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# Uniform asymptotic expansion for a class of polynomials biorthogonal on the unit circle

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An asymptotic expansion including error bounds is given for polynomials $\{P_n, Q_n\}$ that are biorthogonal on the unit circle with respect to the weight function $(1-e^{i\theta})^{\alpha+\beta}(1-e^{-i\theta})^{\alpha-\beta}$. The asymptotic parameter is $n$; the expansion is uniform with respect to $z$ in compact subsets of $\mathbb{C} \setminus \{0\}$. The point $z=1$ is an interesting point, where the asymptotic behaviour of the polynomials strongly changes. The approximants in the expansions are confluent hypergeometric functions. The polynomials are special cases of the Gauss hypergeometric functions. The results of the paper apply in fact on these functions for the case that in ${}_2F_1(-a,b;c;\zeta)$ $a$ is positive and large, $b$ and $c$ are fixed and $\zeta$ is the uniformity parameter with $\zeta=0$ as "transition" point.

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## 1. Introduction

The polynomials considered are

$$\begin{cases} P_n(z;\alpha,\beta) = {}_2F_1\left(\begin{matrix}-n, \alpha+\beta+1 \\ 2\alpha+1\end{matrix};1-z\right) = \dfrac{(\alpha-\beta)_n}{(2\alpha+1)_n}\,{}_2F_1\left(\begin{matrix}-n, \alpha+\beta+1 \\ \beta+1-n-\alpha\end{matrix};z\right) \\ Q_n(z;\alpha,\beta) = P_n(z;\alpha,-\beta). \end{cases} \tag{1.1}$$

The biorthogonality means that there is a weight function

$$\alpha(\theta) = (1-e^{i\theta})^{\alpha+\beta}(1-e^{-i\theta})^{\alpha-\beta} = (2-2\cos\theta)^{\alpha}(-e^{i\theta})^{\beta} \tag{1.2}$$

such that

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} P_n(e^{i\theta};\alpha,\beta)Q_n(e^{-i\theta};\alpha,\beta)\alpha(\theta)d\theta = 0,\ m \neq n, \tag{1.3}$$

$$= \frac{\Gamma(2\alpha+1)}{\Gamma(\alpha+\beta+1)\Gamma(\alpha-\beta+1)}\,\frac{n!}{(2\alpha+1)_n},\quad m=n.$$

A proof of this can be found in [1]. In the same paper the polynomials are considered for large values of $n$, especially for values of $z$ near unity. It is shown that

$$\lim_{n\to\infty} P_n(e^{i\theta/n};\alpha,\beta) = {}_1F_1(\alpha+\beta+1;2\alpha+1;i\theta) \tag{1.4}$$

which is analogous to a well-known asymptotic result for Jacobi-polynomials in terms of Bessel functions. Askey raised the question how to obtain more terms in the asymptotic result (1.4) and to give bounds on the error in the expansion.

In this paper we give the full asymptotic expansion which gives (1.4) as a special case and we give the error bounds. The result is valid for $z$ ranging in compact subsets of $\mathbb{C} \setminus \{0\}$. So, especially our

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expansion is holding in a neighborhood of $z=1$.

The point $z=1$ is interesting, since $\alpha(\theta)$ is vanishing there (that is, $\alpha(\theta)$ vanishes at $\theta=0$, and we write $z = e^{i\theta}$ for $z$ on the unit circle). As Askey remarked, one wants to understand the effect of zeros of the weight function on the asymptotic behaviour of orthogonal or biorthogonal polynomials. Also, it is interesting to obtain information on the location of the zeros of $P_n$ and $Q_n$. A special case gives direct information. Let $\alpha = -\beta = \frac{1}{2}$, then we have

$$\begin{cases} P_n(z;\frac{1}{2},-\frac{1}{2}) = \dfrac{1-z^{n+1}}{1-z}, \\ Q_n(z;\frac{1}{2},-\frac{1}{2}) = z^n. \end{cases} \tag{1.5}$$

In this simple case the zeros of $P_n$ are uniformly distributed over the unit circle, but those of $Q_n$ all concentrate at $z=0$.

The results of the paper do not apply just to $P_n, Q_n$ introduced in (1.1), but also to the more general case of hypergeometric functions

$${}_2F_1\left({-a,b \atop c};\zeta\right)$$

where $a$ is positive and large, $b$ and $c$ are fixed and $\zeta$ is the uniformity parameter with $\zeta=0$ as "transition" point.

## 2. The asymptotic expansion

The standard integral for ${}_2F_1$-functions gives

$$P_n(z;\alpha,\beta) = \frac{\Gamma(2\alpha+1)}{\Gamma(\alpha+\beta+1)} I_n, \tag{2.1}$$

where

$$I_n = \frac{1}{\Gamma(\alpha-\beta)} \int_0^1 t^{\alpha+\beta}(1-t)^{\alpha-\beta-1}(1-t\zeta)^n dt, \quad \zeta = 1-z. \tag{2.2}$$

For convergence of the integral we have the conditions $\alpha+\beta>-1$, $\alpha-\beta>0$, However, the reciprocal gamma function before the integral removes the singularity due to $\alpha = \beta$. So we suppose that

$$\alpha+\beta>-1. \tag{2.3}$$

Put

$$1-t\zeta = z^u = e^{u\ln z},$$

with $\ln z$ the principal branch of the logarithm, which is real when $z > 0$. Then we have

$$\begin{cases} I_n = z^{\alpha-\beta-1}\left[\dfrac{\ln z}{z-1}\right]^{2\alpha} J_n, \\ J_n = \dfrac{1}{\Gamma(\alpha-\beta)} \displaystyle\int_0^1 f(u) u^{\alpha+\beta}(1-u)^{\alpha-\beta-1} e^{\omega u} du, \\ f(u) = \left[\dfrac{1-z^u}{-u\ln z}\right]^{\alpha+\beta}\left[\dfrac{1-z^{u-1}}{(1-u)\ln z}\right]^{\alpha-\beta-1}, \\ \omega = (n+1)\ln z. \end{cases} \tag{2.4}$$

The function $f$ is holomorphic in a neighborhood of $[0,1]$; singularities occur at

$$u_k = \frac{2k\pi i}{\ln z}, \quad u_m = 1+\frac{2m\pi i}{\ln z}, \quad k,m\in\mathbb{Z}\setminus\{0\}. \tag{2.5}$$

So when $z$ ranges through compact subsets of $\mathbb{C}\setminus\{0\}$, the singularities of $f$ are bounded away from $[0,1]$.

Before constructing the uniform asymptotic expansion we first remark that simpler, i.e., non-uniform expansions can be obtained for two separate cases:

(i) $\mathrm{Re}\,\omega = (n+1)\ln|z| < 0$, $z$ fixed.

The dominant point in the integral $J_n$ is $u=0$, and the asymptotic expansion follows by expanding the function $f(u)$ at $u=0$. The result is

$$P_n(z;\alpha,\beta)\sim\frac{\Gamma(2\alpha+1)}{\Gamma(\alpha-\beta)}[(n+1)(1-z)]^{-\alpha-\beta-1}. \tag{2.6}$$

(ii) $\mathrm{Re}\,\omega > 0$, $z$ fixed.

Now the dominant point is $u=1$ and an expansion of $f(u)$ at $u=1$ has to be used. In this case

$$P_n(z;\alpha,\beta) \sim \frac{\Gamma(2\alpha+1)}{\Gamma(\alpha+\beta+1)}z^{n+\alpha-\beta}(z-1)^{\beta-\alpha}(n+1)^{\beta-\alpha}. \tag{2.7}$$

In the uniform expansion contributions from both $u=0$ and $u=1$ will be taken into account. In this way we can allow $\mathrm{Re}\,\omega$ to be negative as well as positive; even $\omega=0$ is accepted.

Observe that $\mathrm{Re}\,\omega > 0$, $\mathrm{Re}\,\omega < 0$ is equivalent with $|z|>1, |z|<1$ respectively, so that in fact all points on the unit circle in the $z$-plane are "transition" points; i.e., points for which the asymptotic behaviour of the polynomials $P_n, Q_n$ will change drastically. For polynomials this is not surprising, of course. However, in (2.2), (2.6), (2.7) and in the following analysis $n$ need not be an integer.

A uniform expansion for $J_n$ of (2.4) is obtained as follows. We write

$$f(u) = \alpha_0+\beta_0 u+u(1-u)g_0(u) \tag{2.8}$$

with $\alpha_0 = f(0)$, $\beta_0 = f(1)-f(0)$. Then $J_n$ of (2.4) equals

$$J_n = \alpha_0\phi_0 + \beta_0\phi_1 + H_n,$$

with

$$\begin{cases} \phi_0 = \dfrac{\Gamma(\alpha+\beta+1)}{\Gamma(2\alpha+1)}\,{}_1F_1(\alpha+\beta+1;2\alpha+1;\omega) \\ \phi_1 = \dfrac{\Gamma(\alpha+\beta+2)}{\Gamma(2\alpha+2)}\,{}_1F_1(\alpha+\beta+2;2\alpha+2;\omega) = \dfrac{d}{d\omega}\phi_0. \end{cases} \tag{2.9}$$

Partial integration gives for $H_n$, when $\omega\neq0$,

$$\begin{aligned} H_n &= \frac{1}{\omega\Gamma(\alpha-\beta)}\int_0^1 g(u)u^{\alpha+\beta+1}(1-u)^{\alpha-\beta}de^{\omega u} \\ &= \frac{1}{\omega\Gamma(\alpha-\beta)}\int_0^1 f_1(u)u^{\alpha+\beta}(1-u)^{\alpha-\beta-1}e^{\omega u}du, \end{aligned}$$

with

$$f_1(u) = [(2\alpha+1)u-\alpha-\beta-1]g_0(u)-u(1-u)g'_0(u). \tag{2.10}$$

This new $f_1$ has the same domain of regularity as $g_0$ and $f$. By repeating the above procedure, we obtain the formal expansion

$$J_n \sim \phi_0 \sum_{m=0}^{\infty} \frac{\alpha_m}{\omega^m} + \phi_1 \sum_{m=0}^{\infty} \frac{\beta_m}{\omega^m} \tag{2.11}$$

with

$$\begin{cases} \alpha_m = f_m(0), \ \beta_m = f_m(1)-f_m(0), \ f_0 = f_1, \\ f_m(u) = [(2\alpha+1)u-\alpha-\beta-1]g_{m-1}(u)-u(1-u)g'_{m-1}(u) \ (m\geq 1) \\ \quad = \alpha_m+\beta_m u+u(1-u)g_m(u) \ (m\geq 0) \end{cases} \tag{2.12}$$

and where $\omega = (n+1)\ln z$.

The restrictions on $\alpha,\beta$ are as in (2.3). For $\omega$ we temporarily suppose $\omega\neq 0$. In the following section we prove that

$$A_m = \frac{\alpha_m}{\ln^m z}, \ B_m = \frac{\beta_m}{\ln^m z}, \ m = 0,1,..., \tag{2.13}$$

are regular functions of $z$ in $\mathbb{C} \setminus \{0\}$. So, the complete expansion for $P_n$ is

$$P_n(z;\alpha,\beta) \sim \frac{\Gamma(2\alpha+1)}{\Gamma(\alpha+\beta+1)} z^{\alpha-\beta-1} [\frac{\ln z}{z-1}]^{2\alpha} \left[\phi_0 \sum_{m=0}^{\infty} \frac{A_m}{(n+1)^m} + \phi_1 \sum_{m=0}^{\infty} \frac{B_m}{(n+1)^m}\right] \tag{2.14}$$

and the same for $Q_n(z;\alpha,\beta)$ with $\beta$ replaced by $-\beta$ (also in (2.3)). So we have for both $P_n$ and $Q_n$ an expansion as in (2.14) when $-\alpha-1<\beta<\alpha+1$

An error bound for the expansion in (2.14) follows easily from the integration by parts procedure. Writing for $J_n$ of (2.4)

$$J_n = \phi_0 \sum_{m=0}^{k-1} \frac{A_m}{(n+1)^m} + \phi_1 \sum_{m=0}^{k-1} \frac{B_m}{(n+1)^m} + R_k, \quad k = 0,1,..., \tag{2.15}$$

we have for $R_k$ the representation

$$R_k = \frac{1}{\omega^k \Gamma(\alpha-\beta)} \int_0^1 f_k(u) u^{\alpha+\beta}(1-u)^{\alpha-\beta-1} e^{\omega u} du. \tag{2.16}$$

Again, $f_k(u)/\ln^k z$ is regular (see the next section) and we define positive numbers $M_k$ not depending on $u$ such that

$$|f_k(u)| \leq |\ln^k z| M_k, \ u \in [0,1], \ z \in \mathbb{C} \setminus \{0\}. \tag{2.17}$$

Then we obtain

$$|R_k| \leq \frac{1}{(n+1)^k} M_k |\bar{\phi}_0|$$

where $\bar{\phi}_0$ is $\phi_0$ of (2.9) with $\omega$ replaced by Re $\omega$. This gives an error bound for the asymptotic expansion and it shows the asymptotic nature of (2.14).

## 3. On the regularity of $A_m, B_m$ at $z = 1$

In this section we show that the coefficients of (2.14) defined in (2.13) are regular functions of $z$, especially when $\lambda := \ln z = 0$. Also, we show that $|f_k(u)|$ can be bounded as in (2.17), again when $\lambda=0$. We suppose in this section that $|\lambda|$ is small, say $|\lambda| \leq \lambda_0$, where $\lambda_0$ is a fixed small positive number.

Before proving the regularity of $A_m$ and $B_m$ we remark that $f(u)$ of (2.4) depends in a crucial way on the uniformity parameter $\ln z$. The result (2.14) is certainly not true for more general functions, say functions just regular on a set in the complex $u$-plane containing $[0,1]$ in its interior.

Inspection of $f$ of (2.4) shows that it can be written as

$$f(u) = \phi(\lambda u)\psi(\lambda(u-1)), \ \lambda = \ln z, \tag{3.1}$$

where

$$\phi(x) = (\frac{e^x-1}{x})^{\alpha+\beta}, \ \ \psi(x) = (\frac{e^x-1}{x})^{\alpha-\beta-1}. \tag{3.2}$$

The fact that in both $\phi(\alpha)$ and $\psi(x)$ the same function of $x$ appears is not so important. Crucial is that the parameter $\lambda$ appears in both $\phi$ and $\psi$ in (3.1). We expand $f(u)$ in powers of $\lambda$. That is, we write

$$f(u) = \sum_{m=0}^{\infty} p_m \lambda^m u^m \sum_{m=0}^{\infty} q_m \lambda^m (u-1)^m = \sum_{m=0}^{\infty} r_m(u)\lambda^m \tag{3.3}$$

where

$$r_m(u) = \sum_{k=0}^{m} p_k q_{m-k} u^k (u-1)^{m-k}.$$

Observe that $r_m$ is a polynomial in $u$ of degree $m$. Since $|\lambda|$ is small, the manipulations of the series in (3.3) holds for $u$-values in a set $U$ in the $u$-plane that contains [0,1] in its interior; the $r_m$-series in (3.3) converges uniformly with respect to $u \in U$.

By using (3.3) and (2.8) we infer that $g_0$ can be written in the form

$$g_0(u) = \sum_{m=0}^{\infty} s_m(u)\lambda^m, \text{ with } s_0 = s_1 = 0, \tag{3.4}$$

and $s_m$ a polynomial in $u$ of degree $m-2$ ($m \geq 2$).

The proof of this is most easily established by using a Cauchy integral representation for $g_0$. From (2.8) it follows that

$$g_0(u) = \frac{1}{2\pi i}\int_C \frac{f(v)}{v(1-v)(v-u)}\, dv, \tag{3.5}$$

where $C$ is a contour in the above mentioned domain $U$; $C$ encloses the interval [0,1]. This representation follows by writing $f(u)$, $f(1)$, $f(0)$ as similar contour integrals. Substituting the $r_m$-series of (3.3) we obtain for $s_m$ in (3.4)

$$s_m(u) = \frac{1}{2\pi i}\int_C \frac{r_m(v)}{v(1-v)(v-u)}\, dv.$$

Since $r_m(v)$ is a polynomial of degree $m$, $s_0 = s_1 = 0$, which establishes (3.4); it is also clear that $s_m(u)$ is a polynomial of degree $m-2$ for $m \geq 2$.

We next consider $f_1$ of (2.10). Writing

$$f_1(u) = \sum_{m=0}^{\infty} r_m^{(1)}(u)\lambda^m$$

we obtain

$$r_m^{(1)}(u) = [(2\alpha+1)u - \alpha - \beta - 1]s_m(u) - u(1-u)s_m{}'(u)].$$

So, $r_m^{(1)}$ is a polynomial of degree $m-1$, $r_0^{(1)} = r_1^{(1)} = 0$. This shows that $f_1(u)/\lambda$ is regular for all $u \in U$ and $|\lambda| < \lambda_0$; so $A_1, B_1$ are regular for $|\lambda| < \lambda_0$. The same procedure can be used for the higher coefficients $A_k, B_k$ in (2.14) and to establish the meaning of (2.17) for $z \to 1$, or $\lambda \to 0$, for all $k \geq 0$.

As remarked earlier, the special form of $\phi$ and $\psi$ in (3.1) is not important. In fact the method applies to more general functions $f$, or $\phi$ and $\psi$, as long as the representation (3.1) remains and $\phi$ and

$\psi$ are regular in some neighbourhood of [0,1]. Use of analytic functions and of Cauchy integrals is not needed. The verification of the regularity of $f_k(u)\,/\ln^k z$ as $z \to 1$ can be proved without using (3.5), but the present proof is rather elegant and short.

## 4. Concluding Remarks

(i) Although the coefficients $\alpha_m$, $\beta_m$ in (2.11) are defined in terms of a recursion relation (2.12), the evaluation of these coefficients, and hence of $A_m$, $B_m$ in (2.12), is a tedious process. Especially the evaluation of $A_m$, $B_m$ for $z$ at or near unity is difficult. A completely different approach to obtain (2.14) can be based on the differential equation of the Gauss functions. By substitution (2.14) into a transformed version of this equation a recursive system is obtained for $A_m$, $B_m$ and their derivatives. Taylor expansions of $A_m$, $B_m$ around $z = 1$ can then be substituted to compute coefficients of these expansions. In Olver [2] asymptotic methods for special functions are usually based on differential equations.

(ii) In the discussion around (2.6), (2.7) we observed that the role of the critical points $u = 0$, $u = 1$ of $J_n$ in (2.4) is interchanged when $\log|z|$ changes sign. The confluent hypergeometric functions $\phi_0$, $\phi_1$ of (2.9) are exponentially large when Re $\omega \to +\infty$. When Re $\omega \to -\infty$ they are of algebraic growth in $\omega$, and not exponentially small. These asymptotic features and the use of confluent hypergeometric functions as approximants in such problems are not discussed earlier in the asymptotic literature, as far as I know.

(iii) The special case $2(\alpha+\beta+1) = 2\alpha+1$ ( or $\beta = -\frac{1}{2}$ ) makes a Bessel function of $\phi_0$ in (2.9). Also, it gives the integrand of $J_n$ in (2.4) some symmetry. In this form the asymptotic problem resembles that of certain Legendre functions, as considered by Ursell in [3]. The simple case $\alpha = \frac{1}{2}, \beta = -\frac{1}{2}$ yields for $\phi_0$ the spherical Bessel function of order zero (see (1.5)).

## References


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