Unified primal formulation-based a priori and a posteriori error analysis of mixed finite element methods

Martin Vohralík

Laboratoire Jacques-Louis Lions
Université Pierre et Marie Curie (Paris 6)

Austin, April 30, 2008

Introduction

Model problem (**S** inhomogeneous and anisotropic)

$$\mathbf{u} = -\mathbf{S}\nabla p, \nabla\cdot\mathbf{u} = f \text{ in } \Omega \qquad -\nabla\cdot(\mathbf{S}\nabla p) = f \text{ in } \Omega$$
$$p = 0 \text{ on } \partial\Omega \qquad p = 0 \text{ on } \partial\Omega$$

Mixed finite elements

$$(\mathbf{S}^{-1}\mathbf{u}_h, \mathbf{v}_h) - (p_h, \nabla\cdot\mathbf{v}_h) = 0 \quad \forall\,\mathbf{v}_h \in \mathbf{V}_h,$$
$$(\nabla\cdot\mathbf{u}_h, \phi_h) = (f, \phi_h) \quad \forall\phi_h \in \Phi_h$$

Traditional analysis

- weak mixed formulation
  $$(\mathbf{S}^{-1}\mathbf{u}, \mathbf{v}) - (p, \nabla\cdot\mathbf{v}) = 0 \quad \forall\,\mathbf{v} \in \mathbf{H}(\mathrm{div}, \Omega),$$
  $$(\nabla\cdot\mathbf{u}, \phi) = (f, \phi) \ \forall\,\phi \in L^2(\Omega)$$
- inf–sup condition
- $\nabla\cdot\mathbf{V}_h = \Phi_h$

Presented analysis

- classical weak formulation
  $$(\mathbf{S}\nabla p, \nabla\varphi) = (f, \varphi) \quad \forall\varphi \in H_0^1(\Omega)$$
- postprocessing and discrete Friedrichs inequality
- $\nabla\cdot\mathbf{V}_h = \Phi_h$

unified and optimal a priori and a posteriori error analysis

Introduction

Model problem (**S** inhomogeneous and anisotropic)

$$\mathbf{u} = -\mathbf{S}\nabla p, \nabla \cdot \mathbf{u} = f \text{ in } \Omega \qquad -\nabla \cdot (\mathbf{S}\nabla p) = f \text{ in } \Omega$$
$$p = 0 \text{ on } \partial\Omega \qquad p = 0 \text{ on } \partial\Omega$$

Mixed finite elements

$$(\mathbf{S}^{-1}\mathbf{u}_h, \mathbf{v}_h) - (p_h, \nabla \cdot \mathbf{v}_h) = 0 \quad \forall \mathbf{v}_h \in \mathbf{V}_h,$$
$$(\nabla \cdot \mathbf{u}_h, \phi_h) = (f, \phi_h) \quad \forall \phi_h \in \Phi_h$$

Traditional analysis

- weak mixed formulation
  $$(\mathbf{S}^{-1}\mathbf{u}, \mathbf{v}) - (p, \nabla \cdot \mathbf{v}) = 0 \quad \forall \mathbf{v} \in \mathbf{H}(\mathrm{div}, \Omega),$$
  $$(\nabla \cdot \mathbf{u}, \phi) = (f, \phi) \ \forall \phi \in L^2(\Omega)$$
- inf–sup condition
- $\nabla \cdot \mathbf{V}_h = \Phi_h$

Presented analysis

- classical weak formulation
  $$(\mathbf{S}\nabla p, \nabla \varphi) = (f, \varphi) \quad \forall \varphi \in H_0^1(\Omega)$$
- postprocessing and discrete Friedrichs inequality
- $\nabla \cdot \mathbf{V}_h = \Phi_h$

unified and optimal a priori and a posteriori error analysis

Introduction

Model problem (**S** inhomogeneous and anisotropic)

$$\mathbf{u} = -\mathbf{S}\nabla p, \nabla \cdot \mathbf{u} = f \text{ in } \Omega \qquad -\nabla \cdot (\mathbf{S}\nabla p) = f \text{ in } \Omega$$
$$p = 0 \text{ on } \partial\Omega \qquad p = 0 \text{ on } \partial\Omega$$

Mixed finite elements

$$(\mathbf{S}^{-1}\mathbf{u}_h, \mathbf{v}_h) - (p_h, \nabla \cdot \mathbf{v}_h) = 0 \quad \forall \mathbf{v}_h \in \mathbf{V}_h,$$
$$(\nabla \cdot \mathbf{u}_h, \phi_h) = (f, \phi_h) \quad \forall \phi_h \in \Phi_h$$

Traditional analysis

- weak mixed formulation
  $$(\mathbf{S}^{-1}\mathbf{u}, \mathbf{v}) - (p, \nabla \cdot \mathbf{v}) = 0 \quad \forall \mathbf{v} \in \mathbf{H}(\mathrm{div}, \Omega),$$
  $$(\nabla \cdot \mathbf{u}, \phi) = (f, \phi) \; \forall \phi \in L^2(\Omega)$$
- inf–sup condition
- $\nabla \cdot \mathbf{V}_h = \Phi_h$

Presented analysis

- classical weak formulation
  $$(\mathbf{S}\nabla p, \nabla \varphi) = (f, \varphi) \quad \forall \varphi \in H_0^1(\Omega)$$
- postprocessing and discrete Friedrichs inequality
- $\nabla \cdot \mathbf{V}_h = \Phi_h$

unified and optimal a priori and a posteriori error analysis

Introduction

Model problem (**S** inhomogeneous and anisotropic)

$$\mathbf{u} = -\mathbf{S}\nabla p, \nabla \cdot \mathbf{u} = f \text{ in } \Omega \qquad -\nabla \cdot (\mathbf{S}\nabla p) = f \text{ in } \Omega$$

$$p = 0 \text{ on } \partial\Omega \qquad p = 0 \text{ on } \partial\Omega$$

Mixed finite elements

$$(\mathbf{S}^{-1}\mathbf{u}_h, \mathbf{v}_h) - (p_h, \nabla \cdot \mathbf{v}_h) = 0 \quad \forall \mathbf{v}_h \in \mathbf{V}_h,$$
$$(\nabla \cdot \mathbf{u}_h, \phi_h) = (f, \phi_h) \quad \forall \phi_h \in \Phi_h$$

Traditional analysis

- weak mixed formulation
  $$(\mathbf{S}^{-1}\mathbf{u}, \mathbf{v}) - (p, \nabla \cdot \mathbf{v}) = 0 \quad \forall \mathbf{v} \in \mathbf{H}(\mathrm{div}, \Omega),$$
  $$(\nabla \cdot \mathbf{u}, \phi) = (f, \phi) \ \forall \phi \in L^2(\Omega)$$
- inf–sup condition
- $\nabla \cdot \mathbf{V}_h = \Phi_h$

Presented analysis

- classical weak formulation
  $$(\mathbf{S}\nabla p, \nabla \varphi) = (f, \varphi) \quad \forall \varphi \in H_0^1(\Omega)$$
- postprocessing and discrete Friedrichs inequality
- $\nabla \cdot \mathbf{V}_h = \Phi_h$

unified and optimal a priori and a posteriori error analysis

Outline

1. General framework
   - An abstract result for the flux variable
   - Postprocessing for the scalar variable
2. A priori error estimates
   - Lowest-order Raviart–Thomas case
   - General case
3. A posteriori error estimates
   - Estimates for the flux
   - Estimates for the potential
   - Local efficiency
4. Remarks
   - Comments on the estimates
   - $L^2(\Omega)$ estimates
   - RT_0 and pure diffusion problems
5. Numerical experiments
6. Conclusions and future work

Outline

Bilinear forms and weak solution

Definition (Bilinear form $\mathcal{B}$)

$$\mathcal{B}(p,\varphi) := \sum_{K\in\mathcal{T}_h} (\mathbf{S}\nabla p, \nabla\varphi)_K, \quad p,\varphi \in H^1(\mathcal{T}_h).$$

Definition (Bilinear form $\mathcal{A}$)

$$\mathcal{A}(\mathbf{u},\mathbf{v}) := \sum_{K\in\mathcal{T}_h} (\mathbf{u}, \mathbf{S}^{-1}\mathbf{v})_K, \quad \mathbf{u},\mathbf{v} \in \mathbf{L}^2(\Omega).$$

Definition (Weak solution)

$$p \in H^1_0(\Omega) \text{ such that } \mathcal{B}(p,\varphi) = (f,\varphi) \qquad \forall \varphi \in H^1_0(\Omega);$$

or

$$p \in H^1_0(\Omega) \text{ such that } \mathcal{A}(\mathbf{S}\nabla p, \mathbf{S}\nabla\varphi) = (f,\varphi) \qquad \forall \varphi \in H^1_0(\Omega).$$

Bilinear forms and weak solution

Definition (Bilinear form $\mathcal{B}$)

$$\mathcal{B}(p, \varphi) := \sum_{K \in \mathcal{T}_h} (\mathbf{S}\nabla p, \nabla \varphi)_K, \quad p, \varphi \in H^1(\mathcal{T}_h).$$

Definition (Bilinear form $\mathcal{A}$)

$$\mathcal{A}(\mathbf{u}, \mathbf{v}) := \sum_{K \in \mathcal{T}_h} (\mathbf{u}, \mathbf{S}^{-1}\mathbf{v})_K, \quad \mathbf{u}, \mathbf{v} \in \mathbf{L}^2(\Omega).$$

Definition (Weak solution)

$$p \in H_0^1(\Omega) \text{ such that } \mathcal{B}(p, \varphi) = (f, \varphi) \qquad \forall \varphi \in H_0^1(\Omega);$$

or

$$p \in H_0^1(\Omega) \text{ such that } \mathcal{A}(\mathbf{S}\nabla p, \mathbf{S}\nabla \varphi) = (f, \varphi) \qquad \forall \varphi \in H_0^1(\Omega).$$

Bilinear forms and weak solution

Definition (Bilinear form $\mathcal{B}$)

$$\mathcal{B}(p,\varphi) := \sum_{K\in\mathcal{T}_h} (\mathbf{S}\nabla p, \nabla\varphi)_K, \quad p,\varphi \in H^1(\mathcal{T}_h).$$

Definition (Bilinear form $\mathcal{A}$)

$$\mathcal{A}(\mathbf{u},\mathbf{v}) := \sum_{K\in\mathcal{T}_h} (\mathbf{u}, \mathbf{S}^{-1}\mathbf{v})_K, \quad \mathbf{u},\mathbf{v} \in \mathbf{L}^2(\Omega).$$

Definition (Weak solution)

$$p \in H^1_0(\Omega) \text{ such that } \mathcal{B}(p,\varphi) = (f,\varphi) \qquad \forall \varphi \in H^1_0(\Omega);$$

or

$$p \in H^1_0(\Omega) \text{ such that } \mathcal{A}(\mathbf{S}\nabla p, \mathbf{S}\nabla\varphi) = (f,\varphi) \qquad \forall \varphi \in H^1_0(\Omega).$$

Energy norms

Definition (Energy semi-norm)

$$|||\varphi|||^2 := \mathcal{B}(\varphi, \varphi), \quad \varphi \in H^1(\mathcal{T}_h).$$

Definition (Energy norm)

$$|||\varphi|||^2 := \mathcal{B}(\varphi, \varphi), \quad \varphi \in W_0^k(\mathcal{T}_h).$$

Definition (Energy norm)

$$|||\mathbf{v}|||_*^2 := \mathcal{A}(\mathbf{v}, \mathbf{v}), \quad \mathbf{v} \in \mathbf{L}^2(\Omega).$$

Definition (Energy–div norm)

$$|||\mathbf{v}|||_{*,\mathrm{div}}^2 := |||\mathbf{v}|||_*^2 + \|\nabla \cdot \mathbf{v}\|^2, \quad \mathbf{v} \in \mathbf{H}(\mathrm{div}, \Omega).$$

Energy norms

Definition (Energy semi-norm)

$$|||\varphi|||^2 := \mathcal{B}(\varphi, \varphi), \quad \varphi \in H^1(\mathcal{T}_h).$$

Definition (Energy norm)

$$|||\varphi|||^2 := \mathcal{B}(\varphi, \varphi), \quad \varphi \in W_0^k(\mathcal{T}_h).$$

Definition (Energy norm)

$$|||\mathbf{v}|||_*^2 := \mathcal{A}(\mathbf{v}, \mathbf{v}), \quad \mathbf{v} \in \mathbf{L}^2(\Omega).$$

Definition (Energy–div norm)

$$|||\mathbf{v}|||_{*,\mathrm{div}}^2 := |||\mathbf{v}|||_*^2 + \|\nabla \cdot \mathbf{v}\|^2, \quad \mathbf{v} \in \mathbf{H}(\mathrm{div}, \Omega).$$

Energy norms

Definition (Energy semi-norm)

$$|||\varphi|||^2 := \mathcal{B}(\varphi, \varphi), \quad \varphi \in H^1(\mathcal{T}_h).$$

Definition (Energy norm)

$$|||\varphi|||^2 := \mathcal{B}(\varphi, \varphi), \quad \varphi \in W_0^k(\mathcal{T}_h).$$

Definition (Energy norm)

$$|||\mathbf{v}|||_*^2 := \mathcal{A}(\mathbf{v}, \mathbf{v}), \quad \mathbf{v} \in \mathbf{L}^2(\Omega).$$

Definition (Energy–div norm)

$$|||\mathbf{v}|||_{*,\mathrm{div}}^2 := |||\mathbf{v}|||_*^2 + \|\nabla \cdot \mathbf{v}\|^2, \quad \mathbf{v} \in \mathbf{H}(\mathrm{div}, \Omega).$$

Energy norms

Definition (Energy semi-norm)

$$|||\varphi|||^2 := \mathcal{B}(\varphi, \varphi), \quad \varphi \in H^1(\mathcal{T}_h).$$

Definition (Energy norm)

$$|||\varphi|||^2 := \mathcal{B}(\varphi, \varphi), \quad \varphi \in W_0^k(\mathcal{T}_h).$$

Definition (Energy norm)

$$|||\mathbf{v}|||_*^2 := \mathcal{A}(\mathbf{v}, \mathbf{v}), \quad \mathbf{v} \in \mathbf{L}^2(\Omega).$$

Definition (Energy–div norm)

$$|||\mathbf{v}|||_{*,\mathrm{div}}^2 := |||\mathbf{v}|||_*^2 + \|\nabla \cdot \mathbf{v}\|^2, \quad \mathbf{v} \in \mathbf{H}(\mathrm{div}, \Omega).$$

Outline

An abstract result for the flux variable

Theorem (Abstract framework (scheme-independent))

Let $\mathbf{v}, \mathbf{w}, \mathbf{t} \in \mathbf{L}^2(\Omega)$ *be arbitrary. Then*

$$|||\mathbf{v} - \mathbf{w}|||_* \leq |||\mathbf{w} - \mathbf{t}|||_* + \left| \mathcal{A}\left(\mathbf{v} - \mathbf{w}, \frac{\mathbf{v} - \mathbf{t}}{|||\mathbf{v} - \mathbf{t}|||_*} \right) \right|.$$

A priori error estimate

$|||\mathbf{u} - \mathbf{u}_h|||_* \leq |||\mathbf{u} - \Pi_h \mathbf{u}|||_*$

A posteriori error estimate

- put $\mathbf{v} = \mathbf{u}$, $\mathbf{w} = \mathbf{u}_h$, $\mathbf{t} = -\mathbf{S}\nabla s$ with $s \in H_0^1(\Omega)$ arbitrary:

$$|||\mathbf{u} - \mathbf{u}_h|||_* \leq |||\mathbf{u}_h + \mathbf{S}\nabla s|||_* + \left| \mathcal{A}\left(\mathbf{u} - \mathbf{u}_h, \frac{\mathbf{u} + \mathbf{S}\nabla s}{|||\mathbf{u} + \mathbf{S}\nabla s|||_*} \right) \right|$$

- notice that $\mathcal{A}(\mathbf{u}, -\mathbf{S}\nabla\varphi) = (f, \varphi)$ (here $\varphi = p - s/|||p - s|||$)
- notice that $\mathcal{A}(\mathbf{u}_h, -\mathbf{S}\nabla\varphi) = (\pi_l(f), \varphi)$
- get $|||\mathbf{u} - \mathbf{u}_h|||_* \leq \inf_{s \in H_0^1(\Omega)} |||\mathbf{u}_h + \mathbf{S}\nabla s|||_* + \left\{ \sum_{K \in \mathcal{T}_h} \frac{C_{\mathrm{P}} h_K^2}{c_{\mathbf{S},K}} \|f - \pi_l(f)\|_K^2 \right\}^{\frac{1}{2}}$

An abstract result for the flux variable

Theorem (Abstract framework (scheme-independent))

Let $\mathbf{v}, \mathbf{w}, \mathbf{t} \in \mathbf{L}^2(\Omega)$ *be arbitrary. Then*

$$|||\mathbf{v} - \mathbf{w}|||_* \leq |||\mathbf{w} - \mathbf{t}|||_* + \left| \mathcal{A}\left(\mathbf{v} - \mathbf{w}, \frac{\mathbf{v} - \mathbf{t}}{|||\mathbf{v} - \mathbf{t}|||_*}\right) \right|.$$

A priori error estimate

- put $\mathbf{v} = \mathbf{u}_h$, $\mathbf{w} = \mathbf{u}$, $\mathbf{t} = \Pi_h \mathbf{u}$:

$$|||\mathbf{u}_h - \mathbf{u}|||_* \leq |||\mathbf{u} - \Pi_h \mathbf{u}|||_* + \left| \mathcal{A}\left(\mathbf{u}_h - \mathbf{u}, \frac{\mathbf{u}_h - \Pi_h \mathbf{u}}{|||\mathbf{u}_h - \Pi_h \mathbf{u}|||_*}\right) \right|$$

- notice that $\mathcal{A}(\mathbf{u}_h - \mathbf{u}, \mathbf{u}_h - \Pi_h \mathbf{u}) = 0$ in MFEs
- get $|||\mathbf{u} - \mathbf{u}_h|||_* \leq |||\mathbf{u} - \Pi_h \mathbf{u}|||_*$

A posteriori error estimate

- put $\mathbf{v} = \mathbf{u}$, $\mathbf{w} = \mathbf{u}_h$, $\mathbf{t} = -\mathbf{S}\nabla s$ with $s \in H_0^1(\Omega)$ arbitrary:

$$|||\mathbf{u} - \mathbf{u}_h|||_* \leq |||\mathbf{u}_h + \mathbf{S}\nabla s|||_* + \left| \mathcal{A}\left(\mathbf{u} - \mathbf{u}_h, \frac{\mathbf{u} + \mathbf{S}\nabla s}{|||\mathbf{u} + \mathbf{S}\nabla s|||_*}\right) \right|$$

An abstract result for the flux variable

Theorem (Abstract framework (scheme-independent))

Let $\mathbf{v}, \mathbf{w}, \mathbf{t} \in \mathbf{L}^2(\Omega)$ *be arbitrary. Then*

$$|||\mathbf{v} - \mathbf{w}|||_* \le |||\mathbf{w} - \mathbf{t}|||_* + \left| \mathcal{A}\left(\mathbf{v} - \mathbf{w}, \frac{\mathbf{v} - \mathbf{t}}{|||\mathbf{v} - \mathbf{t}|||_*}\right) \right|.$$

A priori error estimate

$$|||\mathbf{u} - \mathbf{u}_h|||_* \le |||\mathbf{u} - \Pi_h \mathbf{u}|||_*$$

A posteriori error estimate

- put $\mathbf{v} = \mathbf{u}$, $\mathbf{w} = \mathbf{u}_h$, $\mathbf{t} = -\mathbf{S}\nabla s$ with $s \in H_0^1(\Omega)$ arbitrary:

$$|||\mathbf{u} - \mathbf{u}_h|||_* \le |||\mathbf{u}_h + \mathbf{S}\nabla s|||_* + \left| \mathcal{A}\left(\mathbf{u} - \mathbf{u}_h, \frac{\mathbf{u} + \mathbf{S}\nabla s}{|||\mathbf{u} + \mathbf{S}\nabla s|||_*}\right) \right|$$

- notice that $\mathcal{A}(\mathbf{u}, -\mathbf{S}\nabla\varphi) = (f, \varphi)$ (here $\varphi = p - s/|||p - s|||$)
- notice that $\mathcal{A}(\mathbf{u}_h, -\mathbf{S}\nabla\varphi) = (\pi_l(f), \varphi)$
- get $|||\mathbf{u} - \mathbf{u}_h|||_* \le \inf_{s \in H_0^1(\Omega)} |||\mathbf{u}_h + \mathbf{S}\nabla s|||_* + \left\{ \sum_{K \in \mathcal{T}_h} \frac{C_{\mathrm{P}} h_K^2}{c_{\mathbf{S},K}} \|f - \pi_l(f)\|_K^2 \right\}^{\frac{1}{2}}$

An abstract result for the flux variable

Theorem (Abstract framework (scheme-independent))

Let $\mathbf{v}, \mathbf{w}, \mathbf{t} \in \mathbf{L}^2(\Omega)$ *be arbitrary. Then*

$$|||\mathbf{v} - \mathbf{w}|||_* \leq |||\mathbf{w} - \mathbf{t}|||_* + \left| \mathcal{A}\left(\mathbf{v} - \mathbf{w}, \frac{\mathbf{v} - \mathbf{t}}{|||\mathbf{v} - \mathbf{t}|||_*} \right) \right|.$$

A priori error estimate

$|||\mathbf{u} - \mathbf{u}_h|||_* \leq |||\mathbf{u} - \Pi_h \mathbf{u}|||_*$

A posteriori error estimate

- put $\mathbf{v} = \mathbf{u}$, $\mathbf{w} = \mathbf{u}_h$, $\mathbf{t} = -\mathbf{S}\nabla s$ with $s \in H_0^1(\Omega)$ arbitrary:

$$|||\mathbf{u} - \mathbf{u}_h|||_* \leq |||\mathbf{u}_h + \mathbf{S}\nabla s|||_* + \left| \mathcal{A}\left(\mathbf{u} - \mathbf{u}_h, \frac{\mathbf{u} + \mathbf{S}\nabla s}{|||\mathbf{u} + \mathbf{S}\nabla s|||_*} \right) \right|$$

- notice that $\mathcal{A}(\mathbf{u}, -\mathbf{S}\nabla\varphi) = (f, \varphi)$ (here $\varphi = p - s/|||p - s|||$)
- notice that $\mathcal{A}(\mathbf{u}_h, -\mathbf{S}\nabla\varphi) = (\pi_l(f), \varphi)$
- get $|||\mathbf{u} - \mathbf{u}_h|||_* \leq \inf_{s \in H_0^1(\Omega)} |||\mathbf{u}_h + \mathbf{S}\nabla s|||_* + \left\{ \sum_{K \in \mathcal{T}_h} \frac{C_{\mathrm{P}} h_K^2}{c_{\mathbf{S},K}} \|f - \pi_l(f)\|_K^2 \right\}^{\frac{1}{2}}$

Outline

1. General framework
   - An abstract result for the flux variable
   - Postprocessing for the scalar variable
2. A priori error estimates
   - Lowest-order Raviart–Thomas case
   - General case
3. A posteriori error estimates
   - Estimates for the flux
   - Estimates for the potential
   - Local efficiency
4. Remarks
   - Comments on the estimates
   - $L^2(\Omega)$ estimates
   - RT_0 and pure diffusion problems
5. Numerical experiments
6. Conclusions and future work

Postprocessing for the scalar variable

Postprocessing in mixed finite elements

- Arnold and Brezzi ’85: Mixed and nonconforming finite element methods: implementation, postprocessing and error estimates
- Bramble and Xu ’89: A local post-processing technique for improving the accuracy in mixed finite-element approximations
- Stenberg ’91: Postprocessing schemes for some mixed finite elements
- Arbogast and Chen ’95: On the implementation of mixed methods as nonconforming methods for second-order elliptic problems
- Chen ’96: Equivalence between and multigrid algorithms for nonconforming and mixed methods for second-order elliptic problems

Postprocessing for the scalar variable

Postprocessing in mixed finite elements

- Usually used in order to implement MFEMs and get superconvergence for the postprocessed variable.
- Usually not used in order to get a priori or a posteriori error estimates.

Postprocessing for the scalar variable

Postprocessing in mixed finite elements

- Usually used in order to implement MFEMs and get superconvergence for the postprocessed variable.
- Usually not used in order to get a priori or a posteriori error estimates.

Lowest-order Raviart–Thomas case

Definition (Postprocessed scalar variable $\tilde{p}_h$)

We define $\tilde{p}_h$ such that, separately on each $K \in \mathcal{T}_h$,

- $-\mathbf{S}_K \nabla \tilde{p}_h|_K = \mathbf{u}_h|_K$ (flux of $\tilde{p}_h$ is $\mathbf{u}_h$),
- $(\tilde{p}_h, 1)_K / |K| = p_K$ (mean of $\tilde{p}_h$ on K is p_K).

Properties of $\tilde{p}_h$

- $\tilde{p}_h$ exists and is unique (it is a pw second-order polynomial)
- $\tilde{p}_h$ is nonconforming, $\notin H_0^1(\Omega)$, only $\in H^1(\mathcal{T}_h)$ in general
- means of traces of $\tilde{p}_h$ on the sides continuous, $\tilde{p}_h \in W_0^0(\mathcal{T}_h)$
- the means are equal to the Lagrange multipliers from the hybridization

Remarks

- exact (not weak) connection of $\tilde{p}_h$ and $\mathbf{u}_h$
- only valid in the lowest-order case on simplices or, when $\mathbf{S}$ is diagonal, on rectangular parallelepipeds

Lowest-order Raviart–Thomas case

Definition (Postprocessed scalar variable $\tilde{p}_h$)

We define $\tilde{p}_h$ such that, separately on each $K \in \mathcal{T}_h$,

- $-\mathbf{S}_K \nabla \tilde{p}_h|_K = \mathbf{u}_h|_K$ (flux of $\tilde{p}_h$ is $\mathbf{u}_h$),
- $(\tilde{p}_h, 1)_K / |K| = p_K$ (mean of $\tilde{p}_h$ on K is p_K).

Properties of $\tilde{p}_h$

- $\tilde{p}_h$ exists and is unique (it is a pw second-order polynomial)
- $\tilde{p}_h$ is nonconforming, $\notin H_0^1(\Omega)$, only $\in H^1(\mathcal{T}_h)$ in general
- means of traces of $\tilde{p}_h$ on the sides continuous, $\tilde{p}_h \in W_0^0(\mathcal{T}_h)$
- the means are equal to the Lagrange multipliers from the hybridization

Remarks

- exact (not weak) connection of $\tilde{p}_h$ and $\mathbf{u}_h$
- only valid in the lowest-order case on simplices or, when $\mathbf{S}$ is diagonal, on rectangular parallelepipeds

Lowest-order Raviart–Thomas case

Definition (Postprocessed scalar variable $\tilde{p}_h$)

We define $\tilde{p}_h$ such that, separately on each $K \in \mathcal{T}_h$,

- $-\mathbf{S}_K \nabla \tilde{p}_h|_K = \mathbf{u}_h|_K$ (flux of $\tilde{p}_h$ is $\mathbf{u}_h$),
- $(\tilde{p}_h, 1)_K / |K| = p_K$ (mean of $\tilde{p}_h$ on K is p_K).

Properties of $\tilde{p}_h$

- $\tilde{p}_h$ exists and is unique (it is a pw second-order polynomial)
- $\tilde{p}_h$ is nonconforming, $\notin H_0^1(\Omega)$, only $\in H^1(\mathcal{T}_h)$ in general
- means of traces of $\tilde{p}_h$ on the sides continuous, $\tilde{p}_h \in W_0^0(\mathcal{T}_h)$
- the means are equal to the Lagrange multipliers from the hybridization

Remarks

- exact (not weak) connection of $\tilde{p}_h$ and $\mathbf{u}_h$
- only valid in the lowest-order case on simplices or, when $\mathbf{S}$ is diagonal, on rectangular parallelepipeds

Lowest-order Raviart–Thomas case

Definition (Postprocessed scalar variable $\tilde{p}_h$)

We define $\tilde{p}_h$ such that, separately on each $K \in \mathcal{T}_h$,

- $-\mathbf{S}_K \nabla \tilde{p}_h|_K = \mathbf{u}_h|_K$ (flux of $\tilde{p}_h$ is $\mathbf{u}_h$),
- $(\tilde{p}_h, 1)_K / |K| = p_K$ (mean of $\tilde{p}_h$ on K is p_K).

Properties of $\tilde{p}_h$

- $\tilde{p}_h$ exists and is unique (it is a pw second-order polynomial)
- $\tilde{p}_h$ is nonconforming, $\notin H_0^1(\Omega)$, only $\in H^1(\mathcal{T}_h)$ in general
- means of traces of $\tilde{p}_h$ on the sides continuous, $\tilde{p}_h \in W_0^0(\mathcal{T}_h)$
- the means are equal to the Lagrange multipliers from the hybridization

Proof:
$$\begin{aligned} 0 &= -(\nabla \tilde{p}_h, \mathbf{v}_{\sigma_{K,L}})_{K\cup L} - (\tilde{p}_h, \nabla \cdot \mathbf{v}_{\sigma_{K,L}})_{K\cup L} \\ &= -\langle \mathbf{v}_{\sigma_{K,L}} \cdot \mathbf{n}, \tilde{p}_h \rangle_{\partial K} - \langle \mathbf{v}_{\sigma_{K,L}} \cdot \mathbf{n}, \tilde{p}_h \rangle_{\partial L} \\ &= \langle \mathbf{v}_{\sigma_{K,L}} \cdot \mathbf{n}_K, \tilde{p}_h|_L - \tilde{p}_h|_K \rangle_{\sigma_{K,L}} \end{aligned}$$

Lowest-order Raviart–Thomas case

Definition (Postprocessed scalar variable $\tilde{p}_h$)

We define $\tilde{p}_h$ such that, separately on each $K \in \mathcal{T}_h$,

- $-\mathbf{S}_K \nabla \tilde{p}_h|_K = \mathbf{u}_h|_K$ (flux of $\tilde{p}_h$ is $\mathbf{u}_h$),
- $(\tilde{p}_h, 1)_K / |K| = p_K$ (mean of $\tilde{p}_h$ on K is p_K).

Properties of $\tilde{p}_h$

- $\tilde{p}_h$ exists and is unique (it is a pw second-order polynomial)
- $\tilde{p}_h$ is nonconforming, $\notin H_0^1(\Omega)$, only $\in H^1(\mathcal{T}_h)$ in general
- means of traces of $\tilde{p}_h$ on the sides continuous, $\tilde{p}_h \in W_0^0(\mathcal{T}_h)$
- the means are equal to the Lagrange multipliers from the hybridization

Remarks

- exact (not weak) connection of $\tilde{p}_h$ and $\mathbf{u}_h$
- only valid in the lowest-order case on simplices or, when $\mathbf{S}$ is diagonal, on rectangular parallelepipeds

Lowest-order Raviart–Thomas case

Definition (Postprocessed scalar variable $\tilde{p}_h$)

We define $\tilde{p}_h$ such that, separately on each $K \in \mathcal{T}_h$,

- $-\mathbf{S}_K \nabla \tilde{p}_h|_K = \mathbf{u}_h|_K$ (flux of $\tilde{p}_h$ is $\mathbf{u}_h$),
- $(\tilde{p}_h, 1)_K / |K| = p_K$ (mean of $\tilde{p}_h$ on K is p_K).

Properties of $\tilde{p}_h$

- $\tilde{p}_h$ exists and is unique (it is a pw second-order polynomial)
- $\tilde{p}_h$ is nonconforming, $\notin H_0^1(\Omega)$, only $\in H^1(\mathcal{T}_h)$ in general
- means of traces of $\tilde{p}_h$ on the sides continuous, $\tilde{p}_h \in W_0^0(\mathcal{T}_h)$
- the means are equal to the Lagrange multipliers from the hybridization

Remarks

- exact (not weak) connection of $\tilde{p}_h$ and $\mathbf{u}_h$
- only valid in the lowest-order case on simplices or, when $\mathbf{S}$ is diagonal, on rectangular parallelepipeds

General postprocessing

Definition (Postprocessed scalar variable $\tilde{p}_h$ (Arbogast & Chen))

We define $\tilde{p}_h$ such that, separately on each $K \in \mathcal{T}_h$,

- $(\tilde{p}_h, \phi_h)_K = (p_h, \phi_h)_K \quad \forall \phi_h \in \Phi_h(K)$.
- $(\tilde{p}_h, \mu_h)_\sigma = (\lambda_h, \mu_h)_\sigma \quad \forall \mu_h \in \Lambda_h(\sigma) \quad \forall \sigma \in \mathcal{E}_h^{\text{int}}$.

Properties of $\tilde{p}_h$

- $\tilde{p}_h$ exists and is unique
- $\tilde{p}_h$ is nonconforming, $\notin H_0^1(\Omega)$, but $\tilde{p}_h \in W_0^k(\mathcal{T}_h)$
- $\tilde{p}_h$ is in general a nonconforming polynomial plus a bubble
- $\tilde{p}_h$ satisfies $-(\mathbf{S}^{-1}\mathbf{u}_h, \mathbf{v}_h)_K = (\nabla \tilde{p}_h, \mathbf{v}_h)_K \quad \forall \mathbf{v}_h \in \mathbf{V}_h(K)$

Remarks

- $\mathbf{u}_h$ is a $P_{\mathbf{V}_h, \mathbf{S}^{-1}}$ projection of $-\mathbf{S}\nabla \tilde{p}_h$ onto $\mathbf{V}_h$, weak connection of $\tilde{p}_h$ and $\mathbf{u}_h$
- basis of our a priori and a posteriori error estimates

General postprocessing

Definition (Postprocessed scalar variable $\tilde{p}_h$ (Arbogast & Chen))

We define $\tilde{p}_h$ such that, separately on each $K \in \mathcal{T}_h$,

- $(\tilde{p}_h, \phi_h)_K = (p_h, \phi_h)_K \quad \forall \phi_h \in \Phi_h(K)$.
- $(\tilde{p}_h, \mu_h)_\sigma = (\lambda_h, \mu_h)_\sigma \quad \forall \mu_h \in \Lambda_h(\sigma) \quad \forall \sigma \in \mathcal{E}_h^{\mathrm{int}}$.

Properties of $\tilde{p}_h$

- $\tilde{p}_h$ exists and is unique
- $\tilde{p}_h$ is nonconforming, $\notin H_0^1(\Omega)$, but $\tilde{p}_h \in W_0^k(\mathcal{T}_h)$
- $\tilde{p}_h$ is in general a nonconforming polynomial plus a bubble
- $\tilde{p}_h$ satisfies $-(\mathbf{S}^{-1}\mathbf{u}_h, \mathbf{v}_h)_K = (\nabla \tilde{p}_h, \mathbf{v}_h)_K \quad \forall \mathbf{v}_h \in \mathbf{V}_h(K)$

Remarks

- $\mathbf{u}_h$ is a $P_{\mathbf{V}_h, \mathbf{S}^{-1}}$ projection of $-\mathbf{S}\nabla \tilde{p}_h$ onto $\mathbf{V}_h$, weak connection of $\tilde{p}_h$ and $\mathbf{u}_h$
- basis of our a priori and a posteriori error estimates

General postprocessing

Definition (Postprocessed scalar variable $\tilde{p}_h$ (Arbogast & Chen))

We define $\tilde{p}_h$ such that, separately on each $K \in \mathcal{T}_h$,

- $(\tilde{p}_h, \phi_h)_K = (p_h, \phi_h)_K \quad \forall \phi_h \in \Phi_h(K)$.
- $(\tilde{p}_h, \mu_h)_\sigma = (\lambda_h, \mu_h)_\sigma \quad \forall \mu_h \in \Lambda_h(\sigma) \quad \forall \sigma \in \mathcal{E}_h^{\mathrm{int}}$.

Properties of $\tilde{p}_h$

- $\tilde{p}_h$ exists and is unique
- $\tilde{p}_h$ is nonconforming, $\notin H_0^1(\Omega)$, but $\tilde{p}_h \in W_0^k(\mathcal{T}_h)$
- $\tilde{p}_h$ is in general a nonconforming polynomial plus a bubble
- $\tilde{p}_h$ satisfies $-(\mathbf{S}^{-1}\mathbf{u}_h, \mathbf{v}_h)_K = (\nabla \tilde{p}_h, \mathbf{v}_h)_K \quad \forall \mathbf{v}_h \in \mathbf{V}_h(K)$

Remarks

- $\mathbf{u}_h$ is a $P_{\mathbf{V}_h, \mathbf{S}^{-1}}$ projection of $-\mathbf{S}\nabla \tilde{p}_h$ onto $\mathbf{V}_h$, weak connection of $\tilde{p}_h$ and $\mathbf{u}_h$
- basis of our a priori and a posteriori error estimates

General postprocessing

Definition (Postprocessed scalar variable $\tilde{p}_h$ (Arbogast & Chen))

We define $\tilde{p}_h$ such that, separately on each $K \in \mathcal{T}_h$,

- $(\tilde{p}_h, \phi_h)_K = (p_h, \phi_h)_K \quad \forall \phi_h \in \Phi_h(K)$.
- $(\tilde{p}_h, \mu_h)_\sigma = (\lambda_h, \mu_h)_\sigma \quad \forall \mu_h \in \Lambda_h(\sigma) \quad \forall \sigma \in \mathcal{E}_h^{\mathrm{int}}$.

Properties of $\tilde{p}_h$

- $\tilde{p}_h$ exists and is unique
- $\tilde{p}_h$ is nonconforming, $\notin H_0^1(\Omega)$, but $\tilde{p}_h \in W_0^k(\mathcal{T}_h)$
- $\tilde{p}_h$ is in general a nonconforming polynomial plus a bubble
- $\tilde{p}_h$ satisfies $-(\mathbf{S}^{-1}\mathbf{u}_h, \mathbf{v}_h)_K = (\nabla \tilde{p}_h, \mathbf{v}_h)_K \quad \forall \mathbf{v}_h \in \mathbf{V}_h(K)$

Remarks

- $\mathbf{u}_h$ is a $P_{\mathbf{V}_h, \mathbf{S}^{-1}}$ projection of $-\mathbf{S}\nabla \tilde{p}_h$ onto $\mathbf{V}_h$, weak connection of $\tilde{p}_h$ and $\mathbf{u}_h$
- basis of our a priori and a posteriori error estimates

General postprocessing

Definition (Postprocessed scalar variable $\tilde{p}_h$ (Arbogast & Chen))

We define $\tilde{p}_h$ such that, separately on each $K \in \mathcal{T}_h$,

- $(\tilde{p}_h, \phi_h)_K = (p_h, \phi_h)_K \quad \forall \phi_h \in \Phi_h(K)$.
- $(\tilde{p}_h, \mu_h)_\sigma = (\lambda_h, \mu_h)_\sigma \quad \forall \mu_h \in \Lambda_h(\sigma) \quad \forall \sigma \in \mathcal{E}_h^{\mathrm{int}}$.

Properties of $\tilde{p}_h$

- $\tilde{p}_h$ exists and is unique
- $\tilde{p}_h$ is nonconforming, $\notin H_0^1(\Omega)$, but $\tilde{p}_h \in W_0^k(\mathcal{T}_h)$
- $\tilde{p}_h$ is in general a nonconforming polynomial plus a bubble
- $\tilde{p}_h$ satisfies $-(\mathbf{S}^{-1}\mathbf{u}_h, \mathbf{v}_h)_K = (\nabla \tilde{p}_h, \mathbf{v}_h)_K \quad \forall \mathbf{v}_h \in \mathbf{V}_h(K)$

Remarks

- $\mathbf{u}_h$ is a $P_{\mathbf{V}_h, \mathbf{S}^{-1}}$ projection of $-\mathbf{S}\nabla\tilde{p}_h$ onto $\mathbf{V}_h$, weak connection of $\tilde{p}_h$ and $\mathbf{u}_h$
- basis of our a priori and a posteriori error estimates

General postprocessing

Definition (Postprocessed scalar variable $\tilde{p}_h$ (Arbogast & Chen))

We define $\tilde{p}_h$ such that, separately on each $K \in \mathcal{T}_h$,

- $(\tilde{p}_h, \phi_h)_K = (p_h, \phi_h)_K \quad \forall \phi_h \in \Phi_h(K)$.
- $(\tilde{p}_h, \mu_h)_\sigma = (\lambda_h, \mu_h)_\sigma \quad \forall \mu_h \in \Lambda_h(\sigma) \quad \forall \sigma \in \mathcal{E}_h^{\rm int}$.

Properties of $\tilde{p}_h$

- $\tilde{p}_h$ exists and is unique
- $\tilde{p}_h$ is nonconforming, $\notin H_0^1(\Omega)$, but $\tilde{p}_h \in W_0^k(\mathcal{T}_h)$
- $\tilde{p}_h$ is in general a nonconforming polynomial plus a bubble
- $\tilde{p}_h$ satisfies $-(\mathbf{S}^{-1}\mathbf{u}_h, \mathbf{v}_h)_K = (\nabla \tilde{p}_h, \mathbf{v}_h)_K \quad \forall \mathbf{v}_h \in \mathbf{V}_h(K)$

Remarks

- $\mathbf{u}_h$ is a $P_{\mathbf{V}_h, \mathbf{S}^{-1}}$ projection of $-\mathbf{S}\nabla \tilde{p}_h$ onto $\mathbf{V}_h$, weak connection of $\tilde{p}_h$ and $\mathbf{u}_h$
- basis of our a priori and a posteriori error estimates

Outline

Outline

Lowest-order Raviart–Thomas case

Lowest-order Raviart–Thomas case

- $|||p - \tilde{p}_h||| = |||\mathbf{u} - \mathbf{u}_h|||_* \leq |||\mathbf{u} - \Pi_h \mathbf{u}|||_* \leq Ch$
- $\tilde{p}_h \in W_0^0(\mathcal{T}_h)$: discrete Friedrichs inequality

$$\|p - \tilde{p}_h\| \leq C_{\mathrm{DF}}^{\frac{1}{2}} \left\{ \sum_{K \in \mathcal{T}_h} \|\nabla(p - \tilde{p}_h)\|_K^2 \right\}^{\frac{1}{2}}$$

- optimal value of C_{DF} (only depends on the shape regularity parameter and $\inf_{\mathbf{b} \in \mathbb{R}^d} \{\mathrm{thick}_{\mathbf{b}}(\Omega)\}$): Vohralík, NFAO 2005
- consequently: $\{\sum_{K \in \mathcal{T}_h} \|p - \tilde{p}_h\|_{1,K}^2\}^{\frac{1}{2}} \leq Ch$
- superconvergence: $\|p - \tilde{p}_h\| \leq Ch^2$

Lowest-order Raviart–Thomas case

Lowest-order Raviart–Thomas case

- $|||p - \tilde{p}_h||| = |||\mathbf{u} - \mathbf{u}_h|||_* \leq |||\mathbf{u} - \Pi_h \mathbf{u}|||_* \leq Ch$
- $\tilde{p}_h \in W_0^0(\mathcal{T}_h)$: discrete Friedrichs inequality

$$\|p - \tilde{p}_h\| \leq C_{\mathrm{DF}}^{\frac{1}{2}} \left\{ \sum_{K \in \mathcal{T}_h} \|\nabla(p - \tilde{p}_h)\|_K^2 \right\}^{\frac{1}{2}}$$

- optimal value of C_{DF} (only depends on the shape regularity parameter and $\inf_{\mathbf{b} \in \mathbb{R}^d} \{\mathrm{thick}_{\mathbf{b}}(\Omega)\}$): Vohralík, NFAO 2005
- consequently: $\left\{\sum_{K \in \mathcal{T}_h} \|p - \tilde{p}_h\|_{1,K}^2\right\}^{\frac{1}{2}} \leq Ch$
- superconvergence: $\|p - \tilde{p}_h\| \leq Ch^2$

Lowest-order Raviart–Thomas case

Lowest-order Raviart–Thomas case

- $|||p - \tilde{p}_h||| = |||\mathbf{u} - \mathbf{u}_h|||_* \leq |||\mathbf{u} - \Pi_h \mathbf{u}|||_* \leq Ch$
- $\tilde{p}_h \in W_0^0(\mathcal{T}_h)$: discrete Friedrichs inequality

$$\|p - \tilde{p}_h\| \leq C_{\mathrm{DF}}^{\frac{1}{2}} \left\{ \sum_{K \in \mathcal{T}_h} \|\nabla(p - \tilde{p}_h)\|_K^2 \right\}^{\frac{1}{2}}$$

- optimal value of C_{DF} (only depends on the shape regularity parameter and $\inf_{\mathbf{b} \in \mathbb{R}^d} \{\mathrm{thick}_{\mathbf{b}}(\Omega)\}$): Vohralík, NFAO 2005
- consequently: $\left\{ \sum_{K \in \mathcal{T}_h} \|p - \tilde{p}_h\|_{1,K}^2 \right\}^{\frac{1}{2}} \leq Ch$
- superconvergence: $\|p - \tilde{p}_h\| \leq Ch^2$

Lowest-order Raviart–Thomas case

Lowest-order Raviart–Thomas case

- $|||p - \tilde{p}_h||| = |||\mathbf{u} - \mathbf{u}_h|||_* \leq |||\mathbf{u} - \Pi_h \mathbf{u}|||_* \leq Ch$
- $\tilde{p}_h \in W_0^0(\mathcal{T}_h)$: discrete Friedrichs inequality

$$\|p - \tilde{p}_h\| \leq C_{\mathrm{DF}}^{\frac{1}{2}} \left\{ \sum_{K \in \mathcal{T}_h} \|\nabla(p - \tilde{p}_h)\|_K^2 \right\}^{\frac{1}{2}}$$

- optimal value of C_{DF} (only depends on the shape regularity parameter and $\inf_{\mathbf{b} \in \mathbb{R}^d} \{\mathrm{thick}_{\mathbf{b}}(\Omega)\}$): Vohralík, NFAO 2005
- consequently: $\left\{ \sum_{K \in \mathcal{T}_h} \|p - \tilde{p}_h\|_{1,K}^2 \right\}^{\frac{1}{2}} \leq Ch$
- superconvergence: $\|p - \tilde{p}_h\| \leq Ch^2$

Lowest-order Raviart–Thomas case

Lowest-order Raviart–Thomas case

- $|||p - \tilde{p}_h||| = |||\mathbf{u} - \mathbf{u}_h|||_* \leq |||\mathbf{u} - \Pi_h \mathbf{u}|||_* \leq Ch$
- $\tilde{p}_h \in W_0^0(\mathcal{T}_h)$: discrete Friedrichs inequality

$$\|p - \tilde{p}_h\| \leq C_{\mathrm{DF}}^{\frac{1}{2}} \left\{ \sum_{K \in \mathcal{T}_h} \|\nabla(p - \tilde{p}_h)\|_K^2 \right\}^{\frac{1}{2}}$$

- optimal value of C_{DF} (only depends on the shape regularity parameter and $\inf_{\mathbf{b} \in \mathbb{R}^d}\{\mathrm{thick}_{\mathbf{b}}(\Omega)\}$): Vohralík, NFAO 2005
- consequently: $\left\{\sum_{K \in \mathcal{T}_h} \|p - \tilde{p}_h\|_{1,K}^2\right\}^{\frac{1}{2}} \leq Ch$
- superconvergence: $\|p - \tilde{p}_h\| \leq Ch^2$

Outline

General case

General case

- a little bit more complicated since we only have $\mathbf{u}_h = -P_{\mathbf{V}_h,\mathbf{S}^{-1}}(\mathbf{S}\nabla\tilde{p}_h)$ instead of $\mathbf{u}_h = -\mathbf{S}\nabla\tilde{p}_h$
- one still easily recovers all the known a priori error estimates for mixed finite elements

Outline

What is/should be an a posteriori error estimate

Usual form

- $\|p - p_h\|^2 \lesssim \sum_{K \in \mathcal{T}_h} \eta_K(p_h)^2$.
- Can be used to determine mesh elements with large error.
- We can then refine these elements: mesh adaptivity.

Reliability

- $\|p - p_h\|^2 \leq C \sum_{K \in \mathcal{T}_h} \eta_K(p_h)^2$

Guaranteed upper bound

- $\|p - p_h\|^2 \leq \sum_{K \in \mathcal{T}_h} \eta_K(p_h)^2$

Local efficiency

- $\eta_K(p_h)^2 \leq C^2_{\mathrm{eff},K} \sum_{L \text{ close to } K} \|p - p_h\|^2_L$

Asymptotic exactness

- $\sum_{K \in \mathcal{T}_h} \eta_K(p_h)^2 / \|p - p_h\|^2 \to 1$

Robustness

- independence of the data variation or mesh properties

Negligible evaluation cost

- estimators which can be evaluated locally

What is/should be an a posteriori error estimate

Usual form

- $\|p - p_h\|^2 \lesssim \sum_{K\in\mathcal{T}_h} \eta_K(p_h)^2$.
- Can be used to determine mesh elements with large error.
- We can then refine these elements: mesh adaptivity.

Reliability

- $\|p - p_h\|^2 \leq C \sum_{K\in\mathcal{T}_h} \eta_K(p_h)^2$

Guaranteed upper bound

- $\|p - p_h\|^2 \leq \sum_{K\in\mathcal{T}_h} \eta_K(p_h)^2$

Local efficiency

- $\eta_K(p_h)^2 \leq C^2_{\mathrm{eff},K} \sum_{L \text{ close to } K} \|p - p_h\|^2_L$

Asymptotic exactness

- $\sum_{K\in\mathcal{T}_h} \eta_K(p_h)^2 / \|p - p_h\|^2 \to 1$

Robustness

- independence of the data variation or mesh properties

Negligible evaluation cost

- estimators which can be evaluated locally

What is/should be an a posteriori error estimate

Usual form

- $\|p - p_h\|^2 \lesssim \sum_{K \in \mathcal{T}_h} \eta_K(p_h)^2$.
- Can be used to determine mesh elements with large error.
- We can then refine these elements: mesh adaptivity.

Reliability

- $\|p - p_h\|^2 \leq C \sum_{K \in \mathcal{T}_h} \eta_K(p_h)^2$
- Problems:
  - What is C?
  - What does it depend on?
  - How does it depend on data?

Local efficiency

- $\eta_K(p_h)^2 \leq C_{\mathrm{eff},K}^2 \sum_{L \text{ close to } K} \|p - p_h\|_L^2$

Asymptotic exactness

- $\sum_{K \in \mathcal{T}_h} \eta_K(p_h)^2 / \|p - p_h\|^2 \to 1$

Robustness

- independence of the data variation or mesh properties

Negligible evaluation cost

What is/should be an a posteriori error estimate

Usual form

- $\|p - p_h\|^2 \lesssim \sum_{K\in\mathcal{T}_h} \eta_K(p_h)^2$.
- Can be used to determine mesh elements with large error.
- We can then refine these elements: mesh adaptivity.

Reliability

- $\|p - p_h\|^2 \leq C \sum_{K\in\mathcal{T}_h} \eta_K(p_h)^2$

Guaranteed upper bound

- $\|p - p_h\|^2 \leq \sum_{K\in\mathcal{T}_h} \eta_K(p_h)^2$

Local efficiency

- $\eta_K(p_h)^2 \leq C_{\mathrm{eff},K}^2 \sum_{L \text{ close to } K} \|p - p_h\|_L^2$

Asymptotic exactness

- $\sum_{K\in\mathcal{T}_h} \eta_K(p_h)^2 / \|p - p_h\|^2 \to 1$

Robustness

- independence of the data variation or mesh properties

Negligible evaluation cost

- estimators which can be evaluated locally

What is/should be an a posteriori error estimate

Usual form

- $\|p - p_h\|^2 \lesssim \sum_{K \in \mathcal{T}_h} \eta_K(p_h)^2$.
- Can be used to determine mesh elements with large error.
- We can then refine these elements: mesh adaptivity.

Reliability

- $\|p - p_h\|^2 \leq C \sum_{K \in \mathcal{T}_h} \eta_K(p_h)^2$

Guaranteed upper bound

- $\|p - p_h\|^2 \leq \sum_{K \in \mathcal{T}_h} \eta_K(p_h)^2$

Local efficiency

- $\eta_K(p_h)^2 \leq C_{\mathrm{eff},K}^2 \sum_{L \text{ close to } K} \|p - p_h\|_L^2$

Asymptotic exactness

- $\sum_{K \in \mathcal{T}_h} \eta_K(p_h)^2 / \|p - p_h\|^2 \to 1$

Robustness

- independence of the data variation or mesh properties

Negligible evaluation cost

- estimators which can be evaluated locally

What is/should be an a posteriori error estimate

Usual form

- $\|p - p_h\|^2 \lesssim \sum_{K\in\mathcal{T}_h} \eta_K(p_h)^2$.
- Can be used to determine mesh elements with large error.
- We can then refine these elements: mesh adaptivity.

Reliability

- $\|p - p_h\|^2 \leq C \sum_{K\in\mathcal{T}_h} \eta_K(p_h)^2$

Guaranteed upper bound

- $\|p - p_h\|^2 \leq \sum_{K\in\mathcal{T}_h} \eta_K(p_h)^2$

Local efficiency

- $\eta_K(p_h)^2 \leq C^2_{\mathrm{eff},K} \sum_{L \text{ close to } K} \|p - p_h\|^2_L$

Asymptotic exactness

- $\sum_{K\in\mathcal{T}_h} \eta_K(p_h)^2 / \|p - p_h\|^2 \to 1$

Robustness

- independence of the data variation or mesh properties

Negligible evaluation cost

- estimators which can be evaluated locally

What is/should be an a posteriori error estimate

Usual form

- $\|p - p_h\|^2 \lesssim \sum_{K\in\mathcal{T}_h} \eta_K(p_h)^2$.
- Can be used to determine mesh elements with large error.
- We can then refine these elements: mesh adaptivity.

Reliability

- $\|p - p_h\|^2 \le C \sum_{K\in\mathcal{T}_h} \eta_K(p_h)^2$

Guaranteed upper bound

- $\|p - p_h\|^2 \le \sum_{K\in\mathcal{T}_h} \eta_K(p_h)^2$

Local efficiency

- $\eta_K(p_h)^2 \le C^2_{\mathrm{eff},K} \sum_{L \text{ close to } K} \|p - p_h\|^2_L$

Asymptotic exactness

- $\sum_{K\in\mathcal{T}_h} \eta_K(p_h)^2 / \|p - p_h\|^2 \to 1$

Robustness

- independence of the data variation or mesh properties

Negligible evaluation cost

- estimators which can be evaluated locally

What is/should be an a posteriori error estimate

Usual form

- $\|p - p_h\|^2 \lesssim \sum_{K \in \mathcal{T}_h} \eta_K(p_h)^2$.
- Can be used to determine mesh elements with large error.
- We can then refine these elements: mesh adaptivity.

Reliability

- $\|p - p_h\|^2 \leq C \sum_{K \in \mathcal{T}_h} \eta_K(p_h)^2$

Guaranteed upper bound

- $\|p - p_h\|^2 \leq \sum_{K \in \mathcal{T}_h} \eta_K(p_h)^2$

Local efficiency

- $\eta_K(p_h)^2 \leq C_{\mathrm{eff},K}^2 \sum_{L\,\mathrm{close\ to}\,K} \|p - p_h\|_L^2$

Asymptotic exactness

- $\sum_{K \in \mathcal{T}_h} \eta_K(p_h)^2 / \|p - p_h\|^2 \to 1$

Robustness

- independence of the data variation or mesh properties

Negligible evaluation cost

- estimators which can be evaluated locally

Previous works on a posteriori analysis for MFEMs

Previous works . . .

- Alonso '96
- Braess and Verfürth '96
- Carstensen '97
- Hoppe and Wohlmuth '97, '99
- Kirby '03
- El Alaoui and Ern '04
- Wheeler and Yotov '05
- Lovadina and Stenberg '06

. . . do not cover

- evaluation of the constants (guaranteed upper bound)
- robustness
- asymptotic exactness
- an analysis of the convection–diffusion case

Previous works on a posteriori analysis for MFEMs

Previous works . . .

- Alonso '96
- Braess and Verfürth '96
- Carstensen '97
- Hoppe and Wohlmuth '97, '99
- Kirby '03
- El Alaoui and Ern '04
- Wheeler and Yotov '05
- Lovadina and Stenberg '06

. . . do not cover

- evaluation of the constants (guaranteed upper bound)
- robustness
- asymptotic exactness
- an analysis of the convection–diffusion case

Outline

A first abstract estimate for the flux

Theorem (A first abstract estimate for the flux and its efficiency)

Let $\mathbf{u}$ *be the weak flux and let* $\mathbf{u}_h \in \mathbf{H}(\mathrm{div}, \Omega)$ *be arbitrary. Then*

$$|||\mathbf{u} - \mathbf{u}_h|||_*^2 \leq \inf_{s \in H_0^1(\Omega)} |||\mathbf{u}_h + \mathbf{S}\nabla s|||_*^2 + \frac{C_{\mathrm{F},\Omega} h_\Omega^2}{c_{\mathbf{S},\Omega}} \|f - \nabla \cdot \mathbf{u}_h\|^2$$

$$\leq |||\mathbf{u} - \mathbf{u}_h|||_*^2 + \frac{C_{\mathrm{F},\Omega} h_\Omega^2}{c_{\mathbf{S},\Omega}} \|f - \nabla \cdot \mathbf{u}_h\|^2.$$

Properties

- Guaranteed upper bound (no undetermined constant).
- $|||\mathbf{u}_h + \mathbf{S}\nabla s|||_*$ penalizes $\mathbf{u}_h \neq -\mathbf{S}\nabla s$ for some $s \in H_0^1(\Omega)$.
- Advantage: scheme-independent (promoted by Repin).
- Disadvantage: scheme-independent (no information from the computation used).
- Disadvantage: $C_{\mathrm{F},\Omega}^{1/2} h_\Omega / c_{\mathbf{S},\Omega}^{1/2} \|f - \nabla \cdot \mathbf{u}_h\|$ too big.

A first abstract estimate for the flux

Theorem (A first abstract estimate for the flux and its efficiency)

Let $\mathbf{u}$ *be the weak flux and let* $\mathbf{u}_h \in \mathbf{H}(\mathrm{div}, \Omega)$ *be arbitrary. Then*

$$|||\mathbf{u} - \mathbf{u}_h|||_*^2 \leq \inf_{s \in H_0^1(\Omega)} |||\mathbf{u}_h + \mathbf{S}\nabla s|||_*^2 + \frac{C_{\mathrm{F},\Omega} h_\Omega^2}{c_{\mathbf{S},\Omega}} \|f - \nabla \cdot \mathbf{u}_h\|^2$$

$$\leq |||\mathbf{u} - \mathbf{u}_h|||_*^2 + \frac{C_{\mathrm{F},\Omega} h_\Omega^2}{c_{\mathbf{S},\Omega}} \|f - \nabla \cdot \mathbf{u}_h\|^2.$$

Properties

- Guaranteed upper bound (no undetermined constant).
- $|||\mathbf{u}_h + \mathbf{S}\nabla s|||_*$ penalizes $\mathbf{u}_h \neq -\mathbf{S}\nabla s$ for some $s \in H_0^1(\Omega)$.
- Advantage: scheme-independent (promoted by Repin).
- Disadvantage: scheme-independent (no information from the computation used).
- Disadvantage: $C_{\mathrm{F},\Omega}^{1/2} h_\Omega / c_{\mathbf{S},\Omega}^{1/2} \|f - \nabla \cdot \mathbf{u}_h\|$ too big.

An improved abstract estimate for the flux

Theorem (An improved abstract estimate for the flux and its efficiency)

Let $\mathbf{u}$ *be the weak flux and let* $\mathbf{u}_h \in \mathbf{H}(\mathrm{div}, \Omega)$ *such that* $\nabla \cdot \mathbf{u}_h = \pi_l(f)$ *be arbitrary. Then*

$$|||\mathbf{u} - \mathbf{u}_h|||_*^2 \leq \inf_{s \in H_0^1(\Omega)} |||\mathbf{u}_h + \mathbf{S}\nabla s|||_*^2 + \eta_{\mathrm{R}}^2 \leq |||\mathbf{u} - \mathbf{u}_h|||_*^2 + \eta_{\mathrm{R}}^2,$$

where

$$\eta_{\mathrm{R}} := \left\{ \sum_{K \in \mathcal{T}_h} \frac{C_{\mathrm{P}} h_K^2}{c_{\mathbf{S},K}} \|f - \pi_l(f)\|_K^2 \right\}^{\frac{1}{2}}.$$

Properties

- No global Galerkin orthogonality needed, just local conservativity.
- η_{R} is in general a higher-order term for RT methods.
- η_{R} is not in general a higher-order term for BDM methods.

An improved abstract estimate for the flux

Theorem (An improved abstract estimate for the flux and its efficiency)

Let $\mathbf{u}$ *be the weak flux and let* $\mathbf{u}_h \in \mathbf{H}(\mathrm{div}, \Omega)$ *such that* $\nabla \cdot \mathbf{u}_h = \pi_l(f)$ *be arbitrary. Then*

$$|||\mathbf{u} - \mathbf{u}_h|||_*^2 \le \inf_{s \in H_0^1(\Omega)} |||\mathbf{u}_h + \mathbf{S}\nabla s|||_*^2 + \eta_{\mathrm{R}}^2 \le |||\mathbf{u} - \mathbf{u}_h|||_*^2 + \eta_{\mathrm{R}}^2,$$

where

$$\eta_{\mathrm{R}} := \left\{ \sum_{K \in \mathcal{T}_h} \frac{C_{\mathrm{P}} h_K^2}{c_{\mathbf{S},K}} \|f - \pi_l(f)\|_K^2 \right\}^{\frac{1}{2}}.$$

Properties

- No global Galerkin orthogonality needed, just local conservativity.
- η_{R} is in general a higher-order term for RT methods.
- η_{R} is not in general a higher-order term for BDM methods.

An energy–div norm abstract estimate for the flux

Theorem (An energy–div norm abstract estimate for the flux and its efficiency)

Let $\mathbf{u}$ *be the weak flux and let* $\mathbf{u}_h \in \mathbf{H}(\mathrm{div}, \Omega)$ *such that* $\nabla \cdot \mathbf{u}_h = \pi_l(f)$ *be arbitrary. Then*

$$|||\mathbf{u} - \mathbf{u}_h|||^2_{*,\mathrm{div}} \le \inf_{s \in H^1_0(\Omega)} |||\mathbf{u}_h + \mathbf{S}\nabla s|||^2_* + \|f - \pi_l(f)\|^2 + \eta^2_{\mathrm{R}}$$

$$\le |||\mathbf{u} - \mathbf{u}_h|||^2_{*,\mathrm{div}} + \eta^2_{\mathrm{R}}.$$

Properties

- η_{R} gets always a higher-order term.

An energy–div norm abstract estimate for the flux

Theorem (An energy–div norm abstract estimate for the flux and its efficiency)

Let $\mathbf{u}$ *be the weak flux and let* $\mathbf{u}_h \in \mathbf{H}(\mathrm{div}, \Omega)$ *such that* $\nabla \cdot \mathbf{u}_h = \pi_l(f)$ *be arbitrary. Then*

$$|||\mathbf{u} - \mathbf{u}_h|||^2_{*,\mathrm{div}} \leq \inf_{s \in H^1_0(\Omega)} |||\mathbf{u}_h + \mathbf{S}\nabla s|||^2_* + \|f - \pi_l(f)\|^2 + \eta^2_{\mathrm{R}}$$

$$\leq |||\mathbf{u} - \mathbf{u}_h|||^2_{*,\mathrm{div}} + \eta^2_{\mathrm{R}}.$$

Properties

- η_{R} gets always a higher-order term.

A fully computable estimate for the flux

Theorem (A fully computable estimate for the flux)

Let $\mathbf{u}$ *be the weak flux and let* $\mathbf{u}_h \in \mathbf{H}(\mathrm{div}, \Omega)$ *such that* $\nabla \cdot \mathbf{u}_h = \pi_l(f)$ *be arbitrary. Then*

$$|||\mathbf{u} - \mathbf{u}_h|||_*^2 \leq \sum_{K \in \mathcal{T}_h} \left(\eta_{\mathrm{P},K}^2 + \eta_{\mathrm{R},K}^2 \right),$$

$$|||\mathbf{u} - \mathbf{u}_h|||_{*,\mathrm{div}}^2 \leq \sum_{K \in \mathcal{T}_h} \left(\eta_{\mathrm{P},K}^2 + \eta_{\mathrm{R},K}^2 + \eta_{\mathrm{D},K}^2 \right).$$

- **potential estimator**
  - $\eta_{\mathrm{P},K} := |||\mathbf{u}_h + \mathbf{S}\nabla(\mathcal{I}_{\mathrm{Os}}(\tilde{p}_h))|||_{*,K}$
  - $\mathcal{I}_{\mathrm{Os}}(\tilde{p}_h)$: Oswald interpolate $\mathbb{P}_n(\mathcal{T}_h) \to \mathbb{P}_n(\mathcal{T}_h) \cap H_0^1(\Omega)$
- **residual estimator**
  - $\eta_{\mathrm{R},K} := \frac{C_{\mathrm{P}}^{1/2} h_K}{c_{\mathbf{S},K}^{1/2}} \|f - \pi_l(f)\|_K$
- **divergence estimator**
  - $\eta_{\mathrm{D},K} := \|f - \pi_l(f)\|_K$

A fully computable estimate for the flux

Theorem (A fully computable estimate for the flux)

Let $\mathbf{u}$ *be the weak flux and let* $\mathbf{u}_h \in \mathbf{H}(\mathrm{div}, \Omega)$ *such that* $\nabla \cdot \mathbf{u}_h = \pi_l(f)$ *be arbitrary. Then*

$$|||\mathbf{u} - \mathbf{u}_h|||_*^2 \leq \sum_{K \in \mathcal{T}_h} \left(\eta_{\mathrm{P},K}^2 + \eta_{\mathrm{R},K}^2 \right),$$

$$|||\mathbf{u} - \mathbf{u}_h|||_{*,\mathrm{div}}^2 \leq \sum_{K \in \mathcal{T}_h} \left(\eta_{\mathrm{P},K}^2 + \eta_{\mathrm{R},K}^2 + \eta_{\mathrm{D},K}^2 \right).$$

- **potential estimator**
  - $\eta_{\mathrm{P},K} := |||\mathbf{u}_h + \mathbf{S}\nabla(\mathcal{I}_{\mathrm{Os}}(\tilde{p}_h))|||_{*,K}$
  - $\mathcal{I}_{\mathrm{Os}}(\tilde{p}_h)$: Oswald interpolate $\mathbb{P}_n(\mathcal{T}_h) \to \mathbb{P}_n(\mathcal{T}_h) \cap H_0^1(\Omega)$
- **residual estimator**
  - $\eta_{\mathrm{R},K} := \frac{C_{\mathrm{P}}^{1/2} h_K}{c_{\mathbf{S},K}^{1/2}} \|f - \pi_l(f)\|_K$
- **divergence estimator**
  - $\eta_{\mathrm{D},K} := \|f - \pi_l(f)\|_K$

A fully computable estimate for the flux

Theorem (A fully computable estimate for the flux)

Let $\mathbf{u}$ *be the weak flux and let* $\mathbf{u}_h \in \mathbf{H}(\mathrm{div}, \Omega)$ *such that* $\nabla \cdot \mathbf{u}_h = \pi_l(f)$ *be arbitrary. Then*

$$|||\mathbf{u} - \mathbf{u}_h|||_*^2 \le \sum_{K \in \mathcal{T}_h} \left(\eta_{\mathrm{P},K}^2 + \eta_{\mathrm{R},K}^2 \right),$$

$$|||\mathbf{u} - \mathbf{u}_h|||_{*,\mathrm{div}}^2 \le \sum_{K \in \mathcal{T}_h} \left(\eta_{\mathrm{P},K}^2 + \eta_{\mathrm{R},K}^2 + \eta_{\mathrm{D},K}^2 \right).$$

- **potential estimator**
  - $\eta_{\mathrm{P},K} := |||\mathbf{u}_h + \mathbf{S}\nabla(\mathcal{I}_{\mathrm{Os}}(\tilde{p}_h))|||_{*,K}$
  - $\mathcal{I}_{\mathrm{Os}}(\tilde{p}_h)$: Oswald interpolate $\mathbb{P}_n(\mathcal{T}_h) \to \mathbb{P}_n(\mathcal{T}_h) \cap H_0^1(\Omega)$
- **residual estimator**
  - $\eta_{\mathrm{R},K} := \frac{C_{\mathrm{P}}^{1/2} h_K}{c_{\mathbf{S},K}^{1/2}} \|f - \pi_l(f)\|_K$
- **divergence estimator**
  - $\eta_{\mathrm{D},K} := \|f - \pi_l(f)\|_K$

A fully computable estimate for the flux

Theorem (A fully computable estimate for the flux)

Let $\mathbf{u}$ *be the weak flux and let* $\mathbf{u}_h \in \mathbf{H}(\mathrm{div}, \Omega)$ *such that* $\nabla \cdot \mathbf{u}_h = \pi_l(f)$ *be arbitrary. Then*

$$|||\mathbf{u} - \mathbf{u}_h|||_*^2 \leq \sum_{K \in \mathcal{T}_h} \left(\eta_{\mathrm{P},K}^2 + \eta_{\mathrm{R},K}^2 \right),$$

$$|||\mathbf{u} - \mathbf{u}_h|||_{*,\mathrm{div}}^2 \leq \sum_{K \in \mathcal{T}_h} \left(\eta_{\mathrm{P},K}^2 + \eta_{\mathrm{R},K}^2 + \eta_{\mathrm{D},K}^2 \right).$$

- **potential estimator**
  - $\eta_{\mathrm{P},K} := |||\mathbf{u}_h + \mathbf{S}\nabla(\mathcal{I}_{\mathrm{Os}}(\tilde{p}_h))|||_{*,K}$
  - $\mathcal{I}_{\mathrm{Os}}(\tilde{p}_h)$: Oswald interpolate $\mathbb{P}_n(\mathcal{T}_h) \to \mathbb{P}_n(\mathcal{T}_h) \cap H_0^1(\Omega)$
- **residual estimator**
  - $\eta_{\mathrm{R},K} := \frac{C_{\mathrm{P}}^{1/2} h_K}{c_{\mathbf{S},K}^{1/2}} \|f - \pi_l(f)\|_K$
- **divergence estimator**
  - $\eta_{\mathrm{D},K} := \|f - \pi_l(f)\|_K$

Outline

An abstract estimate for the potential

Theorem (Abstract a posteriori estimate for the potential and its efficiency)

Let p be the weak potential and let $\tilde{p}_h \in H^1(\mathcal{T}_h)$ be arbitrary. Then

$$\begin{aligned}|||p - \tilde{p}_h|||^2 \leq & \inf_{s \in H^1_0(\Omega)} |||\tilde{p}_h - s|||^2 \\ & + \inf_{\mathbf{t} \in \mathbf{H}(\mathrm{div},\Omega)} \sup_{\varphi \in H^1_0(\Omega),\, |||\varphi|||=1} ((f - \nabla\cdot\mathbf{t}, \varphi) - (\mathbf{S}\nabla\tilde{p}_h + \mathbf{t}, \nabla\varphi))^2 \\ \leq & \, 2|||p - \tilde{p}_h|||^2.\end{aligned}$$

Properties

- Guaranteed upper bound, quasi-exact, and robust.
- Holds uniformly for any mesh (anisotropic) and polynomial degree of p_h.

An abstract estimate for the potential

Theorem (Abstract a posteriori estimate for the potential and its efficiency)

Let p be the weak potential and let $\tilde{p}_h \in H^1(\mathcal{T}_h)$ be arbitrary. Then

$$
\begin{aligned}
|||p - \tilde{p}_h|||^2 \leq & \inf_{s \in H_0^1(\Omega)} |||\tilde{p}_h - s|||^2 \\
& + \inf_{\mathbf{t} \in \mathbf{H}(\mathrm{div},\Omega)} \sup_{\varphi \in H_0^1(\Omega),\, |||\varphi|||=1} ((f - \nabla\cdot\mathbf{t}, \varphi) - (\mathbf{S}\nabla\tilde{p}_h + \mathbf{t}, \nabla\varphi))^2 \\
\leq & 2|||p - \tilde{p}_h|||^2.
\end{aligned}
$$

Properties

- Guaranteed upper bound, quasi-exact, and robust.
- Holds uniformly for any mesh (anisotropic) and polynomial degree of p_h.

A first computable estimate for the potential

Theorem (A first computable estimate for the potential)

Let p be the weak potential and let $\tilde{p}_h \in H^1(\mathcal{T}_h)$ be arbitrary. Take any $\mathbf{t}_h \in \mathbf{H}(\mathrm{div}, \Omega)$ and any $s_h \in H_0^1(\Omega)$. Then

$$|||p - \tilde{p}_h|||^2 \leq |||\tilde{p}_h - s_h|||^2 + \left(\frac{C_{\mathrm{F},\Omega}^{1/2} h_\Omega}{c_{\mathbf{S},\Omega}^{1/2}} \|f - \nabla \cdot \mathbf{t}_h\| + |||\mathbf{S}\nabla \tilde{p}_h + \mathbf{t}_h|||_* \right)^2.$$

Properties

- $|||\mathbf{S}\nabla \tilde{p}_h + \mathbf{t}_h|||_*$ penalizes $-\mathbf{S}\nabla \tilde{p}_h \not\in \mathbf{H}(\mathrm{div}, \Omega)$.
- $|||\tilde{p}_h - s_h|||$ penalizes $\tilde{p}_h \not\in H_0^1(\Omega)$.
- Advantage: scheme-independent.
- Disadvantage: $C_{\mathrm{F},\Omega}^{1/2} h_\Omega / c_{\mathbf{S},\Omega}^{1/2} \|f - \nabla \cdot \mathbf{t}_h\|$ too big.

A first computable estimate for the potential

Theorem (A first computable estimate for the potential)

Let p be the weak potential and let $\tilde{p}_h \in H^1(\mathcal{T}_h)$ be arbitrary. Take any $\mathbf{t}_h \in \mathbf{H}(\mathrm{div}, \Omega)$ and any $s_h \in H_0^1(\Omega)$. Then

$$|||p - \tilde{p}_h|||^2 \leq |||\tilde{p}_h - s_h|||^2 + \left(\frac{C_{\mathrm{F},\Omega}^{1/2} h_\Omega}{c_{\mathbf{S},\Omega}^{1/2}} \|f - \nabla \cdot \mathbf{t}_h\| + |||\mathbf{S}\nabla \tilde{p}_h + \mathbf{t}_h|||_* \right)^2.$$

Properties

- $|||\mathbf{S}\nabla \tilde{p}_h + \mathbf{t}_h|||_*$ penalizes $-\mathbf{S}\nabla \tilde{p}_h \notin \mathbf{H}(\mathrm{div}, \Omega)$.
- $|||\tilde{p}_h - s_h|||$ penalizes $\tilde{p}_h \notin H_0^1(\Omega)$.
- Advantage: scheme-independent.
- Disadvantage: $C_{\mathrm{F},\Omega}^{1/2} h_\Omega / c_{\mathbf{S},\Omega}^{1/2} \|f - \nabla \cdot \mathbf{t}_h\|$ too big.

A fully computable estimate for the potential

Theorem (A fully computable estimate for the potential)

Let p be the weak potential and let $\tilde{p}_h \in H^1(\mathcal{T}_h)$ and $\mathbf{u}_h \in \mathbf{H}(\mathrm{div}, \Omega)$ such that $\nabla \cdot \mathbf{u}_h = \pi_l(f)$ be arbitrary. Then

$$|||p - \tilde{p}_h|||^2 \leq \sum_{K \in \mathcal{T}_h} \left\{ \eta_{\mathrm{NC},K}^2 + (\eta_{\mathrm{R},K} + \eta_{\mathrm{DF},K})^2 \right\}.$$

- **nonconformity estimator**
  - $\eta_{\mathrm{NC},K} := |||\tilde{p}_h - \mathcal{I}_{\mathrm{Os}}(\tilde{p}_h)|||_K$
- **diffusive flux estimator**
  - $\eta_{\mathrm{DF},K} := |||\mathbf{u}_h + \mathbf{S}\nabla \tilde{p}_h|||_{*,K}$
- **residual estimator**
  - $\eta_{\mathrm{R},K} := \frac{C_{\mathrm{P}}^{1/2} h_K}{c_{\mathbf{S},K}^{1/2}} \|f - \pi_l(f)\|_K$

A fully computable estimate for the potential

Theorem (A fully computable estimate for the potential)

Let p be the weak potential and let $\tilde{p}_h \in H^1(\mathcal{T}_h)$ and $\mathbf{u}_h \in \mathbf{H}(\mathrm{div}, \Omega)$ such that $\nabla \cdot \mathbf{u}_h = \pi_l(f)$ be arbitrary. Then

$$|||p - \tilde{p}_h|||^2 \leq \sum_{K \in \mathcal{T}_h} \left\{ \eta_{\mathrm{NC},K}^2 + (\eta_{\mathrm{R},K} + \eta_{\mathrm{DF},K})^2 \right\}.$$

- **nonconformity estimator**
  - $\eta_{\mathrm{NC},K} := |||\tilde{p}_h - \mathcal{I}_{\mathrm{Os}}(\tilde{p}_h)|||_K$
- **diffusive flux estimator**
  - $\eta_{\mathrm{DF},K} := |||\mathbf{u}_h + \mathbf{S}\nabla \tilde{p}_h|||_{*,K}$
- **residual estimator**
  - $\eta_{\mathrm{R},K} := \frac{C_{\mathrm{P}}^{1/2} h_K}{c_{\mathbf{S},K}^{1/2}} \|f - \pi_l(f)\|_K$

A fully computable estimate for the potential

Theorem (A fully computable estimate for the potential)

Let p be the weak potential and let $\tilde{p}_h \in H^1(\mathcal{T}_h)$ and $\mathbf{u}_h \in \mathbf{H}(\mathrm{div}, \Omega)$ such that $\nabla \cdot \mathbf{u}_h = \pi_l(f)$ be arbitrary. Then

$$|||p - \tilde{p}_h|||^2 \leq \sum_{K \in \mathcal{T}_h} \left\{ \eta_{\mathrm{NC},K}^2 + (\eta_{\mathrm{R},K} + \eta_{\mathrm{DF},K})^2 \right\}.$$

- **nonconformity estimator**
  - $\eta_{\mathrm{NC},K} := |||\tilde{p}_h - \mathcal{I}_{\mathrm{Os}}(\tilde{p}_h)|||_K$
- **diffusive flux estimator**
  - $\eta_{\mathrm{DF},K} := |||\mathbf{u}_h + \mathbf{S}\nabla \tilde{p}_h|||_{*,K}$
- **residual estimator**
  - $\eta_{\mathrm{R},K} := \frac{C_{\mathrm{P}}^{1/2} h_K}{c_{\mathbf{S},K}^{1/2}} \|f - \pi_l(f)\|_K$

A fully computable estimate for the potential

Theorem (A fully computable estimate for the potential)

Let p be the weak potential and let $\tilde{p}_h \in H^1(\mathcal{T}_h)$ and $\mathbf{u}_h \in \mathbf{H}(\mathrm{div}, \Omega)$ such that $\nabla \cdot \mathbf{u}_h = \pi_l(f)$ be arbitrary. Then

$$|||p - \tilde{p}_h|||^2 \leq \sum_{K \in \mathcal{T}_h} \left\{ \eta_{\mathrm{NC},K}^2 + (\eta_{\mathrm{R},K} + \eta_{\mathrm{DF},K})^2 \right\}.$$

- **nonconformity estimator**
  - $\eta_{\mathrm{NC},K} := |||\tilde{p}_h - \mathcal{I}_{\mathrm{Os}}(\tilde{p}_h)|||_K$
- **diffusive flux estimator**
  - $\eta_{\mathrm{DF},K} := |||\mathbf{u}_h + \mathbf{S}\nabla\tilde{p}_h|||_{*,K}$
- **residual estimator**
  - $\eta_{\mathrm{R},K} := \frac{C_{\mathrm{P}}^{1/2} h_K}{c_{\mathbf{S},K}^{1/2}} \|f - \pi_l(f)\|_K$

Outline

1. General framework
   - An abstract result for the flux variable
   - Postprocessing for the scalar variable
2. A priori error estimates
   - Lowest-order Raviart–Thomas case
   - General case
3. A posteriori error estimates
   - Estimates for the flux
   - Estimates for the potential
   - Local efficiency
4. Remarks
   - Comments on the estimates
   - $L^2(\Omega)$ estimates
   - RT_0 and pure diffusion problems
5. Numerical experiments
6. Conclusions and future work

Local efficiency of the estimates

Theorem (Local efficiency of the estimates)

Let $p, \mathbf{u}$ be the weak potential and flux, respectively, and let $\mathbf{u}_h$ be the MFE flux and $\tilde{p}_h$ the postprocessed potential. Then

$$\eta_{\mathrm{DF},K} \le |||\mathbf{u} - \mathbf{u}_h|||_{*,K} + |||p - \tilde{p}_h|||_K,$$

$$\eta_{\mathrm{P},K} \le \eta_{\mathrm{DF},K} + \eta_{\mathrm{NC},K},$$

$$\eta_{\mathrm{NC},K} \le C\sqrt{\frac{C_{\mathbf{S},K}}{c_{\mathbf{S},\mathcal{T}_K}}}|||p - \tilde{p}_h|||_{\mathcal{T}_K},$$

$$\eta_{\mathrm{R},K} \le C\sqrt{\frac{C_{\mathbf{S},K}}{c_{\mathbf{S},K}}}|||\mathbf{u} - \mathbf{u}_h|||_{*,K},$$

where C depends only on the space dimension d, the maximal polynomial degree n of $\tilde{p}_h$, the shape regularity parameter $\kappa_{\mathcal{T}}$, and the polynomial degree m of f.

Local efficiency of the estimates

Proof for $\eta_{\mathrm{NC},K}$.

- Oswald interpolate (Karakashian and Pascal '03, Burman and Ern '07):

$$\|\nabla(\varphi_h - \mathcal{I}_{\mathrm{Os}}(\varphi_h))\|_K^2 \leq C \sum_{\sigma \in \tilde{\mathcal{E}}_K} h_\sigma^{-1} \|[\![\varphi_h]\!]\|_\sigma^2$$

- Achdou, Bernardi, Coquel '03:

$$h_\sigma^{-\frac{1}{2}} \|[\![\tilde{p}_h]\!]\|_\sigma \leq C \sum_{L;\sigma \in \mathcal{E}_L} \|\nabla(\tilde{p}_h - \varphi)\|_L$$

-

$$\eta_{\mathrm{NC},K}^2 = |||\tilde{p}_h - \mathcal{I}_{\mathrm{Os}}(\tilde{p}_h)|||_K^2 \leq C c_{\mathbf{S},K} \sum_{\sigma \in \tilde{\mathcal{E}}_K} h_\sigma^{-1} \|[\![\tilde{p}_h]\!]\|_\sigma^2$$

$$\leq C c_{\mathbf{S},K} \sum_{L \in \mathcal{T}_K} \|\nabla(p - \tilde{p}_h)\|_L^2 \leq C \frac{c_{\mathbf{S},K}}{c_{\mathbf{S},\mathcal{T}_K}} \sum_{L \in \mathcal{T}_K} |||p - \tilde{p}_h|||_L^2$$

Local efficiency of the estimates

Proof for $\eta_{\mathrm{R},K}$.

- $\|f - \pi_l(f)\|_K = \|f - \nabla \cdot \mathbf{u}_h\|_K \leq C C_{\mathbf{S},K}^{1/2} h_K^{-1} |||\mathbf{u} - \mathbf{u}_h|||_{*,K}$
- element bubble functions
- equivalence of norms on finite-dimensional spaces
- weak solution definition
- Green theorem
- Cauchy–Schwarz inequality
- energy norm definition
- inverse inequality

- residual estimator is always efficient (also for BDM)

Outline

Outline

Comments on the estimates

General comments

- $p \in H^1(\Omega)$, no additional regularity
- no convexity of Ω needed
- no saturation assumption
- no Helmholtz decomposition
- no shape-regularity needed for the upper bounds (only for the efficiency proofs)
- polynomial degree-independent upper bound
- no "monotonicity" hypothesis on inhomogeneities distribution
- the only important tool: optimal Poincaré–Friedrichs and trace inequalities
- holds from diffusion to convection–diffusion–reaction cases

Outline

$L^2(\Omega)$ estimates

Theorem (Estimate for $\tilde{p}_h$ in the $L^2(\Omega)$-norm)

Let p be the weak potential and let $\tilde{p}_h \in W_0^0(\mathcal{T}_h)$ and $\mathbf{u}_h \in \mathbf{H}(\mathrm{div}, \Omega)$ such that $\nabla \cdot \mathbf{u}_h = \pi_l(f)$ be arbitrary. Then

$$\|p - \tilde{p}_h\|^2 \leq \frac{C_{\mathrm{DF}}}{c_{\mathbf{S},\Omega}} \sum_{K \in \mathcal{T}_h} \left\{ \eta_{\mathrm{NC},K}^2 + (\eta_{\mathrm{R},K} + \eta_{\mathrm{DF},K})^2 \right\}.$$

Theorem (Estimate for p_h in the $L^2(\Omega)$-norm)

Let p be the weak potential and let $p_h \in \Phi_h$, $\tilde{p}_h \in W_0^0(\mathcal{T}_h)$, and $\mathbf{u}_h \in \mathbf{H}(\mathrm{div}, \Omega)$ such that $\nabla \cdot \mathbf{u}_h = \pi_l(f)$ be arbitrary. Then

$$\|p - p_h\| \leq \left\{ \frac{C_{\mathrm{DF}}}{c_{\mathbf{S},\Omega}} \sum_{K \in \mathcal{T}_h} \left\{ \eta_{\mathrm{NC},K}^2 + (\eta_{\mathrm{R},K} + \eta_{\mathrm{DF},K})^2 \right\} \right\}^{\frac{1}{2}} + \|\tilde{p}_h - p_h\|.$$

$L^2(\Omega)$ estimates

Theorem (Estimate for $\tilde{p}_h$ in the $L^2(\Omega)$-norm)

Let p be the weak potential and let $\tilde{p}_h \in W_0^0(\mathcal{T}_h)$ and $\mathbf{u}_h \in \mathbf{H}(\mathrm{div}, \Omega)$ such that $\nabla \cdot \mathbf{u}_h = \pi_l(f)$ be arbitrary. Then

$$\|p - \tilde{p}_h\|^2 \leq \frac{C_{\mathrm{DF}}}{c_{\mathbf{S},\Omega}} \sum_{K \in \mathcal{T}_h} \left\{ \eta_{\mathrm{NC},K}^2 + (\eta_{\mathrm{R},K} + \eta_{\mathrm{DF},K})^2 \right\}.$$

Theorem (Estimate for p_h in the $L^2(\Omega)$-norm)

Let p be the weak potential and let $p_h \in \Phi_h$, $\tilde{p}_h \in W_0^0(\mathcal{T}_h)$, and $\mathbf{u}_h \in \mathbf{H}(\mathrm{div}, \Omega)$ such that $\nabla \cdot \mathbf{u}_h = \pi_l(f)$ be arbitrary. Then

$$\|p - p_h\| \leq \left\{ \frac{C_{\mathrm{DF}}}{c_{\mathbf{S},\Omega}} \sum_{K \in \mathcal{T}_h} \left\{ \eta_{\mathrm{NC},K}^2 + (\eta_{\mathrm{R},K} + \eta_{\mathrm{DF},K})^2 \right\} \right\}^{\frac{1}{2}} + \|\tilde{p}_h - p_h\|.$$

Some additional comments

Some additional comments

- We believe that $L^2(\Omega)$ norm is not optimal for a posteriori error estimates in mixed finite elements.
- We believe that trying to directly and only derive estimates for p_h in the $L^2(\Omega)$-norm was the bottleneck of a lot of previous works.
- $|||\mathbf{u}_h + \mathbf{S}\nabla\tilde{p}_h|||_{*,K}$ or $|||\mathbf{u}_h + \mathbf{S}\nabla(\mathcal{I}_{\mathrm{Os}}(\tilde{p}_h))|||_{*,K}$ (our estimates): clear physical meaning
- $h_K|||\mathbf{u}_h + \mathbf{S}\nabla p_h|||_{*,K} = h_K|||\mathbf{u}_h|||_{*,K}$ in RT_0 (some previous works): no good sense

Outline

Pure diffusion problem $-\nabla \cdot (\mathbf{S}\nabla p) = f$, $p = 0$ on $\partial\Omega$

Theorem (Mixed FEM for the diffusion problem)

There holds

$$|||p - \tilde{p}_h||| \le \inf_{s \in H_0^1(\Omega)} |||\tilde{p}_h - s||| + \left\{ \sum_{K \in \mathcal{T}_h} C_{\mathrm{P}} \frac{h_K^2}{c_{\mathbf{S},K}} \|f - f_K\|_K^2 \right\}^{\frac{1}{2}}.$$

Theorem (Galerkin FEM for the diffusion problem)

There holds $|||p - p_h||| \le \inf_{s_h \in V_h} |||p - s_h|||.$

Mixed FEM 1D:

- no nonconformity, $\tilde{p}_h \in H_0^1(\Omega)$
- $|||p - \tilde{p}_h||| \le Ch^2$ when $f \in H^1(\mathcal{T}_h)$
- $\tilde{p}_h = p$, the exact solution, for pw constant $\mathbf{S}$ (arbitrary inhomogeneities) and pw constant f

Galerkin FEM 1D:

- $|||p - \tilde{p}_h||| \le Ch$

Pure diffusion problem $-\nabla \cdot (\mathbf{S}\nabla p) = f$, $p = 0$ on $\partial\Omega$

Theorem (Mixed FEM for the diffusion problem)

There holds

$$|||p - \tilde{p}_h||| \leq \inf_{s \in H_0^1(\Omega)} |||\tilde{p}_h - s||| + \left\{ \sum_{K \in \mathcal{T}_h} C_{\mathrm{P}} \frac{h_K^2}{c_{\mathbf{S},K}} \|f - f_K\|_K^2 \right\}^{\frac{1}{2}}.$$

Theorem (Galerkin FEM for the diffusion problem)

There holds $$|||p - p_h||| \leq \inf_{s_h \in V_h} |||p - s_h|||.$$

Mixed FEM 1D:

- no nonconformity, $\tilde{p}_h \in H_0^1(\Omega)$
- $|||p - \tilde{p}_h||| \leq Ch^2$ when $f \in H^1(\mathcal{T}_h)$
- $\tilde{p}_h = p$, the exact solution, for pw constant $\mathbf{S}$ (arbitrary inhomogeneities) and pw constant f

Galerkin FEM 1D:

- $|||p - \tilde{p}_h||| \leq Ch$

Pure diffusion problem $-\nabla \cdot (\mathbf{S}\nabla p) = f$, $p = 0$ on $\partial\Omega$

Theorem (Mixed FEM for the diffusion problem)

There holds

$$|||p - \tilde{p}_h||| \leq \inf_{s \in H_0^1(\Omega)} |||\tilde{p}_h - s||| + \left\{ \sum_{K \in \mathcal{T}_h} C_{\mathrm{P}} \frac{h_K^2}{c_{\mathbf{S},K}} \|f - f_K\|_K^2 \right\}^{\frac{1}{2}}.$$

Theorem (Galerkin FEM for the diffusion problem)

There holds $|||p - p_h||| \leq \inf_{s_h \in V_h} |||p - s_h|||.$

Mixed FEM 1D:

- no nonconformity, $\tilde{p}_h \in H_0^1(\Omega)$
- $|||p - \tilde{p}_h||| \leq Ch^2$ when $f \in H^1(\mathcal{T}_h)$
- $\tilde{p}_h = p$, the exact solution, for pw constant $\mathbf{S}$ (arbitrary inhomogeneities) and pw constant f

Galerkin FEM 1D:

- $|||p - \tilde{p}_h||| \leq Ch$

Pure diffusion problem $-\nabla \cdot (\mathbf{S}\nabla p) = f$, $p = 0$ on $\partial\Omega$

Theorem (Mixed FEM for the diffusion problem)

There holds

$$|||p - \tilde{p}_h||| \le \inf_{s \in H^1_0(\Omega)} |||\tilde{p}_h - s||| + \left\{ \sum_{K \in \mathcal{T}_h} C_{\mathrm{P}} \frac{h_K^2}{c_{\mathbf{S},K}} \|f - f_K\|_K^2 \right\}^{\frac{1}{2}}.$$

Theorem (Galerkin FEM for the diffusion problem)

There holds $$|||p - p_h||| \le \inf_{s_h \in V_h} |||p - s_h|||.$$

Mixed FEM 1D:

- no nonconformity, $\tilde{p}_h \in H^1_0(\Omega)$
- $|||p - \tilde{p}_h||| \le Ch^2$ when $f \in H^1(\mathcal{T}_h)$
- $\tilde{p}_h = p$, the exact solution, for pw constant $\mathbf{S}$ (arbitrary inhomogeneities) and pw constant f

Galerkin FEM 1D:

- $|||p - \tilde{p}_h||| \le Ch$

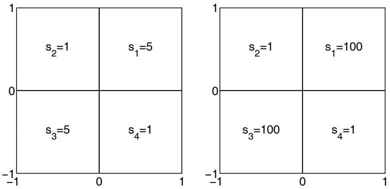
Outline

Problem with discontinuous and inhomogeneous diffusion tensor

- consider the pure diffusion equation

$$-\nabla \cdot (\mathbf{S}\nabla p) = 0 \quad \text{in} \quad \Omega = (-1,1) \times (-1,1)$$

- discontinuous and inhomogeneous **S**, two cases:

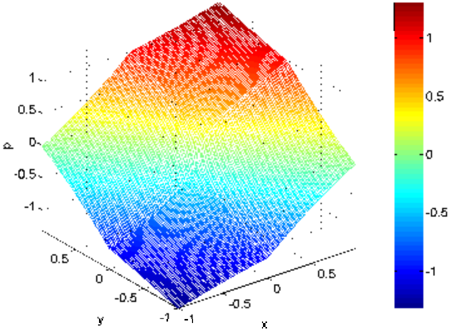


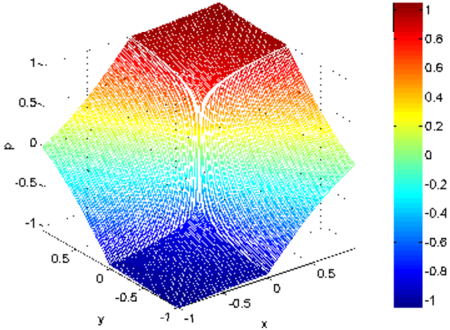
- analytical solution: singularity at the origin

$$p(r,\theta)|_{\Omega_i} = r^{\alpha}(a_i \sin(\alpha\theta) + b_i \cos(\alpha\theta))$$

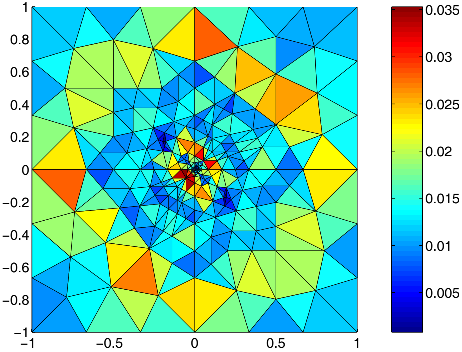
  - (r,θ) polar coordinates in Ω
  - a_i, b_i constants depending on Ω_i
  - α regularity of the solution

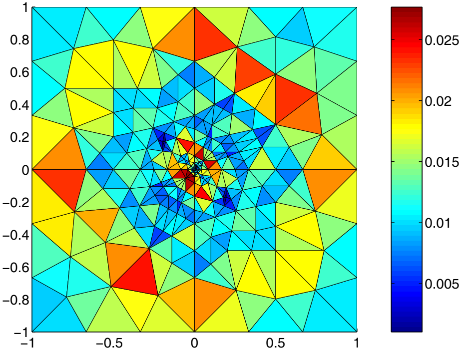
Analytical solutions



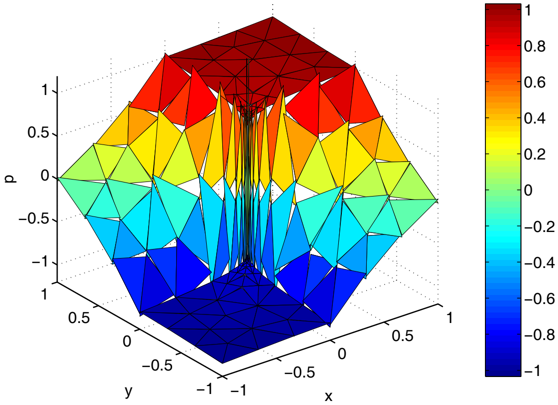


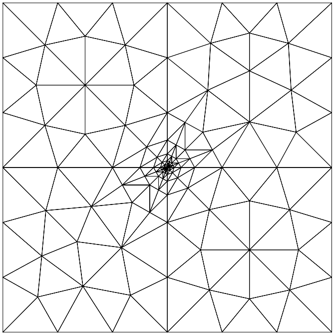
Estimated and actual error distribution on an adaptively refined mesh, case 1



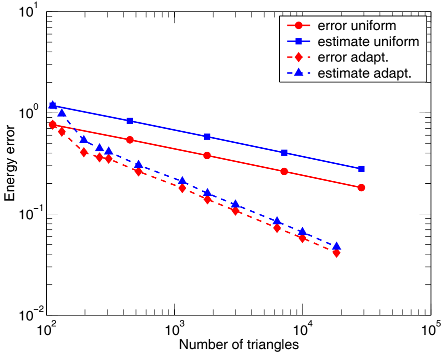


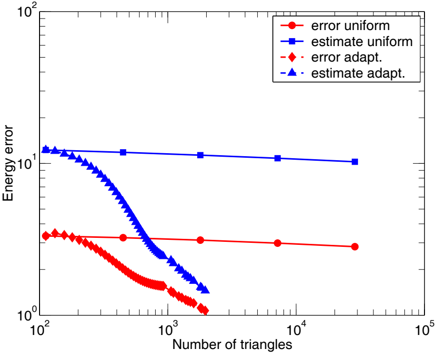
Approximate solution and the corresponding adaptively refined mesh, case 2



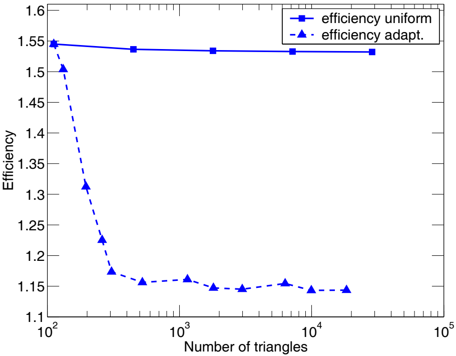


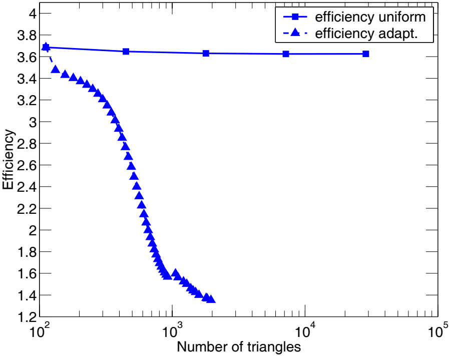
Estimated and actual error against the number of elements in uniformly/adaptively refined meshes





Global efficiency of the estimates





Convection-dominated problem

- consider the convection–diffusion–reaction equation

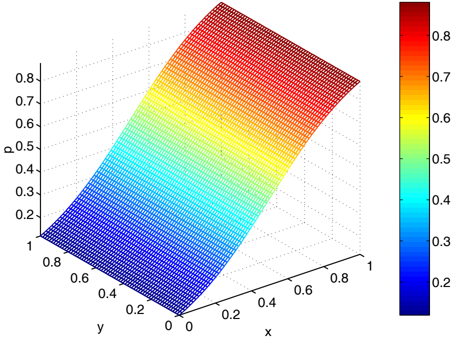
$$-\varepsilon \triangle p + \nabla \cdot (p(0,1)) + p = f \quad \text{in} \quad \Omega = (0,1) \times (0,1)$$

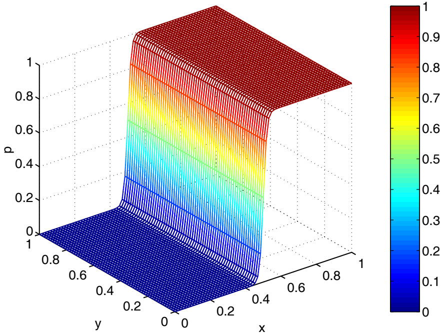
- analytical solution: layer of width a

$$p(x,y) = 0.5 \left(1 - \tanh\left(\frac{0.5 - x}{a}\right)\right)$$

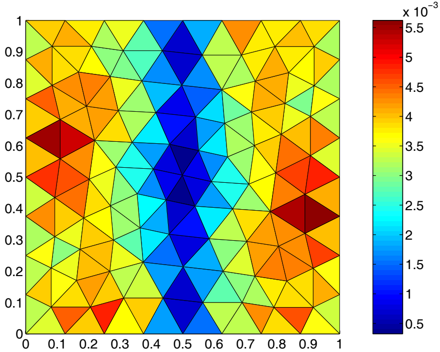
- consider
  - $\varepsilon = 1$, $a = 0.5$
  - $\varepsilon = 10^{-2}$, $a = 0.05$
  - $\varepsilon = 10^{-4}$, $a = 0.02$
- unstructured grid of 46 elements given, uniformly/adaptively refined

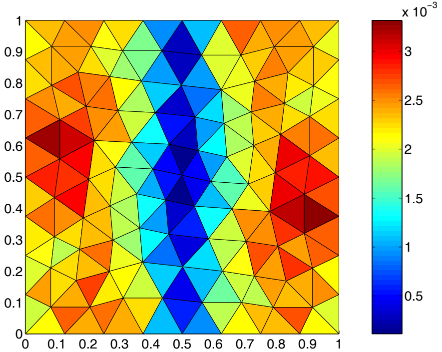
Analytical solutions, $\varepsilon = 1$, $a = 0.5$ and $\varepsilon = 10^{-4}$, $a = 0.02$



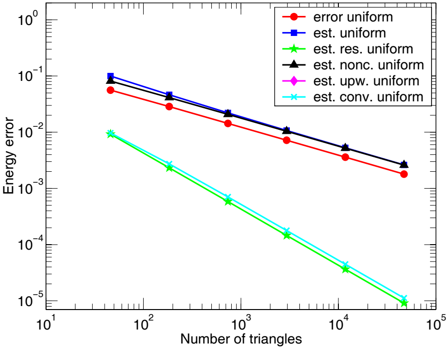


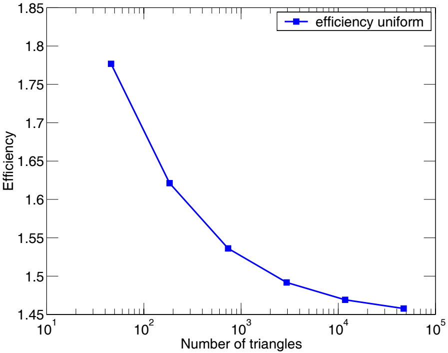
Estimated and actual error distribution, $\varepsilon = 1$, $a = 0.5$



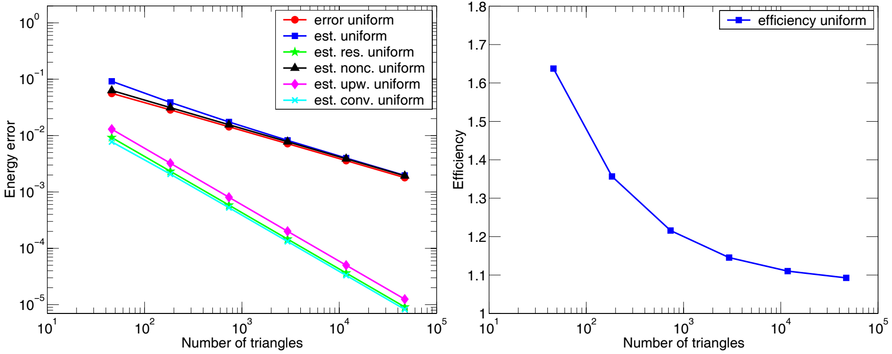


Modified Oswald interpolate: estimated and actual error against the number of elements and global efficiency of the estimates, $\varepsilon = 1$, $a = 0.5$

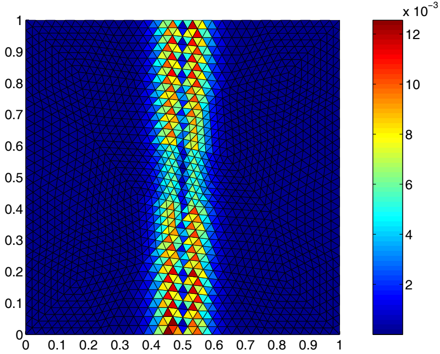


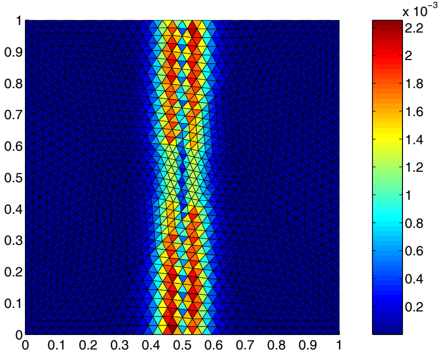


Oswald interpolate: estimated and actual error against the number of elements and global efficiency of the estimates, $\varepsilon = 1$, $a = 0.5$

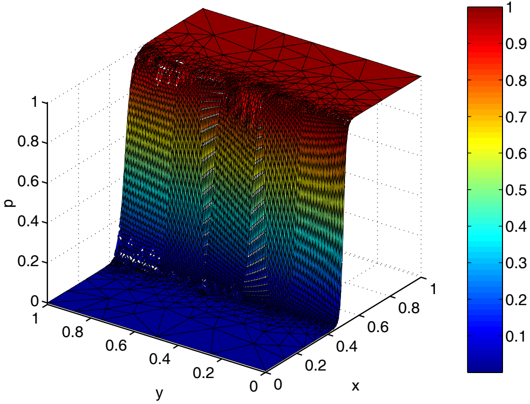


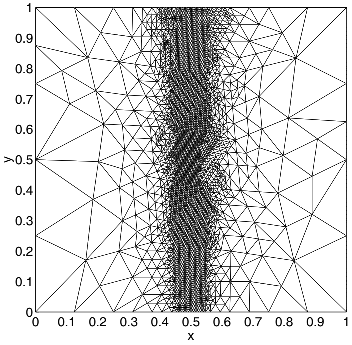
Estimated and actual error distribution, $\varepsilon = 10^{-2}$, $a = 0.05$



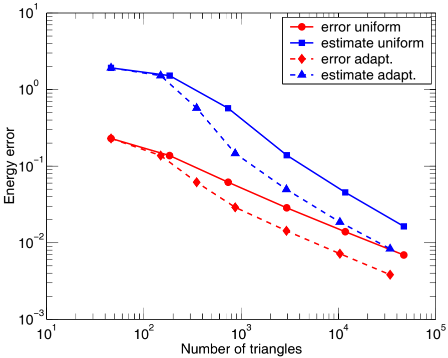


Approximate solution and the corresponding adaptively refined mesh, $\varepsilon = 10^{-4}$, $a = 0.02$

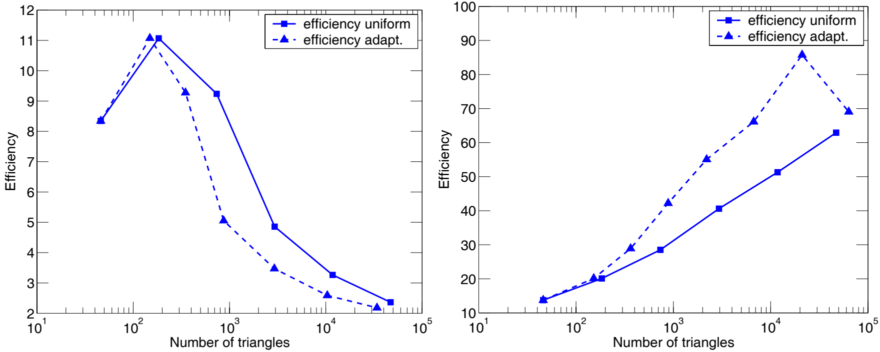




Estimated and actual error against the number of elements in uniformly/adaptively refined meshes, $\varepsilon = 10^{-2}$, $a = 0.05$ and $\varepsilon = 10^{-4}$, $a = 0.02$



Global efficiency of the estimates, $\varepsilon = 10^{-2}$, $a = 0.05$ and $\varepsilon = 10^{-4}$, $a = 0.02$



Outline

Conclusions and future work

Conclusions

- unified framework for a priori and a posteriori error control in mixed finite elements
- optimality of the framework for a posteriori error estimation: guaranteed upper bound, local efficiency, asymptotic exactness, robustness, negligible evaluation cost
- directly implementable—all constants evaluated
- parallel work for finite volumes, discontinuous Galerkin finite elements, and continuous finite elements

Future work

- full asymptotic exactness and robustness
- nonlinear (degenerate) cases
- extensions to other types of problems (Stokes, Navier–Lamé, Maxwell)
- systems of equations

Conclusions and future work

Conclusions

- unified framework for a priori and a posteriori error control in mixed finite elements
- optimality of the framework for a posteriori error estimation: guaranteed upper bound, local efficiency, asymptotic exactness, robustness, negligible evaluation cost
- directly implementable—all constants evaluated
- parallel work for finite volumes, discontinuous Galerkin finite elements, and continuous finite elements

Future work

- full asymptotic exactness and robustness
- nonlinear (degenerate) cases
- extensions to other types of problems (Stokes, Navier–Lamé, Maxwell)
- systems of equations

Bibliography

Papers

- VOHRALÍK M., Unified primal formulation-based a priori and a posteriori error analysis of mixed finite element methods, to be submitted.
- VOHRALÍK M., A posteriori error estimates for lowest-order mixed finite element discretizations of convection–diffusion–reaction equations, *SIAM J. Numer. Anal.* **45** (2007), 1570–1599.
- VOHRALÍK M., Residual flux-based a posteriori error estimates for finite volume discretizations of inhomogeneous, anisotropic, and convection-dominated problems, submitted to *Numer. Math.*
- ERN A., STEPHANSEN, A. F., VOHRALÍK M., Improved energy norm a posteriori error estimation based on flux reconstruction for discontinuous Galerkin methods, submitted to *SIAM J. Numer. Anal*.
- VOHRALÍK M., Guaranteed and fully robust a posteriori error estimates for conforming discretizations of diffusion problems with discontinuous coefficients, submitted to *Math. Comp*.

Thank you for your attention!