type. This important characteristic arises from the fact that all eigenfunctions include a product of the part-function $f(x)$ and the q-Hermite function. The mechanism which causes the part-function $g(x)$ to be proportional to $\cos sx$ is contained explicitly in the second recursion formula (5·23). Therefore, every element in the Hilbert space $\mathcal{H}$ generated by the eigenfunctions $\psi_n(x)$ is considered to include the part-function $g(x)$. In an oscillator system of the Dubna type, the ladder operators, being inversely proportional to $g(x)$, have singularities at the values $x = \frac{\pi}{2s} \times$ (odd number). These singularities are cancelled by the zero-points of the elements of the Hilbert space $\mathcal{H}$ in the inner product. Owing to this cancellation, the raising operator $A^\dagger(x)$ is proved to be adjoint to the lowering operator $A(x)$ in the Hilbert space $\mathcal{H}$.

§7. Discussion

In this constructive approach, q-deformed harmonic oscillators of the Macfarlane and Dubna types were proved to appear as coordinate representations of the same algebra of the q-deformed ladder operators for disconnected sectors $q > 1$ and $q < 1$ of the deformation parameter q. In each case, the eigenvalues of the Hamiltonian given by the q-deformed anti-commutator constitute a spectrum that can be expressed by a single function $E_n(q)$ of the parameter q, irrespective of the choice of the representation. Specifically, the single function $E_n(q)$ represents the energy spectra of oscillators of the Macfarlane type for $q > 1$ and those of the Dubna type for $q < 1$.

In §§4 and 5, the part-functions of the ladder operators were obtained for the two types of oscillators. The part-functions $f(x)$ in the two cases turned out to be identical. Furthermore, the other part-functions $g(x)$ and $h(x)$ in the two cases, which are seemingly different, can be unified by using parametric representations as follows:

$$g(x) = \left(\frac{e^{s^2} - e^{-s^2}}{s^2}\right)^{\frac{1}{4}} \sqrt{\cos tx} \tag{7·1}$$

and

$$h(x) = -2(s+t)x + a_0. \tag{7·2}$$

This sort of unification is possible, because all the differences between the two types of oscillators stem simply from different choices of the s and t parametrization in the part-function D of the momentum operator. Therefore, it is not unnatural to expect that the eigenfunctions derived in §§4 and 5 have a common parametric representation. In fact, the eigenfunctions $\psi_n(x)$ in Eqs. (4·17) and (5·21) have the integrated form

$$\psi_n(x) = K_0 s^n f(x) g(x) \exp\{in[h(x) + (s+t)x]\}$$
$$\times \prod_{m=0}^{n-1} \left\{ e^{(s+t)^2} \left[1 - e^{-2s^2(m+1)}\right] \right\}^{-\frac{1}{2}} H_n(x; e^{-s^2}), \tag{7·3}$$

with the unified q-Hermite function $H_n(x; e^{-s^2})$, which satisfies the two recursion formulas

$$H_{n+1}(x; e^{-s^2}) = \frac{i}{s} \left(e^{\frac{1}{2}(s+t)^2 - isx} - e^{-\frac{1}{2}(s+t)^2 + isx} \right) H_n(x; e^{-s^2})$$

$$-\frac{1}{s^2}(1-e^{-2ns^2})H_{n-1}(x;e^{-s^2}) \qquad (7\cdot4)$$

and

$$is\left\{\exp\left[isx-\frac{1}{2}(s+t)^2\right]+\exp\left[-isx+\frac{1}{2}(s+t)^2\right]\right\}H_{n+1}(x;e^{-s^2})$$
$$=e^{-ns^2}\left\{\exp\left[2isx-(s+t)^2\right]H_n(x-is;e^{-s^2})\right.$$
$$\left.-\exp\left[-2isx+(s+t)^2\right]H_n(x+is;e^{-s^2})\right\}. \qquad (7\cdot5)$$

The power series representation of the unified q-Hermite function is given by

$$H_n(x;e^{-s^2})=\left(\frac{i}{s}\right)^n\sum_{m=0}^{n}(-1)^m\left[\begin{array}{c}n\\m\end{array}\right]_{e^{-s^2}}\exp\left\{(2m-n)\left[isx-\frac{1}{2}(s+t)^2\right]\right\}. \qquad (7\cdot6)$$

The parametric unification realized in this way is a direct proof of the close relationship of the two types of q-deformed oscillator systems.

The main difference between the two types of the q-deformed oscillators appears in the part-function $g(x)$. However, since the ladder operators are more basic than the individual part-functions, we realize that these differences are not essential. In fact, the singularities arising from the factor $1/g(x)$ in the ladder operators do not cause any harmful effect on their action and the redundant factors investigated in the Appendix have no influence to the ladder operators themselves.

The global structure considered in the previous section is also a generic property of the q-deformed harmonic oscillators investigated in the present formalism. The existence of the periodic solution and the infinite number of aperiodic solutions shows a rich structure of our q-deformed oscillator system.

In this way, we have developed a unified theory for seemingly different types of q-deformed harmonic oscillators. In spite of its mathematical beauty, however, the physical implications of this theory have not yet been clarified, and realistic applications of the q-deformed harmonic oscillator [15), 16)] are very limited at the present stage. Nevertheless, it is meaningful to further investigate systems of q-deformed oscillators in view of the important roles played in physics by the harmonic oscillator.

Appendix

Here we solve the difference equation (5·10) for $g(x)^2$. Noting that the function $F(x)$ is proportional to the factor $\exp(-2isx)$, we set

$$g(x)^2=\xi(x)\exp(-isx)+\eta(x)\exp(isx), \qquad (A\cdot1)$$

in which $\xi(x)$ and $\eta(x)$ are assumed not to include the factor $\exp(isx)$. Substitution of this expression into Eq. (5·10) results in the following equations for the unknown functions $\xi(x)$ and $\eta(x)$:

$$\frac{1}{q}\left[q^2\xi(x+is)-\xi(x-is)\right]=-4s^2\xi(x)\xi(x-is)\xi(x+is), \qquad (A\cdot2)$$

$$q\Big[q^2\xi(x-is)-q^{-4}\xi(x+is)\Big]$$
$$= -4s^2\Big[\xi(x)\eta(x-is)\eta(x+is)+q^2\eta(x)\xi(x-is)\eta(x+is) + q^{-2}\eta(x)\eta(x-is)\xi(x+is)\Big], \qquad (\text{A}\cdot 3)$$

$$\frac{1}{q}\Big[q^4\eta(x+is)-q^{-2}\eta(x-is)\Big]$$
$$= -4s^2\Big[\eta(x)\xi(x-is)\xi(x+is)+q^2\xi(x)\xi(x-is)\eta(x+is) + q^{-2}\xi(x)\eta(x-is)\xi(x+is)\Big] \qquad (\text{A}\cdot 4)$$

and

$$q\Big[\eta(x-is)-q^{-2}\eta(x+is)\Big] = -4s^2\eta(x)\eta(x-is)\eta(x+is). \qquad (\text{A}\cdot 5)$$

These simultaneous difference equations have the solutions

$$\xi(x)=\eta(x)=\frac{1}{2}\left(\frac{q^{-1}-q}{s^2}\right)^{\frac{1}{2}}\tanh^{\kappa}\left[\frac{(2\mu+1)\pi}{2s}x\right]\coth^{\lambda}\left[\frac{(2\nu+1)\pi}{2s}x\right], \qquad (\text{A}\cdot 6)$$

where κ and λ are arbitrary real numbers, and μ and ν are arbitrary integers. Consequently we find the solutions of Eq. (5·10) as

$$g^{\kappa,\lambda}_{\mu,\nu}(x)^2 = G^{\kappa,\lambda}_{\mu,\nu}(x)\left(\frac{q^{-1}-q}{s^2}\right)^{\frac{1}{2}}\frac{1}{2}\left[\exp(-isx)\pm\exp(isx)\right], \qquad (\text{A}\cdot 7)$$

where

$$G^{\kappa,\lambda}_{\mu,\nu}(x) = \tanh^{\kappa}\left[\frac{(2\mu+1)\pi}{2s}x\right]\coth^{\lambda}\left[\frac{(2\nu+1)\pi}{2s}x\right]. \qquad (\text{A}\cdot 8)$$

The condition that $g(x)^2$ be real for $x \in \boldsymbol{R}$ selects out the solutions

$$g^{\kappa,\lambda}_{\mu,\nu}(x)^2 = \left(\frac{q^{-1}-q}{s^2}\right)^{\frac{1}{2}} G^{\kappa,\lambda}_{\mu,\nu}(x)\cos sx. \qquad (\text{A}\cdot 9)$$

References

1) A. J. Macfarlane, J. of Phys. A **22** (1989), 4581.
2) G. Szegö, in *Gabor Szegö: Collected Papers 1926-6 Vol 1* (1915-1927), ed. R. A. Askey (Basel:Brikhauser, 1982), p. 793.
3) L. Carlitz, Duke Math. J. **41** (1957), 359.
4) G. Gasper and M. Rahman, *Basic Hypergeometric Series* (Cambridge University Press, Cambridge, 1990).
   R. Koekoek and R. F. Swarttouw, *The Askey-scheme of hypergeometric orthogonal polynomials and its q-analogue* (Report no. 98–17, Delft University of Technology, Delft, 1998).
5) S. V. Shabanov, Phys. Lett. B **293** (1992), 117.
6) A. K. Rajagopal, Phys. Rev. A **47** (1993), R3465.

7) E. D. Kagramanov, R. M. Mir-Kasimov and S. M. Nagiyev, J. Math. Phys. **31** (1990), 1773.
8) N. M. Atakishiev and S. K. Suslov, Teoreticheskaya i Matematicheskaya Fizika **85** (1990), 64.
9) N. M. Atakishiev and S. K. Suslov, Teoreticheskaya i Matematicheskaya Fizika **87** (1991), 154.
10) R. M. Mir-Kasimov, J. of Phys. A **24** (1991), 4283.
11) R. M. Mir-Kasimov, Centre de Recherches Mathematiques CRM Proceedings and Lecture Notes **9** (1996), 209.
12) S. M. Nagiev, Teoreticheskaya i Matematicheskaya Fizika **102** (1995), 247.
13) N. M. Atakishiyev, A. Frank and K. B. Wolf, J. Math. Phys. **35** (1994), 3253.
14) M. K. Atakishiyeva, N. M. Atakishiyev and C. Villegas-Blas, Journal of Computational and Applied Mathematics **99** (1998), 27.
15) Z. Chang and H. Yan, Phys. Rev. A **43** (1991), 6043; Phys. Lett. A **156** (1991), 192.
16) D. V. Anchishkin, A. M. Gravrilik and N. Z. Iorgov, Mod. Phys. Lett. A **15** (2000), 1637; Eur. Phys. J. A **7** (2000), 229.