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Uniform Airy type expansions of integrals

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Uniform Airy Type Expansions of Integrals

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New estimates are obtained for the remainder of uniform Airy type expansions of integrals containing a phase function with two coalescing saddle points. These expansions are obtained by the Bleistein method, in which both saddle points contribute to the expansion. The new estimates are valid as the asymptotic parameter tends to infinity, uniformly with respect to the parameter locating the saddle points as this parameter ranges over a connected unbounded set. Special attention is paid to the case of both the asymptotic parameter and the saddle point parameter tending to infinity. Two examples are worked out in detail, and in a final section it is explained how to apply the method to other types of uniform expansions.

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1. Introduction

Many problems in mathematical physics and special functions lead to integral representations of the form

$$F(z,\alpha) = \int_C e^{zf(x,\alpha)} g(x)dx, \tag{1.1}$$

where C is a contour, z is a large positive parameter, $g(x)$ is an analytic function on a neighbourhood of C, and $f(x,\alpha)$ is an analytic function of x and α, with two saddle points, which depend on α and coalesce with each other as α varies continuously. With the cubic transformation $x \mapsto w$, given by

$$f(x,\alpha) = \tfrac{1}{3}w^3 - b^2 w + c, \tag{1.2}$$

and suggested by CHESTER, FRIEDMAN and URSELL [2], an asymptotic expansion for large z in terms of Airy functions can be obtained, this expansion being uniformly valid with respect to α as α ranges over a given connected set. The coefficients b and c are determined explicitly from the requirement that the transformation (1.2) is analytic on a neighbourhood of the two saddle points. In FRIEDMAN [5] a nice proof is given on the regularity of transformation (1.2).

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For obtaining the expansion we use the following method. The transformation (1.2) yields the standard form

$$\frac{1}{2\pi i}\int_{\mathcal{L}} e^{z(\frac{1}{3}w^3-b^2w)}h_0(w)dw, \tag{1.3}$$

where

$$h_0(w)=g(x(w))\frac{dx}{dw}$$

(we removed the factor e^{zc}); the phase function has two saddle points at $w=\pm b$. Consequently, the integral (1.3) has a turning point character: the behavior changes strongly when b varies from real to imaginary values. When $b=0$ the saddle points coalesce at $w=0$.

We write $h_0(w)=\alpha_0+\beta_0 w+(w^2-b^2)g_0(w)$, such that both saddle points contribute to the expansion. Then the first two terms deliver two terms of the asymptotic expansion. We integrate the remaining integral by parts and it becomes of the form z^{-1} times integral (1.3), where $h_0(w)$ is replaced by

$$h_1(w)=\frac{d}{dw}g_0(w).$$

We refer to this method as being Bleistein's method. This method is used in the literature, for instance in WONG [8, Chap. 7, §5], where an estimate for the remainder of the expansion in terms of Airy functions is given in terms of the first neglected terms. In OLVER [6, Chap. 9, §12 & §13] an estimate for the remainder of an expansion of the Anger function $\mathrm{A}_{-\nu}(a\nu)$ is given, which has a turning point at $a=1$, as $\nu\to\infty$. This estimate is also valid as $a\to\infty$, but the expansion is different from the one considered in our paper. (An integral representation of the form (1.3) is used, and the expansion is obtained by expanding $h_0(w)$ at one saddle point.)

The problem at hand is to obtain estimates for the remainder of the expansion that are valid as $z\to\infty$, uniformly with respect to b as b ranges over a given connected set that is not bounded. We concentrate on the behavior of the remainder for the case of both z and $|b|$ tending to infinity.

In section 2 and 4 we give estimates for the remainder of an Airy function expansion of the integral (1.1) with $f(x,\alpha)=\sinh x-x\cosh\alpha$. This example is also considered in COPSON [3, Chap. 10], but there the parameter α is restricted to a bounded interval. Our estimates are valid as $z\to\infty$, uniformly with respect $\alpha\geq 0$. For these estimates we use a new class of rational functions which are given in section 3.

In section 5 the method of §§2 – 4 will be applied to (1.3) with a general function $h_0(w)$. The contour $\mathcal{L}$ is unbounded, z is a large positive parameter and $b\in[0,\infty)\cup[0,i\infty)$. Again we concentrate on the fact that the b-domain is unbounded. Estimates for the remainder of an Airy function expansion of integral (1.3) are obtained by considering the distance from the singularities of $h_0(w)$ to the saddle points $\pm b$. In section 5 we assume that this distance is at least of order b^θ as $|b|\to\infty$, with $\theta>-\frac{1}{2}$, and we prove that in this case, the obtained expansion has a double asymptotic property.

A boundary case, where $\theta=-\frac{1}{2}$, will be handled in section 6. In that case we give estimates for the remainder of an Airy type expansion of Laguerre polynomials. We compare these results with the estimates given in FRENZEN and WONG [4]. In this case, the double asymptotic property is lost.

The new method presented here is not restricted to Airy type expansions. In the final section we consider some other types of uniform expansions, which are generated by the Bleistein method. In particular, a uniform expansion in terms of Bessel functions is considered.

2. Uniform Airy function expansion of the Bessel function, part 1

We use the following integral representation of the Bessel function.

$$J_\nu(z) = \frac{1}{2\pi i}\int_{\infty-\pi i}^{\infty+\pi i} e^{zf(x,\alpha)}dx, \tag{2.1}$$

where

$$f(x,\alpha) = \sinh x - x\cosh\alpha, \tag{2.2}$$

and $\nu = z\cosh\alpha$, $\alpha \geq 0$. In order to get an Airy function expansion we use the transformation $x \mapsto w$, given by

$$f(x,\alpha) = \tfrac{1}{3}w^3 - b^2w + c. \tag{2.3}$$

Define

$$h_0(w) = \frac{dx}{dw} = \frac{w^2-b^2}{\cosh x - \cosh\alpha}. \tag{2.4}$$

We prescribe that the saddle point $x = \alpha$ must correspond with $w = b$, and $x = -\alpha$ with $w = -b$. It follows from earlier investigations, see for instance [3], that $x(w)$ is an analytic function of w on neighbourhoods of $w = \pm b$. Calculations give

$$b^3 = \tfrac{3}{2}(\alpha\cosh\alpha - \sinh\alpha), \qquad c = 0. \tag{2.5}$$

Since $\alpha \geq 0$, we choose b to be positive. If we regard α as a complex variable, b is an analytic function of α on a neighbourhood of the real axis.

With transformation (2.3) we bring (2.1) into the standard form

$$J_\nu(z) = \frac{1}{2\pi i}\int_{\mathcal{L}} e^{z(\frac{1}{3}w^3-b^2w)}h_0(w)dw, \tag{2.6}$$

where $\mathcal{L}$ is a suitable contour that begins at $\infty e^{-\frac{1}{3}\pi i}$ and ends at $\infty e^{\frac{1}{3}\pi i}$. We take $\mathcal{L}$ the steepest descent contour through b, which is given by

$$\mathcal{L} = \{w = x + iy \in \mathbb{C} \mid y^2 = 3x^2 - 3b^2\}, \tag{2.7}$$

see Figure 2.1, such that $\mathrm{Im}(\frac{1}{3}w^3 - b^2w) = 0$, and $\frac{1}{3}w^3 - b^2w$ attains its maximum on $\mathcal{L}$ at b. In [3] it is shown that $h_0(w)$ is an analytic function on a neighbourhood of $\mathcal{L}$.

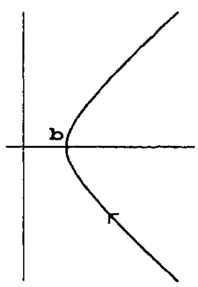

FIGURE 2.1. Steepest descent curve $\mathcal{L}$.

The preceding analysis is given, in more detail, in COPSON [3], where the first term of expansion (2.9) is obtained as an asymptotic approximation. We use the Bleistein method for obtaining an asymptotic expansion. So, we define $g_n(w)$, $h_{n+1}(w)$, $n = 0, 1, 2, \ldots$, by writing

$$\begin{aligned} h_n(w) &= \alpha_n + \beta_n w + (w^2 - b^2) g_n(w), \\ h_{n+1}(w) &= \frac{d}{dw} g_n(w), \end{aligned} \tag{2.8}$$

with $h_0(w)$ given in (2.4), and α_n, β_n following from substitution of $w = \pm b$. A formula, like (2.8), is first given and used in BLEISTEIN [1]. If we use (2.8) in (2.6) and integrate n-times by parts, we obtain

$$J_\nu(z) = \mathrm{Ai}(z^{\frac{2}{3}} b^2) \sum_{k=0}^{n-1} (-1)^k \alpha_k z^{-k-\frac{1}{3}} - \mathrm{Ai}'(z^{\frac{2}{3}} b^2) \sum_{k=0}^{n-1} (-1)^k \beta_k z^{-k-\frac{2}{3}} + \varepsilon_n, \tag{2.9}$$

where

$$\varepsilon_n = (-1)^n z^{-n} \frac{1}{2\pi i} \int_{\mathcal{L}} e^{z(\frac{1}{3} w^3 - b^2 w)} h_n(w)\, dw, \tag{2.10}$$

and where $\mathrm{Ai}(z)$ is the Airy function, and $\mathrm{Ai}'(z)$ its derivative. The functions $h_n(w)$ are, by inheritance, analytic functions on the same neighbourhood of $\mathcal{L}$ where h_0 is an analytic function.

Estimates of $|\varepsilon_n|$, for large values of z and for $b \in [0, \infty)$, given in the literature, are usually of the form

$$|\varepsilon_n| \le \frac{M_n}{z^{n+\frac{1}{3}}} \widetilde{\alpha}_n(\alpha) |\mathrm{Ai}(z^{\frac{2}{3}} b^2)| + \frac{N_n}{z^{n+\frac{2}{3}}} \widetilde{\beta}_n(\alpha) |\mathrm{Ai}'(z^{\frac{2}{3}} b^2)|, \tag{2.11}$$

where M_n and N_n depend on n, and where $\widetilde{\alpha}_n$, $\widetilde{\beta}_n$ are related with the coefficients in (2.9). A proof of an estimate, like (2.11), is given in [4], with

$$\widetilde{\alpha}_n(\alpha) = \begin{cases} 1 & \text{if } 0 < \alpha < \xi, \\ |\alpha_n| & \text{if } \alpha > \xi, \end{cases} \qquad \widetilde{\beta}_n(\alpha) = \begin{cases} 1 & \text{if } 0 < \alpha < \xi, \\ |\beta_n| & \text{if } \alpha > \xi, \end{cases}$$

where ξ is a fixed positive number.

In our analysis we take into account the singularities of $h_0(w)$, in fact the distance from the singularities to the saddle points at $\pm b$. In this way we construct a new bound of the remainder ε_n, and we prove that (2.9) holds uniformly with respect to $b \in [0, \infty)$.

The relevant w-singularities $S_\pm$ of mapping (2.3) come from $x = \alpha \pm 2\pi i$. Calculations give that

$$|b - S_\pm| \sim \tfrac{2}{3}\sqrt{\pi} \frac{b}{\sqrt{\ln b}}, \qquad \text{as} \quad b \to \infty.$$

When $b = 0$ we have $|S_\pm| = (6\pi)^{\frac{1}{3}}$. So, we can choose a fixed number $\delta \in (0, \frac{2}{3}\sqrt{\pi})$, such that $h_0(w)$ is analytic on a disc around b with radius

$$\rho(b) = \frac{\delta(b+1)}{\sqrt{\ln(b+2)}}, \tag{2.12}$$

for all $b \in [0, \infty)$.

In order to obtain the required estimate of $|\varepsilon_n|$, we split up contour $\mathcal{L}$ in $\mathcal{L}'$ and $\mathcal{L}''$, where $\mathcal{L}' = \{w \in \mathcal{L} \mid |w-b| \le \rho(b)\}$, and where $\mathcal{L}'' = \mathcal{L} - \mathcal{L}'$. Furthermore, we define corresponding integrals

$$\begin{aligned}
\varepsilon_{n|_{\mathcal{L}'}} &= (-1)^n z^{-n} \frac{1}{2\pi i} \int_{\mathcal{L}'} e^{z(\frac{1}{3}w^3 - b^2 w)} h_n(w) dw, \\
\varepsilon_{n|_{\mathcal{L}''}} &= (-1)^n z^{-n} \frac{1}{2\pi i} \int_{\mathcal{L}''} e^{z(\frac{1}{3}w^3 - b^2 w)} h_n(w) dw,
\end{aligned} \tag{2.13}$$

For determining the behavior of $h_0(w)$ on this disc around b, for large values of b, we write $w = b + ab$, where $a = a'/\sqrt{(\ln b)}$, $0 < |a'| \le \delta$. Let $x = \alpha + k$ correspond, by (2.3), with this w. Then we have

$$\sinh(\alpha + k) - (\alpha + k)\cosh(\alpha) = \tfrac{1}{3}(b + ab)^3 - b^2(b + ab),$$

and with (2.5) we obtain

$$\sinh(\alpha + k) - \sinh(\alpha) - k\cosh(\alpha) = b^3(a^2 + \tfrac{1}{3}a^3).$$

It follows that $\frac{1}{2}e^{\alpha+k} \sim b^3(a^2 + \frac{1}{3}a^3)$, as $b \to \infty$. Substitution of this result in (2.4) yields

$$\begin{aligned}
h_0(b + ab) &= \frac{ab(2b + ab)}{2\sinh(\frac{k}{2})\sinh(\frac{2\alpha+k}{2})} \sim (2a + a^2)\frac{2b^2}{e^{\alpha+k}} \\
&\sim \frac{2a + a^2}{(a^2 + \frac{1}{3}a^3)b} = \mathcal{O}(b^{-1}\sqrt{\ln b}),
\end{aligned}$$

as $b \to \infty$. Using l'Hôpital's rule, we calculate

$$h_0(b) = \sqrt{\frac{2b}{\sinh \alpha}} \sim 3b^{-1}\sqrt{\ln b}, \qquad \text{as} \quad b \to \infty.$$

So we have proved that

$$\sup_{|w-b| \le \frac{1}{2}\rho(b)} |h_0(w)| \le C_0(\delta)|h_0(b)|, \qquad \text{as} \quad b \to \infty, \tag{2.14}$$

where $C_0(\delta)$ does not depend on b. For obtaining similar estimates for $h_n(w)$, we now introduce a new class of rational functions.

3. Intermezzo: a new class of rational functions

We introduce a class of rational functions, which fulfil the following theorem.

Theorem 3.1. *Let*

$$\begin{aligned}
R_0(u, w, b) &= \frac{1}{u - w}, \\
R_{n+1}(u, w, b) &= \frac{-1}{u^2 - b^2} \frac{d}{du} R_n(u, w, b), \qquad n = 0, 1, 2, \ldots,
\end{aligned} \tag{3.1}$$

where $u, w, b \in \mathbb{C}$. Let $h_n(w)$ be defined by the recursive scheme (2.8), with $h_0(w)$ a given analytic function in a domain G. Then we have

$$h_n(w) = \frac{1}{2\pi i}\int_C R_n(u,w,b)h_0(u)du,$$

where C is a simple closed contour in G which encircles the points w and $\pm b$.

PROOF.

$$\begin{aligned}
h_n(w) &= \frac{1}{2\pi i}\int_C R_0(u,w,b)h_n(u)du = \frac{1}{2\pi i}\int_C R_0(u,w,b)\frac{d}{du}g_{n-1}(u)du \\
&= \frac{1}{2\pi i}\int_C R_1(u,w,b)h_{n-1}(u)du - \frac{1}{2\pi i}\int_C R_1(u,w,b)(\alpha_{n-1}+\beta_{n-1}u)du \\
&=_* \frac{1}{2\pi i}\int_C R_1(u,w,b)h_{n-1}(u)du \\
&\;\vdots \\
&= \frac{1}{2\pi i}\int_C R_n(u,w,b)h_0(u)du
\end{aligned}$$

In * we used that the rational function $R_1(u,w,b)(\alpha_{n-1}+\beta_{n-1}u)$ is $\mathcal{O}(u^{-2})$ as $|u| \to \infty$, and that C encircles the poles of this function. Thus the transformation $u \mapsto u^{-1}$ is well defined at $u = \infty$ and yields an integral with no singularities inside the contour of integration. ∎

In some sense, these rational functions are related to the rational functions introduced in SONI & TEMME [7]. By induction with respect to n, it follows that R_n has an expansion of the form

$$R_n(u,w,b) = \sum_{i=0}^{n-1}\sum_{j=0}^{k_{n,i}} \frac{C_{ij}u^{i-j}}{(u-w)^{n+1-i-j}(u^2-b^2)^{n+i}}, \quad n = 1,2,\ldots, \tag{3.2}$$

with $k_{n,i} = \min(i, n-1-i)$ and where C_{ij} do not depend on u, w and b. Now we can give estimates for R_n, which are needed in the next sections for obtaining estimates of the form (2.14), where, in the left-hand side, $h_0(w)$ is replaced by $h_n(w)$.

Let $w \in \mathcal{L}$ such that $|w-b| = \mathcal{O}(b)$, as $|b| \to \infty$, and let Γ be a simple closed contour that encircles b and w and with $-b$ in its exterior. Because of 0 is not a singularity of R_n we can presume that Γ also encircles 0. See Figure 3.1.

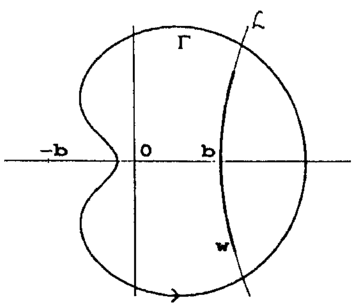

FIGURE 3.1. Contour Γ.

We have for $n = 1, 2, \ldots$,

$$\begin{aligned}
\frac{1}{2\pi i}\int_\Gamma R_n(u,w,b)du &= \frac{1}{2\pi i}\int_{\Gamma^{-1}} \frac{R_n(\frac{1}{z},w,b)}{z^2}dz \\
&= \sum_{i,j} C_{i,j}\frac{1}{2\pi i}\int_{\frac{1}{\Gamma}} \frac{z^{3n-1}}{(1-zw)^{n+1-i-j}b^{2n+2i}(b^{-2}-z^2)^{n+i}}dz \\
&=_* \sum_{i,j} C_{i,j}\frac{d^{n+i-1}}{dz^{n+i-1}}\Big(\frac{z^{3n-1}}{(1-zw)^{n+1-i-j}b^{2n+2i}(b^{-1}-z)^{n+i}}\Big)\Big|_{z=-b^{-1}} \\
&= \mathcal{O}(|b|^{-3n}),
\end{aligned} \tag{3.3}$$

as $|b| \to \infty$, where Γ^{-1} is a simple closed contour that encircles $-b^{-1}$. We have used Cauchy's integral formula in $*$. The following two lemma's are also proved by using (3.2).

Lemma 3.1. *Let $b \in \mathbb{C}$ and $\Omega(b) = \{(u,w) \in \mathbb{C}^2 \mid |u-b| = \rho(b),\ |w-b| \le \frac{1}{2}\rho(b)\}$, such that $\rho(b) \sim \delta|b|^\varepsilon$, as $|b| \to \infty$, where δ, ε are constants, $\delta > 0$, $-\frac{1}{2} < \varepsilon \le 1$. Then,*

$$\sup_{(u,w)\in\Omega(b)} |R_n(u,w,b)| \le A_n(\delta)|b|^{-(1+2\varepsilon)n-\varepsilon},$$

as $|b| \to \infty$, where $A_n(\delta)$ does not depend on b.

Lemma 3.2. *Let $b > 0$ and $\Omega(b) = \{(u,w) \in \mathbb{C}^2 \mid |u-b| = \rho(b),\ |w-b| \le \frac{1}{2}\rho(b)\}$, such that $\rho(b) \sim \delta b/\sqrt{\ln b}$, as $b \to \infty$, where δ is a constant, $\delta > 0$. Then,*

$$\sup_{(u,w)\in\Omega(b)} |R_n(u,w,b)| \le B_n(\delta)(\ln b)^{n+\frac{1}{2}} b^{-3n-1},$$

as $b \to \infty$, where $B_n(\delta)$ does not depend on b.

4. Uniform Airy function expansion of the Bessel function, part 2

We return to the problem of finding an upper bound of ε_n defined in (2.10). Let Γ be a circle around b with radius $\rho(b)$, where $\rho(b)$ is given in (2.12), and let $|w-b| \le \frac{1}{2}\rho(b)$. In the following we use (3.3), and we have

$$\begin{aligned}
h_n(w) &= \frac{1}{2\pi i}\int_\Gamma R_0(u,w,b)h_n(u)du \\
&= \frac{1}{2\pi i}\int_\Gamma R_1(u,w,b)h_{n-1}(u)du - \frac{1}{2\pi i}\int_\Gamma R_1(u,w,b)(\alpha_{n-1}+\beta_{n-1}u)du \\
&=_{(3.3)} \frac{1}{2\pi i}\int_\Gamma R_1(u,w,b)h_{n-1}(u)du + \widetilde{h}_{n-1}\mathcal{O}(b^{-3}) \\
&\ \vdots \\
&=_{(3.3)} \frac{1}{2\pi i}\int_\Gamma R_n(u,w,b)h_0(u)du + \widetilde{h}_{n-1}\mathcal{O}(b^{-3}) + \ldots + \widetilde{h}_0\mathcal{O}(b^{-3n}),
\end{aligned} \tag{4.1}$$

as $b \to \infty$, where $\widetilde{h}_m = \sup_{|w\pm b|\le \frac{1}{2}\rho(b)} |h_m(w)|$. Notice that by (2.14) and by similar estimates in the neighbourhood of $-b$, we have $\widetilde{h}_0 = |h_0(b)|\mathcal{O}(1)$, as $b \to \infty$, and notice that (4.1) also holds for $h_n(w)$, where $|w+b| \le \frac{1}{2}\rho(b)$. So, by induction and lemma 3.2, we have

$$\sup_{|w\pm b|\le \frac{1}{2}\rho(b)} |h_n(w)| \le C_n(\delta)(\ln b)^n b^{-3n}|h_0(b)|, \qquad \text{as} \quad b \to \infty, \tag{4.2}$$

where $C_n(\delta)$ does not depend on w and b. Substituting this relation in (2.13) we obtain

$$\begin{aligned} |\varepsilon_{n|_{\mathcal{L}'}}| &\le C_n(\delta) z^{-n}(\ln b)^n b^{-3n}|h_0(b)|\frac{1}{2\pi i}\int_{\mathcal{L}'} e^{z(\frac{1}{3}w^3-b^2w)}dw \\ &\le C_n(\delta) z^{-n-\frac{1}{3}}(\ln b)^n b^{-3n}|h_0(b)|\mathrm{Ai}(z^{\frac{2}{3}}b^2). \end{aligned} \tag{4.3}$$

In the appendix we shall prove that

$$|\varepsilon_{n|_{\mathcal{L}''}}| \le C'_n z^{-n-\frac{1}{3}} e^{\lambda(2-z)b^3(\ln b)^{-\frac{1}{2}}}|h_0(b)|\mathrm{Ai}(z^{\frac{2}{3}}b^2), \tag{4.4}$$

where the positive C'_n and λ do not depend on b and z.

The main result of the preceding analysis is

$$|\varepsilon_n| \le C_n(\delta) z^{-n-\frac{1}{3}}(\ln b)^n b^{-3n}|h_0(b)|\mathrm{Ai}(z^{\frac{2}{3}}b^2), \tag{4.5}$$

with a slightly different $C_n(\delta)$, due to the influence of the $\mathcal{L}''$−integral. This estimate is valid as $z \to \infty$, uniformly with respect to $b \in [c, \infty)$, $c > 0$ fixed. Notice that with the exception of $C_n(\delta)$, of which we are unable to give an estimate, all functions in the right-hand side of (4.5) are known. So this estimate is less complicated than (2.11), where, in generally, α_n and β_n are complicated functions of b. In the next section we will prove a more general result, which is also valid in a larger b domain.

REMARK 1. In OLVER [6] asymptotic expansions are given of $J_\nu(\nu a)$ as $\nu \to \infty$, uniformly with respect to $a \in [0, \infty)$, and also for complex values of the parameters. In the above sections (and in [3]) z is the large parameter and the expansion holds uniformly with respect to $\nu \in [z, \infty)$.

REMARK 2. As remarked earlier, in this analysis we concentrate on $b \to \infty$. However, by replacing in (4.5) $(\ln b)^n b^{-3n}$ with, say, $(\ln(b+2))^n(b+1)^{-3n}$, it easily follows that (4.5) holds uniformly with respect to $b \in [0, \infty)$.

5. General uniform Airy function expansion

We generalize the analysis by taking in (1.3) a general function $h_0(w)$, and we prescribe that $b \in [0, \infty) \cup [0, i\infty)$. This includes the oscillatory case, since, when $b \in [0, i\infty)$, the argument of the Airy functions is negative.

Let

$$F(z, b) = \frac{1}{2\pi i}\int_{\mathcal{L}} e^{z(\frac{1}{3}w^3-b^2w)} h_0(w)dw, \tag{5.1}$$

where $h_0(w)$ is an analytic function on a neighbourhood of $\mathcal{L}$. In the case that $b \in [0, \infty)$ we take $\mathcal{L}$ as in (2.7), and in the case that $b \in [0, i\infty)$ we take $\mathcal{L} = \{w = x + iy \in \mathbb{C} \mid 3yx^2 = (y \pm ib)^2(y \mp 2ib)\}$, the steepest descent contour through $\pm b$. See Figure 5.1.

REMARK. In fact, it is not needed to restrict our analysis to these contours of integration, but using these steepest descent contours makes the following calculations less complicated.

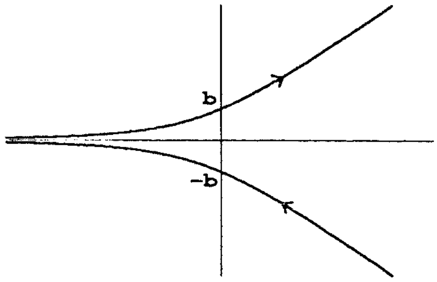

FIGURE 5.1. Steepest descent curve $\mathcal{L}$, as $b \in [0, i\infty)$.

Again, if we use (2.8) in (5.1), we obtain

$$\begin{aligned} F(z,b) =& \mathrm{Ai}(z^{\frac{2}{3}}b^2)\sum_{k=0}^{n-1}(-1)^k\alpha_k z^{-k-\frac{1}{3}} \\ &- \mathrm{Ai}'(z^{\frac{2}{3}}b^2)\sum_{k=0}^{n-1}(-1)^k\beta_k z^{-k-\frac{2}{3}} \\ &+ \varepsilon_n, \end{aligned} \tag{5.2}$$

where ε_n is as in (2.10). We formulate conditions on $h_0(w)$ such that expansion (5.2) (with $z \to \infty$) is uniformly valid with respect to $b \in [0,\infty) \cup [0, i\infty)$. As before, we want to split up the contour of integration. In the case that $b \in [0,\infty)$, we take subcontour $\mathcal{L}' = \{w \in \mathcal{L} \mid |w-b| \le \frac{1}{2}\rho(b)\}$. In the other case, where $b \in [0, i\infty)$, we take $\mathcal{L}' = \{w \in \mathcal{L} \mid |w \pm b| \le \frac{1}{2}\rho(b)\}$; $\rho(b)$ is related with the distance from the singularities of $h_0(w)$ to the saddle points $\pm b$. Consequently, we define $\mathcal{L}'' = \mathcal{L} - \mathcal{L}'$ and $\varepsilon_{n|_{\mathcal{L}'}}$, $\varepsilon_{n|_{\mathcal{L}''}}$ similar to (2.13).

In the appendix we formulate conditions on $h_0(w)$ such that the estimate of $|\varepsilon_{n|_{\mathcal{L}''}}|$, which is given in the appendix, is exponentially small, compared with the estimate of $|\varepsilon_{n|_{\mathcal{L}'}}|$, as $z \to \infty$, uniformly with respect to b.

We define

$$\rho_0(b) = \min\{|w \pm b| \mid w \text{ is a singularity of } h_0(w)\}, \tag{5.3}$$

and we assume that, for large $|b|$, we have $\rho_0(b) > |b|^\theta$, where the constant $\theta > -\frac{1}{2}$. This is the essential assumption on $h_0(w)$ in the neighbourhood of the saddle points.

We take $\rho(b) \le \rho_0(b)$ such that $\rho(b) \sim 2\delta|b|^\varepsilon$ as $|b| \to \infty$, where the constants δ and ε satisfy $\delta > 0$ and $-\frac{1}{2} < \varepsilon \le 1$. We take ε as large as possible. Notice that we concentrate on estimates with $|b| \to \infty$, and that we do not give details for b in compacta.

REMARK 1. We assume that $\varepsilon > -\frac{1}{2}$, in order that the estimate of $|\varepsilon_{n|_{\mathcal{L}''}}|$ is exponentially small, compared with the estimate of $|\varepsilon_{n|_{\mathcal{L}'}}|$, as $z \to \infty$, uniformly with respect to b.

REMARK 2. We require $\varepsilon \le 1$ to ensure that, for large $|b|$, we can draw a circle with center $\pm b$ and radius $\rho(b)$, such that the other saddle point $\mp b$ is outside this circle. This is possible by choosing δ appropriately.

Next we introduce upper bounds for the $h_n(w)$, $n = 0, 1, 2, \ldots$ Thus let

$$\widetilde{h}_n(\delta) = \sup_{|w \pm b| \le \frac{1}{2}\rho(b)} |h_n(w)|. \tag{5.4}$$

Notice that $h_0(w)$ is analytic on $|w \pm b| < \rho(b)$, thus $\widetilde{h}_0(\delta)$ is finite.

Let Γ be a circle around $\pm b$ with radius $\rho(b)$, and let $|w \mp b| \le \frac{1}{2}\rho(b)$. Then, if we use (3.3), we have

$$\begin{aligned}
h_n(w) &= \frac{1}{2\pi i} \int_\Gamma R_0(u, w, b) h_n(u) du \\
&= \frac{1}{2\pi i} \int_\Gamma R_1(u, w, b) h_{n-1}(u) du - \frac{1}{2\pi i} \int_\Gamma R_1(u, w, b)(\alpha_{n-1} + \beta_{n-1} u) du \\
&=_{(3.3)} \frac{1}{2\pi i} \int_\Gamma R_1(u, w, b) h_{n-1}(u) du + \widetilde{h}_{n-1}(\delta) \mathcal{O}(|b|^{-3}) \\
&\vdots \\
&=_{(3.3)} \frac{1}{2\pi i} \int_\Gamma R_n(u, w, b) h_0(u) du + \widetilde{h}_{n-1}(\delta) \mathcal{O}(|b|^{-3}) + \ldots + \widetilde{h}_0(\delta) \mathcal{O}(|b|^{-3n}),
\end{aligned}$$

as $|b| \to \infty$. So, by induction and lemma 3.1 we have

$$\widetilde{h}_n(\delta) \le C_n(\delta) |b|^{-(1+2\varepsilon)n} \widetilde{h}_0(\delta), \qquad \text{as} \quad |b| \to \infty, \tag{5.5}$$

where $C_n(\delta)$ does not depend on b.

Now we shall prove that ε_n can be bounded as follows:

$$|\varepsilon_n| \le C(n, \delta)(|b| + 1)^{-(1+2\varepsilon)n} \widetilde{h}_0(\delta) z^{-n-\frac{1}{3}} \widetilde{\mathrm{Ai}}(z^{\frac{2}{3}} b^2), \tag{5.6}$$

where $C(n, \delta)$ does not depend on b and z, and where

$$\widetilde{\mathrm{Ai}}(u) = \begin{cases} \mathrm{Ai}(u) & \text{if } u \ge 0, \\ \sqrt{\mathrm{Ai}^2(u) + \mathrm{Bi}^2(u)} & \text{if } u < 0. \end{cases} \tag{5.7}$$

When b is bounded then so is $(|b| + 1)^{-(1+2\varepsilon)n} \widetilde{h}_0(\delta)$, and the proof of (5.6) is much simpler. A similar bound (without explicit indication of the role of b, in the form of $(|b| + 1)^{-(1+2\varepsilon)n})$) can be found in [4].

The proof for large b is divided into separate cases: (i) $b \in [0, \infty)$, and (ii) $b \in [0, i\infty)$. We first consider case (i), and with (5.5) we have

$$\begin{aligned}
|\varepsilon_n|_{\mathcal{L}'}| &\le C_n(\delta) z^{-n} |b|^{-(1+2\varepsilon)n} \widetilde{h}_0(\delta) \frac{1}{2\pi i} \int_{\mathcal{L}'} e^{z(\frac{1}{3}w^3 - b^2 w)} dw \\
&\le C_n(\delta) z^{-n-\frac{1}{3}} |b|^{-(1+2\varepsilon)n} \widetilde{h}_0(\delta) \mathrm{Ai}(z^{\frac{2}{3}} b^2).
\end{aligned}$$

In case (ii) we write $w = x + iy$ and we define $\mathcal{L}'_+ = \{y > 0 \mid \exists x \in \mathbb{R} \; : \; x + iy \in \mathcal{L}'\}$. Simple transformations give

$$\begin{aligned}
\frac{1}{2\pi i} \int_{\mathcal{L}'} e^{z(\frac{1}{3}w^3 - b^2 w)} h_n(w) dw &= \frac{1}{2\pi i} \int_{\mathcal{L}'_+} e^{-z(y+ib)^2 f(y)} g(y) (e^{-\frac{2}{3}zb^3} h_n(w) - e^{+\frac{2}{3}zb^3} h_n(\overline{w})) dy \\
&+ \frac{1}{2\pi} \int_{\mathcal{L}'_+} e^{-z(y+ib)^2 f(y)} (e^{-\frac{2}{3}zb^3} h_n(w) + e^{+\frac{2}{3}zb^3} h_n(\overline{w})) dy,
\end{aligned}$$

where $f(y) = \frac{2(2y-ib)^2}{9y}\sqrt{\frac{y-2ib}{3y}}$ and $g(y) = \sqrt{\frac{y-2ib}{3y}} + \frac{ib(y+ib)}{3y^2}\sqrt{\frac{3y}{y-2ib}} \geq 0$. Thus with (5.5) we have

$$\begin{aligned}
\left|\frac{1}{2\pi i}\int_{\mathcal{C}'} e^{z(\frac{1}{3}w^3-b^2w)}h_n(w)dw\right| &\leq \frac{1}{2\pi}\int_{\mathcal{C}'_+} e^{-z(y+ib)^2f(y)}(1+g(y))(|h_n(w)|+|h_n(\overline{w})|)dy \\
&\leq_{(5.5)} C_n(\delta)|b|^{-(1+2\varepsilon)n}\widetilde{h}_0(\delta)\frac{1}{\pi}\int_0^\infty e^{-z(y+ib)^2f(y)}(1+g(y))dy \\
&\leq C'_n(\delta)|b|^{-(1+2\varepsilon)n}\widetilde{h}_0(\delta)\frac{1}{\sqrt{zb/i}} \\
&\sim_* \pi^{\frac{1}{2}}C'_n(\delta)|b|^{-(1+2\varepsilon)n}\widetilde{h}_0(\delta)z^{-\frac{1}{3}}\widetilde{\mathrm{Ai}}(z^{\frac{2}{3}}b^2),
\end{aligned}$$

as $z \to \infty$. In $*$ we used the relation $\widetilde{\mathrm{Ai}}(x) \sim \pi^{-\frac{1}{2}}(-x)^{-\frac{1}{4}}$ as $x \to -\infty$, which can be found in [6, p. 395].

In the appendix we shall prove that

$$|\varepsilon_{n|_{\mathcal{C}''}}| \leq C_n e^{\lambda(\mu-z)|b|^{2\varepsilon+1}}\widetilde{h}_0(\delta)z^{-n-\frac{1}{3}}\widetilde{\mathrm{Ai}}(z^{\frac{2}{3}}b^2), \tag{5.8}$$

where the positive C_n, λ and μ do not depend on b and z, and where $|b| \geq c > 0$. These estimates show that (5.6) is valid. Thus we have proved the following theorem.

Theorem 5.1. *Let $F(z,b)$ be of the form (5.1), where $h_0(w)$ fulfils the conditions mentioned in the beginning of this section. Then we have (5.2) as a uniform asymptotic expansion for $F(z,b)$, where (5.6) is an estimate for $|\varepsilon_n|$ as $z \to \infty$, uniformly with respect to $b \in [0,\infty) \cup [0,i\infty)$, and where $\widetilde{h}_0(\delta)$ is given in (5.4).*

REMARK. With the conditions of this theorem it follows that expansion (5.2) has a double asymptotic property: the roles of b and z can be interchanged. For an example we refer to sections 2 and 4.

The double asymptotic property is lost in the example considered in the next section. In the preceding analysis we had to assume that, for large $|b|$, $\rho_0(b)$, given in (5.3), is at least of order $|b|^\theta$, where $\theta > -\frac{1}{2}$. In the next section we have $\theta = -\frac{1}{2}$.

6. A Boundary case

In this section we show that, in certain circumstances, the condition $\theta > -\frac{1}{2}$ of theorem 5.1 can be replaced with $\theta = -\frac{1}{2}$. We demonstrate this feature by considering a recent expansion for the Laguerre polynomials.

First we summarize the main steps for obtaining an Airy function expansion of the Laguerre polynomials. More details are given in [4] and [8]. Laguerre polynomials have the following integral representation.

$$(-1)^N 2^\alpha e^{-zt/2}L_N^{(\alpha)}(zt) = \frac{1}{2\pi i}\int_{+\infty}^{(1+)} e^{zf(x,t)}(1-x^2)^{\frac{\alpha-1}{2}}dx, \tag{6.1}$$

where the contour of integration begins and ends at $+\infty$ and encircles 1 in the positive direction, and where

$$f(x,t) = \tfrac{1}{4}\ln\left(\frac{1+x}{1-x}\right) - \tfrac{1}{2}xt, \tag{6.2}$$

and $z = 4N + 2\alpha + 2$, $\alpha > -1$ and $t \geq 1$. Again, we use the transformation

$$f(x,t) = \tfrac{1}{3}w^3 - b^2 w. \tag{6.3}$$

We prescribe that the x-saddle points $\pm\sqrt{1-1/t}$ must correspond with $\pm b$. It follows that

$$b^3 = \tfrac{3}{4}(\sqrt{t^2-t} - \operatorname{arccosh}\sqrt{t}). \tag{6.4}$$

With transformation (6.3) we have for (6.1)

$$(-1)^N 2^\alpha e^{-zt/2} L_N^{(\alpha)}(zt) = \frac{1}{2\pi i}\int_{\mathcal{L}} e^{z(\frac{1}{3}w^3 - b^2 w)} h_0(w)\,dw, \tag{6.5}$$

where

$$h_0(w) = (1-x^2)^{\frac{\alpha-1}{2}} \frac{dx}{dw} = 2\frac{(1-x^2)^{\frac{\alpha+1}{2}}(w^2-b^2)}{1-t(1-x^2)}, \tag{6.6}$$

and $\mathcal{L}$ is given in (2.7). Again, using (2.8) in (6.5), we obtain

$$\begin{aligned}(-1)^N 2^\alpha e^{-zt/2} L_N^{(\alpha)}(zt) =& \mathrm{Ai}(z^{\frac{2}{3}} b^2) \sum_{k=0}^{n-1} (-1)^k \alpha_k z^{-k-\frac{1}{3}} \\ &- \mathrm{Ai}'(z^{\frac{2}{3}} b^2) \sum_{k=0}^{n-1} (-1)^k \beta_k z^{-k-\frac{2}{3}} \\ &+ \varepsilon_n,\end{aligned} \tag{6.7}$$

where ε_n is as in (2.10). Notice that $h_0(w)$ is an even analytic function. Consequently, we have

$$\begin{aligned} \alpha_{2m} &= h_{2m}(b), & \alpha_{2m+1} &= 0, \\ \beta_{2m} &= 0, & \beta_{2m+1} &= h_{2m+1}(b)/b. \end{aligned}$$

In order to apply the analysis of the previous section, we locate the relevant singular points of $h_0(w)$. The singularities $S_\pm$, in the w-plane, which are nearest to b, satisfy $\frac{1}{3}S_\pm^3 = \pm\frac{1}{2}\pi i$ as $b = 0$, and

$$S_\pm - b \sim \pm\sqrt{\frac{\pi i}{2b}}, \qquad \text{as} \quad b \to \infty. \tag{6.8}$$

Thus $\rho_0(b)$ of (5.3) is of order $b^{-\frac{1}{2}}$, as $b \to \infty$.

As before, we want to split up $\mathcal{L}$ in $\mathcal{L}'$ and $\mathcal{L}''$, and we define $\varepsilon_{n|_{\mathcal{L}'}}$, $\varepsilon_{n|_{\mathcal{L}''}}$ similar to (2.13). So, define $\mathcal{L}' = \{w \in \mathcal{L} \mid |w-b| \leq \delta b^\varepsilon\}$, where the constants δ and ε satisfy $\delta > 0$ and $-\frac{1}{2} < \varepsilon \leq 1$, in order that the estimate of $|\varepsilon_{n|_{\mathcal{L}''}}|$ is exponentially small, compared with the estimate of $|\varepsilon_{n|_{\mathcal{L}'}}|$, as $z \to \infty$, uniformly with respect to b. We choose ε close to $-\frac{1}{2}$ fixed.

Let Γ_ε be a closed contour, which encircles $\mathcal{L}'$, such that

$$\text{length}\,\Gamma_\varepsilon = \mathcal{O}(b^\varepsilon), \qquad \text{distance}(\Gamma_\varepsilon, \mathcal{L}') \sim cb^{-\frac{1}{2}}, \qquad \text{as} \quad b \to \infty,$$

and such that $h_0(w)$ is analytic on $\overline{I(\Gamma_\varepsilon)}$, where $\overline{I(\Gamma_\varepsilon)}$ is the closure of the interior of Γ_ε. Then straight forward calculations give that

$$\sup_{w\in\overline{I(\Gamma_\varepsilon)}} |h_0(w)| \le C_0(\delta,\varepsilon) b^{(\varepsilon+\frac{1}{2})\alpha} |h_0(b)|, \qquad \text{as} \quad b\to\infty, \tag{6.9}$$

where $C_0(\delta,\varepsilon)$ does not depend on b. Further calculations, which are similar to those in section 5, yield for $n = 1, 2, ...$

$$\sup_{w\in\overline{I(\Gamma_\varepsilon)}} |h_n(w)| \le C_n(\delta,\varepsilon) b^{(\varepsilon+\frac{1}{2})(\alpha+1)} |h_0(b)|, \qquad \text{as} \quad b\to\infty, \tag{6.10}$$

where, here and below, $C_n(\delta,\varepsilon)$ denotes a generic quantity not depending on b and z. Notice that, in contrast to (5.5), the power of b is positive, and is not depending on n. These estimates yield

$$\begin{aligned} |\varepsilon_{n|_{\mathcal{L}'}}| &\le C_n(\delta,\varepsilon) z^{-n-\frac{1}{3}} b^{(\varepsilon+\frac{1}{2})(\alpha+1)} |h_0(b)| \mathrm{Ai}(z^{\frac{2}{3}} b^2) \\ &= C_n(\delta,\varepsilon) z^{-n-\frac{1}{3}} b^{(\varepsilon-1)\alpha+(\varepsilon-\frac{1}{2})} \mathrm{Ai}(z^{\frac{2}{3}} b^2), \end{aligned} \tag{6.11}$$

In the last equation we used

$$h_0(b) = t^{\frac{(1-\alpha)}{2}} \frac{\sqrt{2b}}{(t-1)^{\frac{1}{4}} t^{\frac{3}{4}}}.$$

In the appendix we shall prove that

$$|\varepsilon_{n|_{\mathcal{L}''}}| \le C_n'(\delta,\varepsilon) z^{-n-\frac{1}{3}} e^{\lambda(2-z)b^{2\varepsilon+1}} |h_0(b)| \mathrm{Ai}(z^{\frac{2}{3}} b^2), \tag{6.12}$$

where the positive C_n' and λ do not depend on b and z. Thus we have proved that

$$|\varepsilon_n| \le C_n(\delta,\varepsilon) z^{-n-\frac{1}{3}} (b+1)^{(\varepsilon-1)\alpha+(\varepsilon-\frac{1}{2})} \mathrm{Ai}(z^{\frac{2}{3}} b^2), \tag{6.13}$$

as $z\to\infty$, uniformly with respect to $b\in[c,\infty)$, where $c>0$, c fixed. With some extra work we can prove that (6.13) holds uniformly with respect to $b\in[0,\infty)$. A similar approach can be used for $b\in[0,i\tau]$, where $0<\tau<(\frac{3}{8}\pi)^{\frac{1}{3}}$, τ fixed.

We can compare this estimate with the estimate given in [4] and [8], which is of the form (2.11). Firstly, we notice that (6.13) is not in terms of the first neglected terms of expansion (6.7). But with (6.10) it easily follows that the first neglected terms can be estimated by the right-hand side of (6.13), and in that sense (6.13) clearly shows why expansion (6.7) holds uniformly with respect to b in an unbounded domain. Secondly, in (6.13), the influence of b is more transparent than in the right-hand side of (2.11).

7. Some remarks on other uniform expansions generated by the Bleistein method

The techniques used in section 5 for obtaining an estimate for the remainder, are applied on Airy type expansions. In this section we show that the method is quite general, and can be applied to other uniform expansions of integrals of the form

$$\int_{\mathcal{C}} e^{zf(x,b)} h_0(x)\,dx \tag{7.1}$$

with coinciding saddle points and singularities. By using Bleistein's method, based on integrating by parts, rational functions similar to those of section 3 arise, and again the h_n-functions can be represented as in theorem 3.1.

In the remaining part of this section we work out an example of uniform expansions in terms of Bessel functions. In [4] such an expansion of the Laguerre polynomials is given. Let

$$F(z,A) = \frac{1}{2\pi i}\int_{-\infty}^{(0+)} w^{-\alpha-1}h_0(w)e^{\frac{1}{2}z(w-\frac{A^2}{w})}\,dw, \tag{7.2}$$

where the contour of integration begins and ends at $-\infty$ and encircles the origin in the positive direction. We assume that $h_0(w)$ is analytic on a neighbourhood of the contour of integration, and let $z > 0$, $iA > 0$ and $\alpha > -1$. Notice that $\pm iA$ are the saddle points of the integral. We choose the contour of integration through these saddle points, and the steepest descents path looks like Figure 7.1.

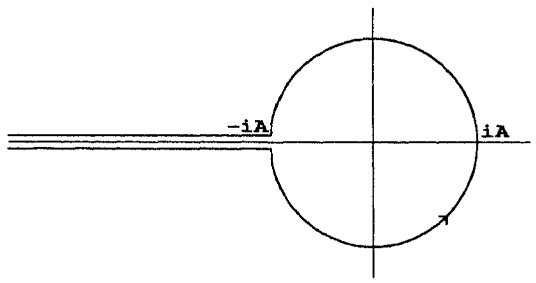

FIGURE 7.1. Steepest descent curve for integral (7.2)

The recursion in connection with integral (7.2) is

$$\begin{aligned} h_n(w) &= \alpha_n + \frac{\beta_n}{w} + (1+\frac{A^2}{w^2})g_n(w), \\ h_{n+1}(w) &= w^{\alpha+1}\frac{d}{dw}(w^{-\alpha-1}g_n(w)), \end{aligned} \tag{7.3}$$

and if we integrate $n-$times by parts, we obtain the expansion

$$\begin{aligned} F(z,A) = &\frac{\mathrm{J}_\alpha(zA)}{A^\alpha}\sum_{k=0}^{n-1}(-1)^k\alpha_k(\frac{2}{z})^k \\ &+ \frac{\mathrm{J}_{\alpha+1}(zA)}{A^{\alpha+1}}\sum_{k=0}^{n-1}(-1)^k\beta_k(\frac{2}{z})^k \\ &+ \varepsilon_n, \end{aligned} \tag{7.4}$$

where

$$\varepsilon_n = (-1)^n(\frac{2}{z})^n\frac{1}{2\pi i}\int_{-\infty}^{(0+)} w^{-\alpha-1}h_n(w)e^{\frac{1}{2}z(w-\frac{A^2}{w})}\,dw, \tag{7.5}$$

and where $\mathrm{J}_\alpha(z)$ and $\mathrm{J}_{\alpha+1}(z)$ are Bessel functions of the first kind. Since zA is purely imaginary, in fact modified Bessel functions occur in the expansion.

The class of rational functions generated by (7.3) is

$$\begin{aligned} Q_0(u,w,A) &= \frac{1}{u-w}, \\ Q_{n+1}(u,w,A) &= \frac{-(\frac{\alpha+1}{u}+\frac{d}{du})Q_n}{(1+\frac{A^2}{u^2})}, \qquad n=0,1,2,\ldots. \end{aligned} \tag{7.6}$$

y induction with respect to n, it follows that Q_n has an expansion of the form

$$Q_n(u,w,A) = \sum_{i=0}^{n-1}\sum_{j=0}^{n-i} \frac{C_{ij}(\frac{A^2}{u^2})^i}{(u-w)^{n+1-i-j}u^{i+j}(1+\frac{A^2}{u^2})^{n+i}}, \qquad n=1,2,\ldots, \tag{7.7}$$

'here C_{ij} do not depend on u, w and A.

Again, we concentrate on the influence of A on the expansion (7.4), especially when $|A|$ is ırge.

If Γ is a simple closed contour that encircles iA and w and with $-iA$ in its exterior, then 'e can prove, just as (3.3), that

$$\frac{1}{2\pi i}\int_\Gamma Q_n(u,w,A)du = \mathcal{O}(|A|^{-n}), \qquad \text{as} \quad |A| \to \infty. \tag{7.8}$$

.s before, we want to split up $\mathcal{L}$ in $\mathcal{L}'$ and $\mathcal{L}''$. We assume that, for large $|A|$, the distance 'om the singularities of $h_0(w)$ to the saddle points $\pm iA$ is at least $2\delta|A|^\varepsilon$, where the constants , ε satisfy $0<\delta$, $\frac{1}{2}<\varepsilon\le 1$. Consequently, we take $\mathcal{L}' = \{w \in \mathcal{L} \mid |w-iA| \le \delta|A|^\varepsilon\}$ nd $\mathcal{L}'' = \mathcal{L}-\mathcal{L}'$, such that the estimate of $|\varepsilon_{n|\mathcal{L}''}|$ is exponentially small, compared with the stimate of $|\varepsilon_{n|\mathcal{L}'}|$, as $z\to\infty$, uniformly with respect to $iA \in [c,\infty)$, where $c>0$ fixed. In fact, 'e need a growth condition on $h_0(w)$ on a prescribed neighbourhood of $\mathcal{L}''$, which is similar to ıe condition mentioned in the appendix.

If we set $\Omega(A) = \{(u,w)\in\mathbb{C}^2 \mid |u-iA| = \frac{3}{2}\delta|A|^\varepsilon,\ |w-iA|\le\delta|A|^\varepsilon\}$, we can prove

$$\sup_{(u,w)\in\Omega(A)} |Q_n(u,w,A)| \le C_n(\delta)|A|^{(1-2\varepsilon)n-\varepsilon}, \tag{7.9}$$

'here $C_n(\delta)$ does not depend on A. Finally we define

$$\widetilde{h}_0(\delta) = \sup_{|w\pm iA|\le\delta|A|^\varepsilon} |h_0(w)|. \tag{7.10}$$

Vith (7.8), (7.9) and straight forward calculations, similar to those leading to (5.5), we obtain ›r $n=1,2,\ldots$

$$\sup_{|w-iA|\le\delta|A|^\varepsilon} |h_n(w)| \le C_n(\delta,\varepsilon)|A|^{(1-2\varepsilon)n}\widetilde{h}_0(\delta), \qquad \text{as} \quad |A|\to\infty, \tag{7.11}$$

'here $C_n(\delta,\varepsilon)$ does not depend on A. With the aid of these estimates we obtain as the main esult of this section

$$|\varepsilon_n| \le C_n'(\delta,\varepsilon)(|A|+1)^{(1-2\varepsilon)n-\alpha}\widetilde{h}_0(\delta)z^{-n}|J_\alpha(zA)|, \tag{7.12}$$

as $z \to \infty$, uniformly with respect to $iA \in [0,\infty)$, where $C_n'(\delta,\varepsilon)$ does not depend on A and z.
A similar approach is possible for real values of A.

Appendix

We formulate conditions on $h_0(w)$ such that the estimate of $|\varepsilon_{n|_{\mathcal{L}''}}|$ is exponentially small, compared with the estimate of $|\varepsilon_{n|_{\mathcal{L}'}}|$, as $z \to \infty$, uniformly with respect to b. We take $\mathcal{L}$, $\mathcal{L}'$, $\mathcal{L}''$, $\rho(b)$, δ, ε, $\varepsilon_{n|_{\mathcal{L}'}}$, $\varepsilon_{n|_{\mathcal{L}''}}$ and $\widetilde{h}_n(\delta)$ as in section 5. Define

$$\mathcal{R}(w,b,p,q,r) = r|w^{-q}e^{p(\frac{1}{3}w^3-b^2w+\frac{2}{3}b^3)}|.$$

We assume that $h_0(w)$ is an analytic function on a neighbourhood $\Omega_0(b)$ of $\mathcal{L}''$, such that for every $w \in \mathcal{L}''$ a disc with center w and radius $\mathcal{R}$ is contained in $\Omega_0(b)$, where $r > 0$ and p, $q \geq 0$ do not depend on b and w. Note that, since $w \in \mathcal{L}''$, $\mathcal{R}$ may be exponentially small, as $|w| \to \infty$. Furthermore, we assume that there are constants $\sigma \geq 0$ and $C_0 > 0$ such that

$$|h_0(w)| \leq C_0\widetilde{h}_0(\delta)|e^{-\sigma(\frac{1}{3}w^3-b^2w+\frac{2}{3}b^3)}|, \qquad \forall w \in \Omega_0(b) \cup \mathcal{L},\ b \in [0,\infty). \tag{A.1}$$

Thus we allow functions $h_0(w)$ being exponentially large as $|w| \to \infty$.

We define recursively neighbourhoods $\Omega_n(b)$ of $\mathcal{L}''$, for $n = 0,1,2,\ldots$. Let $\Omega_{n+1}(b)$ be those $w \in \Omega_n(b)$ such that the disc with center w and radius $2^{-(n+1)}\mathcal{R}$ is contained in $\Omega_n(b)$.

Next, let $w \in \Omega_n(b)$ and let Γ be the circle with center w and radius $2^{-n}\mathcal{R}$. The following two weak asymptotic estimates are simply proved with (3.2):

$$\frac{1}{2\pi i}\int_\Gamma R_n(u,w,b)u^m\,du = \mathcal{O}(|e^{-\frac{1}{3}w^3+b^2w-\frac{2}{3}b^3}|), \tag{A.2}$$

$$\sup_{u\in\Gamma}|R_n(u,w,b)| = \mathcal{O}(|e^{-((n+1)p+\frac{1}{2})(\frac{1}{3}w^3-b^2w+\frac{2}{3}b^3)}|), \tag{A.3}$$

as $|b| \to \infty$, uniformly with respect to $w \in \Omega_n(b)$ and $m \in \{0,1\}$.

Now we can estimate $h_n(w)$ on $\Omega_n(b)$.

$$\begin{aligned}
h_n(w) &= \frac{1}{2\pi i}\int_\Gamma R_0(u,w,b)h_n(u)\,du \\
&= \frac{1}{2\pi i}\int_\Gamma R_1(u,w,b)h_{n-1}(u)\,du - \frac{1}{2\pi i}\int_\Gamma R_1(u,w,b)(\alpha_{n-1}+\beta_{n-1}u)\,du \\
&=_{(A.2)} \frac{1}{2\pi i}\int_\Gamma R_1(u,w,b)h_{n-1}(u)\,du + \widetilde{h}_{n-1}(\delta)\mathcal{O}(|e^{-\frac{1}{3}w^3+b^2w-\frac{2}{3}b^3}|) \\
&\ \ \vdots \\
&=_{(A.2)} \frac{1}{2\pi i}\int_\Gamma R_n(u,w,b)h_0(u)\,du + (\widetilde{h}_{n-1}(\delta)+\ldots+\widetilde{h}_0(\delta))\mathcal{O}(|e^{-\frac{1}{3}w^3+b^2w-\frac{2}{3}b^3}|) \\
&=_{(A.3)\ \&\ (5.5)} \widetilde{h}_0(\delta)\mathcal{O}(|e^{-(np+1+\sigma)(\frac{1}{3}w^3-b^2w+\frac{2}{3}b^3)}|).
\end{aligned}$$

Thus, with (5.5) we have proved that

$$|h_n(w)| \leq C_n \widetilde{h}_0(\delta) |e^{-(np+1+\sigma)(\frac{1}{3}w^3 - b^2 w + \frac{2}{3}b^3)}|, \tag{A.4}$$

for all $w \in \Omega_n(b) \cup \mathcal{L}$ and $b \in [0,\infty) \cup [0, i\infty)$

We choose $z > np + 2 + \sigma$ and estimate $|\varepsilon_{n|_{\mathcal{L}''}}|$ for $b \geq c > 0$.

$$\begin{aligned}
|\varepsilon_{n|_{\mathcal{L}''}}| &\leq_{(A.4)} C_n \widetilde{h}_0(\delta) e^{-\frac{2}{3}zb^3} z^{-n} \frac{1}{2\pi i} \int_{\mathcal{L}''} e^{(z-np-1-\sigma)(\frac{1}{3}w^3 - b^2 w + \frac{2}{3}b^3)} dw \\
&\leq_* \frac{3}{\pi} C_n \widetilde{h}_0(\delta) e^{-\frac{2}{3}zb^3} z^{-n} \int_{\delta' b^{2\varepsilon-1}}^{\infty} e^{(np+1+\sigma-z)(6xb^2+8x^2b+\frac{8}{3}x^3)} \left(\frac{x+b}{\sqrt{3x^2+6xb}}\right) dx \\
&\leq C_n' \widetilde{h}_0(\delta) e^{-\frac{2}{3}zb^3} z^{-n} e^{6\delta'(np+1+\sigma-z)b^{2\varepsilon+1}} \\
&\leq C_n'' \widetilde{h}_0(\delta) \mathrm{Ai}(z^{\frac{2}{3}} b^2) z^{-n-\frac{1}{3}} e^{6\delta'(np+2+\sigma-z)b^{2\varepsilon+1}}.
\end{aligned}$$

In * we substituted $w = x + b + iy$, and we used (2.7) and the fact that $\rho(b) \sim 2\delta |b|^\varepsilon$ as $|b| \to \infty$. The positive constant δ' does not depend on b and z.

With similar estimates for $b \in [ic, i\infty)$, we have proved

$$|\varepsilon_{n|_{\mathcal{L}''}}| \leq C_n \widetilde{h}_0(\delta) \widetilde{\mathrm{Ai}}(z^{\frac{2}{3}} b^2) z^{-n-\frac{1}{3}} e^{6\delta'(np+2+\sigma-z)|b|^{2\varepsilon+1}}, \tag{A.5}$$

where the constants δ' and C_n do not depend on b and z.

REMARK 1. For the special case that has been handled in sections 2 and 4 it is not difficult to prove that $p = \sigma = 0$ and that in $(A.5)$ $|b|^{2\varepsilon+1}$ has to be replaced by $b^3(\ln b)^{-\frac{1}{2}}$. Furthermore, (2.14) shows that in $(A.4)$ and $(A.5)$ $\widetilde{h}_0(\delta)$ can be replaced by $|h_0(b)|$.

REMARK 2. For the boundary case that has been handled in section 6 it is also provable that $p = \sigma = 0$, and (6.10) shows that in $(A.4)$ $\widetilde{h}_0(\delta)$ can be replaced by $(b+1)^{(\varepsilon+\frac{1}{2})(\alpha+1)}|h_0(b)|$, and further calculations show that in $(A.5)$ $\widetilde{h}_0(\delta)$ can be replaced by $|h_0(b)|$.

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