Unified operation approach of generalized closed sets via topological ideals*

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Abstract

The aim of this paper is to unify the concepts of topological ideals, operation functions (= expansions) and generalized closed sets.

1 Introduction

The following three different topological concepts were a major point of research in recent years:

- Topological ideals [4, 9, 10, 11, 12, 14, 15, 16, 32, 33].
- Operation function (= expansion) [17, 26, 27, 28, 29, 30, 35, 36, 37].
- Generalized closed sets [1, 2, 3, 5, 6, 8, 21, 22, 23, 24, 25, 34, 37].

An *ideal* $\mathcal{I}$ on a topological space (X, τ) is a non-void collection of subsets of X satisfying the following two properties: (1) $A \in \mathcal{I}$ and $B \subset A$ implies $B \in \mathcal{I}$ (heredity), and (2) $A \in \mathcal{I}$ and $B \in \mathcal{I}$ implies $A \cup B \in \mathcal{I}$ (finite additivity). The following collections of sets form important ideals on a space (X, τ): $\mathcal{F}$ — the ideal of finite subsets of X, $\mathcal{C}$ — the ideal of countable subsets of X, $\mathcal{CD}$ — the ideal of closed discrete sets in (X, τ), $\mathcal{N}$ — the ideal of nowhere dense sets in (X, τ), $\mathcal{M}$ — the ideal of meager sets in (X, τ), $\mathcal{B}$ — the ideal of all

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bounded sets in (X,τ), $\mathcal{S}$ — the ideal of scattered sets in (X,τ) (here X must be T_0), $\mathcal{K}$ — the ideal of relatively compact sets in (X,τ).

For a topological space $(X,\tau,\mathcal{I})$ and a subset $A \subseteq X$, we denote by $A^*(\mathcal{I}) = \{x \in X : U \cap A \notin \mathcal{I}$ for every $U \in \tau(x)\}$, written simply as A^* in case there is no chance for confusion. In [19], A^* is called the *local function* of A with respect to $\mathcal{I}$ and τ. Recall that $A \subseteq (X,\tau,\mathcal{I})$ is called τ^**-closed* [15] if $A^* \subseteq A$. It is well-known that $\mathrm{Cl}^*(A) = A \cup A^*$ defines a Kuratowski closure operator for a topology $\tau^*(\mathcal{I})$, finer than τ.

An *operation* γ [17, 26] on the topology τ on a given topological space (X,τ) is a function from the topology itself into the power set $\mathcal{P}(X)$ of X such that $V \subseteq V^\gamma$ for each $V \in \tau$, where V^γ denotes the value of γ at V. The following operators are examples of the operation γ: the closure operator γ_{cl} defined by $\gamma(U) = \mathrm{Cl}(U)$, the identity operator γ_{id} defined by $\gamma(U) = U$, the interior-closure operator γ_{ic} defined by $\gamma(U) = \mathrm{Int}(\mathrm{Cl}(U))$. In [35], the γ-operation is called an *expansion*. Another example of the operation γ is the γ_f-operator defined by $(U)^{\gamma_f} = (\mathrm{FrU})^c = X \setminus \mathrm{FrU}$ [35]. Two operators γ_1 and γ_2 are called *mutually dual* [35] if $U^{\gamma_1} \cap U^{\gamma_2} = U$ for each $U \in \tau$. For example the identity operator is mutually dual to any other operator, while the γ_f-operator is mutually dual to the closure operator [35].

The following definition contains the concepts of **generalized closed sets** used throughout this paper. In Theorem 2.2 of Section 2, it is proved that α^{**}g-closedness is same as gα^{**}-closedness.

Definition 1 A subset A of a space (X,τ) is called:

(1) a *generalized closed set* (briefly *g-closed*) [20] if $\overline{A} \subseteq U$ whenever $A \subseteq U$ and U is open,

(2) a *α-generalized closed set* (briefly *αg-closed*) [22] if $\alpha\mathrm{Cl}(A) \subseteq U$ whenever $A \subseteq U$ and U is open,

(3) a *α^{**}-generalized closed set* (briefly *$\alpha^{**}g$-closed*) [22] if $\alpha\mathrm{Cl}(A) \subseteq \mathrm{IntCl}U$ whenever $A \subseteq U$ and U is open,

(4) a *regular generalized closed set* (briefly *r-g-closed*) [31] if $\overline{A} \subseteq U$ whenever $A \subseteq U$ and U is regular open,

(5) a *generalized α^{**}-closed set* (briefly *$g\alpha^{**}$-closed*) [23] if $\alpha\mathrm{Cl}(A) \subseteq \mathrm{IntCl}U$ whenever $A \subseteq U$ and U is α-open.

2 Basic properties of $(\mathcal{I}, \gamma)$-generalized closed sets

Definition 2 A subset A of a topological space (X, τ) is called $(\mathcal{I}, \gamma)$-*generalized closed* if $A^* \subseteq U^\gamma$ whenever $A \subseteq U$ and U is open in (X, τ).

We denote the family of all $(\mathcal{I}, \gamma)$-generalized closed subsets of a space $(X, \tau, \mathcal{I}, \gamma)$ by $IG(X)$ and simply write $\mathcal{I}$-generalized closed (= $\mathcal{I}$-g-closed) in case when γ is the identity operator.

Theorem 2.1 *Every g-closed set is $(\mathcal{I}, \gamma)$-generalized closed but not vice versa.* □

Theorem 2.2 *Let (X, τ) be a topological space and let $A \subseteq X$. Then:*

(1) *A is $\{\emptyset\}$-g-closed if and only if A is g-closed.*

(2) *A is $\mathcal{N}$-g-closed if and only if A is αg-closed.*

(3) *A is $(\mathcal{N}, \gamma_{ic})$-g-closed if and only if A is $\alpha^{**}g$-closed.*

(4) *A is $(\{\emptyset\}, \gamma_{ic})$-g-closed if and only if A is r-g-closed.*

(5) *A is $(\mathcal{N}, \gamma_{ic})$-g-closed if and only if A is $g\alpha^{**}$-closed.*

Proof. Follow from the facts: $A^*(\{\emptyset\}) = \mathrm{Cl}(A)$ and $A^*(\mathcal{N}) = \mathrm{Cl}(\mathrm{Int}(\mathrm{Cl}(A))$ and $A \cup A^*(\mathcal{N}) = \alpha\mathrm{Cl}(A)$ [15, Example 2.10]. □

In the notion of Theorem 2.2, majority of the theorems below generalize well-known results related to the classes of generalized closed sets given in Definition 1.

Theorem 2.3 *If A is $\mathcal{I}$-g-closed and open, then A is τ^*-closed.* □

Lemma 2.4 [13, Theorem II3] *Let $(A_i)_{i\in I}$ be a locally finite family of sets in $(X, \tau, \mathcal{I})$. Then $\cup_{i\in I} A_i^*(\mathcal{I}) = (\cup_{i\in I} A_i)^*(\mathcal{I})$.* □

Theorem 2.5 *Let $(X, \tau, \mathcal{I}, \gamma)$ be a topological space.*

(i) *If $(A_i)_{i\in I}$ is a locally finite family of sets and each $A_i \in IG(X)$, then $\cup_{i\in I} A_i \in IG(X)$.*

(ii) *Countable union of $(\mathcal{I}, \gamma)$-generalized closed sets need not be $(\mathcal{I}, \gamma)$-generalized closed.*

(iii) *Finite intersection of $(\mathcal{I}, \gamma)$-generalized closed sets need not be $(\mathcal{I}, \gamma)$-generalized closed.*

PROOF. (i) Let $\cup_{i\in I} A_i \subseteq U$, where $U \in \tau$. Since $A_i \in IG(X)$ for each $i \in I$, then $A_i^* \subseteq U^\gamma$. Hence $\cup_{i\in I} A_i^* \subseteq U^\gamma$. By Lemma 2.4, $(\cup_{i\in I} A_i)^* \subseteq U^\gamma$. Hence $\cup_{i\in I} A_i \in IG(X)$.

(ii) In the real line with the usual topology $\{\frac{1}{n}\}$ is $\mathcal{F}$-g-closed for each $n \in \omega$, where ω denotes the set of all positive integers. But the set $A = \cup_{n\in\omega}\{\frac{1}{n}\}$ is not $\mathcal{F}$-g-closed. Note that $(0, 2)$ is an open superset of A but the zero point is in the local function of A with respect to the usual topology and $\mathcal{F}$.

(iii) Let $X = \{a, b, c, d, e\}$, $\tau = \{\emptyset, \{a, b\}, \{c\}, \{a, b, c\}, X\}$, $\mathcal{I} = \{\emptyset\}$ and $\gamma = \gamma_{ic}$. Set $A = \{a, c, d\}$ and $B = \{b, c, e\}$. Clearly $A, B \in IG(X)$ but $A \cap B = \{c\} \notin IG(X)$. $\square$

Lemma 2.6 *If A and B are subsets of $(X, \tau, \mathcal{I})$, then $(A \cap B)^*(\mathcal{I}) \subseteq A^*(\mathcal{I}) \cap B^*(\mathcal{I})$.* $\square$

A subset S of a space $(X, \tau, \mathcal{I})$ is a topological space with an ideal $\mathcal{I}_S = \{I \in \mathcal{I} : I \subseteq S\} = \{I \cap S : I \in \mathcal{I}\}$ on S [4].

Lemma 2.7 *Let $(X, \tau, \mathcal{I})$ be a topological space and $A \subseteq S \subseteq X$. Then, $A^*(\mathcal{I}_S, \tau|S) = A^*(\mathcal{I}, \tau) \cap S$ holds.*

Proof. First we prove the following implication: $A^*(\mathcal{I}_S, \tau|S) \subseteq A^*(\mathcal{I}, \tau) \cap S$.

Let $x \notin A^*(\mathcal{I}, \tau) \cap S$. We consider the following two cases:

Case 1. $x \notin S$: Since $A^*(\mathcal{I}_S, \tau|S) \subseteq S$, then $x \notin A^*(\mathcal{I}_S, \tau|S)$.

Case 2. $x \in S$. In this case $x \notin A^*(\mathcal{I}, \tau)$. There exists a set $V \in \tau$ such that $x \in V$ and $V \cap A \in \mathcal{I}$. Since $x \in S$, we have a set $S \cap V \in \tau|S$ such that $x \in S \cap V$ and $(S \cap V) \cap A \in \mathcal{I}$ and hence $(S \cap V) \cap A \in \mathcal{I}_S$. Consequently, $x \notin A^*(\mathcal{I}_S, \tau|S)$.

Both cases show the implication.

Secondly, we prove the converse implication: $A^*(\mathcal{I}, \tau) \cap S \subseteq A^*(\mathcal{I}_S, \tau|S)$. Let $x \notin A^*(\mathcal{I}_S, \tau|S)$. Then, for some open subset $U \cap S$ of $(S, \tau|S)$ containing x, we have $(U \cap S) \cap A \in \mathcal{I}_S$. Since $A \subseteq S$, then $U \cap A \in \mathcal{I}_S \subseteq \mathcal{I}$, i.e., $U \cap A \in \mathcal{I}$ for some $V \in \tau$ containing x. This shows that $x \notin A^*(\mathcal{I}, \tau)$. $\square$

Theorem 2.8 *Let $(X, \tau, \mathcal{I})$ be a topological space and $A \subseteq S \subseteq X$. If A is $\mathcal{I}_S$-g-closed in $(S, \tau|S, \mathcal{I}_S)$ and S is $\mathcal{I}$-g-closed in X, then A is $\mathcal{I}$-g-closed in X.*

Proof. Let $A \subseteq U$ and $U \in \tau$. By assumption and Lemma 2.7, $A^*(\mathcal{I}, \tau) \cap S \subseteq U \cap S$. Then we have $S \subseteq U \cup (X \setminus A^*(\mathcal{I}, \tau))$. Since $X \setminus A^*(\mathcal{I}, \tau) \in \tau$, then $A^*(\mathcal{I}, \tau) \subseteq S^*(\mathcal{I}, \tau) \subseteq U \cup (X \setminus A^*(\mathcal{I}, \tau))$. Therefore, we have that $A^*(\mathcal{I}, \tau) \subseteq U$ and hence A is $\mathcal{I}$-g-closed in X. $\square$

Corollary 2.9 *Let $(X, \tau, \mathcal{I})$ be a topological space and A and F subsets of X. If A is $\mathcal{I}$-g-closed and F is closed in (X, τ), then $A \cap F$ is $\mathcal{I}$-g-closed.*

Proof. Since $A \cap F$ is closed in $(A, \tau|A)$, then $A \cap F$ is $\mathcal{I}_A$-g-closed in $(A, \tau|A, \mathcal{I}_A)$. By Theorem 2.8, $A \cap F$ is $\mathcal{I}$-g-closed. $\square$

Example 2.10 Corollary 2.9 is not necessarily true if γ is an arbitrary operator. Let (X, τ) be the real line and consider any ideal such that $\tau^*(\mathcal{I}) = \tau^\alpha$. Observe that we have this equality in case when $\mathcal{I} = \mathcal{N}$. Define the following γ operation: for any open set U, let $\gamma(U) = X$ if the open interval $(0, 1)$ is contained in U, otherwise let $\gamma(U) = U$. If $A = (0, 1)$, then A is clearly $(\mathcal{I}, \gamma)$-generalized closed. Now, consider the closed set $B = [\frac{1}{2}, 1]$. Then the intersection of A and B is $[\frac{1}{2}, 1)$, which is contained in the open set $V = (\frac{1}{4}, 1)$. Obviously, the local function of $[\frac{1}{2}, 1)$ with respect to $\mathcal{N}$ is $[\frac{1}{2}, 1]$ and is not contained in $\gamma(V) = V$.

Theorem 2.11 *Let $A \subseteq S \subseteq (X, \tau, \mathcal{I}, \gamma)$. If $A \in IG(X)$ and $S \in \tau$, then $A \in IG(S)$.*

Proof. Let U be an open subset of $(S, \tau|S)$ such that $A \subseteq U$. Since $S \in \tau$, then $U \in \tau$. Then $A^*(\mathcal{I}) \subseteq U^\gamma$, since $A \in IG(X)$. Using Lemma 2.7, we have $A^*(\mathcal{I}_S, \tau|S) \subseteq U^{\gamma|S}$, where $U^{\gamma|S}$ means the image of the operation $\gamma|S \colon \tau|S \to \mathcal{P}(S)$, defined by $(\gamma|S)(U) = \gamma(U) \cap S$ for each $U \in \tau|S$. Hence $A \in IG(S)$. $\square$

Theorem 2.12 *Let A be a subset of $(X, \tau, \mathcal{I}, \gamma_{id})$. Then, A is $\mathcal{I}$-g-closed if and only if $A^* \setminus A$ does not contain a non-empty closed subset.*

Proof. (Necessity) Assume that F is a closed subset of $A^* \setminus A$. Note that clearly $A \subseteq X \setminus F$, where A is $\mathcal{I}$-g-closed and $X \setminus F \in \tau$. Thus $A^* \subseteq X \setminus F$, i.e. $F \subseteq X \setminus A^*$. Since due to our assumption $F \subseteq A^*$, $F \subseteq (X \setminus A^*) \cap A^* = \emptyset$.

(Sufficiency) Let U be an open subset containing A. Since A^* is closed [15, Theorem 2.3 (c)] and $A^* \cap (X \setminus U) \subseteq A^* \setminus A$ holds, then $A^* \cap (X \setminus U)$ is a closed set contained in $A^* \setminus A$. By assumption, $A^* \cap (X \setminus U) = \emptyset$ and hence $A^* \subseteq U$. $\square$

Theorem 2.13 *Let $A \subseteq (X, \tau, \mathcal{I})$ and γ_1 and γ_2 be two operations.*

(i) *If A is both $(\mathcal{I}, \gamma_1)$-g-closed and $(\mathcal{I}, \gamma_2)$-g-closed, then A is $(\mathcal{I}, \gamma_1 \wedge \gamma_2)$-g-closed, where $\gamma_1 \wedge \gamma_2$ is an operation defined by $(\gamma_1 \wedge \gamma_2)(U) = \gamma_1(U) \cap \gamma_2(U)$ for each $U \in \tau$.*

(ii) *Under the assumption of (i), if moreover the operators γ_1 and γ_2 are mutually dual, then A is $\mathcal{I}$-g-closed.*

(iii) *Every set $A \subseteq (X, \tau, \mathcal{I})$ is $(\mathcal{I}, \gamma_{cl})$-g-closed.* □

Corollary 2.14 *For a set $A \subseteq (X, \tau, \mathcal{I})$, the following conditions are equivalent:*

(1) *A is $(\mathcal{I}, \gamma_f)$-g-closed.*

(2) *A is $\mathcal{I}$-g-closed.*

Proof. (1) ⇒ (2) By Theorem 2.13 (iii), A is $(\mathcal{I}, \gamma_{cl})$-g-closed. Since γ_f and γ_{cl} are mutually dual due to [35, Proposition 2], then in the notion of Theorem 2.13, A is $\mathcal{I}$-g-closed.

(2) ⇒ (1) is obvious. □

3 γ-$T_{\mathcal{I}}$-spaces and the digital plane

Definition 3 A space $(X, \tau, \mathcal{I}, \gamma)$ is called an *γ-$T_{\mathcal{I}}$-space* if every $(\mathcal{I}, \gamma)$-generalized closed subset of X is τ^*-closed. We use the simpler notation $T_{\mathcal{I}}$-space, in case γ is the identity operator.

Theorem 3.1 *Let $(X, \tau, \mathcal{I}, \gamma)$ be a space and let $A \subseteq X$. Then:*

(1) *X is a $T_{\{\emptyset\}}$-space if and only if X is a $T_{\frac{1}{2}}$-space.*

(2) *X is a $T_{\mathcal{N}}$-space if and only if X is a $T_{\frac{1}{2}}$-space.*

(3) *X is a γ_{ic}-$T_{\mathcal{N}}$-space if and only if X is discrete.*

(4) *X is a γ_{ic}-$T_{\{\emptyset\}}$-space if and only if X is discrete.*

Proof. (1) follows from Theorem 2.2. (2) follows from Theorem 3.9 from [22] and Theorem 2.2, while (3) follows from Theorem 5.3 from [22] and Theorem 2.2. For (4), note that a space is discrete if and only if every r-g-closed set is closed. □

Remark 3.2 Note that when $\mathcal{I}$ is the maximal ideal $\mathcal{P}(X)$, then every space $(X, \tau, \mathcal{I}, \gamma)$ is a γ-$T_{\mathcal{I}}$-space.

Next we consider the case when γ is the identity operator.

Theorem 3.3 *For a space $(X, \tau, \mathcal{I})$, the following conditions are equivalent:*

(1) *X is a $T_{\mathcal{I}}$-space.*

(2) *Each singleton of (X, τ) is either closed or $\tau^*(\mathcal{I})$-open.*

PROOF. (1) $\Rightarrow$ (2) Let $x \in X$. If $\{x\}$ is not closed, then $A = X \setminus \{x\} \notin \tau$ and then A is trivially $\mathcal{I}$-g-closed. By (1), A is τ^*-closed. Hence $\{x\}$ is τ^*-open.

(2) $\Rightarrow$ (1) Let A be $\mathcal{I}$-g-closed and let $x \in \mathrm{Cl}^*(A)$. We have the following two cases:

Case 1. $\{x\}$ is closed. By Theorem 2.12, $A^* \setminus A$ does not contain a non-empty closed subset. This shows that $x \in A$.

Case 2. $\{x\}$ is τ^*-open. Then $\{x\} \cap A \neq \emptyset$. Hence $x \in A$.

Thus in both cases x is in A and so $A = \mathrm{Cl}^*(A)$, i.e. A is τ^*-closed, which shows that X is a $T_{\mathcal{I}}$-space. □

Corollary 3.4 *Every $T_{\frac{1}{2}}$-space is a $T_{\mathcal{I}}$-space.* □

Let $(\mathbf{Z}, \kappa)$ be the digital line (= Khalimsky line) [18]. The topology κ has $\{\{2n-1, 2n, 2n+1\} : n \in \mathbf{Z}\}$ as a subbase. Every singleton is open or closed in $(\mathbf{Z}, \kappa)$. In fact, every singleton $\{2n\}$, $n \in \mathbf{Z}$ is closed and every singleton $\{2m+1\}$, $m \in \mathbf{Z}$ is open. The space $(\mathbf{Z}, \kappa)$ is a typical example of a $T_{\frac{1}{2}}$-space [18] and moreover, it is an example of a $T_{\frac{3}{4}}$-space [5]. Let $(\mathbf{Z}^2, \kappa^2)$ be the digital plane, i.e., the topological product of two digital lines. We define a set $O(\mathbf{Z}^2) = \{(2n+1, 2m+1) \in \mathbf{Z}^2 : n, m \in \mathbf{Z}\}$. Let $\mathcal{I}(O(\mathbf{Z}^2))$ be the ideal of all subsets of $O(\mathbf{Z}^2)$, cf. [15, Example 2.9].

We will show that the digital plane $(\mathbf{Z}^2, \kappa^2)$ is a $T_{\mathcal{I}'}$-space, where $\mathcal{I}' = \mathcal{I}(O(\mathbf{Z}^2))$.

Theorem 3.5 (i) *The space $(\mathbf{Z}^2, \kappa^2, \mathcal{I}')$ is a $T_{\mathcal{I}'}$-space, where $\mathcal{I}' = \mathcal{I}(O(\mathbf{Z}^2))$, and $(\mathbf{Z}^2, \kappa^2)$ is a not $T_{\frac{1}{2}}$.*

(ii) *The induced space $(\mathbf{Z}^2, (\kappa^2)^*)$ from $(\mathbf{Z}^2, \kappa^2)$ is a $T_{\frac{1}{2}}$-space.*

Proof. We will check condition (2) in Theorem 3.3. That is, we will prove that every singleton $\{x, y\}$ is closed or $(\kappa^2)^*$-open. For a subset $A \subseteq \mathbf{Z}^2$, in the proof below, we denote $A^*(\mathcal{I}(O(\mathbf{Z}^2)), \kappa^2)$ by A^*.

We consider the following four cases:

Case 1. $(x, y) = (2n+1, 2m)$: In this case, we claim that the singleton $\{(x, y)\}$ is $(\kappa^2)^*$-open. There exists an open neighborhood $\{2n+1\} \times \{2m-1, 2m, 2m+1\}$ of (x, y), say U, such that $U \cap (\mathbf{Z}^2 \setminus \{(x, y)\}) = \{(2n+1, 2m+1), (2n+1, 2m-1)\} \in \mathcal{I}(O(\mathbf{Z}^2))$. Then, $(x, y) \notin (\mathbf{Z}^2 \setminus \{(x, y)\})^*$ and so $(\mathbf{Z}^2 \setminus \{(x, y)\})^* \subseteq (\mathbf{Z}^2 \setminus \{(x, y)\})$. That is, the singleton $\{(x, y)\}$ is $(\kappa^2)^*$-open in $(\mathbf{Z}^2, \kappa^2)$.

Case 2. $(x, y) = (2n, 2m+1)$: In this case, we claim that the singleton $\{(x, y)\}$ is $(\kappa^2)^*$-open. There exists an open neighborhood $\{2n-1, 2n, 2n+1\} \times \{2m+1\}$ of (x, y), say U, such that $U \cap (\mathbf{Z}^2 \setminus \{(x, y)\}) = \{(2n-1, 2m+1), (2n+1, 2m+1)\} \in \mathcal{I}(O(\mathbf{Z}^2))$. Then, $(x, y) \notin (\mathbf{Z}^2 \setminus \{(x, y)\})^*$ and so $(\mathbf{Z}^2 \setminus \{(x, y)\})^* \subseteq (\mathbf{Z}^2 \setminus \{(x, y)\})$. That is, the singleton $\{(x, y)\}$ is $(\kappa^2)^*$-open in $(\mathbf{Z}^2, \kappa^2)$.

Case 3. $(x, y) = (2n, 2m)$: The singleton $\{(x, y)\}$ is closed and so it is $(\kappa^2)^*$-closed in $(\mathbf{Z}^2, \kappa^2)$.

Case 4. $(x, y) = (2n+1, 2m+1)$: Since $\{2n+1, 2m+1)\}$ is open, it is $(\kappa^2)^*$-open in $(\mathbf{Z}^2, \kappa^2)$.

Therefore, every singleton is closed or $(\kappa^2)^*$-open. By Theorem 3.3, $(\mathbf{Z}^2, \kappa^2, \mathcal{I}')$ is a $T_{\mathcal{I}'}$-space, where $\mathcal{I}' = \mathcal{I}(O(\mathbf{Z}^2))$. Clearly, $(\mathbf{Z}^2, \kappa^2)$ is a not $T_{\frac{1}{2}}$.

(ii) By (i), every singleton is open or closed in $(\mathbf{Z}^2, (\kappa^2)^*)$, Therefore, it is $T_{\frac{1}{2}}$. □

Question. As shown above $T_{\mathcal{N}}$-spaces are precisely the $T_{\frac{1}{2}}$-spaces. Do the classes of $T_{\mathcal{M}}$-, $T_{\mathcal{F}}$-, $T_{\mathcal{C}}$- or $T_{\mathcal{B}}$-spaces coincide with some already known classes of topological spaces (of course weaker than $T_{\frac{1}{2}}$)?

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