Article

Mathematics and Mechanics of Solids
2014, Vol. 19(2) 193–211

DOI: 10.1177/1081286512458109
mms.sagepub.com

Unified magnetomechanical homogenization framework with application to magnetorheological elastomers

George Chatzigeorgiou, Ali Javili and Paul Steinmann
Chair of Applied Mechanics, University of Erlangen–Nuremberg, Erlangen, Germany

Received 2 July 2012; accepted 24 July 2012

Abstract
The aim of this work is to present a general homogenization framework with application to magnetorheological elastomers under large deformation processes. The macroscale and microscale magnetomechanical responses of the composite in the material and spatial description are presented and the conditions for a well-established homogenization problem in Lagrangian description are identified. The connection between the macroscopic magnetomechanical field variables and the volume averaging of the corresponding microscopic variables in the Eulerian description is examined for several types of boundary conditions. It is shown that the use of kinematic and magnetic field potentials instead of kinetic field and magnetic induction potentials provides a more appropriate homogenization process.

Keywords
Homogenization, magnetorheological elastomers, Hill–Mandel condition

1. Introduction

Magnetorheological elastomers (elastomers filled with magnetic particles) are magneto-sensitive composite materials whose mechanical behavior changes with the application of magnetic fields. Recently they have attracted significant research attention due to their interesting applications, which include adaptive engine mounts, vibration absorbers, suspension systems and automotive bushing [1–6] .

The electromechanical and magnetomechanical response of solids has been thoroughly studied in the past. Based on the classical laws of elasticity, electricity and magnetism, the general equations have been derived and several boundary-value problems have been solved [7–13]. In particular, for magnetorheological elastomers, modeling efforts have been presented recently [14–18]. In these efforts the magnetorheological elastomer is examined in a macroscopic level and general magnetomechanical laws are presented.

During the last few decades there have been extensive studies on composite materials. The overall behavior of these materials strongly depends on the properties of the material constituents and the microscopic geometry (volume fraction, shape and orientation of constituents) and the determination of the macroscopic response is a difficult task. The direct simulation of the composites including the microstructural characteristics is restricted by the computational cost, thus alternative approaches, based on micromechanics and homogenization methods, have been developed. A review of the different multi-scale approaches can be found in [19, 20]. The effective mechanical behavior of composites has been studied thoroughly by various researchers for general and periodic microstructure [21–26]. A computational scheme for homogenization of composites with nonlinear constituents has been presented by [27]. Homogenization of composites at finite strains has been considered by many authors

Corresponding author:
Paul Steinmann, Chair of Applied Mechanics, University of Erlangen–Nuremberg, Egerlandstraße 5, 91058 Erlangen, Germany.
Email: paul.steinmann@ltm.uni-erlangen.de

[28–34]. Very recently, homogenization approaches have been considered for the magnetomechanical response of magnetorheological elastomers (see, e.g., [35–39]). In these efforts the effective behavior of the composite is studied under certain assumptions on the constitutive behavior and the boundary conditions. The scope of this work is to present a general homogenization framework for magnetorheological elastomers under finite strains. We are investigating several types of boundary conditions that satisfy the Hill–Mandel conditions, as well as the transition between the material and spatial description of the macroscopic mechanical and magnetic field variables.

The structure of the paper is as follows. After introducing the notation and definitions that are used throughout this work, we define the problem under study in Section 2 and describe the field variables, the balance and the constitutive laws that govern the microstructure and the macrostructure. Sections 3 and 4 present the boundary conditions under which the Hill–Mandel conditions are satisfied in material configuration, while in Section 5 we study the transition of macroscopic variables from material to spatial description through volume averaging over the representative volume element. The paper closes with the conclusions of this work.

Notation and definitions

Direct notation is adopted throughout. Occasional use is made of index notation, the summation convention for repeated indices being implied. The scalar product of two vectors $\boldsymbol{a}$ and $\boldsymbol{b}$, i.e. the single contraction, is denoted $\boldsymbol{a} \cdot \boldsymbol{b} = [\boldsymbol{a}]_i[\boldsymbol{b}]_i$. The scalar product of two second-order tensors $\boldsymbol{A}$ and $\boldsymbol{B}$, i.e. the double contraction, is denoted by $\boldsymbol{A} : \boldsymbol{B} = [\boldsymbol{A}]_{ij}[\boldsymbol{B}]_{ij}$. The action of a second-order tensor $\boldsymbol{A}$ on a vector $\boldsymbol{a}$ is understood as $[\boldsymbol{A} \cdot \boldsymbol{a}]_i = [\boldsymbol{A}]_{ij}[\boldsymbol{a}]_j$ and $[\boldsymbol{a} \cdot \boldsymbol{A}]_i = [\boldsymbol{a}]_j[\boldsymbol{A}]_{ji}$. The double contraction of a third-order tensor $\boldsymbol{\mathcal{A}}$ and a second-order tensor $\boldsymbol{B}$ renders a vector according to $[\boldsymbol{\mathcal{A}} : \boldsymbol{B}]_i = [\boldsymbol{\mathcal{A}}]_{ijk}[\boldsymbol{B}]_{jk}$. The action of a third-order tensor $\boldsymbol{\mathcal{A}}$ on a vector $\boldsymbol{a}$, denoted by $\boldsymbol{\mathcal{A}} \cdot \boldsymbol{a}$, is a second-order tensor with components $[\boldsymbol{\mathcal{A}} \cdot \boldsymbol{a}]_{ij} = [\boldsymbol{\mathcal{A}}]_{ijm}[\boldsymbol{a}]_m$. The composition of two second-order tensors $\boldsymbol{A}$ and $\boldsymbol{B}$, denoted by $\boldsymbol{A} \cdot \boldsymbol{B}$, is a second-order tensor with components $[\boldsymbol{A} \cdot \boldsymbol{B}]_{ij} = [\boldsymbol{A}]_{im}[\boldsymbol{B}]_{mj}$. The tensor product of two vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ is a second-order tensor $\boldsymbol{D} = \boldsymbol{a} \otimes \boldsymbol{b}$ with $[\boldsymbol{D}]_{ij} = [\boldsymbol{a}]_i[\boldsymbol{b}]_j$. The tensor product of two second-order tensors $\boldsymbol{A}$ and $\boldsymbol{B}$ is a fourth-order tensor $\mathsf{D} = \boldsymbol{A} \otimes \boldsymbol{B}$ with $[\mathsf{D}]_{ijkl} = [\boldsymbol{A}]_{ij}[\boldsymbol{B}]_{kl}$. For a vector $\boldsymbol{a}$ and a second-order tensor $\boldsymbol{B}$ the tensor product is understood as $[\boldsymbol{a} \otimes \boldsymbol{B}]_{ijk} = [\boldsymbol{a}]_i[\boldsymbol{B}]_{jk}$ and $[\boldsymbol{B} \otimes \boldsymbol{a}]_{ijk} = [\boldsymbol{a}]_k[\boldsymbol{B}]_{ij}$. The two non-standard tensor products of two second-order tensors $\boldsymbol{A}$ and $\boldsymbol{B}$ are the fourth-order tensors $[\boldsymbol{A} \overline{\otimes} \boldsymbol{B}]_{ijkl} = [\boldsymbol{A}]_{ik}[\boldsymbol{B}]_{jl}$ and $[\boldsymbol{A} \underline{\otimes} \boldsymbol{B}]_{ijkl} = [\boldsymbol{A}]_{il}[\boldsymbol{B}]_{jk}$. The cross-product of two vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ is denoted by $\boldsymbol{a} \times \boldsymbol{b}$ with $[\boldsymbol{a} \times \boldsymbol{b}]_k = [\boldsymbol{\varepsilon}]_{ijk}[\boldsymbol{a}]_i[\boldsymbol{b}]_j$, where $\boldsymbol{\varepsilon}$ denotes the third-order permutation (Levi-Civita) tensor. With the aid of previously introduced operators the cross-product can be expressed in direct notation as $\boldsymbol{a} \times \boldsymbol{b} = [\boldsymbol{a} \otimes \boldsymbol{b}] : \boldsymbol{\varepsilon}$. Analogously, cross-products of a vector $\boldsymbol{a}$ and a second-order tensor $\boldsymbol{B}$ are defined as $\boldsymbol{a} \times \boldsymbol{B} = \boldsymbol{\varepsilon} : [\boldsymbol{a} \otimes \boldsymbol{B}]$ and $\boldsymbol{B} \times \boldsymbol{a} = [\boldsymbol{B} \otimes \boldsymbol{a}] : \boldsymbol{\varepsilon}$ or, alternatively, in index notation, $[\boldsymbol{a} \times \boldsymbol{B}]_{kl} = [\boldsymbol{\varepsilon}]_{ijk}[\boldsymbol{a}]_i[\boldsymbol{B}]_{jl}$ and $[\boldsymbol{B} \times \boldsymbol{a}]_{kl} = [\boldsymbol{\varepsilon}]_{ijl}[\boldsymbol{a}]_j[\boldsymbol{B}]_{ki}$.

Gradient, divergence and curl of an arbitrary quantity $\{\bullet\}$ with respect to the material configuration are defined as

$$\operatorname{Grad}\{\bullet\} := \{\bullet\} \otimes \partial_{\boldsymbol{X}}, \quad \operatorname{Div}\{\bullet\} := \operatorname{Grad}\{\bullet\} : \boldsymbol{I}, \quad \operatorname{Curl}\{\bullet\} := -\operatorname{Grad}\{\bullet\} : \boldsymbol{\varepsilon}.$$

In near-identical fashion, the counterparts of these operators with respect to the spatial configuration can be defined, i.e.

$$\operatorname{grad}\{\bullet\} := \{\bullet\} \otimes \partial_{\boldsymbol{x}}, \quad \operatorname{div}\{\bullet\} := \operatorname{grad}\{\bullet\} : \boldsymbol{i}, \quad \operatorname{curl}\{\bullet\} = -\operatorname{grad}\{\bullet\} : \boldsymbol{\varepsilon}.$$

Here $\boldsymbol{I} = \boldsymbol{i}$ denotes the second-order identity tensor $[\boldsymbol{I}]_{ij} = [\boldsymbol{i}]_{ij} = \delta_{ij}$ with δ_{ij} being the Kronecker delta.

In the following, all quantities corresponding to the magnetic problem are distinguished from those of classical continuum mechanics by using the blackboard font, i.e. $\mathbb{A}$, $\mathbb{B}$. Mechanical and magnetic quantities that present analogy in the unified homogenization framework are denoted with the same letter but different fonts. For instance, Piola stress tensor and material magnetic induction vector are represented with the letters $\boldsymbol{P}$ and $\mathbb{P}$, respectively (see Table 1 for further details). Quantities defined on the macroscopic scale are differentiated from those on the microscopic scale by an accent placed above the symbol. That is, $\{\bar{\bullet}\}$ refers to a macroscopic variable with its microscopic counterpart being $\{\bullet\}$. Unless stated otherwise, capital and small letters denote quantities in the material and spatial configurations respectively.

Table 1. Unified notation for mechanical and magnetic quantities.

$\boldsymbol{F}$	Material deformation gradient	$\mathbb{F}$	Material magnetic field
$\boldsymbol{f}$	Spatial deformation gradient	$\mathbb{f}$	Spatial magnetic field
$\boldsymbol{Y}$	Material deformation map	$\mathbb{Y}$	Material magnetic scalar potential
$\boldsymbol{y}$	Spatial deformation map	$\mathbb{y}$	Spatial magnetic scalar potential
$\boldsymbol{P}$	Piola stress	$\mathbb{P}$	Material magnetic induction
$\boldsymbol{p}$	Cauchy stress	$\mathbb{p}$	Spatial magnetic induction
$\boldsymbol{T}$	Material traction vector	$\mathbb{T}$	Material magnetic flux
$\boldsymbol{t}$	Spatial traction vector	$\mathbb{t}$	Spatial magnetic flux
$\boldsymbol{A}$	Piola stress tensor potential	$\mathbb{A}$	Material magnetic vector potential
$\boldsymbol{a}$	Cauchy stress tensor potential	$\mathbb{a}$	Spatial magnetic vector potential
$\boldsymbol{Z}$	Material position fluctuation function	$\mathbb{Z}$	Fluctuation function of $\mathbb{F}$
$\boldsymbol{z}$	Spatial position fluctuation function	$\mathbb{z}$	Fluctuation function of $\mathbb{f}$
$\boldsymbol{W}$	Piola stress fluctuation function	$\mathbb{W}$	Fluctuation function of $\mathbb{P}$
$\boldsymbol{s}$	Kirchhoff stress	$\mathbb{s}$	Alternative spatial form of $\mathbb{P}$
$\boldsymbol{0}$	Null mechanical quantity	$\mathbb{O}$	Null magnetic quantity

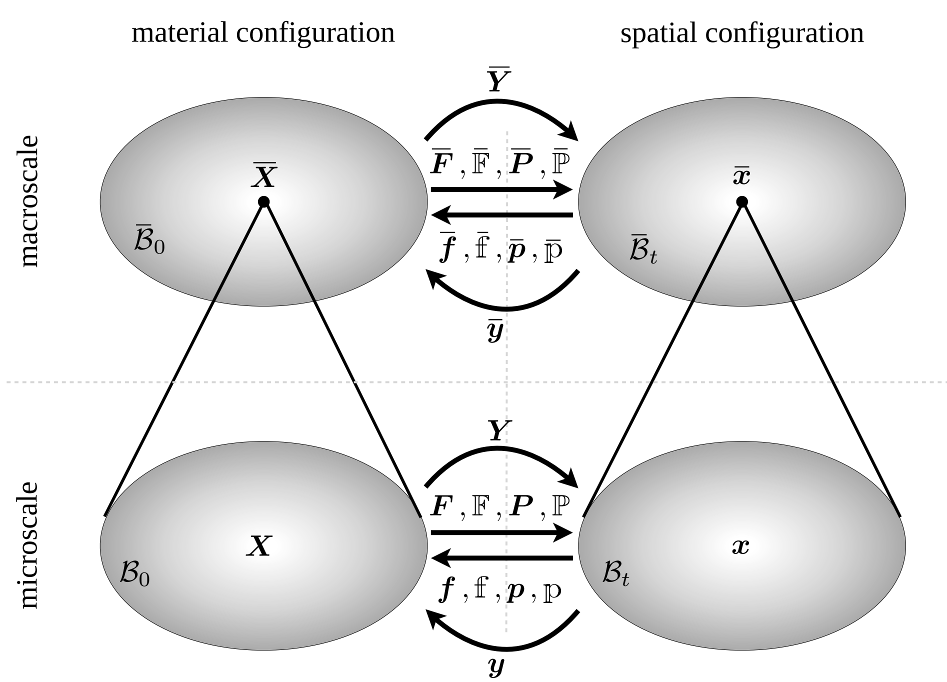

Figure 1. Macroscale and microscale variables.

2. Problem definition

Our intention is to study the behavior of a magnetorheological elastomer subjected to magnetomechanical loading. Since our material is a composite, consisting of an elastomer matrix and magnetic particles, we can consider two separate scales, the macroscale, which describes the continuum body, and the microscale, which describes the representative volume element (RVE) of the microstucture. As depicted in Figure 1, both the macroscale and microscale can be expressed in the material or in the spatial configuration. In the proceeding subsections of this section we are going to present the field variables and the main equations that describe the overall body and its microstructure.

The purpose of this preliminary section is to summarize certain key concepts in nonlinear continuum mechanics coupled to electromagnetics and to introduce the notation adopted here. Detailed expositions on nonlinear continuum mechanics can be found in [40–44], among others. For further details concerning the coupling of continuum mechanics and electromagnetics, see [7, 9, 10].

2.1. Microscopic problem

In the microscale we consider that the RVE in the undeformed (material) configuration (without any magnetic field) occupies the space $\mathcal{B}_0$ with volume V_0 and boundary surface $\partial\mathcal{B}_0$. The deformed (spatial) configuration is denoted by $\mathcal{B}_t$ and the current volume by V_t and boundary surface $\partial\mathcal{B}_t$. For static cases, the position vector $\boldsymbol{x}$ of a point in the spatial configuration $\mathcal{B}_t$ is described in terms of the position vector $\boldsymbol{X}$ of the point in the material configuration $\mathcal{B}_0$ by the nonlinear spatial motion map $\boldsymbol{x} = \boldsymbol{Y}(\boldsymbol{X})$ and the deformation is characterized by the spatial motion deformation gradient $\boldsymbol{F}$

$$\boldsymbol{F} = \mathrm{Grad}\boldsymbol{Y}. \tag{1}$$

In the absence of mechanical micro-body forces, the microscopic equilibrium equation in the material configuration is written

$$\mathrm{Div}\boldsymbol{P} = \boldsymbol{0}, \tag{2}$$

where $\boldsymbol{P}$ is the Piola stress. The equations of magnetostatics in terms of the Lagrangian magnetic induction $\mathbb{P}$ and the Lagrangian magnetic field $\mathbb{F}$ are given by

$$\mathrm{Div}\mathbb{P} = \mathbb{O}, \tag{3}$$

and

$$\mathrm{Curl}\mathbb{F} = \mathbb{O}, \tag{4}$$

respectively. In analogy with (4), the deformation gradient, given by (1), satisfies the compatibility condition

$$\mathrm{Curl}\boldsymbol{F} = \boldsymbol{0}. \tag{5}$$

The form of the coupled magnetomechanical equations, given by Equations (2), (5), (3) and (4), allow us to express the field variables $\boldsymbol{F}$, $\boldsymbol{P}$, $\mathbb{F}$ and $\mathbb{P}$ as functions of appropriate potentials. The deformation gradient can be expressed by the vector field $\boldsymbol{Y}$, as Equation (1) suggests. The magnetic induction is usually expressed in terms of the magnetic vector potential $\mathbb{A}$ through the relation

$$\mathbb{P} = \mathrm{Curl}\mathbb{A}, \tag{6}$$

which guarantees that Equation (3) is satisfied for every magnetic vector potential. In direct analogy, we can define a magnetic scalar potential $\mathbb{Y}$ that satisfies Equation (4) if it is connected with the magnetic field through the relation

$$\mathbb{F} = \mathrm{Grad}\mathbb{Y}. \tag{7}$$

The last field variable, the Piola stress, can be expressed in terms of a stress tensor potential $\boldsymbol{A}$ (second-order tensor) through the relation

$$\boldsymbol{P} = \mathrm{Curl}\boldsymbol{A}. \tag{8}$$

Equation (2) is satisfied for any choice of $\boldsymbol{A}$. Such a tensor potential is uncommon in the engineering literature, since it is the same size as the Piola stress tensor. In the current context, though, it allows us to treat the mechanical stress and the magnetic induction in a similar manner.

The constitutive relations that connect the stress with the deformation gradient and the magnetic field with the magnetic induction in the material configuration are usually provided through the introduction of appropriate energy density functions. We consider an energy density function $W_0^{\boldsymbol{F}\mathbb{P}}(\boldsymbol{F},\mathbb{P})$, such that the total Piola stress (including Maxwell stress and magnetization effect, see [10, 15, 16]) and the Lagrangian magnetic field are given by

$$\boldsymbol{P}(\boldsymbol{F},\mathbb{P}) = \frac{\partial W_0^{\boldsymbol{F}\mathbb{P}}}{\partial \boldsymbol{F}} \quad \text{and} \quad \mathbb{F}(\boldsymbol{F},\mathbb{P}) = \frac{\partial W_0^{\boldsymbol{F}\mathbb{P}}}{\partial \mathbb{P}}. \tag{9}$$

Assuming that the energy function is convex with respect to the deformation gradient and the magnetic induction, we can identify the following energy density functions through Legendre transformations,

$$\begin{aligned} W_0^{\boldsymbol{F}\mathbb{F}}(\boldsymbol{F},\mathbb{F}) &= \inf_{\mathbb{P}}\{W_0^{\boldsymbol{F}\mathbb{P}}(\boldsymbol{F},\mathbb{P}) - \mathbb{F}\cdot\mathbb{P}\}, \\ W_0^{\boldsymbol{P}\mathbb{P}}(\boldsymbol{P},\mathbb{P}) &= \inf_{\boldsymbol{F}}\{W_0^{\boldsymbol{F}\mathbb{P}}(\boldsymbol{F},\mathbb{P}) - \boldsymbol{P}:\boldsymbol{F}\}, \\ W_0^{\boldsymbol{P}\mathbb{F}}(\boldsymbol{P},\mathbb{F}) &= \inf_{\boldsymbol{F},\mathbb{P}}\{W_0^{\boldsymbol{F}\mathbb{P}}(\boldsymbol{F},\mathbb{P}) - \mathbb{F}\cdot\mathbb{P} - \boldsymbol{P}:\boldsymbol{F}\}. \end{aligned} \tag{10}$$

The above energy density functions provide alternative forms of the constitutive relations through the Legendre transformation that are given by

$$\begin{aligned}
\boldsymbol{P}(\boldsymbol{F},\mathbb{F}) &= \frac{\partial W_0^{\boldsymbol{F}\mathbb{F}}}{\partial \boldsymbol{F}} \quad \text{and} \quad \mathbb{P}(\boldsymbol{F},\mathbb{F}) = -\frac{\partial W_0^{\boldsymbol{F}\mathbb{F}}}{\partial \mathbb{F}} \quad \text{for the function} \quad W_0^{\boldsymbol{F}\mathbb{F}}(\boldsymbol{F},\mathbb{F}),\\
\boldsymbol{F}(\boldsymbol{P},\mathbb{P}) &= -\frac{\partial W_0^{\boldsymbol{P}\mathbb{P}}}{\partial \boldsymbol{P}} \quad \text{and} \quad \mathbb{F}(\boldsymbol{P},\mathbb{P}) = \frac{\partial W_0^{\boldsymbol{P}\mathbb{P}}}{\partial \mathbb{P}} \quad \text{for the function} \quad W_0^{\boldsymbol{P}\mathbb{P}}(\boldsymbol{P},\mathbb{P}),\\
\boldsymbol{F}(\boldsymbol{P},\mathbb{F}) &= -\frac{\partial W_0^{\boldsymbol{P}\mathbb{F}}}{\partial \boldsymbol{P}} \quad \text{and} \quad \mathbb{P}(\boldsymbol{P},\mathbb{F}) = -\frac{\partial W_0^{\boldsymbol{P}\mathbb{F}}}{\partial \mathbb{F}} \quad \text{for the function} \quad W_0^{\boldsymbol{P}\mathbb{F}}(\boldsymbol{P},\mathbb{F}).
\end{aligned} \tag{11}$$

The increments of the four energy density functions are written

$$\begin{aligned}
\delta W_0^{\boldsymbol{F}\mathbb{P}} &= \boldsymbol{P} : \delta\boldsymbol{F} + \mathbb{F}\cdot\delta\mathbb{P}, \qquad & \delta W_0^{\boldsymbol{F}\mathbb{F}} &= \boldsymbol{P} : \delta\boldsymbol{F} - \mathbb{P}\cdot\delta\mathbb{F},\\
\delta W_0^{\boldsymbol{P}\mathbb{P}} &= -\boldsymbol{F} : \delta\boldsymbol{P} + \mathbb{F}\cdot\delta\mathbb{P}, \qquad & \delta W_0^{\boldsymbol{P}\mathbb{F}} &= -\boldsymbol{F} : \delta\boldsymbol{P} - \mathbb{P}\cdot\delta\mathbb{F}.
\end{aligned} \tag{12}$$

In the spatial configuration the material position $\boldsymbol{X}$ is described by the deformation map $\boldsymbol{X} = \boldsymbol{y}(\boldsymbol{x})$ and the deformation is characterized by the inverse of the deformation gradient $\boldsymbol{f}$,

$$\boldsymbol{f} = \text{grad}\boldsymbol{y}. \tag{13}$$

The spatial configuration field variables are the inverse of the deformation gradient $\boldsymbol{f}$, the symmetric Cauchy stress tensor $\boldsymbol{p}$, the Eulerian magnetic induction $\mathbb{p}$ and the Eulerian magnetic field $\mathbb{f}$. These are expressed in terms of the deformation gradient $\boldsymbol{F}$, the Piola stress $\boldsymbol{P}$, the Lagrangian magnetic induction $\mathbb{P}$ and the Lagrangian magnetic field $\mathbb{F}$ through the relations

$$\boldsymbol{f} = \boldsymbol{I}\cdot\boldsymbol{F}^{-1}, \quad \boldsymbol{p} = J^{-1}\boldsymbol{P}\cdot\boldsymbol{F}^{\mathrm{t}}, \quad \mathbb{f} = \mathbb{F}\cdot\boldsymbol{F}^{-1}, \quad \mathbb{p} = J^{-1}\mathbb{P}\cdot\boldsymbol{F}^{\mathrm{t}}, \tag{14}$$

where $J = \text{Det}\boldsymbol{F}$. The coupled magnetomechanical system of equations in the material configuration holds in similar form in the spatial configuration. Namely,

$$\text{curl}\boldsymbol{f} = \boldsymbol{0}, \quad \text{div}\boldsymbol{p} = \boldsymbol{0}, \quad \text{curl}\mathbb{f} = \mathbb{O}, \quad \text{div}\mathbb{p} = \mathbb{O}. \tag{15}$$

In an analogous manner with the material description, we can identify appropriate potentials for the spatial description field variables. Specifically, the potential for the inverse of deformation gradient is the vector field $\boldsymbol{y}$ as (13) suggests, the potential for the Cauchy stress is the stress tensor potential $\boldsymbol{a}$, expressed by the relation

$$\boldsymbol{p} = \text{curl}\boldsymbol{a} \quad \text{with } \boldsymbol{a} = \boldsymbol{A}\cdot\boldsymbol{F}^{-1}, \tag{16}$$

the potential for the Eulerian magnetic induction is the magnetic vector potential $\mathbb{a}$, expressed by the relation

$$\mathbb{p} = \text{curl}\mathbb{a} \quad \text{with } \mathbb{a} = \mathbb{A}\cdot\boldsymbol{F}^{-1}, \tag{17}$$

and the potential for the Eulerian magnetic field is the magnetic scalar potential $\mathbb{y}(\boldsymbol{x}) \equiv \mathbb{Y}(\boldsymbol{X})$, expressed by the relation

$$\mathbb{f} = \text{grad}\mathbb{y}. \tag{18}$$

2.2. Macroscopic problem

In the macroscale we consider a continuum body that in the material configuration (without any magnetic field) occupies the space $\overline{\mathcal{B}}_0$ with boundary surface $\partial\overline{\mathcal{B}}_0$. In the spatial configuration the body occupies the space $\overline{\mathcal{B}}_t$ with boundary surface $\partial\overline{\mathcal{B}}_t$. For static cases, the position vector $\bar{\boldsymbol{x}}$ of a point in the spatial configuration is described in terms of the position vector $\overline{\boldsymbol{X}}$ of the point in the material configuration by the nonlinear spatial motion map $\bar{\boldsymbol{x}} = \overline{\boldsymbol{Y}}(\overline{\boldsymbol{X}})$ and the deformation is characterized by the spatial motion macroscopic deformation gradient $\overline{\boldsymbol{F}}$,

$$\overline{\boldsymbol{F}} = \overline{\text{Grad}}\,\overline{\boldsymbol{Y}}. \tag{19}$$

In the absence of mechanical body forces, the macroscopic equilibrium equation in the material configuration is written in terms of the macroscopic Piola stress $\overline{\boldsymbol{P}}$ as

$$\overline{\mathrm{Div}}\overline{\boldsymbol{P}} = \mathbf{0}, \tag{20}$$

The equations of magnetostatics in terms of the macroscopic Lagrangian magnetic induction $\overline{\mathbb{P}}$ and the Lagrangian magnetic field $\overline{\mathbb{F}}$ are given by

$$\overline{\mathrm{Div}}\overline{\mathbb{P}} = \mathbb{O} \quad \text{and} \quad \overline{\mathrm{Curl}}\overline{\mathbb{F}} = \mathbb{O}, \tag{21}$$

respectively. We identify the macroscopic energy density functions $\overline{W}_0^{F\mathbb{P}}$, $\overline{W}_0^{F\mathbb{F}}$, $\overline{W}_0^{P\mathbb{P}}$ and $\overline{W}_0^{P\mathbb{F}}$, whose increments are given by the relations

$$\begin{aligned} \delta\overline{W}_0^{F\mathbb{P}} &= \overline{\boldsymbol{P}} : \delta\overline{\boldsymbol{F}} + \overline{\mathbb{F}} \cdot \delta\overline{\mathbb{P}}, &\qquad \delta\overline{W}_0^{F\mathbb{F}} &= \overline{\boldsymbol{P}} : \delta\overline{\boldsymbol{F}} - \overline{\mathbb{P}} \cdot \delta\overline{\mathbb{F}}, \\ \delta\overline{W}_0^{P\mathbb{P}} &= -\overline{\boldsymbol{F}} : \delta\overline{\boldsymbol{P}} + \overline{\mathbb{F}} \cdot \delta\overline{\mathbb{P}}, &\qquad \delta\overline{W}_0^{P\mathbb{F}} &= -\overline{\boldsymbol{F}} : \delta\overline{\boldsymbol{P}} - \overline{\mathbb{P}} \cdot \delta\overline{\mathbb{F}}. \end{aligned} \tag{22}$$

In order to connect the macroscopic with the microscopic variables we define the following volume and surface integral symbols

$$\begin{aligned} \langle\{\bullet\}\rangle_0 &= \frac{1}{V_0}\int_{\mathcal{B}_0} \{\bullet\}\,\mathrm{d}V, &\quad \langle\{\bullet\}\rangle_t &= \frac{1}{V_t}\int_{\mathcal{B}_t} \{\bullet\}\,\mathrm{d}v, \\ \lceil\{\bullet\}\rfloor_0 &= \frac{1}{V_0}\int_{\partial\mathcal{B}_0} \{\bullet\}\,\mathrm{d}S, &\quad \lceil\{\bullet\}\rfloor_t &= \frac{1}{V_t}\int_{\partial\mathcal{B}_t} \{\bullet\}\,\mathrm{d}s. \end{aligned} \tag{23}$$

Based on classical micromechanics arguments, we define the macroscopic deformation gradient as the volume average of the microscopic deformation gradient over the undeformed RVE,

$$\overline{\boldsymbol{F}} := \langle\boldsymbol{F}\rangle_0 = \lceil\boldsymbol{Y}\otimes\boldsymbol{N}\rfloor_0\,, \tag{24}$$

with $\boldsymbol{N}$ being the normal to the boundary of the RVE in the material configuration. In a similar way we define the macroscopic Piola stress as the volume average of the microscopic Piola stress over the undeformed RVE, given by

$$\overline{\boldsymbol{P}} := \langle\boldsymbol{P}\rangle_0 = \lceil\boldsymbol{T}\otimes\boldsymbol{X}\rfloor_0 \quad \text{with } \boldsymbol{T} := \boldsymbol{P}\cdot\boldsymbol{N}, \tag{25}$$

where $\boldsymbol{T}$ is the traction vector (mechanical flux). Similarly for the magnetic part, the macroscopic Lagrangian magnetic field is identified as the average microscopic magnetic field over the undeformed RVE,

$$\overline{\mathbb{F}} := \langle\mathbb{F}\rangle_0 = \lceil\mathbb{Y}\boldsymbol{N}\rfloor_0\,, \tag{26}$$

and the macroscopic Lagrangian magnetic induction is identified as the volume average of the microscopic Lagrangian magnetic induction over the undeformed RVE,

$$\overline{\mathbb{P}} := \langle\mathbb{P}\rangle_0 = \lceil\mathbb{T}\boldsymbol{X}\rfloor_0 \quad \text{with } \mathbb{T} := \mathbb{P}\cdot\boldsymbol{N}, \tag{27}$$

where $\mathbb{T}$ is the magnetic flux vector.

In the spatial configuration the material position $\overline{\boldsymbol{X}}$ can be described by the deformation map $\overline{\boldsymbol{X}} = \bar{\boldsymbol{y}}(\bar{\boldsymbol{x}})$ and the deformation is characterized by the inverse of the deformation gradient $\bar{\boldsymbol{f}}$,

$$\bar{\boldsymbol{f}} = \overline{\mathrm{grad}}\bar{\boldsymbol{y}}. \tag{28}$$

All four macroscopic field variables in the spatial configuration (the inverse of the deformation gradient $\bar{\boldsymbol{f}}$, the symmetric Cauchy stress tensor $\bar{\boldsymbol{p}}$, the Eulerian magnetic field $\bar{\mathbb{f}}$ and the Eulerian magnetic induction $\bar{\mathbb{p}}$) are connected with the macroscopic material field variables through the relations

$$\bar{\boldsymbol{f}} = \bar{\boldsymbol{I}}\cdot\overline{\boldsymbol{F}}^{-1}, \quad \bar{\boldsymbol{p}} = \bar{J}^{-1}\overline{\boldsymbol{P}}\cdot\overline{\boldsymbol{F}}^{\mathrm{t}}, \quad \bar{\mathbb{f}} = \overline{\mathbb{F}}\cdot\overline{\boldsymbol{F}}^{-1}, \quad \bar{\mathbb{p}} = \bar{J}^{-1}\overline{\mathbb{P}}\cdot\overline{\boldsymbol{F}}^{\mathrm{t}}, \tag{29}$$

where $\bar{J} = \overline{\mathrm{Det}\boldsymbol{F}}$. The coupled magnetomechanical system of equations in the material configuration holds in a similar form for the spatial configuration,

$$\overline{\mathrm{curl}\boldsymbol{f}} = \boldsymbol{0}, \quad \overline{\mathrm{div}\boldsymbol{p}} = \boldsymbol{0}, \quad \overline{\mathrm{curl}\mathbb{f}} = \mathbb{O}, \quad \overline{\mathrm{div}\mathbb{p}} = \mathbb{O}. \tag{30}$$

Using the divergence theorem, the volume averages over the deformed RVE of the spatial microscopic field variables are

$$\begin{aligned} \langle \boldsymbol{f} \rangle_t &= \lceil \boldsymbol{y} \otimes \boldsymbol{n} \rfloor_t, & \langle \boldsymbol{p} \rangle_t &= \lceil \boldsymbol{t} \otimes \boldsymbol{x} \rfloor_t, & \boldsymbol{t} &:= \boldsymbol{p} \cdot \boldsymbol{n}, \\ \langle \mathbb{f} \rangle_t &= \lceil \mathbb{y} \boldsymbol{n} \rfloor_t, & \langle \mathbb{p} \rangle_t &= \lceil \mathbb{t} \boldsymbol{x} \rfloor_t, & \mathbb{t} &:= \mathbb{p} \cdot \boldsymbol{n}, \end{aligned} \tag{31}$$

with $\boldsymbol{n}$ being the normal to the boundary of the RVE in the spatial configuration. In [45] a large deformation process has been defined with meaningful space averages for the mechanical problem. Here we extend their definition for the coupled magnetomechanical problem.

Definition. A large deformation process with meaningful space averages for the magnetomechanical problem is every process for which the following conditions hold:

$$\frac{V_t}{V_0} = \bar{J}, \qquad \langle \boldsymbol{f} \rangle_t = \bar{\boldsymbol{f}}, \qquad \langle \boldsymbol{p} \rangle_t = \frac{V_0}{V_t} \bar{\boldsymbol{P}} \cdot \bar{\boldsymbol{F}}^{\mathrm{t}}, \qquad \langle \mathbb{f} \rangle_t = \bar{\mathbb{f}}, \qquad \langle \mathbb{p} \rangle_t = \frac{V_0}{V_t} \bar{\mathbb{P}} \cdot \bar{\boldsymbol{F}}^{\mathrm{t}}. \tag{32}$$

If all five conditions of Equation (32) are satisfied, then the volume-averaged field variables are identified as macroscopic variables and behave qualitatively like their microscopic counterparts. In such a process the macroscopic spatial field variables are equal to the volume average of the corresponding microscopic spatial field variables over the deformed RVE,

$$\bar{\boldsymbol{f}} = \langle \boldsymbol{f} \rangle_t, \quad \bar{\boldsymbol{p}} = \langle \boldsymbol{p} \rangle_t, \quad \bar{\mathbb{f}} = \langle \mathbb{f} \rangle_t, \quad \bar{\mathbb{p}} = \langle \mathbb{p} \rangle_t. \tag{33}$$

3. Boundary conditions on the RVE: Hill's lemma

The solution of the microscopic problem requires appropriate boundary conditions that will satisfy Hill–Mandel conditions. In the literature several types of boundary conditions have been proposed. In [35] small strain fields were considered and prescribed displacements or tractions and an applied magnetic field at the boundary of the body were used. A magnetic field as a loading condition has also been considered in other micromechanics models in conjunction with boundary tractions [38] or far-field strains [36]. In [39] a prescribed deformation gradient tensor and a magnetic induction vector at the boundary were considered. The homogenization of magnetostrictive particle-filled elastomers under periodic boundary conditions and constant magnetostrictive eigen-deformation in the ferromagnetic particles has been studied by [37].

In this work we want to identify and investigate several cases of boundary conditions (uniform or periodic fields) on the RVE under which the Hill–Mandel condition holds, i.e. the volume averages of the increments of the magnetomechanical energy density functions (12) in the RVE are equal to the macroscopic increments of the energy density functions. In order to achieve our goal, we first express Hill's lemma in the material configuration for both the mechanical and the magnetic problem.

3.1. Mechanical problem

In terms of the spatial coordinates and tractions, Hill's lemma is derived using Equations (1), (2) and it takes its classical form [29, 46]

$$\langle \boldsymbol{P} : \boldsymbol{F} \rangle_0 - \bar{\boldsymbol{P}} : \bar{\boldsymbol{F}} = \left\lceil [\boldsymbol{Y} - \bar{\boldsymbol{F}} \cdot \boldsymbol{X}] \cdot [\boldsymbol{P} \cdot \boldsymbol{N} - \bar{\boldsymbol{P}} \cdot \boldsymbol{N}] \right\rfloor_0. \tag{34}$$

We note that [46] uses a transposed version of Equation (34). In terms of the potential $\boldsymbol{A}$ and boundary deformation tensor, Hill's lemma is derived using Equations (5), (8) and it takes the form

$$\langle \boldsymbol{P} : \boldsymbol{F} \rangle_0 - \bar{\boldsymbol{P}} : \bar{\boldsymbol{F}} = \left\lceil \left[\boldsymbol{A} - \tfrac{1}{2} \bar{\boldsymbol{P}} \times \boldsymbol{X}\right] : [\boldsymbol{F} \times \boldsymbol{N} - \bar{\boldsymbol{F}} \times \boldsymbol{N}] \right\rfloor_0. \tag{35}$$

From these results we have the following possible types of boundary conditions that satisfy the Hill–Mandel condition for the mechanical problem:

$$\begin{aligned} &1)\, \boldsymbol{x} = \boldsymbol{Y}(\boldsymbol{X}) = \bar{\boldsymbol{F}} \cdot \boldsymbol{X} && \text{on } \partial\mathcal{B}_0, && 2)\, \boldsymbol{P} \cdot \boldsymbol{N} = \bar{\boldsymbol{P}} \cdot \boldsymbol{N} && \text{on } \partial\mathcal{B}_0, \\ &3)\, \boldsymbol{A} = \tfrac{1}{2} \bar{\boldsymbol{P}} \times \boldsymbol{X} && \text{on } \partial\mathcal{B}_0, && 4)\, \boldsymbol{F} \times \boldsymbol{N} = \bar{\boldsymbol{F}} \times \boldsymbol{N} && \text{on } \partial\mathcal{B}_0. \end{aligned}$$

As we will see later, another option is to apply periodicity conditions. Also, one could consider cases where the fields are uniform inside the whole RVE (for instance, $\boldsymbol{x} = \overline{\boldsymbol{F}} \cdot \boldsymbol{X}$ or $\boldsymbol{P} \cdot \boldsymbol{N} = \overline{\boldsymbol{P}} \cdot \boldsymbol{N}$ on $\mathcal{B}_0$). These cases can be treated similarly to the cases where the corresponding fields are uniform on the boundary of the RVE.

3.2. Magnetic problem

In terms of the magnetic potential $\mathbb{Y}$ and boundary magnetic induction, Hill's lemma is derived using Equations (3), (7) and it is written

$$\langle \mathbb{P} \cdot \mathbb{F} \rangle_0 - \overline{\mathbb{P}} \cdot \overline{\mathbb{F}} = \left\lceil [\mathbb{Y} - \overline{\mathbb{F}} \cdot \boldsymbol{X}][\mathbb{P} \cdot \boldsymbol{N} - \overline{\mathbb{P}} \cdot \boldsymbol{N}] \right\rfloor_0 . \tag{36}$$

In terms of the magnetic potential $\mathbb{A}$ and boundary magnetic field, Hill's lemma is derived using Equations (4), (6) and it is written

$$\langle \mathbb{P} \cdot \mathbb{F} \rangle_0 - \overline{\mathbb{P}} \cdot \overline{\mathbb{F}} = \left\lceil \left[\mathbb{A} - \tfrac{1}{2} \overline{\mathbb{P}} \times \boldsymbol{X} \right] \cdot [\mathbb{F} \times \boldsymbol{N} - \overline{\mathbb{F}} \times \boldsymbol{N}] \right\rfloor_0 . \tag{37}$$

From these results we have the following possible types of boundary conditions that satisfy the Hill–Mandel condition for the magnetostatic problem:

$$\begin{array}{llll} 1)\, \mathbb{Y} = \overline{\mathbb{F}} \cdot \boldsymbol{X} & \text{on } \partial\mathcal{B}_0, & 2)\, \mathbb{P} \cdot \boldsymbol{N} = \overline{\mathbb{P}} \cdot \boldsymbol{N} & \text{on } \partial\mathcal{B}_0, \\ 3)\, \mathbb{A} = \frac{1}{2} \overline{\mathbb{P}} \times \boldsymbol{X} & \text{on } \partial\mathcal{B}_0, & 4)\, \mathbb{F} \times \boldsymbol{N} = \overline{\mathbb{F}} \times \boldsymbol{N} & \text{on } \partial\mathcal{B}_0. \end{array}$$

In a similar fashion to the mechanical problem, another option is to apply periodicity conditions. Also, one could consider cases where the fields are uniform inside the whole RVE (for instance, $\mathbb{Y} = \overline{\mathbb{F}} \cdot \boldsymbol{X}$ or $\mathbb{P} \cdot \boldsymbol{N} = \overline{\mathbb{P}} \cdot \boldsymbol{N}$ on $\mathcal{B}_0$). These cases can be treated similarly with the cases where the corresponding fields are uniform on the boundary of the RVE.

The proofs of the Hill's lemma expressions for both mechanical and magnetic problems are given in the Appendix.

4. Hill–Mandel conditions for the magnetomechanical problem

We want to study the conditions under which the mechanical energy increments $\boldsymbol{P} : \delta\boldsymbol{F}$, $\boldsymbol{F} : \delta\boldsymbol{P}$ and the magnetic energy increments $\mathbb{F} \cdot \delta\mathbb{P}$, $\mathbb{P} \cdot \delta\mathbb{F}$ satisfy the Hill–Mandel condition that correlates macroscopic energy increments with the volume average of energy increments over the undeformed RVE.

4.1. Mechanical energy increment $\boldsymbol{P} : \delta\boldsymbol{F}$

In this energy increment the Piola stress, which satisfies Equation (2), is work conjugated with the increment of the deformation gradient. Since the deformation gradient can be expressed in terms of $\boldsymbol{x}$ through Equation (1), the studied energy increment presents similarities with the Hill's energy expression (34).

4.1.1. First case We assume that the spatial position $\boldsymbol{x}$ has linear relation with the macroscopic deformation gradient. This is written

$$\boldsymbol{x} = \boldsymbol{Y}(\boldsymbol{X}) = \overline{\boldsymbol{F}} \cdot \boldsymbol{X} + \boldsymbol{Z}(\boldsymbol{X}), \tag{38}$$

where $\boldsymbol{Z}$ is a fluctuation vector field. In this expression a macroscopic (constant in microlevel) term can be added without disturbing the obtained deformation gradient and Piola stress. Equations (1) and (38) yield

$$\boldsymbol{F} = \mathrm{Grad}\boldsymbol{Y} = \overline{\boldsymbol{F}} + \mathrm{Grad}\boldsymbol{Z}. \tag{39}$$

Based on Equations (2), (25), (38) and the divergence theorem, we can write

$$\begin{aligned} \langle \boldsymbol{P} : \delta\boldsymbol{F} \rangle_0 &= \langle \boldsymbol{P} \rangle_0 : \delta\overline{\boldsymbol{F}} + \langle \boldsymbol{P} : \mathrm{Grad}\delta\boldsymbol{Z} \rangle_0 \\ &= \langle \boldsymbol{P} \rangle_0 : \delta\overline{\boldsymbol{F}} + \langle \boldsymbol{P} : \mathrm{Grad}\delta\boldsymbol{Z} + \delta\boldsymbol{Z} \cdot \mathrm{Div}\boldsymbol{P} \rangle_0 \\ &= \langle \boldsymbol{P} \rangle_0 : \delta\overline{\boldsymbol{F}} + \left\langle \mathrm{Div}(\boldsymbol{P}^{\mathrm{t}} \cdot \delta\boldsymbol{Z}) \right\rangle_0 = \langle \boldsymbol{P} \rangle_0 : \delta\overline{\boldsymbol{F}} + \lceil \delta\boldsymbol{Z} \cdot \boldsymbol{P} \cdot \boldsymbol{N} \rfloor_0 . \end{aligned}$$

The surface integral vanishes for one of the following two types of boundary conditions,

1. $\boldsymbol{Z}=\boldsymbol{0}$ on $\partial\mathcal{B}_0$ (linear displacement boundary conditions);
2. $\boldsymbol{Z}$ is periodic and the tractions $\boldsymbol{T}=\boldsymbol{P}\cdot\boldsymbol{N}$ are antiperiodic for geometrically periodic RVE (periodic boundary conditions).

These conditions also guarantee that $\langle\boldsymbol{F}\rangle_0=\overline{\boldsymbol{F}}$. Considering $\langle\boldsymbol{P}\rangle_0=\overline{\boldsymbol{P}}$ we finally get the Hill–Mandel condition $\langle\boldsymbol{P}:\delta\boldsymbol{F}\rangle_0=\overline{\boldsymbol{P}}:\delta\overline{\boldsymbol{F}}$.

4.1.2. Second case We assume that $\boldsymbol{P}\cdot\boldsymbol{N}=\overline{\boldsymbol{P}}\cdot\boldsymbol{N}$ on $\partial\mathcal{B}_0$. Using the divergence theorem and Equation (2) we see that

$$\langle\boldsymbol{P}\rangle_0=\lceil[\boldsymbol{P}\cdot\boldsymbol{N}]\otimes\boldsymbol{X}\rfloor_0=\left\lceil[\overline{\boldsymbol{P}}\cdot\boldsymbol{N}]\otimes\boldsymbol{X}\right\rfloor_0=\overline{\boldsymbol{P}}\cdot\lceil\boldsymbol{N}\otimes\boldsymbol{X}\rfloor_0=\overline{\boldsymbol{P}}.$$

Moreover, using the divergence theorem and Equations (2), (5) and (24), we have

$$\begin{aligned}\langle\boldsymbol{P}:\delta\boldsymbol{F}\rangle_0&=\langle\boldsymbol{P}:\delta\boldsymbol{F}+\delta\boldsymbol{Y}\cdot\mathrm{Div}\boldsymbol{P}\rangle_0=\left\langle\mathrm{Div}(\boldsymbol{P}^{\mathrm{t}}\cdot\delta\boldsymbol{Y})\right\rangle_0\\&=\lceil\delta\boldsymbol{Y}\cdot\boldsymbol{P}\cdot\boldsymbol{N}\rfloor_0=\left\lceil\delta\boldsymbol{Y}\cdot\overline{\boldsymbol{P}}\cdot\boldsymbol{N}\right\rfloor_0\\&=\overline{\boldsymbol{P}}:\lceil\delta\boldsymbol{Y}\otimes\boldsymbol{N}\rfloor_0=\overline{\boldsymbol{P}}:\langle\delta\boldsymbol{F}\rangle_0\,.\end{aligned}$$

Considering $\langle\boldsymbol{F}\rangle_0=\overline{\boldsymbol{F}}$ we finally get the Hill–Mandel condition $\langle\boldsymbol{P}:\delta\boldsymbol{F}\rangle_0=\overline{\boldsymbol{P}}:\delta\overline{\boldsymbol{F}}$.

4.2. Mechanical energy increment $\boldsymbol{F}:\delta\boldsymbol{P}$

In this energy increment the deformation gradient, which satisfies Equation (5), is work conjugated with the increment of the Piola stress. Since the Piola stress can be expressed in terms of $\boldsymbol{A}$ through Equation (8), the studied energy increment presents similarities with the Hill's energy expression (35).

4.2.1. First case We assume that the stress tensor potential $\boldsymbol{A}$ has linear relation with the macroscopic Piola stress. This is written

$$\boldsymbol{A}=\tfrac{1}{2}\overline{\boldsymbol{P}}\times\boldsymbol{X}+\boldsymbol{W}(\boldsymbol{X}),\tag{40}$$

where $\boldsymbol{W}$ is a fluctuation second-order tensor field. In this expression a macroscopic (constant in microlevel) term can be added without disturbing the obtained deformation gradient and Piola stress. Equations (8) and (40) yield

$$\boldsymbol{P}=\mathrm{Curl}\boldsymbol{A}=\overline{\boldsymbol{P}}+\mathrm{Curl}\boldsymbol{W}.\tag{41}$$

Based on Equations (5), (24), (40) and the divergence theorem, we can write

$$\begin{aligned}\langle\boldsymbol{F}:\delta\boldsymbol{P}\rangle_0&=\langle\boldsymbol{F}\rangle_0:\delta\overline{\boldsymbol{P}}+\langle\boldsymbol{F}:\mathrm{Curl}\delta\boldsymbol{W}\rangle_0\\&=\langle\boldsymbol{F}\rangle_0:\delta\overline{\boldsymbol{P}}+\langle\boldsymbol{F}:\mathrm{Curl}\delta\boldsymbol{W}-\delta\boldsymbol{W}:\mathrm{Curl}\boldsymbol{F}\rangle_0\\&=\langle\boldsymbol{F}\rangle_0:\delta\overline{\boldsymbol{P}}+\left\langle\boldsymbol{I}:\mathrm{Curl}\left(\boldsymbol{F}^{\mathrm{t}}\cdot\delta\boldsymbol{W}\right)\right\rangle_0=\langle\boldsymbol{F}\rangle_0:\delta\overline{\boldsymbol{P}}+\lceil\delta\boldsymbol{W}:[\boldsymbol{F}\times\boldsymbol{N}]\rfloor_0\,.\end{aligned}$$

The surface integral vanishes for one of the following two types of boundary conditions,

1. $\boldsymbol{W}=\boldsymbol{0}$ on $\partial\mathcal{B}_0$ (linear displacement boundary conditions);
2. $\boldsymbol{W}$ is periodic and $\boldsymbol{F}\times\boldsymbol{N}$ is antiperiodic for geometrically periodic RVE (periodic boundary conditions).

These conditions also guarantee that $\langle\boldsymbol{P}\rangle_0=\overline{\boldsymbol{P}}$. Considering $\langle\boldsymbol{F}\rangle_0=\overline{\boldsymbol{F}}$ we finally get the Hill–Mandel condition $\langle\boldsymbol{F}:\delta\boldsymbol{P}\rangle_0=\overline{\boldsymbol{F}}:\delta\overline{\boldsymbol{P}}$.

4.2.2. Second case We assume that $\boldsymbol{F}\times\boldsymbol{N}=\overline{\boldsymbol{F}}\times\boldsymbol{N}$ on $\partial\mathcal{B}_0$. We introduce the skew-symmetric second-order tensor $\boldsymbol{M}:=-\frac{1}{2}\boldsymbol{\varepsilon}\cdot\boldsymbol{X}$. By observing that[1] $\mathrm{Curl}\boldsymbol{M}=\boldsymbol{I}$, we can use Equation (5) and the divergence theorem and get

$$\begin{aligned}\langle\boldsymbol{F}\rangle_0&=\left\langle\boldsymbol{F}\cdot[\mathrm{Curl}\boldsymbol{M}]^{\mathrm{t}}-\mathrm{Curl}\boldsymbol{F}\cdot\boldsymbol{M}^{\mathrm{t}}\right\rangle_0=\left\lceil[\boldsymbol{F}\times\boldsymbol{N}]\cdot\boldsymbol{M}^{\mathrm{t}}\right\rfloor_0\\&=\left\lceil[\overline{\boldsymbol{F}}\times\boldsymbol{N}]\cdot\boldsymbol{M}^{\mathrm{t}}\right\rfloor_0=\overline{\boldsymbol{F}}\cdot\left\langle[\mathrm{Curl}\boldsymbol{M}]^{\mathrm{t}}\right\rangle_0=\overline{\boldsymbol{F}}.\end{aligned}$$

Moreover, using Equations (5), (8), (25) and the divergence theorem, we have

$$\begin{aligned}\langle \boldsymbol{F} : \delta \boldsymbol{P} \rangle_0 &= \langle \boldsymbol{F} : \mathrm{Curl}\delta \boldsymbol{A} - \delta \boldsymbol{A} : \mathrm{Curl}\boldsymbol{F} \rangle_0 = \left\langle \boldsymbol{I} : \mathrm{Curl}\left(\boldsymbol{F}^{\mathrm{t}} \cdot \delta \boldsymbol{A}\right)\right\rangle_0 \\ &= \lceil \delta \boldsymbol{A} : [\boldsymbol{F} \times \boldsymbol{N}] \rfloor_0 = \left\lceil \delta \boldsymbol{A} : [\overline{\boldsymbol{F}} \times \boldsymbol{N}] \right\rfloor_0 = -\overline{\boldsymbol{F}} : \lceil \delta \boldsymbol{A} \times \boldsymbol{N} \rfloor_0 \\ &= \overline{\boldsymbol{F}} : \langle \mathrm{Curl}\delta \boldsymbol{A} \rangle_0 = \overline{\boldsymbol{F}} : \langle \delta \boldsymbol{P} \rangle_0 .\end{aligned} \tag{42}$$

Considering $\langle \boldsymbol{P} \rangle_0 = \overline{\boldsymbol{P}}$ we finally get the Hill–Mandel condition $\langle \boldsymbol{F} : \delta \boldsymbol{P} \rangle_0 = \overline{\boldsymbol{F}} : \delta \overline{\boldsymbol{P}}$.

4.3. *Magnetic energy increment* $\overline{\mathbb{P}} \cdot \delta \overline{\mathbb{F}}$

In this energy increment the magnetic induction, which satisfies Equation (3), is work conjugated with the increment of magnetic field. Since the magnetic field can be expressed in terms of $\mathbb{Y}$ through Equation (7), the studied energy increment presents similarities with the Hill's energy expression (36).

4.3.1. First case We assume that the magnetic potential $\mathbb{Y}$ has a linear relation with the macroscopic magnetic field. This is written

$$\mathbb{Y} = \overline{\mathbb{F}} \cdot \boldsymbol{X} + \mathbb{Z}(\boldsymbol{X}), \tag{43}$$

where $\mathbb{Z}$ is a fluctuation scalar field. In this expression a macroscopic (constant in microlevel) term can be added without disturbing the obtained magnetic field and magnetic induction. Equations (7) and (43) yield

$$\mathbb{F} = \mathrm{Grad}\mathbb{Y} = \overline{\mathbb{F}} + \mathrm{Grad}\mathbb{Z}. \tag{44}$$

Based on Equations (3), (27) and (43), we can write

$$\begin{aligned}\langle \mathbb{P} \cdot \delta \mathbb{F} \rangle_0 &= \langle \mathbb{P} \rangle_0 \cdot \delta \overline{\mathbb{F}} + \langle \mathbb{P} \cdot \mathrm{Grad}\delta \mathbb{Z} \rangle_0 \\ &= \langle \mathbb{P} \rangle_0 \cdot \delta \overline{\mathbb{F}} + \langle \mathbb{P} \cdot \mathrm{Grad}\delta \mathbb{Z} + \delta \mathbb{Z}\mathrm{Div}\mathbb{P} \rangle_0 \\ &= \langle \mathbb{P} \rangle_0 \cdot \delta \overline{\mathbb{F}} + \langle \mathrm{Div}(\mathbb{P}\delta \mathbb{Z}) \rangle_0 = \langle \mathbb{P} \rangle_0 \cdot \delta \overline{\mathbb{F}} + \lceil \delta \mathbb{Z}[\mathbb{P} \cdot \boldsymbol{N}] \rfloor_0 .\end{aligned}$$

The surface integral vanishes for one of the following two types of boundary conditions:

1. $\mathbb{Z} = \mathbb{O}$ on $\partial \mathcal{B}_0$ (linear magnetic potential boundary conditions);
2. $\mathbb{Z}$ is periodic and $\mathbb{T} = \mathbb{P} \cdot \boldsymbol{N}$ is antiperiodic for geometrically periodic RVE (periodic boundary conditions).

These conditions also guarantee that $\langle \mathbb{F} \rangle_0 = \overline{\mathbb{F}}$. Considering $\langle \mathbb{P} \rangle_0 = \overline{\mathbb{P}}$ we finally get the Hill–Mandel condition $\langle \mathbb{P} \cdot \delta \mathbb{F} \rangle_0 = \overline{\mathbb{P}} \cdot \delta \overline{\mathbb{F}}$.

4.3.2. Second case We assume that $\mathbb{P} \cdot \boldsymbol{N} = \overline{\mathbb{P}} \cdot \boldsymbol{N}$ on $\partial \mathcal{B}_0$. Using the divergence theorem and Equation (3) we see that

$$\langle \mathbb{P} \rangle_0 = \lceil [\mathbb{P} \cdot \boldsymbol{N}]\boldsymbol{X} \rfloor_0 = \left\lceil [\overline{\mathbb{P}} \cdot \boldsymbol{N}]\boldsymbol{X} \right\rfloor_0 = \lceil \boldsymbol{X} \otimes \boldsymbol{N} \rfloor_0 \cdot \overline{\mathbb{P}} = \overline{\mathbb{P}}.$$

Moreover, using Equations (3), (7), (26) and the divergence theorem, we have

$$\begin{aligned}\langle \mathbb{P} \cdot \delta \mathbb{F} \rangle_0 &= \langle \mathbb{P} \cdot \delta \mathbb{F} + \delta \mathbb{Y}\mathrm{Div}\mathbb{P} \rangle_0 = \langle \mathrm{Div}(\mathbb{P}\delta \mathbb{Y}) \rangle_0 = \lceil \delta \mathbb{Y}[\mathbb{P} \cdot \boldsymbol{N}] \rfloor_0 \\ &= \left\lceil \delta \mathbb{Y}[\overline{\mathbb{P}} \cdot \boldsymbol{N}] \right\rfloor_0 = \overline{\mathbb{P}} \cdot \lceil \delta \mathbb{Y}\boldsymbol{N} \rfloor_0 = \overline{\mathbb{P}} \cdot \langle \delta \mathbb{F} \rangle_0 .\end{aligned}$$

Considering $\langle \mathbb{F} \rangle_0 = \overline{\mathbb{F}}$ we finally get the Hill–Mandel condition $\langle \mathbb{P} \cdot \delta \mathbb{F} \rangle_0 = \overline{\mathbb{P}} \cdot \delta \overline{\mathbb{F}}$.

4.4. *Magnetic energy increment* $\overline{\mathbb{F}} \cdot \delta \overline{\mathbb{P}}$

In this energy increment the magnetic field, which satisfies Equation (4), is work conjugated with the increment of magnetic induction. Since the magnetic induction can be expressed in terms of $\mathbb{A}$ through Equation (6), the studied energy increment presents similarities with the Hill's energy expression (37).

4.4.1. First case We assume that the magnetic potential $\mathbb{A}$ has linear relation with the macroscopic magnetic induction. This is written

$$\mathbb{A} = \tfrac{1}{2}\overline{\mathbb{P}} \times \boldsymbol{X} + \mathbb{W}(\boldsymbol{X}), \tag{45}$$

where $\mathbb{W}$ is a fluctuation vector field. In this expression a macroscopic (constant in microlevel) term can be added without disturbing the obtained magnetic field and magnetic induction. Equations (6) and (45) yield

$$\mathbb{P} = \mathrm{Curl}\mathbb{A} = \overline{\mathbb{P}} + \mathrm{Curl}\mathbb{W}. \tag{46}$$

Based on Equations (4), (26) and (45), we can write

$$\begin{aligned}\langle \mathbb{F} \cdot \delta\mathbb{P}\rangle_0 &= \langle \mathbb{F}\rangle_0 \cdot \delta\overline{\mathbb{P}} + \langle \mathbb{F} \cdot \mathrm{Curl}\delta\mathbb{W}\rangle_0 = \langle \mathbb{F}\rangle_0 \cdot \delta\overline{\mathbb{P}} + \langle \mathbb{F} \cdot \mathrm{Curl}\delta\mathbb{W} - \delta\mathbb{W} \cdot \mathrm{Curl}\mathbb{F}\rangle_0\\ &= \langle \mathbb{F}\rangle_0 \cdot \delta\overline{\mathbb{P}} + \langle \boldsymbol{I} : \mathrm{Curl}(\mathbb{F} \otimes \delta\mathbb{W})\rangle_0 = \langle \mathbb{F}\rangle_0 \cdot \delta\overline{\mathbb{P}} + \lceil \delta\mathbb{W} \cdot [\mathbb{F} \times \boldsymbol{N}]\rfloor_0 .\end{aligned}$$

The surface integral vanishes for one of the following two types of boundary conditions:

1. $\mathbb{W} = \mathbb{O}$ on $\partial\mathcal{B}_0$ (linear magnetic potential boundary conditions);
2. $\mathbb{W}$ is periodic and $\mathbb{F} \times \boldsymbol{N}$ is antiperiodic for geometrically periodic RVE (periodic boundary conditions).

These conditions also guarantee that $\langle \mathbb{P}\rangle_0 = \overline{\mathbb{P}}$. Considering $\langle \mathbb{F}\rangle_0 = \overline{\mathbb{F}}$ we finally get the Hill–Mandel condition $\langle \mathbb{F} \cdot \delta\mathbb{P}\rangle_0 = \overline{\mathbb{F}} \cdot \delta\overline{\mathbb{P}}$.

4.4.2. Second case We assume that $\mathbb{F} \times \boldsymbol{N} = \overline{\mathbb{F}} \times \boldsymbol{N}$ on $\partial\mathcal{B}_0$. Using the second-order tensor $\boldsymbol{M} = -\frac{1}{2}\boldsymbol{\varepsilon} \cdot \boldsymbol{X}$, Equation (4) and the divergence theorem we get

$$\begin{aligned}\langle \mathbb{F}\rangle_0 &= \langle \mathrm{Curl}\boldsymbol{M} \cdot \mathbb{F} - \boldsymbol{M} \cdot \mathrm{Curl}\mathbb{F}\rangle_0 = \lceil \boldsymbol{M} \cdot [\mathbb{F} \times \boldsymbol{N}]\rfloor_0\\ &= \left\lceil \boldsymbol{M} \cdot [\overline{\mathbb{F}} \times \boldsymbol{N}]\right\rfloor_0 = \langle \mathrm{Curl}\boldsymbol{M}\rangle_0 \cdot \overline{\mathbb{F}} = \overline{\mathbb{F}}.\end{aligned}$$

Moreover, using Equations (4), (6), (27) and the divergence theorem, we have

$$\begin{aligned}\langle \mathbb{F} \cdot \delta\mathbb{P}\rangle_0 &= \langle \mathbb{F} \cdot \mathrm{Curl}\delta\mathbb{A} - \delta\mathbb{A} \cdot \mathrm{Curl}\mathbb{F}\rangle_0 = \langle \boldsymbol{I} : \mathrm{Curl}(\mathbb{F} \otimes \delta\mathbb{A})\rangle_0\\ &= \lceil \delta\mathbb{A} \cdot [\mathbb{F} \times \boldsymbol{N}]\rfloor_0 = \left\lceil \delta\mathbb{A} \cdot [\overline{\mathbb{F}} \times \boldsymbol{N}]\right\rfloor_0 = \overline{\mathbb{F}} \cdot \lceil \boldsymbol{N} \times \delta\mathbb{A}\rfloor_0\\ &= \overline{\mathbb{F}} \cdot \langle \mathrm{Curl}\delta\mathbb{A}\rangle_0 = \overline{\mathbb{F}} \cdot \langle \delta\mathbb{P}\rangle_0 .\end{aligned}$$

Considering $\langle \mathbb{P}\rangle_0 = \overline{\mathbb{P}}$ we finally get the Hill–Mandel condition $\langle \mathbb{F} \cdot \delta\mathbb{P}\rangle_0 = \overline{\mathbb{F}} \cdot \delta\overline{\mathbb{P}}$.

Table 2 summarizes the different types of boundary conditions that satisfy the Hill–Mandel condition for each type of energy increment. It is interesting to observe that, if the incremental forms of the magnetomechanical equations are considered, i.e.

$$\mathrm{Div}\delta\boldsymbol{P} = \boldsymbol{0}, \quad \mathrm{Curl}\delta\boldsymbol{F} = \boldsymbol{0}, \quad \mathrm{Div}\delta\mathbb{P} = \mathbb{O}, \quad \mathrm{Curl}\delta\mathbb{F} = \mathbb{O}, \tag{47}$$

then one can satisfy the Hill–Mandel condition for all energy increments (12) once we choose one set of appropriate mechanical and magnetic boundary conditions. Indeed, let us for example choose the mechanical boundary condition of the first case of Section 4.1 and the magnetic boundary condition of the first case of Section 4.3. Under these conditions we clearly have $\delta\overline{W}_0^{\boldsymbol{F}\mathbb{F}} = \left\langle \delta W_0^{\boldsymbol{F}\mathbb{F}}\right\rangle_0$. Considering Equation $(47)_1$ and following similar steps as in Section 4.1, we can prove that the mechanical boundary condition satisfies $\overline{\boldsymbol{F}} : \delta\overline{\boldsymbol{P}} = \langle \boldsymbol{F} : \delta\boldsymbol{P}\rangle_0$, which means that $\delta\overline{W}_0^{\boldsymbol{P}\mathbb{F}} = \left\langle \delta W_0^{\boldsymbol{P}\mathbb{F}}\right\rangle_0$ also holds. Moreover, using Equation $(47)_3$ and following similar steps as in Section 4.3, we can prove that the magnetic boundary condition satisfies $\overline{\mathbb{F}}\cdot\delta\overline{\mathbb{P}} = \langle \mathbb{F} \cdot \delta\mathbb{P}\rangle_0$, which leads to the relations $\delta\overline{W}_0^{\boldsymbol{F}\mathbb{P}} = \left\langle \delta W_0^{\boldsymbol{F}\mathbb{P}}\right\rangle_0$ and $\delta\overline{W}_0^{\boldsymbol{P}\mathbb{P}} = \left\langle \delta W_0^{\boldsymbol{P}\mathbb{P}}\right\rangle_0$. In a similar way we can prove that any choice of mechanical and magnetic boundary conditions leads to equivalence between all the volume averages over the undeformed RVE of the energy increments (12) and the macroscopic energy increments (22).

Table 2. Boundary conditions for satisfying the Hill–Mandel condition. The cases $\{\bullet\} = \mathbf{0}$ or $\{\bullet\} = \mathbb{O}$ in $\mathcal{B}_0$ can be treated in a similar manner with the cases $\{\bullet\} = \mathbf{0}$ or $\{\bullet\} = \mathbb{O}$ on $\partial\mathcal{B}_0$.

Mechanical energy increment	
$\overline{\boldsymbol{P}} : \delta\overline{\boldsymbol{F}} = \langle \boldsymbol{P} : \delta\boldsymbol{F}\rangle_0$	$\overline{\boldsymbol{F}} : \delta\overline{\boldsymbol{P}} = \langle \boldsymbol{F} : \delta\boldsymbol{P}\rangle_0$
1) $\boldsymbol{Y} = \overline{\boldsymbol{F}} \cdot \boldsymbol{X} + \boldsymbol{Z}$ • $\boldsymbol{Z} = \mathbf{0}$ on $\partial\mathcal{B}_0$ • $\boldsymbol{Z}$ periodic and $\boldsymbol{P} \cdot \boldsymbol{N}$ antiperiodic 2) $\boldsymbol{P} \cdot \boldsymbol{N} = \overline{\boldsymbol{P}} \cdot \boldsymbol{N}$ on $\partial\mathcal{B}_0$	1) $\boldsymbol{A} = \frac{1}{2}\overline{\boldsymbol{P}} \times \boldsymbol{X} + \boldsymbol{W}$ • $\boldsymbol{W} = \mathbf{0}$ on $\partial\mathcal{B}_0$ • $\boldsymbol{W}$ periodic and $\boldsymbol{F} \times \boldsymbol{N}$ antiperiodic 2) $\boldsymbol{F} \times \boldsymbol{N} = \overline{\boldsymbol{F}} \times \boldsymbol{N}$ on $\partial\mathcal{B}_0$
Magnetic energy increment	
$\overline{\mathbb{P}} \cdot \delta\overline{\mathbb{F}} = \langle \mathbb{P} \cdot \delta\mathbb{F}\rangle_0$	$\overline{\mathbb{F}} \cdot \delta\overline{\mathbb{P}} = \langle \mathbb{F} \cdot \delta\mathbb{P}\rangle_0$
1) $\mathbb{Y} = \overline{\mathbb{F}} \cdot \boldsymbol{X} + \mathbb{Z}$ • $\mathbb{Z} = \mathbb{O}$ on $\partial\mathcal{B}_0$ • $\mathbb{Z}$ periodic and $\mathbb{P} \cdot \boldsymbol{N}$ antiperiodic 2) $\mathbb{P} \cdot \boldsymbol{N} = \overline{\mathbb{P}} \cdot \boldsymbol{N}$ on $\partial\mathcal{B}_0$	1) $\mathbb{A} = \frac{1}{2}\overline{\mathbb{P}} \times \boldsymbol{X} + \mathbb{W}$ • $\mathbb{W} = \mathbb{O}$ on $\partial\mathcal{B}_0$ • $\mathbb{W}$ periodic and $\mathbb{F} \times \boldsymbol{N}$ antiperiodic 2) $\mathbb{F} \times \boldsymbol{N} = \overline{\mathbb{F}} \times \boldsymbol{N}$ on $\partial\mathcal{B}_0$

5. Macroscopic spatial description

The homogenization process in material description does not guarantee a priori that the homogenized field variables can be transformed into their spatial counterparts through volume averaging in the spatial description. We need to examine whether or not the transition holds for the boundary conditions of Table 2. Since the transition of the magnetic variables from the undeformed to the deformed configuration depends on the deformation gradient, as (29) indicates, we should first examine the transition of the mechanical variables. In order to transform volume and surface integrals from material to spatial description we recall the identities

$$\mathrm{d}v = J\,\mathrm{d}V \quad \text{and} \quad \boldsymbol{n}\,\mathrm{d}s = J\boldsymbol{F}^{-\mathrm{t}} \cdot \boldsymbol{N}\,\mathrm{d}S. \tag{48}$$

5.1. Mechanical type of boundary conditions

5.1.1. First case We first consider the boundary conditions presented in the first case of Section 4.1, i.e. that the position vector $\boldsymbol{x}$ in the spatial description is given by $\boldsymbol{x} = \overline{\boldsymbol{F}} \cdot \boldsymbol{X} + \boldsymbol{Z}(\boldsymbol{X})$, where $\boldsymbol{Z}$ is null on $\partial\mathcal{B}_0$ or periodic. In [45] it was shown that, under such a representation of the spatial position vector, the conditions $(32)_1$ and $(32)_2$ hold,

$$\langle \boldsymbol{f}\rangle_t = \overline{\boldsymbol{f}} \quad \text{and} \quad \frac{V_t}{V_0} = \overline{J}.$$

In addition they have proven that $\boldsymbol{Z}$ being a periodic function guarantees that any periodic or antiperiodic field in the undeformed configuration remains periodic or antiperiodic, respectively, in the deformed configuration. For linear displacement ($\boldsymbol{Z} = \mathbf{0}$) or periodic ($\boldsymbol{Z}$ periodic and tractions antiperiodic) boundary conditions the condition $(32)_3$ also holds [33, 45],

$$\langle \boldsymbol{p}\rangle_t = \frac{V_0}{V_t}\overline{\boldsymbol{P}} \cdot \overline{\boldsymbol{F}}^{\mathrm{t}} = \overline{J}^{-1}\overline{\boldsymbol{P}} \cdot \overline{\boldsymbol{F}}^{\mathrm{t}}.$$

In [47] it was also shown that the periodicity conditions yield equivalence between the two forms of the macroscopic power density expressed in the Lagrangian (using Piola stress and rate of deformation gradient) and Eulerian (using Cauchy stress and velocity gradient) formulations.

By expressing the periodic function $\boldsymbol{Z}$ in terms of the spatial position vector $\boldsymbol{x}$, i.e. $\tilde{\boldsymbol{Z}}(\boldsymbol{x}) \equiv \boldsymbol{Z}(\boldsymbol{X})$, we can invert Equation (38) and write the material position vector in the form

$$\boldsymbol{X} = \boldsymbol{y}(\boldsymbol{x}) = \overline{\boldsymbol{f}} \cdot \boldsymbol{x} + \boldsymbol{z}(\boldsymbol{x}), \tag{49}$$

where $\boldsymbol{z}(\boldsymbol{x}) := -\overline{\boldsymbol{f}} \cdot \tilde{\boldsymbol{Z}}(\boldsymbol{x})$, which is either null on $\partial\mathcal{B}_t$ or periodic in the spatial configuration. Equations $(14)_3$, (27) and (38) allow to rewrite Equation $(31)_4$ of the volume average of the microscopic spatial magnetic induction as

$$\begin{aligned}\langle \mathbb{p} \rangle_t &= \lceil [\mathbb{p} \cdot \boldsymbol{n}] \boldsymbol{x} \rfloor_t = \frac{V_0}{V_t} \lceil [\mathbb{P} \cdot \boldsymbol{N}] \boldsymbol{Y} \rfloor_0 \\ &= \frac{V_0}{V_t} \overline{\boldsymbol{F}} \cdot \lceil [\mathbb{P} \cdot \boldsymbol{N}] \boldsymbol{X} \rfloor_0 + \frac{V_0}{V_t} \lceil [\mathbb{P} \cdot \boldsymbol{N}] \boldsymbol{Z} \rfloor_0 \\ &= \frac{V_0}{V_t} \overline{\mathbb{P}} \cdot \overline{\boldsymbol{F}}^{\mathrm{t}} + \frac{V_0}{V_t} \lceil [\mathbb{P} \cdot \boldsymbol{N}] \boldsymbol{Z} \rfloor_0 .\end{aligned}$$

The second term vanishes and we get $\langle \mathbb{p} \rangle_t = \dfrac{V_0}{V_t} \overline{\mathbb{P}} \cdot \overline{\boldsymbol{F}}^{\mathrm{t}}$ when one of the following boundary conditions are considered:

1. $\boldsymbol{Z} = \boldsymbol{0}$ on $\partial\mathcal{B}_0$;
2. $\boldsymbol{Z}$ periodic and $\mathbb{P} \cdot \boldsymbol{N}$ antiperiodic;
3. $\boldsymbol{Z}$ periodic and $\mathbb{P} \cdot \boldsymbol{N} = \overline{\mathbb{P}} \cdot \boldsymbol{N}$ on $\partial\mathcal{B}_0$.

5.1.2. Second case We consider the boundary conditions presented in the second case of Section 4.1, i.e. $\boldsymbol{P} \cdot \boldsymbol{N} = \overline{\boldsymbol{P}} \cdot \boldsymbol{N}$ on $\partial\mathcal{B}_0$. Then, using Equations (1), (2), $(14)_2$, (24) and the divergence theorem, we can write

$$\begin{aligned}\langle \boldsymbol{p} \rangle_t &= \frac{V_0}{V_t} \left\langle \boldsymbol{P} \cdot \boldsymbol{F}^{\mathrm{t}} \right\rangle_0 = \frac{V_0}{V_t} \left\langle \boldsymbol{P} \cdot \boldsymbol{F}^{\mathrm{t}} + [\mathrm{Div} \boldsymbol{P}] \otimes \boldsymbol{Y} \right\rangle_0 = \\ &= \frac{V_0}{V_t} \left\langle [\mathrm{Div}(\boldsymbol{Y} \otimes \boldsymbol{P})]^{\mathrm{t}} \right\rangle_0 = \frac{V_0}{V_t} \lceil [\boldsymbol{P} \cdot \boldsymbol{N}] \otimes \boldsymbol{Y} \rfloor_0 \\ &= \frac{V_0}{V_t} \left\lceil [\overline{\boldsymbol{P}} \cdot \boldsymbol{N}] \otimes \boldsymbol{Y} \right\rfloor_0 = \frac{V_0}{V_t} \overline{\boldsymbol{P}} \cdot \lceil \boldsymbol{N} \otimes \boldsymbol{Y} \rfloor_0 = \frac{V_0}{V_t} \overline{\boldsymbol{P}} \cdot \left\langle \boldsymbol{F}^{\mathrm{t}} \right\rangle_0 = \frac{V_0}{V_t} \overline{\boldsymbol{P}} \cdot \overline{\boldsymbol{F}}^{\mathrm{t}}.\end{aligned}$$

For this type of boundary conditions no information about the volume average of the inverse of the deformation gradient was obtained.

5.1.3. Third case We consider the boundary conditions presented in the first case of Section 4.2, i.e. $\boldsymbol{A} = \frac{1}{2} \overline{\boldsymbol{P}} \times \boldsymbol{X} + \boldsymbol{W}(\boldsymbol{X})$, where $\boldsymbol{W}$ is null on $\partial\mathcal{B}_0$ or periodic with antiperiodic $\boldsymbol{F} \times \boldsymbol{N}$. Then, using Equations (5), (8), $(14)_2$, (24) and the divergence theorem, we have

$$\begin{aligned}\langle \boldsymbol{p} \rangle_t &= \frac{V_0}{V_t} \left\langle \boldsymbol{P} \cdot \boldsymbol{F}^{\mathrm{t}} \right\rangle_0 = \frac{V_0}{V_t} \left\langle \overline{\boldsymbol{P}} \cdot \boldsymbol{F}^{\mathrm{t}} + [\mathrm{Curl} \boldsymbol{W}] \cdot \boldsymbol{F}^{\mathrm{t}} \right\rangle_0 \\ &= \frac{V_0}{V_t} \overline{\boldsymbol{P}} \cdot \overline{\boldsymbol{F}}^{\mathrm{t}} + \frac{V_0}{V_t} \left\langle [\mathrm{Curl} \boldsymbol{W}] \cdot \boldsymbol{F}^{\mathrm{t}} - \boldsymbol{W} \cdot [\mathrm{Curl} \boldsymbol{F}]^{\mathrm{t}} \right\rangle_0 \\ &= \frac{V_0}{V_t} \overline{\boldsymbol{P}} \cdot \overline{\boldsymbol{F}}^{\mathrm{t}} + \frac{V_0}{V_t} \left\lceil \boldsymbol{W} \cdot [\boldsymbol{F} \times \boldsymbol{N}]^{\mathrm{t}} \right\rfloor_0 = \frac{V_0}{V_t} \overline{\boldsymbol{P}} \cdot \overline{\boldsymbol{F}}^{\mathrm{t}},\end{aligned}$$

since the last integral vanishes for both types of boundary conditions. For this type of boundary condition, no information about the volume average of the inverse of the deformation gradient was obtained.

5.1.4. Fourth case We consider the boundary conditions presented in the second case of Section 4.2, i.e. $\boldsymbol{F} \times \boldsymbol{N} = \overline{\boldsymbol{F}} \times \boldsymbol{N}$ on $\partial\mathcal{B}_0$. Using Equations (5), (8), (16), (25), the identity $J\varepsilon_{jmn} = \varepsilon_{qrs} F_{jq} F_{mr} F_{ns}$ and the divergence theorem, we can write in indicial notation

$$\begin{aligned}\left\langle p_{ij} \right\rangle_t &= \left\langle -\frac{\partial a_{in}}{\partial x_m} \varepsilon_{nmj} \right\rangle_t = \left\lceil a_{in} n_m \varepsilon_{mnj} \right\rfloor_t = \frac{V_0}{V_t} \left\lceil \varepsilon_{jmn} A_{ik} F^{-1}_{kn} J F^{-\mathrm{t}}_{mu} N_u \right\rfloor_0 \\ &= \frac{V_0}{V_t} \left\lceil [J \varepsilon_{jmn}] A_{ik} F^{-1}_{kn} F^{-1}_{um} N_u \right\rfloor_0 = \frac{V_0}{V_t} \left\lceil \varepsilon_{qrs} F_{jq} F_{mr} F_{ns} A_{ik} F^{-1}_{kn} F^{-1}_{um} N_u \right\rfloor_0\end{aligned}$$

$$= \frac{V_0}{V_t}\left\lceil A_{ik}\varepsilon_{qrs}F_{jq}\delta_{ur}\delta_{ks}N_u\right\rfloor_0 = \frac{V_0}{V_t}\left\lceil A_{ik}[\boldsymbol{F}\times\boldsymbol{N}]_{jk}\right\rfloor_0 = \frac{V_0}{V_t}\left\lceil A_{ik}[\overline{\boldsymbol{F}}\times\boldsymbol{N}]_{jk}\right\rfloor_0$$
$$= \frac{V_0}{V_t}\overline{F}_{jq}\left\lceil A_{ik}\varepsilon_{quk}N_u\right\rfloor_0 = \frac{V_0}{V_t}\overline{F}^{\mathrm{t}}_{qj}\left\langle -\frac{\partial A_{ik}}{\partial X_u}\varepsilon_{kuq}\right\rangle_0 = \frac{V_0}{V_t}\overline{F}^{\mathrm{t}}_{qj}\left\langle P_{iq}\right\rangle_0 = \frac{V_0}{V_t}\overline{P}_{iq}\overline{F}^{\mathrm{t}}_{qj},$$

or, in vectorial notation, $\langle \boldsymbol{p}\rangle_t = \dfrac{V_0}{V_t}\overline{\boldsymbol{P}}\cdot\overline{\boldsymbol{F}}^{\mathrm{t}}$. In addition, using Equations (5), (6), (17) and (27), we can write in indicial notation

$$\left\langle \mathbb{p}_j\right\rangle_t = \left\langle -\frac{\partial \mathbb{a}_n}{\partial x_m}\varepsilon_{nmj}\right\rangle_t = \left\lceil \mathbb{a}_n n_m\varepsilon_{mnj}\right\rfloor_t = \frac{V_0}{V_t}\left\lceil \varepsilon_{jmn}\mathbb{A}_k F^{-1}_{kn}JF^{-\mathrm{t}}_{mu}N_u\right\rfloor_0$$
$$= \frac{V_0}{V_t}\left\lceil [J\varepsilon_{jmn}]\mathbb{A}_k F^{-1}_{kn}F^{-1}_{um}N_u\right\rfloor_0 = \frac{V_0}{V_t}\left\lceil \varepsilon_{qrs}F_{jq}F_{mr}F_{ns}\mathbb{A}_k F^{-1}_{kn}F^{-1}_{um}N_u\right\rfloor_0$$
$$= \frac{V_0}{V_t}\left\lceil \mathbb{A}_k\varepsilon_{qrs}F_{jq}\delta_{ur}\delta_{ks}N_u\right\rfloor_0 = \frac{V_0}{V_t}\left\lceil \mathbb{A}_k[\boldsymbol{F}\times\boldsymbol{N}]_{jk}\right\rfloor_0 = \frac{V_0}{V_t}\left\lceil \mathbb{A}_k[\overline{\boldsymbol{F}}\times\boldsymbol{N}]_{jk}\right\rfloor_0$$
$$= \frac{V_0}{V_t}\overline{F}_{jq}\left\lceil \mathbb{A}_k\varepsilon_{quk}N_u\right\rfloor_0 = \frac{V_0}{V_t}\overline{F}^{\mathrm{t}}_{qj}\left\langle -\frac{\partial \mathbb{A}_k}{\partial X_u}\varepsilon_{kuq}\right\rangle_0 = \frac{V_0}{V_t}\overline{F}^{\mathrm{t}}_{qj}\left\langle \mathbb{P}_q\right\rangle_0 = \frac{V_0}{V_t}\overline{\mathbb{P}}_q\overline{F}^{\mathrm{t}}_{qj},$$

or, in vectorial notation, $\langle \mathbb{p}\rangle_t = \dfrac{V_0}{V_t}\overline{\mathbb{P}}\cdot\overline{\boldsymbol{F}}^{\mathrm{t}}$. For this type of boundary conditions no information about the volume average of the inverse of the deformation gradient was obtained.

5.2. Magnetic type of boundary conditions

5.2.1. First case We consider the boundary conditions presented in the first case of Section 4.3, i.e. $\mathbb{Y} = \overline{\mathbb{F}}\cdot\boldsymbol{X} + \mathbb{Z}(\boldsymbol{X})$, where $\mathbb{Z}$ is null on $\partial\mathcal{B}_0$ or periodic with antiperiodic $\mathbb{P}\cdot\boldsymbol{N}$. In order to proceed to the spatial description we need to express $\mathbb{Y}$ in terms of the spatial position vector $\boldsymbol{x}$. This is possible if we assume that the mechanical boundary conditions are those presented in the first case of Section 4.1, i.e. that the position vector $\boldsymbol{x}$ in the spatial description is given by $\boldsymbol{x} = \overline{\boldsymbol{F}}\cdot\boldsymbol{X} + \boldsymbol{Z}(\boldsymbol{X})$, where $\boldsymbol{Z}$ is null on $\partial\mathcal{B}_0$ or periodic. Then the scalar $\mathbb{y}(\boldsymbol{x})$ can be expressed using Equations $(29)_2$, (43) and (49) as

$$\mathbb{y} = \overline{\mathbb{f}}\cdot\boldsymbol{x} + \mathbb{z}(\boldsymbol{x}), \tag{50}$$

with $\mathbb{z}(\boldsymbol{x}) = -\overline{\mathbb{F}}\cdot\boldsymbol{z}(\boldsymbol{x}) + \tilde{\mathbb{Z}}(\boldsymbol{x})$, $\tilde{\mathbb{Z}}(\boldsymbol{x}) \equiv \mathbb{Z}(\boldsymbol{X})$. We mention that $\mathbb{z}$ is either null on $\partial\mathcal{B}_t$ or periodic in the spatial configuration [45]. Under these conditions and using Equation (7) we get

$$\langle \mathbb{f}\rangle_t = \langle \mathrm{grad}\,\mathbb{y}\rangle_t = \overline{\mathbb{f}}. \tag{51}$$

As indicated in the first case of Section 5.1, the condition $(32)_5$ holds, either when $\boldsymbol{Z}$ is null on $\partial\mathcal{B}_0$ or when we consider periodicity for the spatial position vector $\boldsymbol{x}$ and antiperiodicity for the term $\mathbb{P}\cdot\boldsymbol{N}$.

5.2.2. Second case We consider the boundary conditions presented in the second case of Section 4.3, i.e. $\mathbb{P}\cdot\boldsymbol{N} = \overline{\mathbb{P}}\cdot\boldsymbol{N}$ on $\partial\mathcal{B}_0$. Then, using Equations (1), (3), $(14)_3$, (24) and the divergence theorem we get

$$\langle \mathbb{p}\rangle_t = \frac{V_0}{V_t}\left\langle \mathbb{P}\cdot\boldsymbol{F}^{\mathrm{t}}\right\rangle_0 = \frac{V_0}{V_t}\left\langle \mathbb{P}\cdot\boldsymbol{F}^{\mathrm{t}} + \boldsymbol{Y}\mathrm{Div}\mathbb{P}\right\rangle_0$$
$$= \frac{V_0}{V_t}\left\langle \mathrm{Div}(\boldsymbol{Y}\otimes\mathbb{P})\right\rangle_0 = \frac{V_0}{V_t}\left\lceil \boldsymbol{Y}[\mathbb{P}\cdot\boldsymbol{N}]\right\rfloor_0$$
$$= \frac{V_0}{V_t}\left\lceil \boldsymbol{Y}[\overline{\mathbb{P}}\cdot\boldsymbol{N}]\right\rfloor_0 = \frac{V_0}{V_t}\left\lceil \boldsymbol{Y}\otimes\boldsymbol{N}\right\rfloor_0\cdot\overline{\mathbb{P}} = \frac{V_0}{V_t}\overline{\mathbb{P}}\cdot\overline{\boldsymbol{F}}^{\mathrm{t}}.$$

The examined case is more general than the third type of boundary conditions presented in the first case of Section 5.1 for the magnetic induction, since it does not require any assumption about the form of the spatial position vector. For this type of boundary conditions no information about the volume average of the magnetic field was obtained.

5.2.3. Third case We consider the boundary conditions presented in the first case of Section 4.4, i.e. $\mathbb{A} = \frac{1}{2}\overline{\mathbb{P}} \times \boldsymbol{X} + \mathbb{W}(\boldsymbol{X})$, where $\mathbb{W}$ is null on $\partial\mathcal{B}_0$ or periodic with antiperiodic $\mathbb{F} \times \boldsymbol{N}$. Using Equations (5), $(14)_3$, (24) and the divergence theorem we have

$$\begin{aligned}
\langle \mathbb{p} \rangle_t &= \frac{V_0}{V_t}\left\langle \overline{\mathbb{P}} \cdot \overline{\boldsymbol{F}}^{\mathrm{t}} \right\rangle_0 = \frac{V_0}{V_t}\langle \boldsymbol{F} \rangle_0 \cdot \overline{\mathbb{P}} + \frac{V_0}{V_t}\langle \boldsymbol{F} \cdot \mathrm{Curl}\mathbb{W} \rangle_0 \\
&= \frac{V_0}{V_t}\overline{\mathbb{P}} \cdot \overline{\boldsymbol{F}}^{\mathrm{t}} + \frac{V_0}{V_t}\langle \boldsymbol{F} \cdot \mathrm{Curl}\mathbb{W} - [\mathrm{Curl}\boldsymbol{F}] \cdot \mathbb{W} \rangle_0 \\
&= \frac{V_0}{V_t}\overline{\mathbb{P}} \cdot \overline{\boldsymbol{F}}^{\mathrm{t}} + \frac{V_0}{V_t}\langle \mathrm{Curl}\,(\boldsymbol{F} \otimes \mathbb{W}) : \boldsymbol{I} \rangle_0 = \frac{V_0}{V_t}\overline{\mathbb{P}} \cdot \overline{\boldsymbol{F}}^{\mathrm{t}} + \frac{V_0}{V_t}\lceil \mathbb{W} \cdot [\boldsymbol{F} \times \boldsymbol{N}] \rfloor_0 .
\end{aligned}$$

The integral of the second term vanishes and we get $\langle \mathbb{p} \rangle_t = \dfrac{V_0}{V_t}\overline{\mathbb{P}} \cdot \overline{\boldsymbol{F}}^{\mathrm{t}}$ under one of the following conditions:

1. $\mathbb{W} = \mathbb{O}$ on $\partial\mathcal{B}_0$;
2. $\mathbb{W}$ periodic and $\boldsymbol{F} \times \boldsymbol{N}$ antiperiodic;
3. $\mathbb{W}$ periodic and $\boldsymbol{F} \times \boldsymbol{N} = \overline{\boldsymbol{F}} \times \boldsymbol{N}$ on $\partial\mathcal{B}_0$.

The last condition is less general than the boundary conditions presented in the fourth case of Section 5.1 for the deformation gradient, which does not require any assumption about the form of the potential $\mathbb{W}$. For the examined types of boundary conditions no information about the volume average of the magnetic field was obtained.

5.2.4. Fourth case We consider the boundary conditions presented in the second case of Section 4.4, i.e. $\mathbb{F} \times \boldsymbol{N} = \overline{\mathbb{F}} \times \boldsymbol{N}$ on $\partial\mathcal{B}_0$. For the examined type of boundary conditions no information about the volume average of the magnetic field or the magnetic induction was obtained.

Table 3 summarizes the results of this section. Even though the stress condition $(32)_3$ and the magnetic induction condition $(32)_5$ are satisfied for many cases, as Table 3 indicates, $\langle \boldsymbol{p} \rangle_t \neq \overline{\boldsymbol{p}}$ and $\langle \mathbb{p} \rangle_t \neq \overline{\mathbb{p}}$, since $\dfrac{V_t}{V_0}$ does not always represent the determinant of the macroscopic deformation gradient. For the mechanical problem one could consider the microscopic Kirchhoff stress $\boldsymbol{s} := J\boldsymbol{p} = \boldsymbol{P} \cdot \boldsymbol{F}^{\mathrm{t}}$ and the macroscopic Kirchhoff stress $\overline{\boldsymbol{s}} := \overline{\boldsymbol{P}} \cdot \overline{\boldsymbol{F}}^{\mathrm{t}}$ as the spatial representation of the stress tensor in the RVE and the body respectively [28]. For all mechanical boundary conditions it holds

$$\langle \boldsymbol{s} \rangle_0 = \frac{V_t}{V_0}\langle \boldsymbol{p} \rangle_t = \overline{\boldsymbol{P}} \cdot \overline{\boldsymbol{F}}^{\mathrm{t}} = \overline{\boldsymbol{s}}.$$

In the case of incompressible materials $\overline{J} = \dfrac{V_t}{V_0} = 1$ and $\overline{\boldsymbol{p}} = \overline{\boldsymbol{s}} = \langle \boldsymbol{s} \rangle_0 = \langle \boldsymbol{p} \rangle_t$.

In a similar manner one could identify a microscopic spatial magnetic induction $\mathbb{s} := J\mathbb{p} = \mathbb{P} \cdot \boldsymbol{F}^{\mathrm{t}}$ and the corresponding macroscopic spatial magnetic induction $\overline{\mathbb{s}} := \overline{\mathbb{P}} \cdot \overline{\boldsymbol{F}}^{\mathrm{t}}$ and get for all boundary conditions of the second column in magnetic variables of Table 3

$$\langle \mathbb{s} \rangle_0 = \frac{V_t}{V_0}\langle \mathbb{p} \rangle_t = \overline{\mathbb{P}} \cdot \overline{\boldsymbol{F}}^{\mathrm{t}} = \overline{\mathbb{s}}.$$

In the case of incompressible materials $\overline{J} = \dfrac{V_t}{V_0} = 1$ and $\overline{\mathbb{p}} = \overline{\mathbb{s}} = \langle \mathbb{s} \rangle_0 = \langle \mathbb{p} \rangle_t$.

Table 3. Macroscopic transition from material to spatial description.

Mechanical variables	
$\langle \boldsymbol{f} \rangle_t = \overline{\boldsymbol{f}}, \frac{V_t}{V_0} = \overline{J}$	$\langle \boldsymbol{p} \rangle_t = \frac{V_0}{V_t} \overline{\boldsymbol{P}} \cdot \overline{\boldsymbol{F}}^{\mathrm{t}}, \langle \boldsymbol{s} \rangle_0 = \overline{\boldsymbol{s}}$
1) $\boldsymbol{Y} = \overline{\boldsymbol{F}} \cdot \boldsymbol{X} + \boldsymbol{Z}$ • $\boldsymbol{Z} = \boldsymbol{0}$ on $\partial \mathcal{B}_0$ • $\boldsymbol{Z}$ periodic	1) $\boldsymbol{Y} = \overline{\boldsymbol{F}} \cdot \boldsymbol{X} + \boldsymbol{Z}$ • $\boldsymbol{Z} = \boldsymbol{0}$ on $\partial \mathcal{B}_0$ • $\boldsymbol{Z}$ periodic and $\boldsymbol{P} \cdot \boldsymbol{N}$ antiperiodic 2) $\boldsymbol{P} \cdot \boldsymbol{N} = \overline{\boldsymbol{P}} \cdot \boldsymbol{N}$ on $\partial \mathcal{B}_0$ 3) $\boldsymbol{A} = \frac{1}{2} \overline{\boldsymbol{P}} \times \boldsymbol{X} + \boldsymbol{W}$ • $\boldsymbol{W} = \boldsymbol{0}$ on $\partial \mathcal{B}_0$ • $\boldsymbol{W}$ periodic and $\boldsymbol{F} \times \boldsymbol{N}$ antiperiodic 4) $\boldsymbol{F} \times \boldsymbol{N} = \overline{\boldsymbol{F}} \times \boldsymbol{N}$ on $\partial \mathcal{B}_0$
Magnetic variables	
$\langle \mathbb{f} \rangle_t = \overline{\mathbb{f}}$	$\langle \mathbb{p} \rangle_t = \frac{V_0}{V_t} \overline{\mathbb{P}} \cdot \overline{\boldsymbol{F}}^{\mathrm{t}}, \langle \mathbb{s} \rangle_0 = \overline{\mathbb{s}}$
1) $\boldsymbol{Y} = \overline{\boldsymbol{F}} \cdot \boldsymbol{X} + \boldsymbol{Z}$ and $\mathbb{Y} = \overline{\mathbb{F}} \cdot \boldsymbol{X} + \mathbb{Z}$ • $\boldsymbol{Z} = \boldsymbol{0}$ and $\mathbb{Z} = \mathbb{O}$ on $\partial \mathcal{B}_0$ • $\boldsymbol{Z} = \boldsymbol{0}$ on $\partial \mathcal{B}_0$ and $\mathbb{Z}$ periodic • $\mathbb{Z} = \mathbb{O}$ on $\partial \mathcal{B}_0$ and $\boldsymbol{Z}$ periodic • $\boldsymbol{Z}$ and $\mathbb{Z}$ periodic	1) $\boldsymbol{Y} = \overline{\boldsymbol{F}} \cdot \boldsymbol{X} + \boldsymbol{Z}$ • $\boldsymbol{Z} = \boldsymbol{0}$ on $\partial \mathcal{B}_0$ • $\boldsymbol{Z}$ periodic and $\mathbb{P} \cdot \boldsymbol{N}$ antiperiodic 2) $\mathbb{P} \cdot \boldsymbol{N} = \overline{\mathbb{P}} \cdot \boldsymbol{N}$ on $\partial \mathcal{B}_0$ 3) $\mathbb{A} = \frac{1}{2} \overline{\mathbb{P}} \times \boldsymbol{X} + \mathbb{W}$ • $\mathbb{W} = \mathbb{O}$ on $\partial \mathcal{B}_0$ • $\mathbb{W}$ periodic and $\boldsymbol{F} \times \boldsymbol{N}$ antiperiodic 4) $\boldsymbol{F} \times \boldsymbol{N} = \overline{\boldsymbol{F}} \times \boldsymbol{N}$ on $\partial \mathcal{B}_0$

6. Conclusion

A general magnetomechanical homogenization framework under large deformation processes has been presented. The boundary conditions under which the homogenization problem of a magnetorheological elastomer is well defined are summarized in Table 2. Any set of mechanical and magnetic boundary conditions guarantees that the Hill–Mandel condition is satisfied for the magnetomechanical energy increments, when the problem is described in the material configuration. The analysis in this work also shows that a large deformation process with meaningful space averages can exist under one of the following pairs of conditions:

1. $\boldsymbol{x} = \overline{\boldsymbol{F}} \cdot \boldsymbol{X}$ and $\mathbb{Y} = \overline{\mathbb{F}} \cdot \boldsymbol{X}$ on $\partial \mathcal{B}_0$;
2. $\boldsymbol{x} = \overline{\boldsymbol{F}} \cdot \boldsymbol{X}$ on $\partial \mathcal{B}_0$ and $\mathbb{Y} = \overline{\mathbb{F}} \cdot \boldsymbol{X} + \mathbb{Z}$ with $\mathbb{Z}$ periodic and $\mathbb{P} \cdot \boldsymbol{N}$ antiperiodic;
3. $\boldsymbol{x} = \overline{\boldsymbol{F}} \cdot \boldsymbol{X} + \boldsymbol{Z}$ with $\boldsymbol{Z}$ periodic and $\boldsymbol{P} \cdot \boldsymbol{N}$ antiperiodic and $\mathbb{Y} = \overline{\mathbb{F}} \cdot \boldsymbol{X} + \mathbb{Z}$ with $\mathbb{Z}$ periodic and $\mathbb{P} \cdot \boldsymbol{N}$ antiperiodic.

These types of boundary conditions guarantee that the macroscopic mechanical and magnetic field variables in the deformed and undeformed configuration are given by volume averaging of the corresponding microscopic variables over the deformed and undeformed RVE, respectively.

This contribution shows that the use of kinematic and magnetic field potentials instead of kinetic field and magnetic induction potentials provides a more appropriate homogenization process, in which averaging over the RVE in material and spatial description renders equivalent counterparts. The choice of kinematic and magnetic field potentials has additional advantages in the numerical implementation procedure.

Notes

1. For the proof one needs to recall the identity $\varepsilon_{ikl}\varepsilon_{ikm} = 2\delta_{lm}$.

Funding

This work was supported by the ERC (advanced grant MOCOPOLY).

Conflict of interest

None declared.

References

[1] Kordonsky, W. Magneto-rheological effects as a base of new devices and technologies. *Journal of Magnetism and Magnetic Materials* 1993; 122: 395–398.
[2] Jolly, MR, Carlson, JD, and Munoz, BC. A model of the behavior of magneto-rheological materials. *Smart Mater Structures* 1996; 5: 607–614.
[3] Carlson, JD, and Jolly, MR. MR fluid, foam and elastomer devices. *Mechatronics* 2000; 10: 555–569.
[4] Ginder, JM, Clark, SM, Schlotter, WF, and Nichols, ME. Magnetostrictive phenomena in magnetorheological elastomers. *Intl J Mod Phys B* 2002; 16: 2412–2418.
[5] Bellan, C, and Bossis, G. Field dependence of viscoelastic properties of MR elastomers. *Int J Mod Phys B* 2002; 16: 2447–2453.
[6] Varga, Z, Filipcsei, G, and Zrínyi, M. Magnetic field sensitive functional elastomers with tuneable elastic modulus. *Polymer* 2006; 47: 227–233.
[7] Pao, YH. Electromagnetic forces in deformable continua. *Mech Today* 1978; 4: 209–305.
[8] Eringen, AC, and Maugin, GA. *Electrodynamics of Continua I: Foundations and Solid Media*. New York: Springer-Verlag, 1989.
[9] Maugin, GA. *Material Inhomogeneities in Elasticity*, vol. 3. London: Chapman & Hall/CRC, 1993.
[10] Kovetz, A. *Electromagnetic Theory*. New York: Oxford University Press, 2000.
[11] Vu, DK, and Steinmann, P. Nonlinear electro- and magneto-elastostatics: Material and spatial settings. *Intl J Solids Structures* 2007; 44: 7891–7905.
[12] Bustamante, R, Dorfmann, A, and Ogden, RW. On variational formulations in nonlinear magnetoelastostatics. *Maths Mech Solids* 2008; 13: 725–745.
[13] Vu, DK, and Steinmann, P. Material and spatial motion problems in nonlinear electro- and magneto-elastostatics. *Maths Mech Solids* 2010; 15: 239–257.
[14] Brigadnov, IA, and Dorfmann, A. Mathematical modeling of magneto-sensitive elastomers. *Intl J Solids Structures* 2003; 40: 4659–4674.
[15] Dorfmann, A, and Ogden, RW. Nonlinear magnetoelastic deformations on elastomers. *Acta Mechanica* 2004; 167: 13–28.
[16] Steigmann, DJ. Equilibrium theory for magnetic elastomers and magnetoelastic membranes. *Intl J Non-Lin Mech* 2004; 39: 1193–1216.
[17] Kankanala, SV, and Triantafyllidis, N. On finitely strained magnetorheological elastomers. *J Mech Phys Solids* 2004; 52: 2869–2908.
[18] Bustamante, R. Transversely isotropic nonlinear magneto-active elastomers. *Acta Mechanica* 2010; 210: 183–214.
[19] Kanouté, P, Boso, DP, Chaboche, JL, and Schrefler, BA. Multiscale methods for composites: a review. *Arch Computat Meth Engng* 2009; 16: 31–75.
[20] Charalambakis, N. Homogenization techniques and micromechanics. a survey and perspectives. *Appl Mech Rev* 2010; 63(3): 030803.
[21] Hashin, Z, and Shtrikman, S. A variational approach to the theory of the elastic behaviour of multiphase materials. *J Mech Phys Solids* 1963; 11: 127–140.
[22] Hill, R. Elastic properties of reinforced solids: Some theoretical principles. *J Mech Phys Solids* 1963; 11: 357–372.
[23] Bensoussan, A, Lions, J, and Papanicolaou, G. *Asymptotic Methods for Periodic Structures*. Amsterdam: North-Holland, 1978.
[24] Sanchez-Palencia, E. *Non-homogeneous Media and Vibration Theory* (*Lecture Notes in Physics*, vol. 127). Berlin: Springer-Verlag, 1978.
[25] Mura, T. *Micromechanics of Defects in Solids*. Amsterdam: Martinus Nijhoff Publishers, 1987.
[26] Miehe, C, and Koch, A. Computational micro-to-macro transitions of discretized microstructures undergoing small strains. *Arch Appl Mech* 2002; 72: 300–317.
[27] Suquet, PM. *Elements of Homogenization for Inelastic Solid Mechanics* (*Lecture Notes in Physics*, vol. 272). Berlin: Springer-Verlag, 1987, pp. 193–278.
[28] Hill, R. On constitutive macro-variables for heterogeneous solids at finite strain. *Proceedings of the Royal Society of London A* 1972; 326: 131–147.
[29] Nemat-Nasser, S. Averaging theorems in finite deformation plasticity. *Mech Mater* 1999; 31: 493–523.
[30] Miehe, C, Schröder, J, and Schotte, J. Computational homogenization analysis in finite plasticity Simulation of texture development in polycrystalline materials. *Comput Meth Appl Mech Engng* 1999; 171: 387–418.
[31] Miehe, C. Computational micro-to-macro transitions for discretized micro-structures of heterogeneous materials at finite strains based on the minimization of averaged incremental energy. *Comput Meth Appl Mech Engng* 2003; 192: 559–591.

[32] Hirschberger, CB, Sukumar, N, and Steinmann, P. Computational homogenization of material layers with micromorphic mesostructure. *Philos Mag* 2008; 88: 3603–3631.
[33] Ricker, S, Mergheim, J, Steinmann, P, and Müller, R. A comparison of different approaches in the multi-scale computation of configurational forces. *Intl J Fracture* 2010; 166: 203–214.
[34] McBride, A, Mergheim, J, Javili, A, Steinmann, P, and Bargmann, S. Micro-to-macro transitions for heterogeneous material layers accounting for in-plane stretch. *J Mech Phys Solids* 2012; 60: 1221–1239.
[35] Borcea, L, and Bruno, O. On the magneto-elastic properties of elastomer–ferromagnet composites. *J Mech Phys Solids* 2001; 49: 2877–2919.
[36] Yin, HM, Sun, LZ, and Chen, JS. Micromechanics-based hyperelastic constitutive modeling of magnetostrictive particle-filled elastomers. *Mech Mater* 2002; 34: 505–516.
[37] Wang, D, Chen, J-S, and Sun, L. Homogenization of magnetostrictive particle-filled elastomers using an interface-enriched reproducing kernel particle method. *Finite Elements Anal Des* 2003; 39: 765–782.
[38] Yin, HM, Sun, LZ, and Chen, JS. Magneto-elastic modeling of composites containing chain-structured magnetostrictive particles. *J Mech Phys Solids* 2006; 54: 975–1003.
[39] Ponte Castañeda, P, and Galipeau, E. Homogenization-based constitutive models for magneto-rheological elastomers at finite strain. *J Mech Phys Solids* 2011; 59: 194–215.
[40] Truesdell, C, and Noll, W. *The Non-Linear Field Theories of Mechanics*, 3rd edn. Berlin: Springer-Verlag, 2004.
[41] Gurtin, ME, Fried, E, and Anand, L. *The Mechanics and Thermodynamics of Continua*. Cambridge: Cambridge University Press, 2009.
[42] Marsden, JE, and Hughes, TJR. *Mathematical Foundations of Elasticity*. New York: Dover Publications, 1994.
[43] Ogden, RW. *Non-Linear Elastic Deformations*. New York: Dover Publications, 1997.
[44] Maugin, GA. *Configurational Forces: Thermomechanics, Physics, Mathematics, and Numerics*. Boca Raton, FL: CRC Press Inc., 2010.
[45] Costanzo, F, Gray, GL, and Andia, PC. On the definitions of effective stress and deformation gradient for use in MD: Hill's macro-homogeneity and the virial theorem. *Intl J Engng Sci* 2005; 43: 533–555.
[46] Hill, R. On macroscopic effects of heterogeneity in elastoplastic media at finite strain. *Math Proc Cambridge Philos Soc* 1984; 95: 481–494.
[47] Bettaieb, MB, Débordes, O, Dogui, A, and Dûchene, L. Averaging properties for periodic homogenization and large deformation. *J Multiscale Computat Engng* 2012; 10: 281–293.

Appendix: Hill's lemma for magnetomechanics

A.1. Hill's lemma for mechanics

In order to prove Equation (34), we start from the integral of the right-hand side. Using Equations (1), (2), (24) and the divergence theorem we have

$$\begin{aligned}\big\lceil [\boldsymbol{Y} - \overline{\boldsymbol{F}} \cdot \boldsymbol{X}] \cdot [\boldsymbol{P} \cdot \boldsymbol{N} - \overline{\boldsymbol{P}} \cdot \boldsymbol{N}] \big\rfloor_0 &= \lceil \boldsymbol{Y} \cdot \boldsymbol{P} \cdot \boldsymbol{N} \rfloor_0 - \overline{\boldsymbol{P}} : \lceil \boldsymbol{Y} \otimes \boldsymbol{N} \rfloor_0 - \overline{\boldsymbol{F}} : \lceil [\boldsymbol{P} \cdot \boldsymbol{N}] \otimes \boldsymbol{X} \rfloor_0 \\ &\quad + [\overline{\boldsymbol{F}}^{\mathrm{t}} \cdot \overline{\boldsymbol{P}}] : \lceil \boldsymbol{X} \otimes \boldsymbol{N} \rfloor_0 \\ &= \langle \boldsymbol{P} : \boldsymbol{F} + \boldsymbol{Y} \cdot \mathrm{Div}\boldsymbol{P} \rangle_0 - \overline{\boldsymbol{P}} : \overline{\boldsymbol{F}} \\ &\quad - \overline{\boldsymbol{F}} : \langle \boldsymbol{P} + \boldsymbol{X} \cdot \mathrm{Div}\boldsymbol{P} \rangle_0 + \overline{\boldsymbol{P}} : \overline{\boldsymbol{F}} \\ &= \langle \boldsymbol{P} : \boldsymbol{F} \rangle_0 - \overline{\boldsymbol{P}} : \overline{\boldsymbol{F}}.\end{aligned}$$

In order to prove Equation (35), we start from the integral of the right-hand side. We have

$$\begin{aligned}\Big\lceil \Big[\boldsymbol{A} - \tfrac{1}{2}\overline{\boldsymbol{P}} \times \boldsymbol{X}\Big] : [\boldsymbol{F} \times \boldsymbol{N} - \overline{\boldsymbol{F}} \times \boldsymbol{N}] \Big\rfloor_0 &= \lceil \boldsymbol{A} : [\boldsymbol{F} \times \boldsymbol{N}] \rfloor_0 - \big\lceil \boldsymbol{A} : [\overline{\boldsymbol{F}} \times \boldsymbol{N}] \big\rfloor_0 \\ &\quad - \tfrac{1}{2} \big\lceil [\overline{\boldsymbol{P}} \times \boldsymbol{X}] : [\boldsymbol{F} \times \boldsymbol{N}] \big\rfloor_0 \\ &\quad + \tfrac{1}{2} \big\lceil [\overline{\boldsymbol{P}} \times \boldsymbol{X}] : [\overline{\boldsymbol{F}} \times \boldsymbol{N}] \big\rfloor_0 .\end{aligned}$$

Using the divergence theorem and Equations (5), (8) and (25), the first term is written

$$\lceil \boldsymbol{A} : [\boldsymbol{F} \times \boldsymbol{N}] \rfloor_0 = \big\langle \boldsymbol{I} : \mathrm{Curl}\, \big(\boldsymbol{F}^{\mathrm{t}} \cdot \boldsymbol{A}\big) \big\rangle_0 = \langle \boldsymbol{F} : \mathrm{Curl}\boldsymbol{A} - \boldsymbol{A} : \mathrm{Curl}\boldsymbol{F} \rangle_0 = \langle \boldsymbol{P} : \boldsymbol{F} \rangle_0 ,$$

the second term is written

$$- \big\lceil \boldsymbol{A} : [\overline{\boldsymbol{F}} \times \boldsymbol{N}] \big\rfloor_0 = - \big\langle \boldsymbol{I} : \mathrm{Curl}\, \big(\overline{\boldsymbol{F}}^{\mathrm{t}} \cdot \boldsymbol{A}\big) \big\rangle_0 = - \big\langle \overline{\boldsymbol{F}} : \mathrm{Curl}\boldsymbol{A} \big\rangle_0 = - \langle \boldsymbol{P} \rangle_0 : \overline{\boldsymbol{F}} = -\overline{\boldsymbol{P}} : \overline{\boldsymbol{F}},$$

and the third term is written

$$-\tfrac{1}{2}\left\lceil [\overline{\boldsymbol{P}}\times\boldsymbol{X}]:[\boldsymbol{F}\times\boldsymbol{N}]\right\rfloor_0 = -\tfrac{1}{2}\left\langle \boldsymbol{I}:\mathrm{Curl}\left(\boldsymbol{F}^{\mathrm{t}}\cdot[\overline{\boldsymbol{P}}\times\boldsymbol{X}]\right)\right\rangle_0$$
$$= -\tfrac{1}{2}\left\langle \boldsymbol{F}:\mathrm{Curl}(\overline{\boldsymbol{P}}\times\boldsymbol{X}) - [\overline{\boldsymbol{P}}\times\boldsymbol{X}]:\mathrm{Curl}\boldsymbol{F}\right\rangle_0$$
$$= -\left\langle \boldsymbol{F}:\overline{\boldsymbol{P}}\right\rangle_0 = -\overline{\boldsymbol{P}}:\overline{\boldsymbol{F}},$$

where we have used the identity $\varepsilon_{ikl}\varepsilon_{ikm} = 2\delta_{lm}$. Finally, the fourth term is written

$$\tfrac{1}{2}\left\lceil [\overline{\boldsymbol{P}}\times\boldsymbol{X}]:[\overline{\boldsymbol{F}}\times\boldsymbol{N}]\right\rfloor_0 = \overline{\boldsymbol{F}}:\tfrac{1}{2}\left\langle \mathrm{Curl}(\overline{\boldsymbol{P}}\times\boldsymbol{X})\right\rangle_0 = \overline{\boldsymbol{P}}:\overline{\boldsymbol{F}}.$$

Combining all of the expressions we get

$$\left\lceil \left(\boldsymbol{A}-\tfrac{1}{2}\overline{\boldsymbol{P}}\times\boldsymbol{X}\right):[\boldsymbol{F}\times\boldsymbol{N}-\overline{\boldsymbol{F}}\times\boldsymbol{N}]\right\rfloor_0 = \langle\boldsymbol{P}:\boldsymbol{F}\rangle_0 - \overline{\boldsymbol{P}}:\overline{\boldsymbol{F}}.$$

□

A.2. Hill's lemma for magnetostatics

In order to prove Equation (36), we start from the integral of the right-hand side. Using indicial notation, Equations (3), (7), (27) and the divergence theorem, we get

$$\left\lceil [\mathbb{Y}-\overline{\mathbb{F}}\cdot\boldsymbol{X}][\mathbb{P}\cdot\boldsymbol{N}-\overline{\mathbb{P}}\cdot\boldsymbol{N}]\right\rfloor_0 = \lceil\mathbb{Y}\mathbb{P}\cdot\boldsymbol{N}\rfloor_0 - \overline{\mathbb{P}}\cdot\lceil\mathbb{Y}\boldsymbol{N}\rfloor_0 - \overline{\mathbb{F}}\cdot\lceil[\boldsymbol{X}\otimes\mathbb{P}]\cdot\boldsymbol{N}\rfloor_0$$
$$+ [\overline{\mathbb{F}}\otimes\overline{\mathbb{P}}]:\lceil\boldsymbol{X}\otimes\boldsymbol{N}\rfloor_0$$
$$= \langle\mathbb{Y}\mathrm{Div}\mathbb{P}+\mathbb{P}\cdot\mathbb{F}\rangle_0 - \overline{\mathbb{P}}\cdot\overline{\mathbb{F}} - \overline{\mathbb{F}}\cdot\langle\mathbb{P}+\boldsymbol{X}\mathrm{Div}\mathbb{P}\rangle_0 + \overline{\mathbb{F}}\cdot\overline{\mathbb{P}}$$
$$= \langle\mathbb{P}\cdot\mathbb{F}\rangle_0 - \overline{\mathbb{P}}\cdot\overline{\mathbb{F}}.$$

In order to prove Equation (37), we start from the integral of the right-hand side. We have

$$\left\lceil \left[\mathbb{A}-\tfrac{1}{2}\overline{\mathbb{P}}\times\boldsymbol{X}\right]\cdot[\mathbb{F}\times\boldsymbol{N}-\overline{\mathbb{F}}\times\boldsymbol{N}]\right\rfloor_0 = \lceil\mathbb{A}\cdot[\mathbb{F}\times\boldsymbol{N}]\rfloor_0 - \left\lceil\mathbb{A}\cdot[\overline{\mathbb{F}}\times\boldsymbol{N}]\right\rfloor_0$$
$$-\tfrac{1}{2}\left\lceil[\overline{\mathbb{P}}\times\boldsymbol{X}]\cdot[\mathbb{F}\times\boldsymbol{N}]\right\rfloor_0$$
$$+\tfrac{1}{2}\left\lceil[\overline{\mathbb{P}}\times\boldsymbol{X}]\cdot[\overline{\mathbb{F}}\times\boldsymbol{N}]\right\rfloor_0.$$

Using the divergence theorem and Equations (4), (6) and (26), the first two terms are written

$$\langle\boldsymbol{I}:\mathrm{Curl}\,(\mathbb{F}\otimes\mathbb{A})\rangle_0 - \overline{\mathbb{F}}\cdot\langle\mathrm{Curl}\mathbb{A}\rangle_0 = \langle\mathbb{F}\cdot\mathrm{Curl}\mathbb{A}-\mathbb{A}\cdot\mathrm{Curl}\mathbb{F}\rangle_0 - \overline{\mathbb{F}}\cdot\overline{\mathbb{P}} = \langle\mathbb{F}\cdot\mathbb{P}\rangle_0 - \overline{\mathbb{F}}\cdot\overline{\mathbb{P}},$$

the third term is written

$$-\tfrac{1}{2}\left\lceil[\overline{\mathbb{P}}\times\boldsymbol{X}]\cdot[\mathbb{F}\times\boldsymbol{N}]\right\rfloor_0 = -\tfrac{1}{2}\left\langle\boldsymbol{I}:\mathrm{Curl}\left(\mathbb{F}\otimes[\overline{\mathbb{P}}\times\boldsymbol{X}]\right)\right\rangle_0$$
$$= -\tfrac{1}{2}\left\langle\mathbb{F}\cdot\mathrm{Curl}(\overline{\mathbb{P}}\times\boldsymbol{X}) - [\overline{\mathbb{P}}\times\boldsymbol{X}]\cdot\mathrm{Curl}\mathbb{F}\right\rangle_0$$
$$= -\left\langle\mathbb{F}\cdot\overline{\mathbb{P}}\right\rangle_0 = -\overline{\mathbb{P}}\cdot\overline{\mathbb{F}},$$

and the fourth term is written

$$\tfrac{1}{2}\left\lceil[\overline{\mathbb{P}}\times\boldsymbol{X}]\cdot[\overline{\mathbb{F}}\times\boldsymbol{N}]\right\rfloor_0 = \tfrac{1}{2}\left\langle\overline{\mathbb{F}}\cdot\mathrm{Curl}(\overline{\mathbb{P}}\times\boldsymbol{X})\right\rangle_0 = \overline{\mathbb{P}}\cdot\overline{\mathbb{F}}.$$

Combining all of the previous expressions we get

$$\left\lceil \left[\mathbb{A}-\tfrac{1}{2}\overline{\mathbb{P}}\times\boldsymbol{X}\right]\cdot[\mathbb{F}\times\boldsymbol{N}-\overline{\mathbb{F}}\times\boldsymbol{N}]\right\rfloor_0 = \langle\mathbb{P}\cdot\mathbb{F}\rangle_0 - \overline{\mathbb{P}}\cdot\overline{\mathbb{F}}.$$

□