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Unified Theory of Zeta-Functions Allied to Epstein Zeta-Functions and Associated with Maass Forms

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Abstract: In this paper, we shall establish a hierarchy of functional equations (as a G-function hierarchy) by unifying zeta-functions that satisfy the Hecke functional equation and those corresponding to Maass forms in the framework of the ramified functional equation with (essentially) two gamma factors through the Fourier–Whittaker expansion. This unifies the theory of Epstein zeta-functions and zeta-functions associated to Maass forms and in a sense gives a method of construction of Maass forms. In the long term, this is a remote consequence of generalizing to an arithmetic progression through perturbed Dirichlet series.

Keywords: Fourier–Whittaker expansion; Chowla–Selberg integral formula; Maass forms; Epstein zeta-function; ramified functional equation

MSC: 11R45; 11N41; 11F30

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1. Introduction

As is well known ([1]), Hecke [2,3] made a revolution in the theory of automorphic forms in pursuit of a generalization of Hamburger's theorem [4], cf. Lemma 7. The established correspondence between automorphic forms and zeta-functions with a functional equation is referred to as the Riemann–Hecke–Bochner correspondence, or RHB correspondence [5,6].

Both [7,8] are devoted to a generalization of Hecke's work, the RHB correspondence, but their contents are rather different. The work in [7] is devoted to the study of Eisenstein–Maass series $E(\tau, s)$ in (9) (analytic continuation thereof, cf. ([9], p. 4)), while Maass lecture notes [8] contain results on the theory of Hecke Eisenstein series $G(\tau, \bar{\tau}; a, b)$ in (44).

Maass theory has created a new research horizon, and there are enormous numbers of works related to it. However, [10] seems to have shown lately that the Maass theory of real analytic automorphic forms and the Epstein zeta-functions are closely related and that the latter may be accommodated in the former. The link is the Fourier–Whittaker expansion, which is perceived as the Chowla–Selberg integral formula and has been accommodated in the Fourier–Bessel expansion for zeta-functions satisfying the Hecke-type functional equation. Another link is the Epstein-type Eisenstein series $\zeta_{\mathbb{Z}^2}(s, \tau)$ in (10), which gives rise to the Fourier expansion for real-analytic Eisenstein series by multiplying by y^s. The underlying unifying principle is the ramified functional equation, which is not in RHB correspondence with the modular relation (Fourier series) but with the Fourier–Whittaker expansion. In addition, it reduces to an unramified one for Hecke-type zeta-functions and is in RHB correspondence with the theta transformation.

Our objective is to unify the theories of zeta-functions allied to Epstein zeta-functions with positive definite quadratic forms and those of zeta-functions associated to real-analytic automorphic forms (including holomorphic modular forms etc.) in the framework of the RHB correspondence between the ramified functional equation and the Fourier–Whittaker expansion.

For this, we show that the following theorem governs both the theory of Epstein zeta-functions and that of zeta-functions associated with real-analytic Maass forms as symbolized by the extended G-function hierarchy (3). This theorem is a rather special case of ([6], Theorem 4.3, pp. 115–119), ([6], Theorem 10.1, p. 269), cf. Appendix A. Throughout, we use the Meijer G-function defined by (20). For notation etc., cf. Definition 1 below. Hopefully, this will liquidate the situation about Maass forms as described in [11].

Theorem 1. *Let $\chi(s)$ be a meromorphic function satisfying the ramified functional equation*

$$\chi(s) = \begin{cases} G^{2,2}_{2,2}\left(2 \,\middle|\, \begin{matrix} 1+\mu-s, \frac{1}{2}+\varkappa \\ \mu, -\mu \end{matrix} \right) \sum_{k=1}^{\infty} \frac{\alpha_k}{\lambda_k^s} \\ \quad + G^{2,2}_{2,2}\left(2 \,\middle|\, \begin{matrix} 1+\mu-s, \frac{1}{2}-\varkappa \\ \mu, -\mu \end{matrix} \right) \sum_{k=1}^{\infty} \frac{\alpha'_k}{\lambda'^{s}_k}, \\ G^{2,2}_{2,2}\left(2 \,\middle|\, \begin{matrix} 1+\mu-s, \frac{1}{2}+\varkappa \\ \mu, -\mu \end{matrix} \right) \sum_{k=1}^{\infty} \frac{\beta_k}{\mu_k^{r-s}} \\ \quad + G^{2,2}_{2,2}\left(2 \,\middle|\, \begin{matrix} 1+\mu-s, \frac{1}{2}-\varkappa \\ \mu, -\mu \end{matrix} \right) \sum_{k=1}^{\infty} \frac{\beta'_k}{\mu'^{r-s}_k}, \end{cases} \tag{1}$$

where the first and second rows are valid for $\sigma > \sigma_a^*$ and $\sigma < r - \sigma_a^*$, respectively. Then, the functional Equation (1) and the modular relation are equivalent:

$$X(z) = \begin{cases} \sum_{k=1}^{\infty} \alpha_k G^{2,1}_{1,2}\left(2z\lambda_k \,\middle|\, \begin{matrix} \frac{1}{2}+\varkappa \\ \mu, -\mu \end{matrix} \right) e^{-z\lambda_k} \\ \quad + \sum_{k=1}^{\infty} \alpha'_k G^{2,1}_{1,2}\left(2z\lambda'_k \,\middle|\, \begin{matrix} \frac{1}{2}-\varkappa \\ \mu, -\mu \end{matrix} \right) e^{-z\lambda'_k}, \\ z^{-r} \sum_{k=1}^{\infty} \beta_k G^{2,1}_{1,2}\left(\frac{2\mu_k}{z} \,\middle|\, \begin{matrix} \frac{1}{2}+\varkappa \\ \mu, -\mu \end{matrix} \right) e^{-\frac{\mu_k}{z}} \\ \quad + z^{-r} \sum_{k=1}^{\infty} \beta'_k G^{2,1}_{1,2}\left(\frac{2\mu'_k}{z} \,\middle|\, \begin{matrix} \frac{1}{2}-\varkappa \\ \mu, -\mu \end{matrix} \right) e^{-\frac{\mu'_k}{z}} + \mathrm{P}(z). \end{cases} \tag{2}$$

Proof. Proof is given in [12], ([6], pp. 217–220) of the most general case of ramified functional equations (cf. Appendix A), which shows that the functional Equation (1) implies the modular relation (2). Since this is an *unprocessed modular relation*, i.e., without the processing gamma factor $\Gamma(w \mid \Delta)$, (A4), the proof hinges on the Mellin transform pair in Lemma 2. $\square$

The G-function hierarchy extended from Kuzumaki [13] reads

$$G^{1,0}_{0,1}\left(z\,\middle|\,\begin{matrix}-\\ \mu\end{matrix}\right)=\sqrt{\frac{2}{\pi}}z^{\mu+\frac{1}{2}}K_{\pm\frac{1}{2}}(z)=z^{\mu}e^{-z} \tag{3}$$
$$\searrow$$
$$G^{2,0}_{0,2}\left(z^2\,\middle|\,\begin{matrix}-\\ \frac{\mu}{2},-\frac{\mu}{2}\end{matrix}\right)=2K_{\mu}(2z)=\sqrt{\pi}e^{-2z}(4z)^{\mu}\Psi\left(\mu+\frac{1}{2},\,2\mu+1;\,4z\right)$$
$$=\sqrt{\frac{\pi}{z}}W_{0,\mu}(4z)$$
$$=2\sqrt{\pi}e^{2z}G^{2,0}_{1,2}\left(4z\,\middle|\,\begin{matrix}\frac{1}{2}\\ \mu,\ -\mu\end{matrix}\right)$$
$$=\frac{2\sqrt{\pi}e^{-2z}}{\Gamma\left(\mu+\frac{1}{2}\right)\Gamma\left(-\mu+\frac{1}{2}\right)}G^{2,1}_{1,2}\left(4z\,\middle|\,\begin{matrix}\frac{1}{2}\\ \mu,\ -\mu\end{matrix}\right)$$
$$\searrow$$
$$W_{\varkappa,\mu}(2z)=z^{\mu+\frac{1}{2}}W\left(z;\mu+\frac{1}{2}+\varkappa,\mu+\frac{1}{2}-\varkappa\right)$$
$$=\frac{(2z)^{\frac{1}{2}}}{\Gamma\left(\frac{1}{2}+\mu-\varkappa\right)\Gamma\left(\frac{1}{2}-\mu+\varkappa\right)}G^{2,1}_{1,2}\left(2z\,\middle|\,\begin{matrix}\frac{1}{2}+\varkappa\\ \mu,-\mu\end{matrix}\right)e^{-z}$$
$$\searrow$$

The main formula H,

where the arrow $\searrow$ indicates the uplift.

For notation, we refer to Section 2, especially Definition 1, cf. also Definition 2. We make a convention that the various factors of the G-function in (3) are included in the coefficients α_k etc., e.g., the penultimate one is multiplied by $\frac{\sqrt{2}}{\Gamma\left(\frac{1}{2}-\varkappa+\mu\right)\Gamma\left(\frac{1}{2}-\varkappa-\mu\right)}$ etc. The residual function $\mathrm{P}(z)$, (24), thus may differ at each occurrence and is to be found accordingly.

Under Lemma 2, Theorem 1 leads to a generalization of ([8], Theorem 35, p. 228).

Corollary 1. *The following modular relation (Fourier–Whittaker expansion)*

$$X(z)=\begin{cases}\displaystyle\sum_{k=1}^{\infty}\alpha_k W(z\lambda_k;a,b)+\sum_{k=1}^{\infty}\alpha'_k W(z\lambda'_k;b,a),\\ \displaystyle z^{-r}\sum_{k=1}^{\infty}\beta_k W\left(\frac{\mu_k}{z};a,b\right)+z^{-r}\sum_{k=1}^{\infty}\beta'_k W\left(\frac{\mu'_k}{z};b,a\right)+\mathrm{P}(z).\end{cases} \tag{4}$$

and the ramified functional equation

$$\chi(s)=\begin{cases}\displaystyle\Gamma(s;a,b)\sum_{k=1}^{\infty}\frac{\alpha_k}{\lambda_k^s}+\Gamma(s;b,a)\sum_{k=1}^{\infty}\frac{\alpha'_k}{\lambda_k'^{\,s}},\\ \Gamma(r-s;a,b)\sum_{k=1}^{\infty}\frac{\beta_k}{\mu_k^{r-s}}+\Gamma(r-s;b,a)\sum_{k=1}^{\infty}\frac{\beta'_k}{\mu_k'^{\,r-s}}.\end{cases} \tag{5}$$

are equivalent. Further, they are in RHB correspondence: (non-holomorphic) modular form $\leftrightarrow$ zeta-function. Here,

$$a=\frac{1}{2}+\mu+\varkappa,\quad b=\frac{1}{2}+\mu-\varkappa. \tag{6}$$

Proof. To show the last statement, we show that (4) is the Fourier expansion of the relevant non-holomorphic modular form. This is done in Maass [8], and the statement will be given in Lemma 5 with a simplified proof in Section 3 below. □

We note that the index $\varkappa$ comes from the multiplier system (57). Section 2 is devoted to the elaboration of this corollary.

Corollary 2. *In the case $\varkappa = 0$, $\varphi(s) = \varphi'(s)$ and $\psi(s) = \psi'(s)$, Theorem 1 reads: The unramified functional equation*

$$\chi(s) = \begin{cases} G_{2,2}^{2,2}\left(2 \,\middle|\, \begin{matrix} 1+\mu-s, \frac{1}{2} \\ \mu, -\mu \end{matrix} \right) \sum_{k=1}^{\infty} \frac{\alpha_k}{\lambda_k^s}, \\ G_{2,2}^{2,2}\left(2 \,\middle|\, \begin{matrix} 1+\mu-s, \frac{1}{2}+\varkappa \\ \mu, -\mu \end{matrix} \right) \sum_{k=1}^{\infty} \frac{\beta_k}{\mu_k^{r-s}}, \end{cases} \tag{7}$$

where the first and the second rows are valid for $\sigma > \sigma_a^*$ and $\sigma < r - \sigma_a^*$, respectively, and the following modular relations are equivalent:

$$X(z) = \begin{cases} \sum_{k=1}^{\infty} \alpha_k G_{1,2}^{2,1}\left(2z\lambda_k \,\middle|\, \begin{matrix} \frac{1}{2} \\ \mu, -\mu \end{matrix} \right) e^{-z\lambda_k}, \\ z^{-r} \sum_{k=1}^{\infty} \beta_k G_{1,2}^{2,1}\left(\frac{2\mu_k}{z} \,\middle|\, \begin{matrix} \frac{1}{2} \\ \mu, -\mu \end{matrix} \right) e^{-\frac{\mu_k}{z}} + \mathrm{P}(z). \end{cases} \tag{8}$$

Section 3 is devoted to the elaboration of this corollary.

We denote the upper half-plane $\{\tau = x + iy \in \mathbb{C} | y > 0\}$ by $\mathcal{H}$. Let Γ denote the Fuchsian group of the first kind, typically $\Gamma = \Gamma(1) = \mathrm{PSL}_2(\mathbb{Z})$ acting on $\mathcal{H}$ through linear fractional transformations. In what follows, we write $\tau = x + iy \in \mathcal{H}$ and $s = \sigma + it$ for the complex variable.

In most of the literature on the spectral theory of automorphic forms, the *Eisenstein–Maass series* is used. Γ is assumed to have only one cusp at ∞, and it is defined for $\sigma > 1$ by [14,15], ([16], p.20), etc.

$$E(\tau, s) = \sum_{\gamma \in \Gamma_\infty \backslash \Gamma} y^s(\gamma\tau), \quad y(\tau) = \mathrm{Im}\,\tau, \tag{9}$$

where Γ_∞ is the stabilizer of ∞. $E(\tau, s)$ was first introduced by [7] for $\Gamma = \Gamma(1)$ (cf. (14) for a more general case).

The *Epstein-type Eisenstein series* $\zeta_{\mathbb{Z}^2}(s, \tau)$ is defined by

$$\zeta_{\mathbb{Z}^2}(s, \tau) = \sum_{m,n=-\infty}^{\infty}{}' |m\tau + n|^{-2s} \tag{10}$$

where $\mathrm{Re}\, s = \sigma > 1$, $\tau \in \mathcal{H}$ and the prime on the summation sign means that $m = n = 0$ is excluded (cf., e.g., [13,17] etc.).

It follows that

$$E(\tau, s) = y^s \sum_{\substack{c,d=-\infty \\ (c,d)=1}}^{\infty}{}' |c\tau + d|^{-2s} = \frac{y^s}{\zeta(2s)} \zeta_{\mathbb{Z}^2}(s, \tau), \tag{11}$$

where $\zeta(s)$ denotes the Riemann zeta-function.

The first equality is due to the following. If $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z})$, $\det\gamma = ad - bc = 1$, then the y-part is

$$y(\gamma\tau) = \frac{y}{|c\tau + d|^2}. \tag{12}$$

Note that the condition $(c, d) = 1$ is involved. The second equality follows from

Lemma 1. *Suppose $h(m, n) \neq 0$ save for the case $m = n = 0$ that it is a homogeneous function $\ell h(m, n) = h(\ell m, \ell n)$ and that the series*

$$H(s) = \sum_{m,n=-\infty}^{\infty}{}' h(m, n)^{-s}$$

is absolutely convergent for $\sigma > 1$. Then,

$$\frac{1}{\zeta(s)} \sum_{m,n=-\infty}^{\infty}{}' h(m, n)^{-s} = \sum_{\substack{c,d=-\infty \\ (c,d)=1}}^{\infty}{}' h(c, d)^{-s}$$

Proof. Let $\mu(n)$ be the Möbius function defined by $\frac{1}{\zeta(s)} = \sum_{n=1}^{\infty} \frac{\mu(n)}{n^s}$, $\sigma > 1$. Then,

$$\begin{aligned} \frac{1}{\zeta(s)} H(s) &= \sum_{\ell,m,n=-\infty}^{\infty}{}' \mu(\ell)(h(\ell m, \ell n))^{-s} = \sum_{m,n=-\infty}^{\infty}{}' \sum_{\ell|(m,n)} \mu(\ell) h(m, n)^{-s} \\ &= \sum_{\substack{c,d=-\infty \\ (c,d)=1}}^{\infty}{}' h(c, d)^{-s}, \end{aligned} \tag{13}$$

where the last equality follows from the Möbius inversion formula to the effect that $\sum_{d|n} \mu(d) = 1$ if and only if $n = 1$. □

For a positive integer N and integers μ, ν, Maass ([7], (106), p. 162) introduces the *Eisenstein series of Stufe N* (in slightly different notation)

$$E(\tau, s; \mu, \nu, N) = \sum_{\substack{m \equiv \mu \bmod N \\ n \equiv \nu \bmod N}}{}' y^s |m\tau + n|^{-2s}, \tag{14}$$

where the prime on the summation sign means the omission of the case $(m, n) = (0, 0)$ and (11) is a special case of this with $N = 1$ and with the factor $\frac{1}{\zeta(2s)}$. The Fourier expansion is given as ([7], (112), p. 163) (which corresponds to the q-expansion). This is a basis of analytic continuation and the functional equation as stated in [18,19], etc. See (45) for the kth Hecke Eisenstein series $E_k(\tau, s; \mu, \nu; N)$.

In the same year, Part I of [20] appeared, and its complete form [21] was published in 1967, both of which contain the Chowla–Selberg integral formula. It is the Fourier–Bessel expansion equivalent to the functional equation for the Epstein zeta-function. However, Maass theory and the Chowla–Selberg integral formula have been thought of as independent of each other.

We note, however, that (11) implies *the non-holomorphic Eisenstein–Maass series is the y^s times the Epstein-type Eisenstein series $\frac{1}{\zeta(2s)} \zeta_{\mathbb{Z}^2}(s, \tau)$.*

A year following the release of [7], Bellman published [22] and derived many illustrative non-holomorphic automorphic functions by using the Hardy transform [10] in which it is revealed that Maass' method of using the DE is one of the approaches leading to the Fourier–Whittaker expansion.

Proposition 1. *The Epstein-type Eisenstein series and the Epstein zeta-function associated with a positive definite binary quadratic form defined by* (18) *are essentially the same:*

$$\zeta_{\mathbb{Z}^2}(s,\tau) = Z(s,Q_1), \quad Q_1(m,n) = (x^2+y^2)n^2 + 2xmn + m^2 \tag{15}$$
$$Z(s,Q) = a^{-s}\zeta_{\mathbb{Z}^2}(s,\tau),$$

where

$$Q = Q(m,n) = am^2 + 2bmn + cn^2 \tag{16}$$

is a positive definite binary quadratic form, so that $a, c > 0$ *and the discriminant*

$$\Delta = 4(b^2 - ac) < 0 \tag{17}$$

and where the Epstein zeta-function $Z(s,Q)$ *(associated with* Q*) is defined by*

$$Z(s,Q) = \sum_{m,n=-\infty}^{\infty}{}' \frac{1}{Q(m,n)^s} = \sum_{m,n}{}' \frac{1}{(am^2+2bmn+cn^2)^s} \tag{18}$$

for $\sigma > 1$.

See the works of Siegel [23,24], Terras [25], Kanemitsu and Tsukada [6] etc. For the Epstein zeta-function, there are many results known, and the most remarkable one is the Chowla–Selberg integral formula, which is nothing other than the Fourier expansion (in x), and so the theory of Fourier–Bessel expansion ([6], Chapter 4) is immediately translated to that of the non-holomorphic Eisenstein series as has been done in [10]. Zhang and Williams [17] *reduce the Epstein zeta-function to an Epstein-type Eisenstein series* and apply the Poisson summation formula to deduce the Chowla–Selberg integral formula while [13] applied the beta-transform to derive the same. For Epstein zeta-functions with characters, we refer to [26] and references therein.

2. Maass Forms and Rudiments

As mentioned above, the Maass lecture notes ([8], pp. 160–217) contain some results on the non-holomorphic Eisenstein series (44) below.

One of the main results is ([8], Theorem 35, p. 228), whose main body is the RHB correspondence between the Fourier–Whittaker expansion (of real-analytic forms) and associated Dirichlet series. Its special case $b = 0, a_0 = 0$—analytic modular forms—is stated on ([8], p. 240), which amounts to Hecke's original form of the functional equation. One missing point here is that *in the real-analytic case* $a = b = s$, $G(\tau,\bar{\tau};s,s)$ *in* (44) *is the Epstein zeta-function analytic in* s.

Research on (44) has been done by several authors including Knopp [5] and Pasles [27], etc.

The relation does not seem to be studied between two non-holomorphic Eisenstein series (44) and (10). *Our aim is to clarify the relation and show that* (44) *entails the Epstein zeta function case, which gives rise to the Fourier expansion of the Eisenstein–Maass series.*

We make frequent use of the G-function according to the notation of [6,28,29], which has a slight difference from that of Erdélyi [30] in that z is to read z^{-1}.

The Fox H-function is defined for $(0 \le n \le p, 0 \le m \le q, A_j, B_j > 0)$ by

$$H^{m,n}_{p,q}\left(z \,\middle|\, \begin{matrix} (1-a_1,A_1),\ldots,(1-a_n,A_n),(a_{n+1},A_{n+1}),\ldots,(a_p,A_p) \\ (b_1,B_1),\ldots,(b_m,B_m),(1-b_{m+1},B_{m+1}),\ldots,(1-b_q,B_q) \end{matrix}\right)$$
$$= \frac{1}{2\pi i}\int_L \frac{\prod_{j=1}^{m}\Gamma(b_j+B_js)\prod_{j=1}^{n}\Gamma(a_j-A_js)}{\prod_{j=n+1}^{p}\Gamma(a_j+A_js)\prod_{m+1}^{q}\Gamma(b_j-B_js)} z^{-s}\mathrm{d}s, \tag{19}$$

under a certain convergence condition, where L is a (deformed) Bromwich contour $\sigma = c$. The Meijer G-functions is a special case with $A_j = 1, B_j = 1$:

$$\begin{aligned} & G_{p,q}^{m,n}\left(z\middle|\begin{array}{c} a_1,\ldots,a_n,a_{n+1},\ldots,a_p \\ b_1,\ldots,b_m,b_{m+1},\ldots,b_q \end{array}\right) \\ &= H_{p,q}^{m,n}\left(z\middle|\begin{array}{c} (a_1,1),\ldots,(a_n,1),(a_{n+1},1),\ldots,(a_p,1) \\ (b_1,1),\ldots,(b_m,1),(b_{m+1},1),\ldots,(b_q,1) \end{array}\right) \\ &= \frac{1}{2\pi i}\int_L \frac{\prod_{j=1}^{m}\Gamma(b_j+s)\prod_{j=1}^{n}\Gamma(1-a_j-s)}{\prod_{j=n+1}^{p}\Gamma(a_j+s)\prod_{m+1}^{q}\Gamma(1-b_j-s)} z^{-s}\mathrm{d}s. \end{aligned} \tag{20}$$

The integrals are absolutely convergent if $m+n > \frac{1}{2}(p+q)$—a condition that is satisfied in all the cases appearing below.

Maass theory depends essentially on the Whittaker function as a generalization of the *K-Bessel* function. Here, we employ the G-function expressions.

In ([30], p. 216), Equation (6) reads

$$G_{1,2}^{2,0}\left(x\middle|\begin{array}{c} a \\ b,\ c \end{array}\right) = x^{\frac{1}{2}(b+c-1)}\, e^{-\frac{1}{2}x}\, W_{\kappa,\mu}(x), \tag{21}$$

where $\kappa = \frac{1}{2}(b+c+1) - a,\ \mu = \frac{1}{2}(b-c)$ ([30], p. 216, (8))

$$G_{1,2}^{2,1}\left(x\middle|\begin{array}{c} a \\ b,\ c \end{array}\right) = \Gamma(b-a+1)\Gamma(c-a+1)x^{\frac{1}{2}(b+c-1)}\, e^{\frac{1}{2}x}\, W_{\kappa,\mu}(x), \tag{22}$$

where $\kappa = a - \frac{1}{2}(b+c+1),\ \mu = \frac{1}{2}(b-c)$.

It is indicated in ([6], pp. 269–270) that ([8], Theorem 35) can be interpreted as a modular relation for zeta-functions satisfying the ramified functional equation. Our aim in this section is to show how Maass' Theorem 35 may be accommodated in this framework as Corollary 1 above.

We state a special case of Definition A1 below following [6].

Definition 1. *Suppose $\varphi(s) = \sum_{k=1}^{\infty}\frac{\alpha_k}{\lambda_k^s}$ and $\varphi'(s) = \sum_{k=1}^{\infty}\frac{\alpha'_k}{\lambda_k'^s}$ or $\psi(s) = \sum_{k=1}^{\infty}\frac{\beta_k}{\mu_k^s}$ and $\psi'(s) = \sum_{k=1}^{\infty}\frac{\beta_k}{\mu_k'^s}$ are absolutely convergent for $\sigma > \sigma_a^*$ resp. $\sigma < r - \sigma_a^*$, where r is a real number. We say that $\varphi, \varphi', \psi, \psi'$ satisfy the ramified functional Equation (1) if there exists a meromorphic function $\chi(s)$ that is convex in the sense that $\lim_{|v|\to\infty}\chi(u+iv) = 0$ in any finite interval and such that*

$$\chi(s) = \begin{cases} G_{2,2}^{2,2}\left(\frac{1}{2}\middle|\begin{array}{c} 1, a+b \\ s, b \end{array}\right)\varphi(s) + G_{2,2}^{2,2}\left(\frac{1}{2}\middle|\begin{array}{c} 1, a+b \\ s, a \end{array}\right)\varphi'(s), \\ G_{2,2}^{2,2}\left(\frac{1}{2}\middle|\begin{array}{c} 1, a+b \\ r-s, b \end{array}\right)\psi(r-s) + G_{2,2}^{2,2}\left(\frac{1}{2}\middle|\begin{array}{c} 1, a+b \\ r-s, a \end{array}\right)\psi'(r-s). \end{cases} \tag{23}$$

Further, suppose that all the poles of $\chi(s)$ lie in a compact set $\mathcal{S}$ with a boundary $\mathcal{C}$ contained in the strip $r - \sigma_a^* < \sigma < \sigma_a^*$. Let

$$\mathrm{P}(z) = \frac{1}{2\pi i}\int_{\mathcal{C}}\chi(w)\, z^{-w}\mathrm{d}w \tag{24}$$

be the residual function (the sum of all residues) and let

$$X(z) = \frac{1}{2\pi i}\int_{L_1} \chi(w)\, z^{-w} \mathrm{d}w \tag{25}$$

be the key-function, where L_1 is an indented Bromwitch path such that all poles of $\Gamma(-w)\Gamma(1-a-b-w)$ lie on the right or all poles of $\Gamma(s+w)\Gamma(b+w)$ and $\mathcal{S}$ lie to the left of it.

Maass introduced the Gamma factor as the Mellin transform of a Whittaker function ([8], (1), p. 218)

$$\Gamma(s;a,b) = \int_0^\infty x^s W(x;a,b)\,\frac{\mathrm{d}x}{x}, \tag{26}$$

where W is the modified Whittaker function ([8], (1), p. 182)

$$W(\varepsilon y;a,b) = y^{-\frac{1}{2}(a+b)} W_{\varkappa,\mu}(2y), \quad \varkappa = \varepsilon\frac{1}{2}(a-b), \mu = \frac{1}{2}(a+b-1), \tag{27}$$

where $y > 0$.

Maass notes ([8], p. 213) that

$$W(-x;a,b) = W(x;b,a). \tag{28}$$

This is a consequence of the formula ([30], I, p. 265, (8))

$$W_{\varkappa,\mu}(x) = W_{\varkappa,-\mu}(x), \tag{29}$$

which in turn is a consequence of the transformation formula ([30], I, p. 257, (6))

$$\Psi(a,c;x) = x^{1-c}\Psi(a-c+1, 2-c; x). \tag{30}$$

We note that (28) *necessitates the division of the Fourier series in Lemma 5 into two parts, positive and negative, yielding the ramified modular relation entailing the ramified functional equation in Corollary 1.*

In ([10], Lemma 3) the G-function expression for $\Gamma(s;a,b)$ respectively $W(z;a,b)$ is

$$\Gamma(s;a,b) = \frac{2^{\frac{1}{2}(a+b)}}{\Gamma(b)\Gamma(1-a)} G^{2,2}_{2,2}\left(\frac{1}{2} \,\middle|\, \begin{matrix} 1, a+b \\ s, b \end{matrix} \right) = \frac{2^{\frac{1}{2}(a+b)}}{\Gamma(b)\Gamma(1-a)} G^{2,2}_{2,2}\left(2 \,\middle|\, \begin{matrix} 1-s, 1-b \\ 0, 1-a-b \end{matrix} \right)$$

or

$$W(z;a,b) = \frac{2^{\frac{1}{2}(a+b)}}{\Gamma(b)\Gamma(1-a)} G^{2,1}_{1,2}\left(2z \,\middle|\, \begin{matrix} 1-b \\ 0, 1-a-b \end{matrix} \right) e^{-z}.$$

These are not in a format that is compatible with the hierarchy. This is because (22) is applied to deduce (33). It is (31) that completes the hierarchy ([28], p. 714, Formula 3).

$$\begin{aligned} W_{\varkappa,\mu}(2z) &= \frac{1}{\Gamma\left(\frac{1}{2}-\varkappa-\mu\right)\Gamma\left(\frac{1}{2}-\varkappa+\mu\right)} G^{2,1}_{1,2}\left(2z \,\middle|\, \begin{matrix} 1+\varkappa \\ \frac{1}{2}-\mu, \frac{1}{2}+\mu \end{matrix} \right) e^{-z} \\ &= \frac{(2z)^{\frac{1}{2}}}{\Gamma\left(\frac{1}{2}-\varkappa-\mu\right)\Gamma\left(\frac{1}{2}-\varkappa+\mu\right)} G^{2,1}_{1,2}\left(2z \,\middle|\, \begin{matrix} \frac{1}{2}+\varkappa \\ \mu, -\mu \end{matrix} \right) e^{-z} \end{aligned} \tag{31}$$

Lemma 2. *In view of the expressions*

$$\Gamma\left(s;\mu+\frac{1}{2}+\varkappa,\mu+\frac{1}{2}-\varkappa\right) = \frac{\sqrt{2}}{\Gamma\left(\frac{1}{2}-\varkappa+\mu\right)\Gamma\left(\frac{1}{2}-\varkappa-\mu\right)} G^{2,2}_{2,2}\left(2 \,\middle|\, \begin{matrix} 1+\mu-s,\frac{1}{2}+\varkappa \\ \mu,-\mu \end{matrix}\right). \tag{32}$$

and

$$W\left(z;\mu+\frac{1}{2}+\varkappa,\mu+\frac{1}{2}-\varkappa\right) = \frac{\sqrt{2}z^{-\mu}}{\Gamma\left(\frac{1}{2}-\varkappa+\mu\right)\Gamma\left(\frac{1}{2}-\varkappa-\mu\right)} G^{2,1}_{1,2}\left(2z \,\middle|\, \begin{matrix} \frac{1}{2}+\varkappa \\ \mu,-\mu \end{matrix}\right)e^{-z} \tag{33}$$

(26) is a special case of the Mellin transform pair $e^{-x}G^{m,n}_{p,q}(ax|\cdot) \leftrightarrow G^{m,n+1}_{p+1,q}(a|\cdot)$ *([31], (16), p. 338):*

$$\int_0^\infty x^{s-1}e^{-x}G^{m,n}_{p,q}\left(ax \,\middle|\, \begin{matrix} a_1,\cdots,a_p \\ b_1,\cdots,b_q \end{matrix}\right)\mathrm{d}x = G^{m,n+1}_{p+1,q}\left(a \,\middle|\, \begin{matrix} 1-s,a_1,\cdots,a_p \\ b_1,\cdots,b_q \end{matrix}\right). \tag{34}$$

This is proved in ([10], Lemma 3). The works presented in ([8], p. 222) and ([29], p. 82) imply (32), while ([8], p. 222) follows from ([31], p. 337) and ([30], (22), p. 64).

The following lemma reduces the ramified to unramified functional equation in Corollary 2. We appeal to the special case of (27) with $a=b$, which appears in (3):

Lemma 3. ([8], (6), p. 220) *The special case of* (27) *with* $a=b$ *reads*

$$W(x;a,a) = x^{-a}W_{0,a-\frac{1}{2}}(2x) = \sqrt{\frac{2}{\pi}}x^{\frac{1}{2}-a}K_{a-\frac{1}{2}}(x). \tag{35}$$

The Mellin transform of this amounts to ([8], (6), p. 220)

$$\Gamma(s;a,a) = \frac{2^{s-a-1}}{\sqrt{\pi}}\Gamma\left(\frac{s}{2}\right)\Gamma\left(\frac{s}{2}-a+\frac{1}{2}\right). \tag{36}$$

Corollary 3. *Corollary 1 with* $b=a$ *boils down to an analog of ([32], p. 324): The functional equation*

$$\Gamma(2s+\nu;a,a)\varphi(2s+\nu) = \Gamma(r-2s-\nu;a,a)\psi(r-2s-\nu) \tag{37}$$

and the modular relation

$$\sum_{k=1}^\infty \alpha_k K_\nu(2\sqrt{\lambda_k z}) = z^{-r}\sum_{k=1}^\infty \beta_k K_\nu\left(2\sqrt{\frac{\mu_k}{z}}\right) + \mathrm{P}(z) \tag{38}$$

are equivalent.

Equations (22), (43), ([30], (4)–(6), p. 216) and (31) lead to the G-function hierarchy (3). The following lemma is for Section 3.1.

Lemma 4. ([8], (7), p. 220) *The special case of* (27) *with* $b=0$ *reads*

$$W(x;a,0) = 2^{\frac{a}{2}}\Gamma(z). \tag{39}$$

Notation and formulas. The assembled data on special functions that are frequently used throughout. The modified Bessel function of the third kind of index ν, referred to as the K-

Bessel function $K_\nu(z)$, is defined by ([33], (15), p.183). We assume its well-known properties as presented in (43).

The Whittaker function $w = W_{\kappa,\mu}$ is one of two independent solutions to the Whittaker differential equation satisfying the boundary condition that $w(z) \to 0$ as $z \to \infty$ (cf. [30], p. 248, (4)):

$$\frac{d^2w}{dx^2} + \left(-\frac{1}{4} + \frac{\kappa}{x} + \frac{\frac{1}{4} - \mu^2}{x^2}\right) w = 0. \tag{40}$$

In view of the identity (([30], p. 264, (4)), ([28], p. 797))

$$\Psi(a, b; x) = x^{-\frac{1}{2}-\mu}\, e^{\frac{1}{2}x}\, W_{\kappa,\mu}(x), \tag{41}$$

where $\kappa = \frac{1}{2}b - a$, and $\mu = \frac{1}{2}b - \frac{1}{2}$, we may deduce the reduction to the confluent hypergeometric functions $\Psi(a, b; z)$, which may be defined by ($\mathrm{Re}\, a > 0$)

$$\Gamma(a)\Psi(a, b; z) \;=\; \int_0^\infty e^{-zt} t^{a-1} (1+t)^{b-a-1}\, \mathrm{d}t. \tag{42}$$

From ([30], p. 265, (13) and (14)), we also have

$$\begin{aligned} W_{0,\nu}(x) &= \frac{1}{\sqrt{\pi}} x^{\frac{1}{2}} K_\nu\left(\frac{1}{2}x\right) \\ &= e^{-\frac{1}{2}x} x^{\nu+\frac{1}{2}}\, \Psi\left(\nu + \frac{1}{2}, 2\nu + 1;\; x\right), \end{aligned} \tag{43}$$

Equations (22) and (43) lead to the G-function hierarchy (3), cf. [13].

Here, we assemble some concrete examples scattered around in literature with comments.

On ([8], p. 169), the *Hecke Eisenstein series*

$$G(\tau, \bar{\tau}; a, b) = \sum_{(m,n)\neq(0,0)} \frac{1}{(m\tau + n)^a (m\bar{\tau} + n)^b}, \quad \mathrm{Re}(a + b) > 2 \tag{44}$$

is defined for $\tau \in \mathcal{H}$, where $k := a - b$ is an even integer (in Maass, k is replaced by $2k$). The special case with $a = 2$ was studied by Hecke [34] and explicated in ([35], pp. 63–68). It is a prototype of a non-holomorphic modular form for $\{\Gamma(1), a, b, 1\}$; see Example 1. A slightly more general case than (44) was developed by Miyake ([36], pp. 268–293): For a positive integer N and integers μ, ν, the Hecke Eisenstein series $E_k(\tau, s; \mu, \nu; N)$ with congruence conditions is defined ([36], (7.2.52), p. 289) similarly to (14) and

$$E_k(\tau, s; 1, 1; 1) = G(\tau, \bar{\tau}; s + k, s) \tag{45}$$

Both are non-holomorphic modular forms for $\{\Gamma(1), s + k.s, 1\}$. The result is stated for the Eisenstein series with characters ([36], (7.2.1), p. 274) (where it is referred to as the Eisenstein series with parameter s)

$$E_k(\tau, s; \chi, \psi) = \sum_{m,n=-\infty}^{\infty}{}' \chi(m)\psi(n)(m\tau + n)^{-k} |m\tau + n|^{-2s} \tag{46}$$

for Dirichlet characters χ, ψ mod N. The main result is ([36], Theorem 7.2.9, p. 284) giving rise to the Fourier expansion of $E_k(\tau, s; \chi, \psi)$ in terms of confluent hypergeometric functions; see Example 1 below.

There is no mention of Hecke [34] or Maass [8], although they are very closely connected.

For non-negative even integer k, the kth Eisenstein–Maass series is defined by

$$E_k(\tau,s)=\sum_{\gamma\in\Gamma_\infty\backslash\Gamma}(c\tau+d)^{-k}y^s(\gamma\tau) \tag{47}$$

so that $E_0(\tau,s)=E(\tau,s)$. This was studied by Noda [37], whose main results depend on the expression of Fourier coefficients in terms of the confluent hypergeometric functions, which in turn depends on Miyake's results through $E_k(\tau,s)=\frac{y^s}{\zeta(2s+k)}E_k(\tau,s,1,1,1)$.

Similarly, $E_k(\tau,s)$ is also the product of (44) by $\frac{y^s}{\zeta(2s+k)}$:

$$E_k(\tau,s)=\frac{y^s}{\zeta(2s+k)}G(\tau,\bar{\tau},s+k,s). \tag{48}$$

We refer to $E_k(\tau,s)$ as the Hecke–Maass series or as the k-th Eisenstein–Maass series.

It turns out that both Eisenstein series (9) and (46) have Fourier expansions in terms of the x-part on which their functional equations are based. Maass' results ([8], p. 212) (Whittaker function (27)) and Miyake's results ([36], Section 7.2) (confluent hypergeometric function (42)) are directly translated into the case of Hecke–Maass series. Of course, these two functions W and Ψ are nearly the same, cf. (41).

Research on the special case ([8], p. 209)

$$f(\tau,\bar{\tau};a,b)=\sum_{n--\infty}^{\infty}\frac{1}{(\tau+n)^a(\bar{\tau}+n)^b},\quad \mathrm{Re}(a+b)>2$$

of (44) or rather its generalization

$$f(\tau,\bar{\tau};a,b,\mu)=\sum_{n--\infty}^{\infty}\frac{e^{-2\pi i\mu n}}{(\tau+n)^a(\bar{\tau}+n)^b} \tag{49}$$

has been done by several authors including Knopp etc. [5,27,38–40]. It is stated in [27] that (49) first appeared in [5] as a result of John Hawkins. In [27], more general summation formulas are developed with one more factor in the denominator, containing information on other works.

In [41,42], RHB correspondence for non-analytic automorphic integrals on the Hecke group is presented.

We establish the Fourier–Whittaker expansion for the Hecke–Maass functions in Lemma 5, thus identifying the index $\varkappa$ and rendering the RHB correspondence visible with the ramified functional equation in Corollary 1. In the process, we may slightly simplify Maass' treatment to express the Fourier coefficient in terms of the Whittaker function. We may deduce results on $E_k(\tau,s)$ from Maass' results (Example 1) on the Hecke–Eisenstein series (44) and conversely. For the Eisenstein–Maass series and the Epstein-type Eisenstein series, we state Fourier–Bessel expansions for several related zeta-functions by accommodating them in the Fourier–Whittaker expansion by Example 1.

Let Γ be the horocyclic group ([8], p. 185). As on ([8], p. 186), let $\{\Gamma,a,b,v\}$ denote the space of functions $f(\tau,\bar{\tau})$ real-analytic on $\mathcal{H}$ satisfying automorphy:

$$f(\tau,\bar{\tau})|_\gamma=v(\gamma)f(\gamma\tau,\gamma\bar{\tau}),\gamma\in\Gamma, \tag{50}$$

with v being a multiplier system for Γ of dimension $-r$, the DE (with the operator in (53))

$$\Omega_{ab}f(\tau,\bar{\tau})=0, \tag{51}$$

and the growth condition

$$f(\tau,\bar{\tau})|_{\alpha^{-1}}=O(y^K),\quad K>0,\quad y\to\infty, \tag{52}$$

uniformly in x provided that $\alpha^{-1}\infty$ is a parabolic cusp of Γ.

In the above, for a concrete example. cf. (57) and

$$\Omega_{ab} = \Delta + i(a-b)y\frac{\partial}{\partial x} - (a+b)y\frac{\partial}{\partial y} = \Delta + iry\frac{\partial}{\partial x} - qy\frac{\partial}{\partial y}, \tag{53}$$

for example, where

$$r = a-b, \quad q = a+b \tag{54}$$

and where

$$\Delta = -y^2\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) \tag{55}$$

is the non-Euclidean Laplace operator. As on ([8], p. 186) let $\{a,b\}$ denote the space of functions $f(\tau,\bar{\tau})$ real-analytic on $\mathcal{H}$ satisfying the DE (51). Therefore, the space $\{\Gamma,a,b,v\}$ consists of functions $f(\tau,\bar{\tau}) \in \{a,b\}$ satisfying (50) and (52) (See Table 1).The Hecke–Eisenstein series (44) belongs to $\{\Gamma,a,b,1\}$ and satisfies the invariance property $G(\tau,\bar{\tau},a,b)|_{\gamma_{ab}} = G(\tau,\bar{\tau},a,b)$, where

$$f(\tau,\bar{\tau})|_{\gamma_{ab}} = (c\tau+d)^{-a}(c\bar{\tau}+d)^{-b}v(\gamma)f(\gamma(\tau),\gamma(\bar{\tau})), \quad \gamma = \begin{pmatrix} * & * \\ c & d \end{pmatrix} \in \Gamma.$$

Maass' main result ([8], Theorem 35, pp. 228–229) is stated for the *Hecke group* $\mathfrak{G}(\lambda)$, ([2], p. 671), ([8], p. 226), generated by

$$T^\lambda = \begin{pmatrix} 1 & \lambda \\ 0 & 1 \end{pmatrix}, \quad S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \tag{56}$$

where $\lambda > 0$ but only for $\lambda \geq 2$ or $\lambda = 2\cos\frac{\pi}{\ell}$, $\ell = 3,5,\cdots$, $\mathfrak{G}(\lambda)$ is a discontinuous group (including the full modular group $\Gamma(1)$).

Then, (50) reads for generators of $\mathfrak{G}(\lambda)$

$$\begin{aligned} f(\tau+\lambda,\bar{\tau}+\lambda) &= e^{2\pi i\varkappa} f(\tau,\bar{\tau}), \\ f\left(-\frac{1}{\tau},-\frac{1}{\bar{\tau}}\right) &= \gamma(-i\tau)^a(-i\bar{\tau})^b f(\tau,\bar{\tau}), \end{aligned} \tag{57}$$

where $\lambda > 0, 0 \leq \varkappa < 1, \gamma = \pm 1$ and $\lambda = 1, \varkappa = 0, a = b = 0$ does not occur and (52) reads

$$f(\tau,\bar{\tau}) = O(y^{K_1}), \quad y \to \infty, \quad f(\tau,\bar{\tau}) = O(y^{K_2}), \quad y \to +0, \quad K_j > 0, \tag{58}$$

uniformly in x. We refer to the function that satisfies automorphy (57) and is a solution of (51) with growth condition (58) as a *Hecke Maass function*.

Table 1. Spaces of real-analytic functions.

Space	Condition (s)
$\{a,b\}$	(51)
$\{\Gamma,a,b,v\} \subset \{a,b\}$	(50), (52)
$\{\mathfrak{G},a,b,v\} \subset \{\Gamma,a,b,v\}$	(57), (58)

Lemma 5. *The Hecke–Maass function $f(\tau,\bar{\tau}) \in [\mathfrak{G}(\lambda),a,b,v]$, with v described by* (57)*, admits the Fourier–Whittaker expansion*

$$f(\tau,\bar{\tau}) = a_0 u(y,q) + b_0 + \sum_{n+\varkappa\neq 0} a_{n+\varkappa} W\left(\frac{2\pi(n+\varkappa)}{\lambda}y; a, b\right) e^{\frac{2\pi i(n+\varkappa)}{\lambda}x}, \tag{59}$$

which leads to the Fourier–Whittaker expansion (4)*. Here, W is a modified Whittaker function* (27)*, $\varepsilon^2 = 1$, and (with q as in* (54)*)*

$$u(y,q) = \frac{y^{1-q} - 1}{1-q}. \tag{60}$$

Proof is given in Section 3.

Example 1. *By Lemma 5, the Hecke–Eisenstein series* (44) *admits the Fourier–Whittaker expansion with $\lambda = 1$, $\varkappa = 0$. We determine the coefficients a_n as follows.*

$$G(\tau,\bar{\tau};a,b) = \varphi_{\frac{r}{2}}(y,q) + 2(-1)^{\frac{r}{2}}(\sqrt{2}\pi)^q \sum_{n\neq 0} \frac{\sigma_{q-1}(n)}{\Gamma(\frac{q}{2}+\varepsilon\frac{r}{2})} W(2\pi ny;a,b)e^{2\pi inx}, \tag{61}$$

where $\varepsilon = \operatorname{sgn} n$ and $q = a+b, r = a-b$ even, $\sigma_{q-1}(n) = \sum_{d|n} d^{q-1}$ and

$$\begin{aligned}\varphi_{\frac{r}{2}}(y,q) =& 2\zeta(q) + (-1)^{\frac{r}{2}} 2^{3-q}\pi \frac{\Gamma(q-1)}{\Gamma\left(\frac{q+r}{2}\right)\Gamma\left(\frac{q-r}{2}\right)}\zeta(q-1) \\ &+ (-1)^{\frac{r}{2}} 2^{3-q}\pi \frac{\Gamma(q-1)}{\Gamma\left(\frac{q+r}{2}\right)\Gamma\left(\frac{q-r}{2}\right)}\zeta(q-1)(1-q)u(y,q).\end{aligned} \tag{62}$$

Hence,

$$\varphi_{\frac{r}{2}}(y,q) = 2\zeta(q) + 4(-1)^{\frac{r}{2}}(2y)^{1-q}\pi \frac{\Gamma(q-1)}{\Gamma\left(\frac{q+r}{2}\right)\Gamma\left(\frac{q-r}{2}\right)}\zeta(q-1). \tag{63}$$

In the form of $G(\tau,\bar{\tau};s+k,s)$

$$\begin{aligned}G(\tau,\bar{\tau};s+k,s) =& 2\zeta(2s+k) + 4(-1)^{\frac{k}{2}}(2y)^{1-2s-k}\pi \frac{\Gamma(2s+k-1)}{\Gamma(s+k)\Gamma(s)}\zeta(2s+k-1) \\ &+ 2(-1)^{\frac{k}{2}}(\sqrt{2}\pi)^{2s+k}\sum_{n=1}^{\infty}\frac{\sigma_{2s+k-1}(n)}{\Gamma(s+k)}W(2\pi ny;s+k,s)e^{2\pi inx} \\ &+ 2(-1)^{\frac{k}{2}}(\sqrt{2}\pi)^{2s+k}\sum_{n=1}^{\infty}\frac{\sigma_{2s+k-1}(n)}{\Gamma(s)}W(-2\pi ny;s+k,s)e^{-2\pi inx}.\end{aligned} \tag{64}$$

By (27)*,* (41)*, for $n \geq 1$*

$$\begin{aligned}W(2\pi ny;s+k,s) &= 2^{\frac{2s+k}{2}}e^{-2\pi ny}\Psi(s,2s+k;4\pi ny), \\ W(-2\pi ny;s+k,s) &= 2^{\frac{2s+k}{2}}e^{-2\pi ny}\Psi(s+k,2s+k;4\pi ny),\end{aligned}$$

whence (64) *reads*

$$\begin{aligned}G(\tau,\bar{\tau};s+k,s) =& 2\zeta(2s+k) + 4(-1)^{\frac{k}{2}}(2y)^{1-2s-k}\pi \frac{\Gamma(2s+k-1)}{\Gamma(s+k)\Gamma(s)}\zeta(2s+k-1) \\ &+ 2\sum_{n\neq 0}\sigma_{2s+k-1}(n)a_n(y,s)e^{2\pi inx},\end{aligned} \tag{65}$$

where

$$a_n(y,s) = \begin{cases} (-1)^{\frac{k}{2}}(2\pi)^{2s+k}e^{-2\pi ny}\frac{1}{\Gamma(s+k)}\Psi(s,2s+k;4\pi ny) & n>0, \\ (-1)^{\frac{k}{2}}(2\pi)^{2s+k}e^{2\pi ny}\frac{1}{\Gamma(s)}\Psi(s+k,2s+k;-4\pi ny) & n<0. \end{cases}$$

(65) *is in conformity with Doi-Miyake ([36], Theorem 7.2.9, p. 284) for* $E_k(\tau, s; 1, 1)$ *defined by* (46). *In view of* (48), *the result of* [37] *on* $E_k(\tau, s)$ *follows with the correction to the effect that it is to be multiplied by* 2.

Remark 1. *Equation* (27) *makes the results of Doi-Miyake and Maass inconsistent, i.e., the variables of the Whittaker function in Maass results are multiplied by* $\frac{1}{2}$.

3. Zeta-Functions with Fourier–Whittaker Expansion

The objectives of this section are to establish Lemma 5 and accommodate the zeta-functions satisfying Hecke's functional equation in the framework of the ramified functional equations.

For the Hecke group, we have a slightly simplified ([8], Lemma 6, p. 182):

Lemma 6. *Let* $f(\tau) = f(x, y)$ *and* $g(y)$ *be differentiable functions. Then,*

$$\Omega_{ab}^{\nu'}(f(\tau)g(y)) = -y^2 f(\tau)\frac{\mathrm{d}^2 g}{\mathrm{d}y^2} - y\left(2y\frac{\partial f}{\partial y} + qf(\tau)\right)\frac{\mathrm{d}g}{\mathrm{d}y} + g(y)\Omega_{ab}^{\nu'} f(\tau). \tag{66}$$

In the case $f(\tau)g(y) \in \{a, b\}$ *with* $f(\tau) = e^{i\varepsilon x}y^p \in \{a, b\}$ *and* $\varepsilon^2 = 1$, (66) *amounts to the DE*

$$\begin{aligned} &\Omega_{ab}^{\nu'}(f(\tau)g(y)) \\ &= -f(\tau)\left(y^2\frac{\mathrm{d}^2}{\mathrm{d}y^2} + y(2p+q)\frac{\mathrm{d}}{\mathrm{d}y} - y^2 + \varepsilon r y + p(p+q-1) - \nu'\right)g(y) = 0. \end{aligned} \tag{67}$$

If $g(y)$ *is of polynomial growth as* $y \to \infty$, *then the solution of* (67) *with* $p = -\frac{q}{2}$ *of polynomial growth is constant times* $g(y)$:

$$g(y) = W_{\frac{1}{2}\varepsilon r, \frac{1}{2}\sqrt{(q-1)^2+4\nu'}}(2y), \tag{68}$$

which in the case $\nu' = 0$ *reduces to Cf.* (27).

$$g(y) = W_{\frac{1}{2}\varepsilon r, \frac{1}{2}(q-1)}(2y) = y^{\frac{q}{2}} W(\varepsilon y; a, b). \tag{69}$$

Once formulated, it is not difficult to prove, and we omit the proof.

To apply Lemma 6 to (48), we note that $G(\tau, \bar{\tau}, s+k, s)$ is in $\{a, b\}$ with $a = s+k, b = s$, so that $r = k$, $q = 2s+k$. See Example 1.

Remark 2. *If* $\Omega_{ab} f = 0$ *and* $f(\tau) = y^{-\frac{q}{2}}h(x)$, *then* (66) *reads*

$$\Omega_{ab}(f(\tau)g(y)) = -y^2 f(\tau)\frac{\mathrm{d}^2 g}{\mathrm{d}y^2}, \tag{70}$$

in particular,

$$\Omega_{ab}(f(\tau)y^s) = -s(s-1)f(\tau)y^s, \tag{71}$$

i.e., $f(\tau)y^s$ *is an eigenfunction of* Ω_{ab} *with eigenvalue* $\nu = -s(s-1)$. *In this case,* (40) *holds true with*

$$\mu = \frac{1}{2}\sqrt{(q-1)^2 + 4\nu'} = \frac{q-1}{2} \tag{72}$$

leading to (69).

Proof of Lemma 5. Proof is given in ([8], p. 187) for a horocyclic group, which can lead to the proof of the expansion on ([8], p. 230) for a Hecke group $\mathfrak{G}(\lambda)$. The first formula on ([8], p. 187) for a Hecke–Maass function $f(\tau, \bar{\tau})$ reads, with N replaced by λ,

$$f(\tau, \bar{\tau}) = \sum_{n=-\infty}^{\infty} \alpha_{n+\varkappa}(y) e^{\frac{2\pi i(n+\varkappa)}{\lambda} x}, \tag{73}$$

where

$$\alpha_{n+\varkappa}(y) = \frac{1}{\lambda} \int_0^\lambda f(\tau, \bar{\tau}) e^{-\frac{2\pi i(n+\varkappa)}{\lambda} x} \, \mathrm{d}x. \tag{74}$$

(67) with $p = 0, \nu' = 0$ reads

$$\begin{aligned} \Omega_{ab}(e^{i\varepsilon x} g(y)) &= -e^{i\varepsilon x} \left(y^2 \frac{\mathrm{d}^2}{\mathrm{d}y^2} + qy \frac{\mathrm{d}}{\mathrm{d}y} - y^2 + \varepsilon r y \right) g(y) \\ &= e^{i\varepsilon} \left(\Omega_{ab} + y^2 - \varepsilon r y \right) g(y). \end{aligned} \tag{75}$$

Applying (75) on (74) for $n + \varkappa \neq 0$, we see that $\alpha_{n+\varkappa}(y) e^{\frac{2\pi i(n+\varkappa)}{\lambda} x}$ satisfies the condition in Lemma 6, and so (69) applies. For $n + \varkappa = 0$, we use the first equality of (75) and remark that the two independent solutions are given by $1, u(y, q)$, where $u(y, q)$ is defined by (60). This completes the proof. $\square$

3.1. Hecke's Functional Equation

For comparison's sake with Definition 1 as well as convenience of reference, we state a prototype version of Definition A1.

Definition 2. *For increasing sequences $\{\lambda_n\}$, $\{\mu_n\}$ of positive real numbers and complex sequences $\{a_n\}$, $\{b_n\}$ form the Dirichlet series*

$$\varphi(s) = \sum_{n=1}^{\infty} \frac{a_n}{\lambda_n^s} \quad \text{and} \quad \psi(s) = \sum_{n=1}^{\infty} \frac{b_n}{\mu_n^s} \tag{76}$$

which we assume have the abscissa of absolute convergence σ_φ^* and σ_ψ^*, respectively. Then, $\varphi(s)$ and $\psi(s)$ are said to satisfy Hecke's functional equation

$$A^{-s} \Gamma(s) \varphi(s) = A^{-(r-s)} \Gamma(r-s) \psi(r-s), \tag{77}$$

where $A > 0$ is a constant, if there exists a regular function $\chi(s)$ outside of a compact set $\mathcal{S}$ such that $\chi(s) = A^{-s}\Gamma(s)\varphi(s)$ for $\sigma > \sigma_\varphi^*$ and $\chi(s) = A^{-(r-s)}\Gamma(r-s)\psi(r-s)$ for $\sigma < r - \sigma_\psi^*$ and such that $\chi(s)$ is convex in the sense that

$$e^{-\varepsilon|t|} \chi(\sigma + it) = O(1), \;\; 0 < \varepsilon < \frac{\pi}{2},$$

uniformly in σ, $\sigma_1 \leq \sigma \leq \sigma_2$, $|t| \to \infty$.

The residual function is defined in the sense of Bochner [43] by

$$\mathrm{P}(x) = \frac{1}{2\pi i} \int_C \chi(s) x^{-s} ds, \tag{78}$$

where C encircles all the singularities of $\chi(s)$ in $\mathcal{S}$.

We introduce the modular type functions (Fourier expansion) corresponding to Dirichlet series (76)

$$f(\tau) = \sum_{n=1}^{\infty} a_n e^{A i n \tau} \quad \text{and} \quad g(\tau) = \sum_{n=1}^{\infty} b_n e^{A i n \tau}, \quad \tau \in \mathcal{H} \tag{79}$$

which are absolutely convergent.

Lemma 7. (Hecke) *The Dirichlet series* (76) *satisfies the condition that* $A^{-s}\Gamma(s)\varphi(s) + \frac{a_0}{s} + \frac{Cb_0}{r-s}$ *is BEV (bounded in every vertical strip) and satisfies the functional Equation* (77) *with* $\psi(r-s)$ *replaced by* $C\psi(r-s) = \sum_{n=1}^{\infty} \frac{Cb_n}{\mu_n^{r-s}}$*, which is equivalent to and satisfies the (theta) transformation formula*

$$f(\tau) = C\left(\frac{\tau}{i}\right)^{-r} g\left(-\frac{1}{\tau}\right). \tag{80}$$

Lemma 7 is a slight modification of ([44], Theorem 1, p. I-5), which is a useful statement of Hecke's epoch-making discovery [2,3] mentioned in the beginning.

The following theorem constitutes the basis of results related to Hecke's functional equation, cf. [45], ([6], p. 10).

Theorem 2. *The functional Equation* (77)*, the theta transformation formula* (80)*, Bochner's modular relation* (81) *below and the Fourier–Bessel expansion* (82) *below are all equivalent:*

$$\sum_{n=1}^{\infty} a_n e^{-\lambda_n x} = \left(\frac{A}{x}\right)^r \sum_{n=1}^{\infty} b_n e^{-\mu_n \frac{A^2}{x}} + \mathrm{P}\left(\frac{x}{A}\right) \tag{81}$$

with $\operatorname{Re} x > 0$ *and*

$$\begin{aligned} A^{-s}\Gamma(s)\varphi(s,\alpha) =& 2\alpha^{\frac{r-s}{2}} \sum_{n=1}^{\infty} b_n \mu_n^{-\frac{r-s}{2}} K_{s-r}(2A\sqrt{\alpha\mu_n}) \\ &+ A^{-s}\int_0^{\infty} e^{-\alpha u} u^{s-1} \mathrm{P}\left(\frac{u}{A}\right) du \end{aligned} \tag{82}$$

with $\alpha > 0, \sigma > \max\{r - \frac{1}{2}, -1\}, s \neq 0$*, where*

$$\varphi(s,\alpha) = \sum_{n=1}^{\infty} \frac{a_n}{(\lambda_n + \alpha)^s} \tag{83}$$

denotes the (Hurwitz type) **perturbed Dirichlet series** *associated with* φ*.*

Remark 3. *The contents of Theorem 2 are expounded in respective chapters of [6]. The Bochner modular relation* (81) *is a right-half plane version of the Hecke correspondence, Lemma 7. There are two more equivalent assertions. The Ewald expansion is in Chapter* 5*, and the Riesz sum in Chapter* 6*. It is the Ewald expansion that is adopted in [46] to express weak Maass forms, cf. ([47], pp. 63–65). In place of Whittaker functions, the incomplete gamma function is used.*

The Fourier expansion (79) *is intrinsic to a holomorphic modular form that used to be known as the q-expansion, the Laurent expansion at* ∞*. There is no counterpart for Maass forms since they are not meromorphic at* ∞*. Thus, instead of Fourier expansion, the Fourier–Whittaker expansion (generalization of Fourier–Bessel expansion) is adopted in Maass theory, and this makes it comparable to the theory of Epstein type zeta-functions since the main ingredient in the latter is also the Fourier–Bessel expansion or the Chowla–Selberg integral formula.*

3.2. General Epstein Zeta-Functions

In this subsection, we refer to part of the results in ([45], §6.5, pp. 125–130) for general Epstein zeta-functions and deduce the Chowla–Selberg integral formula from the Fourier–Bessel expansion. This gives a smooth shift from the Fourier–Bessel expansion (82) to Theorem 5 in a more general way.

Notation. Let $\boldsymbol{g}$, $\boldsymbol{h} \in \mathbb{R}^n$ be n-dimensional real vectors that (in the first place) give rise to the perturbation and the (additive) characters, respectively.

Let $Y = (y_{ij})$ be a positive definite $n \times n$ real symmetric matrix. Define the Epstein zeta-function associated to the quadratic form

$$Y[\boldsymbol{a}] = \boldsymbol{a} \cdot Y\boldsymbol{a} = {}^t\boldsymbol{a}Y\boldsymbol{a} = \sum_{i,j=1}^{n} y_{ij}a_i a_j, \tag{84}$$

where $\boldsymbol{a} = (a_1, \ldots, a_n) \in \mathbb{R}^n$ and "$\cdot$" means the scalar product, by

$$Z(Y, \boldsymbol{o}, \boldsymbol{o}, s) = \sum_{\substack{\boldsymbol{a} \in \mathbb{Z}^n \\ \boldsymbol{a} \neq \boldsymbol{o}}} \frac{1}{Y[\boldsymbol{a}]^s}, \quad \sigma > \frac{n}{2}, \tag{85}$$

where $\sigma = \mathrm{Re}\, s$.

For $\boldsymbol{g},\ \boldsymbol{h} \in \mathbb{R}^n$, define the general *Epstein zeta-function* (with perturbation) by

$$Z(Y, \boldsymbol{g}, \boldsymbol{h}, s) = \sum_{\substack{\boldsymbol{a} \in \mathbb{Z}^n \\ \boldsymbol{a}+\boldsymbol{g} \neq \boldsymbol{o}}} \frac{e^{2\pi i \boldsymbol{h} \cdot \boldsymbol{a}}}{Y[\boldsymbol{a}+\boldsymbol{g}]^s}, \quad \sigma > \frac{n}{2}, \tag{86}$$

and incorporate the *completion*

$$\Lambda(Y, \boldsymbol{g}, \boldsymbol{h}, s) = \pi^{-s}\Gamma(s)\, Z(Y, \boldsymbol{g}, \boldsymbol{h}, s), \tag{87}$$

which satisfies the functional equation of the form (77) with an additional factor and replacement of parameters (which follows from Theorem 4):

$$\Lambda(Y, \boldsymbol{g}, \boldsymbol{h}, s) = \frac{1}{\sqrt{|Y|}}\, e^{-2\pi i \boldsymbol{g} \cdot \boldsymbol{h}}\, \Lambda\Big(Y^{-1}, \boldsymbol{h}, -\boldsymbol{g}, \frac{n}{2} - s\Big). \tag{88}$$

Theorem 3 (Fourier–Bessel expansion)**.** *With the notation above, we have for $b > 0$,*

$$\begin{aligned}
&\pi^{-s}\,\Gamma(s) \sum_{\boldsymbol{a} \in \mathbb{Z}^n} \frac{e^{2\pi i \boldsymbol{h} \cdot \boldsymbol{a}}}{(Y[\boldsymbol{a}+\boldsymbol{g}]+b)^s} \\
&= \frac{2}{\sqrt{|Y|}} \sum_{\substack{\boldsymbol{a} \in \mathbb{Z}^n \\ \boldsymbol{a}+\boldsymbol{h} \neq \boldsymbol{o}}} e^{-2\pi i \boldsymbol{g} \cdot (\boldsymbol{a}+\boldsymbol{h})} \sqrt{\frac{Y^{-1}[\boldsymbol{a}+\boldsymbol{h}]}{b}}^{\,s-\frac{n}{2}} K_{s-\frac{n}{2}}\Big(2\sqrt{Y^{-1}[\boldsymbol{a}+\boldsymbol{h}]b}\,\pi\Big) \\
&\quad + \delta(\boldsymbol{h})\, \frac{1}{\sqrt{|Y|}}\, \frac{\Gamma(s-\frac{n}{2})}{\pi^{s-\frac{n}{2}}}\, \frac{1}{b^{s-\frac{n}{2}}}.
\end{aligned} \tag{89}$$

To state Theorem 4, we introduce new notation.

Let $Y = \left(\begin{array}{c|c} A & B \\ \hline {}^tB & C \end{array}\right)$ be a block decomposition with A an $n \times n$ matrix and C an $m \times m$ matrix. Set

$$D = C - {}^tBA^{-1}B.$$

In accordance with this decomposition, we decompose the vectors $\boldsymbol{g} = \begin{pmatrix} \boldsymbol{g}_1 \\ \boldsymbol{g}_2 \end{pmatrix}, \boldsymbol{h} = \begin{pmatrix} \boldsymbol{h}_1 \\ \boldsymbol{h}_2 \end{pmatrix}$, $\boldsymbol{g}_1, \boldsymbol{h}_1 \in \mathbb{Z}^n, \boldsymbol{g}_2, \boldsymbol{h}_2 \in \mathbb{Z}^m$.

Theorem 4. (general Chowla-Selberg integral formula) *Under the above notation, we have*

$$\Lambda(Y, \boldsymbol{g}, \boldsymbol{h}, s) = \delta(\boldsymbol{g}_2)\, e^{-2\pi i \boldsymbol{g}_2 \cdot \boldsymbol{h}_2} \Lambda(A, \boldsymbol{g}_1, \boldsymbol{h}_1, s) + \delta(\boldsymbol{h}_1) \frac{1}{\sqrt{|A|}} \Lambda\Big(D, \boldsymbol{g}_2, \boldsymbol{h}_2, s - \frac{n}{2}\Big) \tag{90}$$
$$+ \frac{2e^{-2\pi i \boldsymbol{g}_1 \cdot \boldsymbol{h}_1}}{\sqrt{|A|}} \sum_{\substack{\boldsymbol{a} \in \mathbb{Z}^n \\ \boldsymbol{a}+\boldsymbol{h}_1 \neq \boldsymbol{o}}} \sum_{\substack{\boldsymbol{b} \in \mathbb{Z}^m \\ \boldsymbol{b}+\boldsymbol{g}_2 \neq \boldsymbol{o}}} e^{2\pi i(-\boldsymbol{g}_1 \cdot \boldsymbol{a} + \boldsymbol{h}_2 \cdot \boldsymbol{b})} e^{-2\pi i A^{-1} B(\boldsymbol{b}+\boldsymbol{g}_2)\cdot(\boldsymbol{a}+\boldsymbol{h}_1)}$$
$$\times \sqrt{\frac{A^{-1}[\boldsymbol{a}+\boldsymbol{h}_1]}{D[\boldsymbol{b}+\boldsymbol{g}_2]}}^{\,s-\frac{n}{2}} K_{s-\frac{n}{2}}\Big(2\sqrt{A^{-1}[\boldsymbol{a}+\boldsymbol{h}_1]\, D[\boldsymbol{b}+\boldsymbol{g}_2]}\,\pi\Big).$$

As the proof dependent on Theorem 3 is similar to that of Theorem 6.2 from Theorem 6.1 in [45], it is omitted.

Proof of the case $m = 1$ based on Theorem 3 is due to Berndt [48].

3.3. From Epstein-Type to Real-Analytic Eisenstein Series

Definition 3. *The real-analytic Eisenstein series is defined by the following (cf. [19]):*

$$E^*(z, s) = \frac{\Gamma(s)}{2\pi^s} \sum_{\substack{m,n \in \mathbb{Z} \\ (m,n) \neq (0,0)}} \frac{y^s}{|mz+n|^{2s}} = \frac{\Gamma(s)}{2\pi^s} y^s \zeta_{\mathbb{Z}^2}(s, z). \tag{91}$$

By (11),

$$E^*(z, s) = \frac{1}{2}\xi(2s) E(z, s), \tag{92}$$

where

$$\xi(s) = \pi^{-\frac{s}{2}} \Gamma\Big(\frac{s}{2}\Big) \zeta(s), \tag{93}$$

which is a completion of the Riemann zeta-function and satisfies the functional equation

$$\xi(s) = \xi(1-s). \tag{94}$$

In Table 2, A is the constant in Hecke's functional Equation (77), and $\mathfrak{C}$ is an absolute ideal class $\in I/P$ of an imaginary quadratic field K and

$$\zeta(s, \mathfrak{C}) = \frac{1}{w} Z(s), \quad Z(s) = \sum_{m,n}{}' \frac{1}{Q(m,n)^s} = a^{-s} \zeta_{\mathbb{Z}^2}(s, \tau), \tag{95}$$
$$\zeta_{\mathbb{Z}^2}(s, \tau) = \sum_{m,n}{}' \frac{1}{|m+n\tau|^{2s}},$$

where w indicates the number of roots of unity contained in K. See Proposition 1 and ([49], §2.1).

Table 2. Epstein zeta-functions and non-holom. Eisenstein ser.

Author (s)	Epstein	Non-Holom.
[18]	-	$\frac{1}{\xi(2s)} E^*(z,s) = E(z,s)$
[13]	$\zeta_{\mathbb{Z}^2}(s;\alpha) = Z(s, Q_1)$	$\frac{y^s}{2\zeta(2s)} Z(s, Q_1)$
[17]	$Z(s, Q) = \sum'_{m,n} \frac{1}{Q(m,n)^s}$	$\frac{y^s}{2\zeta(2s)} Z(s, Q)$
[14]	-	$\frac{1}{\xi(2s)} E^*(z,s) = E(z,s)$

Theorem 5. (Chowla-Selberg integral formula) (i) *All five Dirichlet series in Table 3 have the Fourier–Bessel expansion*

$$\Gamma(s)\zeta(s,\mathfrak{C}) = \frac{1}{w}\Bigg(2\Gamma(s)\zeta(2s)a^{-s} + \frac{2^{2s}a^{s-1}\sqrt{\pi}}{|d|^{s-\frac{1}{2}}}\Gamma\left(s-\frac{1}{2}\right)\zeta(2s-1) \\ + \frac{4\pi^s(2a)^{s-\frac{1}{2}}}{|d|^{\frac{s}{2}-\frac{1}{4}}}\sum_{n=1}^{\infty} n^{s-\frac{1}{2}}\sigma_{1-2s}(n)\cos 2\pi nx K_{s-\frac{1}{2}}(2\pi yn)\Bigg), \tag{96}$$

$$Z(s,Q) = a^{-s}\zeta_{\mathbb{Z}^2}(s,\tau) = 2\zeta(2s)a^{-s} + \frac{2\sqrt{\pi}\Gamma(s-\frac{1}{2})}{\Gamma(s)}\zeta(2s-1)y^{1-2s}a^{-s} \\ + \frac{8a^{-s}}{\Gamma(s)}\pi^s y^{\frac{1}{2}-s}\sum_{n=1}^{\infty} n^{s-\frac{1}{2}}\sigma_{1-2s}(n)\cos 2\pi nx K_{s-\frac{1}{2}}(2\pi yn), \tag{97}$$

for the Eisenstein–Maass series or the real-analytic Eisenstein series

$$E(z,s) = 2y^s + \frac{2\sqrt{\pi}\Gamma(s-\frac{1}{2})}{\Gamma(s)}\frac{\zeta(2s-1)}{\zeta(2s)}y^{-s+1} \\ + \frac{8y^{\frac{1}{2}}\pi^s}{\Gamma(s)\zeta(2s)}\sum_{n=1}^{\infty} n^{s-\frac{1}{2}}\sigma_{1-2s}(n)\cos 2\pi nx K_{s-\frac{1}{2}}(2\pi yn) \tag{98}$$

or

$$E^*(z,s) = \pi^{-s}\Gamma(s)\zeta(2s)y^s + \pi^{s-1}\Gamma(1-s)\zeta(2-2s)y^{1-s} \\ + 4\sqrt{y}\sum_{n=1}^{\infty} n^{s-\frac{1}{2}}\sigma_{1-2s}(n)\cos 2\pi nx K_{s-\frac{1}{2}}(2\pi yn). \tag{99}$$

as well as that for the Epstein-type Eisenstein series.
(ii) *Save for $E^*(s,z)$, they satisfy Hecke's functional Equation (77) with $A = \frac{2\pi}{\sqrt{|\Delta|}}$ for $\zeta_K(s)$ (K being an imaginary quadratic field with discriminant Δ) and $Z(s,Q)$ resp. $A = \frac{\pi}{y}$ for others.*

Table 3. Epstein zeta-functions and non-holomorphic Eisenstein series.

Zeta	Symbol	A
Partial Dedekind zeta	$\zeta_K(s,\mathfrak{C}) = \sum_{\mathfrak{C}\ni\mathfrak{a}\neq 0}\frac{1}{(N\mathfrak{a})^s}$	$\frac{2\pi}{\sqrt{\lvert\Delta\rvert}}$
Epstein zeta (18)	$Z(s,Q) = \sum'_{m,n}\frac{1}{Q(m,n)^s}$	$\frac{2\pi}{\sqrt{\lvert\Delta\rvert}}$
Epstein-type Eisenstein (9)	$\zeta_{\mathbb{Z}^2}(s;\alpha) = Z(s,Q_1)$	$\frac{\pi}{y}$
Eisenstein–Maass (9)	$E(\tau,s) = E_0(\tau,s) = \frac{y^s}{\zeta(2s)}\zeta_{\mathbb{Z}^2}(s,\ \tau)$	$\frac{\pi}{y}$
Real-analytic Eisenstein (91)	$E^*(z,s) = \frac{1}{2}\left(\frac{\pi}{y}\right)^{-s}\Gamma(s)\zeta_{\mathbb{Z}^2}(s,z)$	1
Hecke–Eisenstein (44)	$G(\tau,\bar{\tau};a,b)$	-
kth Eisenstein-Maass (47)	$E_k(z,s) = \frac{y^s}{\zeta(2s+k)}G(\tau,\bar{\tau},s+k,s)$	-

Q_1 is defined in (15).

This is immediate from Corollary 2, Lemma 3 and Example 1 in view of

$$\zeta_{\mathbb{Z}^2}(s,\tau) = G(\tau,\bar{\tau};s,s). \tag{100}$$

A direct proof of (99) is given in [18,19].
A more general case of (97) is given in Theorem 4.

Remark 4. *Since $E^*(z,s)$ involves the Hamma factor, the functional equation reads*

$$E^*(z,s) = E^*(z,1-s). \tag{101}$$

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Appendix A. The Main Formula H

In this section, we state the main formula H ([6], Theorem 4.3, p. 118) in its full generality, first proved by [12].

Definition A1. *We have two sets of Dirichlet series $\{\varphi_h(s)\}, 1 \le h \le H$ and $\{\psi_i(s)\}, 1 \le i \le I$ that satisfy the generalized functional Equation (A3) in the following sense.*

With increasing sequences (of positive numbers) $\{\lambda_k^{(h)}\}_{k=1}^\infty, (1 \le h \le H)$, $\{\mu_k^{(i)}\}_{k=1}^\infty$ $(1 \le i \le I)$ and complex sequences $\{\alpha_k^{(h)}\}_{k=1}^\infty$ $(1 \le h \le H)$, $\{\beta_k^{(i)}\}_{k=1}^\infty$ $(1 \le i \le I)$, we form the Dirichlet series

$$\varphi_h(s) = \sum_{k=1}^\infty \frac{\alpha_k^{(h)}}{\lambda_k^{(h)s}} \qquad (1 \le h \le H) \tag{A1}$$

and

$$\psi_i(s) = \sum_{k=1}^\infty \frac{\beta_k^{(i)}}{\mu_k^{(i)s}} \qquad (1 \le i \le I), \tag{A2}$$

which we suppose to have finite abscissa of absolute convergence $\sigma_{\varphi_h} (1 \le h \le H), \sigma_{\psi_i} (1 \le i \le I)$, respectively.

We assume the existence of the meromorphic function χ, which satisfies, for a real number r, the functional equation

$$\chi(s) = \begin{cases} \displaystyle\sum_{h=1}^{H} \frac{\prod_{j=1}^{M^{(h)}} \Gamma\Big(d_j^{(h)} + D_j^{(h)} s\Big) \prod_{j=1}^{N^{(h)}} \Gamma\Big(c_j^{(h)} - C_j^{(h)} s\Big)}{\prod_{j=N^{(h)}+1}^{P^{(h)}} \Gamma\Big(c_j^{(h)} + C_j^{(h)} s\Big) \prod_{j=M^{(h)}+1}^{Q^{(h)}} \Gamma\Big(d_j^{(h)} - D_j^{(h)} s\Big)} \varphi_h(s), \\ \qquad\qquad \mathrm{Re}(s) > \max\limits_{1\le h\le H} \sigma_{\varphi_h} \\ \displaystyle\sum_{i=1}^{I} \frac{\prod_{j=1}^{\tilde N^{(i)}} \Gamma\Big(e_j^{(i)} + E_j^{(i)} (r-s)\Big) \prod_{j=1}^{\tilde M^{(i)}} \Gamma\Big(f_j^{(i)} - F_j^{(i)} (r-s)\Big)}{\prod_{j=\tilde M^{(i)}+1}^{\tilde Q^{(i)}} \Gamma\Big(f_j^{(i)} + F_j^{(i)} (r-s)\Big) \prod_{j=\tilde N^{(i)}+1}^{\tilde P^{(i)}} \Gamma\Big(e_j^{(i)} - E_j^{(i)} (r-s)\Big)} \psi_i(r-s), \\ \qquad\qquad \mathrm{Re}(s) < \min\limits_{1\le i\le I} \big(r - \sigma_{\psi_i}\big) \end{cases} \tag{A3}$$

$\left(C_j^{(h)}, D_j^{(h)}, E_j^{(i)}, F_j^{(i)} > 0\right)$.

We assume further that only finitely many of the poles s_k $(1 \leq k \leq L)$ *of* $\chi(s)$ *are neither a pole of*

$$\frac{\prod_{j=1}^{N^{(h)}} \Gamma\left(c_j^{(h)} - C_j^{(h)} s\right)}{\prod_{j=N^{(h)}+1}^{P^{(h)}} \Gamma\left(c_j^{(h)} + C_j^{(h)} s\right) \prod_{j=M^{(h)}+1}^{Q^{(h)}} \Gamma\left(d_j^{(h)} - D_j^{(h)} s\right)}$$

nor a pole of

$$\frac{\prod_{j=1}^{\tilde{M}^{(i)}} \Gamma\left(f_j^{(i)} - F_j^{(i)} r + F_j^{(i)} s\right)}{\prod_{j=\tilde{N}^{(i)}+1}^{\tilde{P}^{(i)}} \Gamma\left(e_j^{(i)} - E_j^{(i)} r + E_j^{(i)} s\right) \prod_{j=\tilde{M}^{(i)}+1}^{\tilde{Q}^{(i)}} \Gamma\left(f_j^{(i)} + F_j^{(i)} r - F_j^{(i)} s\right)}.$$

We introduce the *processing Gamma factor*

$$\Gamma(w \mid \Delta) = \frac{\prod_{j=1}^{m} \Gamma(b_j + B_j w) \prod_{j=1}^{n} \Gamma(a_j - A_j w)}{\prod_{j=n+1}^{p} \Gamma(a_j + A_j w) \prod_{j=m+1}^{q} \Gamma(b_j - B_j w)} \quad (A_j, B_j > 0), \tag{A4}$$

and suppose that for any real numbers $u_1, u_2 \quad (u_1 < u_2)$

$$\lim_{|v| \to \infty} \Gamma(u + iv - s \mid \Delta)\, \chi(u + iv) = 0, \tag{A5}$$

uniformly in $u_1 \leq u \leq u_2$.

We choose $L_1(s)$ so that the poles of

$$\frac{\prod_{j=1}^{n} \Gamma(a_j + A_j s - A_j w) \prod_{j=1}^{N^{(h)}} \Gamma\left(c_j^{(h)} - C_j^{(h)} w\right)}{\prod_{j=n+1}^{p} \Gamma(a_j - A_j s + A_j w) \prod_{j=N^{(h)}+1}^{P^{(h)}} \Gamma\left(c_j^{(h)} + C_j^{(h)} w\right)} \times \frac{1}{\prod_{j=m+1}^{q} \Gamma(b_j + B_j s - B_j w) \prod_{j=M^{(h)}+1}^{Q^{(h)}} \Gamma\left(d_j^{(h)} - D_j^{(h)} w\right)}$$

lie on the right of $L_1(s)$, and those of

$$\frac{\prod_{j=1}^{m} \Gamma(b_j - B_j s + B_j w) \prod_{j=1}^{M^{(h)}} \Gamma\left(d_j^{(h)} + D_j^{(h)} w\right)}{\prod_{j=m+1}^{q} \Gamma(b_j + B_j s - B_j w) \prod_{j=M^{(h)}+1}^{Q^{(h)}} \Gamma\left(d_j^{(h)} - D_j^{(h)} w\right)} \times \frac{1}{\prod_{j=n+1}^{p} \Gamma(a_j - A_j s + A_j w) \prod_{j=N^{(h)}+1}^{P^{(h)}} \Gamma\left(c_j^{(h)} + C_j^{(h)} w\right)}$$

lie on the left of $L_1(s)$, and choose $L_2(s)$ so that the poles of

$$\frac{\prod_{j=1}^{m}\Gamma\big(b_j - B_j s + B_j w\big)\prod_{j=1}^{\tilde{M}^{(i)}}\Gamma\Big(f_j^{(i)} - F_j^{(i)} r + F_j^{(i)} w\Big)}{\prod_{j=m+1}^{q}\Gamma\big(b_j + B_j s - B_j w\big)\prod_{j=\tilde{M}^{(i)}+1}^{\tilde{Q}^{(i)}}\Gamma\Big(f_j^{(i)} + F_j^{(i)} r - F_j^{(i)} w\Big)} \times \frac{1}{\prod_{j=n+1}^{p}\Gamma\big(a_j - A_j s + A_j w\big)\prod_{j=\tilde{N}^{(i)}+1}^{\tilde{P}^{(i)}}\Gamma\Big(e_j^{(i)} - E_j^{(i)} r + E_j^{(i)} w\Big)}$$

lie on the left of $L_2(s)$, and those of

$$\frac{\prod_{j=1}^{n}\Gamma\big(a_j + A_j s - A_j w\big)\prod_{j=1}^{\tilde{N}^{(i)}}\Gamma\Big(e_j^{(i)} + E_j^{(i)} r - E_j^{(i)} w\Big)}{\prod_{j=n+1}^{p}\Gamma\big(a_j - A_j s + A_j w\big)\prod_{j=\tilde{N}^{(i)}+1}^{\tilde{P}^{(i)}}\Gamma\Big(e_j^{(i)} - E_j^{(i)} r + E_j^{(i)} w\Big)} \times \frac{1}{\prod_{j=m+1}^{q}\Gamma\big(b_j + B_j s - B_j w\big)\prod_{j=\tilde{M}^{(i)}+1}^{\tilde{Q}^{(i)}}\Gamma\Big(f_j^{(i)} + F_j^{(i)} r - F_j^{(i)} w\Big)}$$

lie on the right of $L_2(s)$. Further, they squeeze a compact set $\mathcal{S}$ such that $s_k \in \mathcal{S}$ $(1 \le k \le L)$.

Under these conditions, we define the χ-function, key-function , $\mathrm{X}(z, s \mid \Delta)$ by

$$\mathrm{X}(z, s \mid \Delta) = \frac{1}{2\pi i}\int_{L_1(s)} \Gamma(w - s \mid \Delta)\chi(w)\, z^{-w} \mathrm{d}w. \tag{A6}$$

where $\Gamma(s \mid \Delta)$ is the processing gamma factor (A4).

Then, we have the following modular relation, equivalent to the functional Equation (A3):

The Main Formula H.

$$\mathrm{X}(z, s \mid \Delta) =$$

$$\begin{cases} \displaystyle\sum_{h=1}^{H}\sum_{k=1}^{\infty}\alpha_k^{(h)} H^{m+M^{(h)},n+N^{(h)}}_{p+P^{(h)},q+Q^{(h)}}\left(z\lambda_k^{(h)} \,\middle|\, \begin{matrix} \{(1-a_j-A_j s, A_j)\}_{j=1}^{n}, \{(1-c_j^{(h)}, C_j^{(h)})\}_{j=1}^{N^{(h)}}, \\ \{(b_j - B_j s, B_j)\}_{j=1}^{m}, \{(d_j^{(h)}, D_j^{(h)})\}_{j=1}^{M^{(h)}}, \end{matrix} \begin{matrix} \{(a_j - A_j s, A_j)\}_{j=n+1}^{p}, \{(c_j^{(h)}, C_j^{(h)})\}_{j=N^{(h)}+1}^{P^{(h)}} \\ \{(1-b_j-B_j s, B_j)\}_{j=m+1}^{q}, \{(1-d_j^{(h)}, D_j^{(h)})\}_{j=M^{(h)}+1}^{Q^{(h)}} \end{matrix}\right) \\ \qquad \text{if } L_1(s) \text{ can be taken to the right of } \sigma_\varphi \\ \displaystyle z^{-r}\sum_{i=1}^{I}\sum_{k=1}^{\infty}\beta_k^{(i)} H^{n+\tilde{N}^{(i)},m+\tilde{M}^{(i)}}_{q+\tilde{Q}^{(i)},p+\tilde{P}^{(i)}}\left(\frac{\mu_k^{(i)}}{z} \,\middle|\, \begin{matrix} \{(1-b_j-B_j(r-s), B_j)\}_{j=1}^{m}, \{(1-f_j^{(i)}, F_j^{(i)})\}_{j=1}^{\tilde{M}^{(i)}}, \\ \{(a_j - A_j(r-s), A_j)\}_{j=1}^{n}, \{(e_j^{(i)}, E_j^{(i)})\}_{j=1}^{\tilde{N}^{(i)}}, \end{matrix} \begin{matrix} \{(b_j - B_j(r-s), B_j)\}_{j=m+1}^{q}, \{(f_j^{(i)}, F_j^{(i)})\}_{j=\tilde{M}^{(i)}+1}^{\tilde{Q}^{(i)}} \\ \{(1-a_j-A_j(r-s), A_j)\}_{j=n+1}^{p}, \{(1-e_j^{(i)}, E_j^{(i)})\}_{j=\tilde{N}^{(i)}+1}^{\tilde{P}^{(i)}} \end{matrix}\right) \\ \displaystyle + \sum_{k=1}^{L}\mathrm{Res}\Big(\Gamma(w-s \mid \Delta)\chi(w)\, z^{-w}, w = s_k\Big) \\ \qquad \text{if } L_2(s) \text{ can be taken to the left of } \min\limits_{1 \le i \le I}\big(r - \sigma_{\psi_i}\big), \end{cases}$$

$$z^s \, \mathrm{X}(z, s \mid \Delta) =$$

$$\begin{cases} \displaystyle\sum_{h=1}^{H} \sum_{k=1}^{\infty} \frac{\alpha_k^{(h)}}{\left(\lambda_k^{(h)}\right)^s} H^{m+M^{(h)}, n+N^{(h)}}_{p+P^{(h)}, q+Q^{(h)}} \left(z\lambda_k^{(h)} \,\middle|\, \begin{array}{l} \{(1-a_j, A_j)\}_{j=1}^{n}, \{(1-c_j^{(h)} + C_j^{(h)} s, C_j^{(h)})\}_{j=1}^{N^{(h)}}, \\ \{(b_j, B_j)\}_{j=1}^{m}, \{(d_j^{(h)} + D_j^{(h)} s, D_j^{(h)})\}_{j=1}^{M^{(h)}}, \end{array} \right. \\ \qquad\qquad \left. \begin{array}{l} \{(a_j, A_j)\}_{j=n+1}^{p}, \{(c_j^{(h)} + C_j^{(h)} s, C_j^{(h)})\}_{j=N^{(h)}+1}^{P^{(h)}} \\ \{(1-b_j, B_j)\}_{j=m+1}^{q}, \{(1-d_j^{(h)} + D_j^{(h)} s, D_j^{(h)})\}_{j=M^{(h)}+1}^{Q^{(h)}} \end{array} \right) \\ \qquad\qquad \text{if } L_1(s) \text{ can be taken to the right of } \sigma_\varphi \\[2ex] \displaystyle\sum_{i=1}^{I} \sum_{k=1}^{\infty} \frac{\beta_k^{(i)}}{\left(\mu_k^{(i)}\right)^{r-s}} H^{n+\tilde{N}^{(i)}, m+\tilde{M}^{(i)}}_{q+\tilde{Q}^{(i)}, p+\tilde{P}^{(i)}} \left(\frac{\mu_k^{(i)}}{z} \,\middle|\, \begin{array}{l} \{(1-b_j, B_j)\}_{j=1}^{m}, \{(1-f_j^{(i)} + F_j^{(i)}(r-s), F_j^{(i)})\}_{j=1}^{\tilde{M}^{(i)}}, \\ \{(a_j, A_j)\}_{j=1}^{n}, \{(e_j^{(i)} + E_j^{(i)}(r-s), E_j^{(i)})\}_{j=1}^{\tilde{N}^{(i)}}, \end{array} \right. \\ \qquad\qquad \left. \begin{array}{l} \{(b_j, B_j)\}_{j=m+1}^{q}, \{(f_j^{(i)} + F_j^{(i)}(r-s), F_j^{(i)})\}_{j=\tilde{M}^{(i)}+1}^{\tilde{Q}^{(i)}} \\ \{(1-a_j, A_j)\}_{j=n+1}^{p}, \{(1-e_j^{(i)} + E_j^{(i)}(r-s), E_j^{(i)})\}_{j=\tilde{N}^{(i)}+1}^{\tilde{P}^{(i)}} \end{array} \right) \\ \displaystyle + \sum_{k=1}^{L} \mathrm{Res}\Big(\Gamma(w-s \mid \Delta)\chi(w)\, z^{s-w}, w = s_k\Big) \\ \qquad\qquad \text{if } L_2(s) \text{ can be taken to the left of } \min_{1\le i\le I}\left(r - \sigma_{\psi_i}\right), \end{cases} \tag{A7}$$

Further research on (generalizations of) Maass forms will be conducted elsewhere based on the two variable G-functions, cf. [50], as well as in the direction to a counterpart of the Epstein zeta-functions with indefinite quadratic forms including real quadratic fields.

It is also possible to deduce the Fourier–Bessel expansion for the Eisenstein–Maass series $E(\tau, s)$ (9) from more general series, e.g., the Poincaré series as on ([14], pp. 5–6). We return to this aspect elsewhere.

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