Uniform Asymptotic Expansions of Integrals with Many Nearby Stationary Points and Algebraic Singularities

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1. Introduction. In this paper a method will be developed to obtain the uniform asymptotic expansions of two classes of functions defined by integrals. One class of functions is of the form

$$I(k;\alpha) = \int_\Gamma g(t)e^{-kf(t;\alpha)}\,dt; \qquad \boldsymbol{\alpha} = (\alpha_1, \alpha_2, \cdots, \alpha_p); \qquad k \gg 1. \tag{1.1}$$

Here Γ is some contour in the complex t plane. The elements of $\boldsymbol{\alpha}$ label the stationary points of f; *i.e.*,

$$f_t(\alpha_\nu\,; \alpha) = 0, \qquad \nu = 1, 2, \cdots, p. \tag{1.2}$$

(The term stationary point is usually reserved for integrals where Re f = const. on Γ and then one uses saddle point in other cases. We shall refer to all points where f_t vanishes as stationary points.) When the elements of $\boldsymbol{\alpha}$ are fixed (whether or not they are distinct) one could obtain an asymptotic expansion of I by (i) the method of stationary phase (when Re f = const. on Γ), (ii) Laplace's method (when Im f = const. on Γ) or (iii) the method of steepest descents [4]. One shortcoming of these methods is that if an asymptotic expansion is obtained for some fixed α and then distinct elements of $\boldsymbol{\alpha}$ are allowed to coalesce and/or approach endpoints of Γ, this espansion will, in general, fail. Our goal is to obtain an asymptotic expansion as $k \to \infty$ which remains uniformly valid in α, even when its distinct elements are allowed to approach one another and/or boundary points. In part, the method used is based on results for the case $p = 2$ obtained in a series of papers by C. Chester, B. Friedman and F. Ursell [2], B. Friedman [8] and F. Ursell [16] and also on results of N. Levinson [10, 11, 12].

The simplest examples of I occur where $p = 1$ or 2. In the former case, when Γ has a finite endpoint of integration one must adjust the above mentioned

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methods in order to allow for the possibility that $\alpha = \alpha_1$ might approach an endpoint. This problem has been discussed by Felsen [6], Felsen and Marcuvitz [7], Lewis [13] and the author [1].

The case $p = 2$ arises when one considers the integral representation of any solution, $Z_{ka}(kr)$, of Bessel's equation with $k \to \infty$ and $a/r \approx 1$. In general, for $p = 2$, Chester, Friedman and Ursell obtain a result which can be described in the following way: when the contributions from the endpoints of Γ can be neglected, or are zero (as when Γ only has endpoints at ∞), the uniform asymptotic expansion is given by an expression of the form

$$I(k; \alpha_1, \alpha_2) = e^{-k\theta}[k^{-1/3}A(-k^{2/3}\rho)g + k^{-2/3}A(-k^{2/3}\rho)h]. \tag{1.3}$$

Here A is an Airy function;

$$A''(t) = tA(t); \tag{1.4}$$

θ and ρ are functions of $\boldsymbol{\alpha} = (\alpha_1, \alpha_2)$ with $\rho^{3/2}$ being proportional to the distance between the stationary points when this distance is small. The functions g and h are asymptotic series in k^{-1} with coefficients which are functions of $\boldsymbol{\alpha}$ and remain finite even where $\alpha_1 = \alpha_2$. By determining the leading terms of g and h, say g_0 and h_0, one obtains the leading terms of the uniform expansion. In addition one can extract the leading terms of the non-uniform expansion, either for ρ finite (α_1 and α_2 separated) or $\rho = 0$ ($\alpha_1 = \alpha_2$), from this leading term of the uniform expansion.

As a consequence of section 2, we find that when the endpoints of Γ give a non-negligible contribution, one must replace A and A' by an incomplete Airy function and incomplete derivative whose integral representations have finite endpoints but the same integrands as the representations of A and A'. In addition, one must append to (1.3) an asymptotic series of the form

$$k^{-1} \sum_{\Gamma} Fe^{-kf}. \tag{1.5}$$

Here $\sum_\Gamma$ is taken to mean that the expression is to be evaluated at the endpoints of Γ with appropriate signs. F is an asymptotic series in k^{-1} with coefficients which are functions of $\boldsymbol{\alpha}$ and t and remain finite when $\alpha_1 = \alpha_2$. We show in section 2 how the coefficients of g, h and F are determined. For any p we obtain an expansion involving a generalized Airy function and its derivatives to order $p - 1$. This function satisfies an ordinary differential equation of order p. In the asymptotic expansion the function and its derivatives are each multiplied by a series in k^{-1} and an appropriate factor $k^{-\nu/(p+1)}$, $1 \leqq \nu \leqq p$, as appear in (1.3). In addition our result includes a series of endpoint contributions similar to (1.5). This is carried out in section 2.

As the number of stationary points increases, so does the difficulty in carrying out the procedures described in section 2. Therefore, the practicality of our results is open to question, but they are still of value in that they clearly indicate the greatest simplicity one can hope to attain if it is desired to retain the full generality and uniformity of the original integral I in its asymptotic expansion. From recent work in geometrical optics (see [13, 14, 18]) it has become apparent that an understanding of such uniform expansions is closely related to the extensions of that subject to cover regions of anomalies of the ray solution.

The second class of functions defined by integrals are of the form

$$(1.6) \qquad I(k; \boldsymbol{\alpha}, \boldsymbol{\beta}) = \int_\Gamma \prod_{\mu=1}^{q} (t - \beta_\mu)^{r_\mu} g(t) e^{-kf(t;\boldsymbol{\alpha})}\, dt; \qquad \boldsymbol{\beta} = (\beta_1, \cdots, \beta_q); \qquad k \to \infty.$$

Here t, $\boldsymbol{\alpha}$ and Γ are as above. The integrand now has a branch point, zero or pole at each point $t = \beta_\mu$ depending on the value of r_μ. (For brevity, we shall often refer to these points as branch points although we do not mean thereby to exclude the other possibilities, zero and pole.) When the elements of $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ are fixed, the asymptotic expansion of $I(k; \boldsymbol{\alpha}, \boldsymbol{\beta})$ as $k \to \infty$ can be obtained by standard methods, whether or not some elements of $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ coincide. However, as above, these methods fail when the elements of $\boldsymbol{\alpha}$ and/or $\boldsymbol{\beta}$ are allowed to coalesce. In this case we shall obtain an asymptotic expansion of I as $k \to \infty$ which remains uniformly valid even when the elements of $\boldsymbol{\alpha}$ and/or $\boldsymbol{\beta}$ are allowed to coalesce and/or approach endpoints of Γ.

As an example, let us take $p = q = 1$, *i.e.*, an integral with one stationary point and one branch point. For special choices of r_1, this problem was treated by Felsen [6], Felsen and Marcuvitz [7] and Lewis [13]. This case for any r_1 was treated by the author [1]. In all cases, β_1 was either an endpoint of integration or an interior point of an infinite interval. The resulting expansion is of the form (1.3) with the Airy function and derivative replaced by a Weber function [17] and derivative of argument $k^{1/2}\rho$. The factors $k^{-1/3}$, $k^{-2/3}$ are replaced by $k^{-1/2}$, k^{-1} and ρ^2 is proportional to the distance between α_1 and β_1 when this distance is small. When r_1 is an interger, further simplification is possible since the Weber function reduces to a Fresnel integral ($r_1 \geqq 0$) or Hermite polynomial ($r_1 < 0$) and the derivative can be written in terms of the function itself.

For arbitrary p and q we must introduce a generalized Weber function and its derivatives to order $p + q - 1$. This function satisfies an ordinary differential equation of order $p + q$. The function and its derivatives are each multiplied by a series in k^{-1} and an appropriate factor $k^{-\nu/(p+1)}$, $1 \leqq \nu \leqq p + q$. When Γ is finite, we must replace these functions by an incomplete generalized Weber function and its incomplete derivatives and append a series of the form of (1.5).

The results of sections 2 and 3 are not simply power series in k^{-1}. Instead they involve k in a more complicated manner and must be interpreted in some generalized sense such as the expansions discussed by Erdelyi and Wyman [5]. At the end of section 2 we state explicitly the way in which our result should be interpreted. In section 4 we verify this interpretation by obtaining a bound on the error in an n term expansion in terms of the $(n + 1)$st term. In doing so, we apply a theorem due to Levinson [10, 11, 12] (stated in section 2) in the large, while the theorem is a result in the small. We assume this theorem in the large even though it is not proven. It would be possible under this same assumption to obtain a similar estimate for the asymptotic expansion obtained in section 3.

In section 5 we apply the methods of sections 2 and 3 to a specific example. An integral of the type studied here might arise in an electromagnetic problem in which the rays of geometrical optics exhibit both a shadow and a caustic intersecting the shadow. A special case of this integral reduces to the uniform asymptotic expansion of $H_{k\mu}^{(1)}(k\rho)$ with $k \gg 1$, $\mu/\rho \approx 1$.

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2. Integrals with many stationary points. We shall derive the uniform asymptotic expansion of an integral with p stationary points whose positions are denoted by a vector parameter $\boldsymbol{\alpha}$. The large parameter is denoted by k. We define

$$I(k; \boldsymbol{\alpha}) = \int_\Gamma g(t) e^{-kf(t;\boldsymbol{\alpha})}\, dt. \tag{2.1}$$

The contour Γ may be of infinite extent but it must approach infinity in those regions where $\operatorname{Re} f \to \infty$; *i.e.*, in the valleys of $-f$. (For appropriate choices of g we could have $\operatorname{Re} f = \text{const.}$ at ∞ or $\operatorname{Re} f = \text{const.}$ on Γ. In these cases Γ has the boundary of a valley of $-f$ as asymptote. We must deform into that valley before proceeding to use the method to be described here.) The functions $f(t; \boldsymbol{\alpha})$ and $g(t)$ are assumed to be entire and sufficiently well behaved near ∞ that the integral converges for $\boldsymbol{\alpha}$ in some given domain. The stationary points of f are those points where f_t vanishes;

$$f_t(\alpha_i, \boldsymbol{\alpha}) = 0, \qquad i = 1, 2, \cdots, p. \tag{2.2}$$

It is assumed that all the stationary points are simple if the elements of $\boldsymbol{\alpha}$ are distinct; *i.e.*,

$$f_{tt}(\alpha_i, \boldsymbol{\alpha}) \neq 0 \quad \text{if} \quad \alpha_i \neq \alpha_j \quad \text{when} \quad i \neq j. \tag{2.3}$$

One can show that if m elements of $\boldsymbol{\alpha}$ coalesce ($m \leqq p$) on some point then it is at least a stationary point of order m. We assume that this order is exactly m and therefore that the $(m + 1)$st derivative is non-zero there. This eliminates such possibilities as

$$f(t; \boldsymbol{\alpha}) = (\alpha_1 - \alpha_2) h(t; \boldsymbol{\alpha})$$

in which case $f \equiv 0$ for $\alpha_1 = \alpha_2$ and h regular.

The simplest function having these properties is a polynomial of degree $p + 1$. If we could make a change of variables which would reduce the function

f to such a polynomial then we shall have achieved a simplification of the integral I. Therefore we introduce the function Φ by first defining its derivative to be

$$\Phi'(z; \mathbf{a}) = \prod_{\nu=1}^{p} (z - a_\nu); \qquad \mathbf{a} = (a_1, \cdots, a_p) \tag{2.4}$$

and then

$$\Phi(z; \mathbf{a}) = x_0 + \int_0^z \Phi'(s; \mathbf{a})\, ds. \tag{2.5}$$

The vector $\mathbf{a}$ will be defined below.

It will at times be convenient to speak of Φ in terms of the coefficients in the polynomial and therefore we define

$$\Phi(z; \mathbf{a}) = P(z; \mathbf{x}) = \frac{z^{p+1}}{p+1} + \sum_{\mu=1}^{p} \frac{x_\mu}{\mu} z^\mu + x_0\,; \quad \mathbf{x} = (x_0, x_1, \cdots, x_p). \tag{2.6}$$

Our change of variables from t to z should be constructed to make

$$f(t; \alpha) = \Phi(z; \mathbf{a}) = P(z; \mathbf{x}). \tag{2.7}$$

By differentiating this equation we find

$$\frac{dt}{dz} = \frac{\Phi(z; \mathbf{a})}{f_t(t; \boldsymbol{\alpha})}. \tag{2.8}$$

In order that dt/dz be finite and non-zero, we must at least require that

$$z = a_\nu \leftrightarrow t = \alpha_\nu\,; \qquad \nu = 1, 2, \cdots, p. \tag{2.9}$$

When the elements of $\boldsymbol{\alpha}$ are distinct, (2.9) can be viewed as p conditions for the $p + 1$ unknowns $(x_0, \mathbf{a})$ or $\mathbf{x}$. For one more condition we might require one of the following:

$$t = 0 \leftrightarrow z = 0; \qquad x_0 = f(0; \alpha); \tag{2.10}$$

$$\sum_{\nu=1}^{p} a_\nu = x_p = 0. \tag{2.11}$$

The former case was used by the author in [1] with $p = 1$, while the latter case was used by Chester, Friedman and Ursell in [2, 8, 16] with $p = 2$. We assume that (2.7), (2.9) and (2.10) or (2.11) define a one-one analytic transformation from $(t; \boldsymbol{\alpha})$ to $(z; \mathbf{a})$ for $\boldsymbol{\alpha}$ in its prescribed domain and t in some domain containing Γ and the stationary points. Under these assumptions one can show that the derivative (2.8) remains finite and nonzero at the stationary points even when elements of $\boldsymbol{\alpha}$ coalesce. This is done by repeated application of L'Hospital's rule in (2.8).

N. Levinson [10, 11, 12] has proven the following theorem which is pertinent to this discussion:

Theorem. *Let $f(t;\alpha)$ be analytic in $(t;\alpha)$ for small $|t|$ and $|\alpha|$. Let*

(i) $$\frac{d^k f}{dt^k}(0,0) = 0, \qquad 1 \leqq k \leqq p$$

and let

(ii) $$\frac{d^{p+1}f(0;0)}{dt^{p+1}} \neq 0.$$

Then there is a function g of $(z;\alpha)$ analytic for small $|z|$ and $|\alpha|$ such that setting

(iii) $$t = z + z^2 g(z;\alpha)$$

in $f(t;\alpha)$ yields $f(t;\alpha) = P(z;\alpha)$ where P is a polynomial in z,

$$f(t;\alpha) = P(z;\alpha) = \sum_{j=0}^{p+1} q_j(\alpha) z^j.$$

The $q_j(\alpha)$ are analytic for small α,

$$q_j(0) = 0 \qquad 1 \leqq j \leqq p$$

and $q_{p+1}(0) \neq 0$. Condition (iii) *implies $z = t + t^2 h(t;\alpha)$ where h is analytic for small $|\alpha|$ and $|t|$. Thus for any small α there is a one to one analytic correspondence between t and z for small $|t|$ and $|z|$.*

It is noted by Levinson that the proof in the case when α is replaced by a vector goes in much the same way as the scalar case. The function f which we are considering satisfies the criteria (i) and (ii) in the neighborhood of every point in some domain of the t plane. Therefore we conclude that in the neighborhood of each point in the t plane we can transform f to a polynomial Φ as in (2.7) and that the associated change of variables is locally (1-1) analytic in all variables.

We cannot conclude that there exists *one* change of variables and *one* polynomial for which the theorem applies globally. We leave this as an open question and proceed with our formal process under the assumption that our change of variables (2.7) is one-one analytic where required.

Under this change of variables, I becomes

$$I(k;\alpha) = \int_C G(z;\mathrm{a}) e^{-k\Phi(z;\mathrm{a})}\, dz = \int_C G(z;\mathrm{a}) e^{-kP(z;\mathrm{x})}\, dz. \tag{2.12}$$

Here C is the image of Γ and

$$G(z;\mathrm{a}) = g(t)\frac{d\iota}{dz}. \tag{2.13}$$

Let us now set

$$G(z;\mathrm{a}) = H_0(z;\mathrm{a}) + \Phi'(z;\mathrm{a}) F_0(z;\mathrm{a}). \tag{2.14}$$

Here $H_0(z;\, \mathrm{a})$ is a polynomial of degree $p - 1$ obtained as remainder after division of G by Φ' and F_0 is the quotient of that division. To motivate this expansion of G we note that the second term in (2.14) vanishes at each of the stationary points of Φ. Therefore one might expect that the contribution to I from this term will always be of lower order in k than I itself; *i.e.*, the major contribution to the asymptotic expansion of I must come from the polynomial $H_0(z;\, \mathrm{a})$. It will become apparent that this is the case but for the addition of endpoint contributions arising from the integral of $\Phi' F_0$.

We therefore set

$$H_0(z;\, \mathrm{a}) = \sum_{\nu=0}^{p-1} \gamma_\nu^{(0)} z. \tag{2.15}$$

The functions H_0 and F_0 can also be represented by contour integrals. These representations are

$$H_0(z;\, \mathrm{a}) = \frac{1}{2\pi i} \oint \frac{\Phi'(\xi;\, \mathrm{a}) - \Phi'(z;\, \mathrm{a})}{(\xi - z)\Phi'(\xi;\, \mathrm{a})} G(\xi;\, \mathrm{a})\, d\xi \tag{2.16}$$

and

$$F_0(z;\, \mathrm{a}) = \frac{1}{2\pi i} \oint \frac{G(\xi;\, \mathrm{a})}{(\xi - z)\Phi'(\xi;\, \mathrm{a})}\, d\xi. \tag{2.17}$$

Here the contour of integration must enclose all of the points z, $\mathbf{a}$.

Let us now substitute (2.14) and (2.15) into (2.12). The result is

$$I(k;\, \alpha) = \sum_{\nu=0}^{p-1} \frac{\gamma_\nu^{(0)}}{k^{(\nu+1)/(p+1)}} U_p^{(\nu)}(\mathrm{y};\, C') + R_0(k;\, \alpha) \tag{2.18}$$

The function $U_p(\mathrm{y};\, C')$ is defined in Appendix I; $U_p^{(\nu)}$ is the (incomplete) derivative function of order ν and is also defined in the appendix. If C' has endpoints only at ∞ or if its finite endpoints are independent of y_1, then

$$U_p^{(\nu)}(\mathrm{y};\, C') = \left(-\frac{\partial}{\partial y_1}\right)^\nu U_p(\mathrm{y};\, C'). \tag{2.19}$$

The vector $\mathbf{y}$ is related to the vector $\mathbf{x}$ in (2.12) by the equations

$$y_\nu = x_\nu k^{1-\nu/(p+1)} \tag{2.20}$$

and

$$R_0(k;\, \alpha) = \int_C \Phi'(z;\, \mathrm{a}) F_0(z;\, \mathrm{a}) e^{-k\Phi(z;\mathrm{a})}\, dz. \tag{2.21}$$

We can simplify notation in (2.18) and equations appearing below by defining the vectors

$$\mathbf{\Gamma}^{(0)} = \left(\frac{\gamma_0^{(0)}}{k^{1/(p+1)}}, \frac{\gamma_1^{(0)}}{k^{2/(p+1)}}, \cdots \frac{\gamma_{p-1}^{(0)}}{k^{p/(p+1)}}\right) \tag{2.22}$$

and

$$\mathbf{U}_p = (U_p, U_p^{(1)}, \cdots, U_p^{(p-1)}). \tag{2.23}$$

Now (2.18) can be written as

$$I(k; \alpha) = \mathbf{\Gamma}^{(0)} \cdot \mathbf{U}_p + R_0(k; \alpha). \tag{2.24}$$

The dot product here is just the usual Euclidean scalar product and simply replaces the sum in (2.18). Integration by parts in R_0, (2.21), yields

$$R_0(k; \alpha) = B_0(\mathrm{a})/k + I_1(k; \alpha)/k \tag{2.25}$$

where

$$I_1(k; \alpha) = \int_C G_1(z; \mathrm{a}) e^{-k\Phi(z;\mathrm{a})} \, dz; \qquad G_1(z; \mathrm{a}) = \frac{\partial F_0}{\partial z}(z; \mathrm{a}). \tag{2.26}$$

The function $B_0(\mathrm{a})/k$ is the contribution from the endpoints of C. We denote this term symbolically by

$$B_0(\mathrm{a}) = -\sum_C F_0(z; \mathrm{a}) e^{-k\Phi(z;\mathrm{a})} \tag{2.27}$$

Comparison of (2.26) and (2.12) reveals that $I_1(k; \alpha)$ and $I(k; \alpha)$ are of exactly the same form, but in the expansion of I, I_1 is multiplied by k^{-1}. Therefore we conclude that the major contribution to I arises from $\mathbf{\Gamma}^{(0)} \cdot \mathbf{U}_p$ and B_0. This will be verified in section 4.

The procedure described in (2.12)–(2.27) can now be applied to I_1, thereby yielding functions H_1, R_1, B_1, G_2 and I_2 and a vector $\mathbf{\Gamma}^{(1)}$. By continuing to repeat this process we obtain the uniform asymptotic expansion of I in terms of $U_p(\mathrm{y}, C')$. The final result is

$$I(k; \alpha) \sim \sum_{n=0}^{\infty} k^{-n} [\mathbf{\Gamma}^{(n)} \cdot \mathbf{U}_p(\mathrm{y}; C') + B_n(\mathrm{a})/k]. \tag{2.28}$$

Here $\mathbf{\Gamma}^{(n)}$ and B_n are defined recursively via the system of equations

$$H_n(z; \mathrm{a}) = \sum_{\nu=0}^{p-1} \gamma_\nu^{(n)} z^\nu = \frac{1}{2\pi i} \oint \frac{\Phi'(\xi; \mathrm{a}) - \Phi'(z; \mathrm{a})}{(\xi - z)\Phi'(\xi; \mathrm{a})} G_n(\xi; \mathrm{a}) \, d\xi; \tag{2.29}$$

$$G_n(z; \mathrm{a}) = \frac{\partial F_{n-1}(z; \mathrm{a})}{\partial z}, \qquad n \geqq 1; \qquad G_0(z; \mathrm{a}) = G(z; \mathrm{a}) \tag{2.30}$$

$$F_n(z; \mathrm{a}) = \frac{1}{2\pi i} \oint \frac{G_n(\xi; \mathrm{a})}{(\xi - z)\Phi'(\xi; \mathrm{a})} \, d\xi; \tag{2.31}$$

$$\mathbf{\Gamma}^{(n)} = \left(\frac{\gamma_0^{(n)}}{k^{1/(p+1)}}, \frac{\gamma_1^{(n)}}{k^{2/(p+1)}}, \cdots, \frac{\gamma_{p-1}^{(n)}}{k^{p/(p+1)}} \right); \tag{2.32}$$

$$B_n(\mathrm{a}) = -\sum_C F_n(z; \mathrm{a}) e^{-k\Phi(z;\mathrm{a})}: \qquad n = 0, 1, 2, \cdots. \tag{2.33}$$

We claim that (2.28) is uniform in the sense that the error after N terms of the sum is of the order of the next term, independent of the position of the elements of (a) within their admissible range as defined by the range of (α). Therefore, there exist constants $C_0^{(N)}, C_1^{(N)}, \cdots, C_{p-1}^{(N)}, B^N$ such that for k sufficiently large

$$(2.35)\qquad \left| I(k;\alpha) - \sum_{n=0}^{N} k^{-n}[\mathbf{\Gamma}^{(n)}\cdot\mathbf{U}_p/k^{1/(p+1)} + B_n/k] \right|$$
$$< k^{-(N+1)}\left\{ \sum_{\nu=0}^{p-1} C^{(N)}\, |U_p^{(\nu)}(\mathrm{y}; C')|/k^{(\nu+1)/(p+1)} + B^N/k \sum_{C}{}' e^{-k \operatorname{Re} \Phi(z;\mathrm{a})} \right\}.$$

Here $\sum'$ denotes that the terms in the sum should all be taken with positive sign independent of the orientation of C.

We see in (2.28) that U_p and each of its derivatives is multiplied by an asymptotic series in $1/k$ and the expansion includes an asymptotic series of endpoint contributions. To emphasize this fact we write

$$(2.36)\qquad I(k;\alpha) \sim \sum_{\nu=0}^{p-1} f_\nu(k) U_p^{(\nu)}(\mathrm{y}; C') + B(k)$$

with

$$(2.37)\qquad f_\nu(k) = k^{-(\nu+1)/(p+1)} \sum_{n=0}^{\infty} \gamma_\nu^{(n)}/k^n$$

and

$$(2.38)\qquad B(k) = \sum_{n=0}^{\infty} B_n(\mathrm{a})/k^{n+1}$$

3. Integrals with stationary points and branch points, zeroes and poles. The methods of the previous section can be applied with little more difficulty to the problem in which the amplitude of the integrand has branch points, zeroes and/or poles. We define

$$(3.1)\qquad I(k;\alpha,\beta,\mathrm{r}) = \int_\Gamma \prod_{\mu=1}^{q} (t-\beta_\mu)^{r_\mu} g(t) e^{-kf(t;\alpha)}\, dt;$$
$$\beta = (\beta_1, \cdots, \beta_q), \qquad \mathrm{r} = (r_1, \cdots, r_q).$$

The exponent f is assumed to be as in section 2. Each point $t = \beta_\mu$ is a branch point, zero or pole of order r_μ. The case $r_\mu = 0$ is of interest only if β_μ is an endpoint of integration. All procedures of section 2 up to equation (2.12) can be applied to the present problem. In place of (2.12) we obtain

$$(3.2)\qquad I(k;\alpha,\beta,\mathrm{r}) = \int_C \prod_{\mu=1}^{q} (z-b_\mu)^{r_\mu} G(z;\mathrm{a},\mathrm{b}) e^{-k\Phi(z;\mathrm{a})}\, dz.$$

Here

$$(3.3)\qquad G(z;\mathrm{a},\mathrm{b}) = \prod_{\mu=1}^{q} \left(\frac{t-\beta_\mu}{z-b_\mu}\right)^{r_\mu} g(t)\, \frac{dt}{dz}$$

and the vector $\mathbf{b} = (b_1, \cdots, b_q)$ is defined by using (2.7) and requiring

$$f(\beta_\mu; \alpha) = \Phi(b_\mu; \mathbf{a}); \qquad \mu = 1, \cdots, q. \tag{3.4}$$

If the transformation from t to z is analytic and one-one and $g(t)$ is analytic in t, $G(z; \mathbf{a}, \mathbf{b})$ will be analytic in z. We now expand G in the form

$$G(z; \mathbf{a}, \mathbf{b}) = H_0(z; \mathbf{a}, \mathbf{b}) + \Psi(z; \mathbf{a}, \mathbf{b}) F_0(z; \mathbf{a}, \mathbf{b}) \tag{3.5}$$

where H_0 is a polynomial of degree $p + q - 1$ and

$$\Psi(z; \mathbf{a}, \mathbf{b}) = \Phi'(z; \mathbf{a}) \prod_{\mu=1}^{q} (z - b_\mu). \tag{3.6}$$

The motivation here is similar to that of section 2 except that we now want the second term in (3.5) to vanish at the points $z = b_\mu$ as well as at the stationary points. As in the previous section we can write down integral representations for H_0 and F_0. They are

$$H_0(z; \mathbf{a}, \mathbf{b}) = \frac{1}{2\pi i} \oint \frac{\Psi(\xi; \mathbf{a}, \mathbf{b}) - \Psi(z; \mathbf{a}, \mathbf{b})}{(\xi - z)\Psi(\xi; \mathbf{a}, \mathbf{b})} G(\xi; \mathbf{a}, \mathbf{b})\, d\xi \tag{3.7}$$

and

$$F_0(z; \mathbf{a}, \mathbf{b}) = \frac{1}{2\pi i} \oint \frac{G(\xi; \mathbf{a}, \mathbf{b})}{(\xi - z)\Psi(\xi; \mathbf{a}, \mathbf{b})}\, d\xi. \tag{3.8}$$

The paths of integration in (3.7) and (3.8) should enclose all of the points z, $\mathbf{a}$, $\mathbf{b}$. We may also write H_0 in the form

$$H_0(a; \mathbf{a}, \mathbf{b}) = \sum_{\nu=0}^{p+q-1} \gamma_\nu^{(0)} z^\nu \tag{3.9}$$

and then introduce the vector $\mathbf{\Gamma}^{(0)}$ by setting

$$\mathbf{\Gamma}^{(0)} = \left(\frac{\gamma_0^{(0)}}{k^{([r]+1)/(p+1)}}, \frac{\gamma_0^{(1)}}{k^{([r]+2)/(p+1)}}, \cdots, \frac{\gamma_0^{(p+q-1)}}{k^{([r]+p+q)/(p+1)}} \right). \tag{3.10}$$

By substituting (3.5) into (3.2) and using (3.9) and (3.10), we can express I in terms of the function V_{pq} discussed in the appendix. The result is

$$I(k; \boldsymbol{\alpha}, \boldsymbol{\beta}, \mathbf{r}) = \mathbf{\Gamma}^{(0)} \cdot \mathbf{V}_{pq}(\mathbf{y}; \tilde{\mathbf{b}}, \mathbf{r}; C') + R_0(k; \boldsymbol{\alpha}, \boldsymbol{\beta}, \mathbf{r}). \tag{3.11}$$

Here $\mathbf{y}$ plays the same role as in section 2 and is defined by (2.18) and (2.6). The vector $\mathbf{V}_{pq}$ is given by

$$\mathbf{V}_{pq} = (V_{pq}^{(0)}, V_{pq}^{(1)}, \cdots, V_{pq}^{(p+q-1)}) \tag{3.12}$$

and

$$\tilde{\mathbf{b}} = k^{1/(p+1)} \mathbf{b}; \qquad [r] = \sum_{\mu=1}^{q} r_\mu . \tag{3.13}$$

The remainder in (3.11) is

$$R_0(k; \boldsymbol{\alpha}, \boldsymbol{\beta}, \mathbf{r}) = \int_C \prod_{\mu=1}^{q} (z - b_\mu)^{r_\mu} \Psi(z; \mathbf{a}, \mathbf{b}) F_0(z; \mathbf{a}, \mathbf{b}) e^{-k\Phi(z;\mathbf{a})} \, dz. \tag{3.14}$$

Here we can integrate by parts just as in section 2 and apply the above procedure to obtain the complete uniform asymptotic expansion of I. The final result is

$$I(k; \boldsymbol{\alpha}, \boldsymbol{\beta}, \mathbf{r}) \sim \sum_{n=0}^{\infty} k^{-n} [\boldsymbol{\Gamma}^{(n)} \cdot \mathbf{V}_{pq} + B_n/k]. \tag{3.15}$$

Here $\boldsymbol{\Gamma}^{(n)}$ and B_n are defined recursively by the system of equations

$$H_n(z; \mathbf{a}, \mathbf{b}) = \sum_{\nu=0}^{p+q-1} \gamma_\nu^{(n)} z^\nu = \frac{1}{2\pi i} \oint \frac{\Psi(\xi; \mathbf{a}, \mathbf{b}) - \Psi(z; \mathbf{a}, \mathbf{b})}{(\xi - z)\Psi(\xi; \mathbf{a}, \mathbf{b})} G_n(\xi; \mathbf{a}, \mathbf{b}) \, d\xi; \tag{3.16}$$

$$G_n(z, \mathbf{a}, \mathbf{b}) = \prod_{\mu=1}^{q} (z - b_\mu)^{-r_\mu} \frac{\partial}{\partial z} \left\{ \prod_{\mu=1}^{q} (z - b_\mu)^{r_\mu+1} F_{n-1}(z; \mathbf{a}, \mathbf{b}) \right\}; \; G_0 = G; \tag{3.17}$$

$$F_n(z, \mathbf{a}, \mathbf{b}) = \frac{1}{2\pi i} \oint \frac{G_n(\xi; \mathbf{a}, \mathbf{b})}{(\xi - z)\Psi(\xi; \mathbf{a}, \mathbf{b})} \, d\xi; \tag{3.18}$$

$$B_n = -\sum_C \prod_{\mu=1}^{q} (z - b_\mu)^{r_\mu+1} F_n(z; \mathbf{a}, \mathbf{b}) e^{-k\Phi(z;\mathbf{a},\mathbf{b})}; \tag{3.19}$$

$$\boldsymbol{\Gamma}^{(n)} = \left(\frac{\gamma_0^{(n)}}{k^{([r]+1)/(p+1)}}, \frac{\gamma_1^{(n)}}{k^{([r]+2)/(p+1)}}, \cdots, \frac{\gamma_{p+q-1}^{(n)}}{k^{([r]+p+q)/(p+1)}} \right); \tag{3.20}$$

$$n = 0, 1, 2, \cdots.$$

4. An error estimate. We shall verify the estimate (2.35) for I defined by (2.12). We note first that the difference in (2.35) is exactly of the form $k^{-(N+1)}I$ except that G must be replaced by G_N. Therefore we need only show that there exist constants $C_0, C_1, \cdots, C_{p-1}$ and B such that

$$|I(k; \boldsymbol{\alpha})| \leqq \sum_{\nu=0}^{p-1} C_\nu \, |U_p^{(\nu)}(\mathbf{y}; C')|/k^{(1+\nu)/(p+1)} + B/k \sum_C{}' e^{-k \operatorname{Re} P(z;\mathbf{x})}. \tag{4.1}$$

It is assumed that

(i) $G(z; \mathbf{a})$ is analytic in z in some domain containing C so long as the elements of $\mathbf{a}$ are restricted to their prescribed domain $D_\mathbf{a}$. This domain should be sufficiently large to allow for any necessary deformations of C.

(ii) By restricting z to $\tilde{D} \subset D$ there should exist a bound M, a function of $\tilde{D}$ but not of $\mathbf{a}$ in $D_\mathbf{a}$, such that $|G(z; \mathbf{a})| \leqq M$.

The inequality (4.1) can be written in the equivalent form

$$I(k; \boldsymbol{\alpha}) \leqq \sum_{\nu=1}^{p-1} C_\nu \, |I_p^{(\nu)}(k; \mathbf{x})| + B/k \sum_C{}' e^{-k \operatorname{Re} P(z;\mathbf{x})}. \tag{4.2}$$

Here

$$I_p^{(\nu)}(k; \mathrm{x}) = \int_C z^\nu e^{-kP(z;\mathrm{x})}\, dz, \qquad \nu = 0, 1, 2, \cdots, p-1 \tag{4.3}$$

and is obtained by undoing the change of variables used to define U_p in the appendix. To verify our estimate, we must study I in the case where all roots of P (which are the stationary points) are near one another and alternatively, when they are not. Equivalently, we shall consider the positions of the roots relative to some fixed point, say zero, requiring that at least one of the roots, say a_1, is always nearby. The basis of our estimate is the following set of theorems.

Theorem I. If

$$|a_i| \leqq a_0 k^{-1/p+1}, \qquad a_i \varepsilon D_a, \qquad i = 1, 2, \cdots, p \tag{4.4}$$

for any $a_0 > 0$, *then* (4.1) *is true for* $k \geqq k(a_0)$.

Theorem II. There exists a number $R(q)$ *with the following property. If*

$$\begin{aligned} &|a_i| \leqq Rk^{-1/(p+1)}, \qquad i = 1, 2, \cdots, q; \\ &|a_i| > 2Rk^{-1/(p+1)}, \qquad i = q+1, \cdots, p; \qquad R \geqq R(q), \qquad k \geqq k(q), \\ &\qquad a_i \varepsilon D_a, \qquad i = 1, 2, \cdots, p, \end{aligned} \tag{4.5}$$

then (4.1) *is true. Here* $p > q \geqq 1$.

As a consequence of Theorems I and II we have

Theorem III. The estimate (4.1) *is true for* $a_i \varepsilon D_a$, $i = 1, 2, \cdots, p$.

Theorems I and II will be proved below. Assuming them for the present, we prove Theorem III.

Proof of Theorem III. Set $\rho = \max_{1 \leq q < p} (R(q))$. We partition the plane into regions

$$\begin{aligned} &D_1 : |z| \leqq \rho k^{-1/(p+1)}; \\ &A_n : 2^n \rho k^{-1/(p+1)} < |z| \leqq 2^{n+1} \rho k^{-1/(p+1)}, \qquad 1 \leqq n \leqq p-1; \\ &D_2 : 2^{p+1} \rho k^{-1/(p+1)} < |z|. \end{aligned} \tag{4.6}$$

We now choose

$$a_0 = 2^p \rho \tag{4.7}$$

and set

$$k_0 = \max_{1 \leq q < p} (k(q), k(a_0)) \tag{4.8}$$

If all of the roots of $P(z; \mathrm{x})$ are bounded by a_0 then (4.1) follows from Theorem I. If not all roots are bounded by a_0 then some annulus A_j, $1 \leqq j \leqq p-1$, must not have a root. Therefore the region $|z| \leqq 2^{j-1}\rho k^{-1/(p+1)}$ contains q roots, $1 \leqq q < p$ while the remainder of the roots are in the region $|z| > 2^{j}\rho k^{-1/p+1}$. Since $2^{j-1}\rho \geqq R(q)$ for any $j \geqq 1$, and $q \geqq 1$, (4.1) follows from Theorem II. Q.E.D.

We shall now prove Theorems I and II.

Proof of Theorem I. Let us first assume that C is infinite in extent. For convergence, the endpoints of C must be in the valleys of the exponent. For z large enough, say $|z| > 2a_0$ we may take C to be along a path of steepest descent away from the points on C where $|z| = 2a_0$.

Let us define $\bar{C}$ as the truncation of C. By this we mean that $\bar{C}$ has finite endpoints of integration, but we pick these endpoints so large that any error introduced by integrating over $\bar{C}$ instead of C is exponentially small when compared to either the integral over $\bar{C}$ or the integral over C. We write

$$I = \int_C G(z; \mathrm{a})e^{-kP(z;\mathrm{x})}\, dz \approx \int_{\bar{C}} G(z; \mathrm{a})e^{-kP(z;\mathrm{x})}\, dz. \tag{4.9}$$

We expand G as in section 2:

$$G(z; \mathrm{a}) = \gamma_0 + \gamma_1 z + \cdots + \gamma_{p-1}z^{p-1} + P'(z; \mathrm{x})F_0(z; \mathrm{a}). \tag{4.10}$$

From the integral representation (2.17) we can estimate both F_0 and F_0'. To do this, let us assume that

$$\tau = \max |z|, \qquad z \,\varepsilon\, \bar{C}. \tag{4.11}$$

We choose the contour of integration in (2.17) so that $|\xi| > 2\tau$. Then

$$|F_0| \leqq \frac{2M}{Q}, \qquad |F_0'| \leqq \frac{2M}{\tau Q} = m. \tag{4.12}$$

Here Q is the minimum of Φ on the path of integration in (2.17). Since the roots of Φ are bounded, Q is clearly bounded away from zero for k large enough. With an additional error of the same order as in (4.9) we obtain

$$I \approx \mathbf{\Gamma}^{(0)}\cdot\mathbf{U}_p + k^{-1}\int_{\bar{C}'} F_0'(\zeta k^{-1/(p+1)}; \mathrm{a})e^{-P(\zeta;\mathrm{y})}\, d\zeta. \tag{4.13}$$

Here ζ and $\mathbf{y}$ are the stretched variables of the appendix, $\bar{C}'$ is the image in the ζ plane of $\bar{C}$ and $\Gamma^{(0)}$ is defined by (2.32). The integrand on the right is at most $O(1)$ in k as a result of the estimate on F_0 and the range of $\mathbf{y}$ as derivable from the range of $\mathbf{x}$. The integral is therefore at most $O(1)$ in k. The multiplicative factor assures us that the entire second term is $O(k^{-1})$.

We also know that U_p and its derivatives cannot all vanish simultaneously since U_p is a non-trivial solution of a homogeneous ordinary differential equation of degree p. Therefore, for k large enough, we may bound the error of approximate equality and the second term of (4.13) by an expression of the form

$$|I - \mathbf{\Gamma}^{(0)}\cdot\mathbf{U}_p| < \sum_{\nu=0}^{p-1} \delta_\nu \,|U_p^{(\nu)}|\, k^{-(\nu+1)/(p-1)}. \tag{4.14}$$

Since there are no endpoint contributions when C is doubly infinite, Theorem I follows (with $C_\nu \geqq |\gamma_\nu| + \delta$) for this special case.

Now let us assume that C has at least one finite endpoint. Extend C to infinity along paths of steepest descent away from the endpoint(s). Call this new path C_E (extension of C) and call the corresponding integral J;

$$J = \int_{C_E} G(z; \mathbf{a})e^{-kP(z;\mathbf{x})}\, dz, \tag{4.15}$$

and in like manner

$$J_p^{(\nu)}(k; \mathbf{x}) = \int_{C_E} z^\nu e^{-kP(z;\mathbf{x})}\, dz. \tag{4.16}$$

We define a trivial curve as one which is simply connected in the finite plane and has two endpoints at ∞ in the same valley of $-P$. Also the integrand should be analytic in the domain enclosed by the curve, so that the integral over such a curve is zero. We also define a sum of trivial curves as trivial.

Let us suppose that $C + C_E$ is trivial. In this case $I + J = 0$ and $I_p^{(\nu)} + J_p^{(\nu)} = 0$. Then it follows that

$$J = \sum_{\nu=0}^{p-1} \gamma_\nu J_p^{(\nu)} + \int_{C_E} P'(z; \mathbf{x})F_0(z; \mathbf{a})e^{-kP(z;\mathbf{x})}\, dz. \tag{4.17}$$

On C_E the integral is simply of Laplace type and therefore we conclude that

$$I = -J = \sum_{\nu=0}^{p-1} \gamma I_p^{(\nu)} + k^{-1} \sum_{C_E} F_0(z; \mathbf{a})[1 + O(k^{-1})]e^{-kP(z;\mathbf{x})} \tag{4.18}$$

from which the estimate (4.2) immediately follows.

Let us now assume that $C + C_E$ is not as described above. Then from the case where C is infinite we conclude that

$$I = \sum_{\nu=0}^{p-1} \gamma_\nu[1 + O(k^{-1})][I_p^{(\nu)} + J_p^{(\nu)}] - J. \tag{4.19}$$

In this case we conclude by using (4.17) that for k large enough

$$I \leqq 2 \sum_{\nu=0}^{p-1} |\gamma_\nu|\, |I_p^{(\nu)}| + 2\, k^{-1} \sum_{C_E} |F_0(z; \mathbf{a})|\, |e^{-kP(z;\mathbf{x})}| \tag{4.20}$$

from which the estimate (4.2) again follows. Q.E.D.

In order to prove Theorem II we shall use the following

Induction hypothesis. *The estimate* (4.1) *or* (4.2) *is true for all numbers* p', $1 \leqq p' < p$. *That is, suppose in* (2.12) *that* P *has* p' *stationary points. Then we may introduce functions* $U_{p'}^{(\nu)}$ *and* $I_{p'}^{(\nu)}$ *and assume the estimate* (4.1) *or* (4.2) *with* p *replaced by* p'.

For $p = 2$ our result is equivalent to repeating the proof given by Chester, Friedman and Ursell [2] with the added detail of keeping track of endpoint contributions. Therefore the result (4.1) or (4.2) for $p = 2$ has essentially been verified.

This hypothesis is the essence of the proof. We shall stretch coordinates so that q stationary points are confined to the interior of the unit circle and $p - q$ stationary points are confined to the exterior of the circle of radius 2. This will introduce a new large parameter which is independent of k and a function of R alone. The new integral is then written as a sum of two integrals, one of which can be estimated in terms of functions $I_q^{(\nu)}$, the other in terms of $I_{p-q}^{(\nu)}$. By adding these results we obtain an estimate for the complete integral in terms of $I_p^{(\nu)}$ as in (4.2).

The separation imposed by (4.4) on the two groups of roots is indeed small. This exemplifies an important feature of uniform asymptotic expansions: Non-uniform expansions (an expansion of I in terms of $U_q^{(\nu)}$ and $U_{p-q}^{(\nu)}$ is non-uniform) remain valid for surprisingly small values of the distance between the stationary points. D. Ludwig has shown this explicitly in the case where $p = 2$. See [15].

Proof of Theorem II. We assume again that C is infinite in extent and non-trivial. The function G is expanded as in (4.10) and this leads to the result

$$I = \sum_{\nu=0}^{p-1} \gamma_\nu I_p^{(\nu)} + k^{-1} I_1 . \tag{4.21}$$

Here

$$I_1 = \int_C G_1(z; \mathrm{a}) e^{-kP(z;\mathrm{x})}\, dz. \tag{4.22}$$

We introduce the stretched variable t by the equation

$$t = z(k/k')^{1/(p+1)}, \qquad k' = R^{p+1}, \tag{4.23}$$

and then

$$I_1 = (k'/k)^{1/(p+1)} \int_{\tilde{C}} \tilde{G}_1(t; \tilde{\mathrm{a}}) e^{-k'Q}\, dt; \qquad Q = P(t, \tilde{\mathrm{x}}) \tag{4.24}$$

Here $\tilde{C}$ is the image of C and

$$\begin{aligned} \tilde{x}_\nu &= x_\nu (k/k')^{1-\nu/(p+1)}, \qquad \nu = 0, 1, \cdots, p; \\ \tilde{a}_\nu &= a_\nu (k/k')^{1/(p+1)}, \qquad \nu = 1, 2, \cdots, p; \end{aligned} \tag{4.25}$$

$$\tilde{G}_1(t; \tilde{\mathrm{a}}) = G_1(z; \tilde{\mathrm{a}}) = G_1([k'/k]^{1/(p+1)} t; [k'/k]^{1/(p+1)} \mathrm{a}). \tag{4.26}$$

We define $C^{(1)}$ and $C^{(2)}$:

$$C^{(1)} : |z| \leqq \tfrac{3}{2}(k'/k)^{1/(p+1)}; \quad C^{(2)} : |z| > \tfrac{3}{2}(k'/k)^{1/(p+1)}; \quad C^{(1)} + C^{(2)} = C \tag{4.27}$$

and then in the t plane we have the images $\tilde{C}^{(1)}$ and $\tilde{C}^{(2)}$, respectively,

$$\tilde{C}^{(1)} : |t| \leqq \tfrac{3}{2}; \qquad \tilde{C}^{(2)} : |t| > \tfrac{3}{2}. \tag{4.28}$$

Corresponding to these contours we have the contour integrals

$$(4.29)\qquad I^{(\mu)} = \int_{C^{(\mu)}} G_1(z; \mathrm{a}) e^{-kP(z;\mathrm{x})}\, dz = (k'/k)^{1/(p+1)} \int_{\tilde{C}^{(\mu)}} \tilde{G}_1(t; \tilde{\mathrm{a}}) e^{-k'Q}\, dt; \qquad \mu = 1, 2.$$

From (4.4) and (4.25) it follows that the new stationary points are bounded by

$$(4.30)\qquad |\tilde{a}_\nu| \leqq 1, \quad \nu = 1, 2, \cdots, q; \qquad |\tilde{a}_\nu| > 2, \quad \nu = q + 1, \cdots, p.$$

We observe that in the t plane P has q stationary points in $|t| \leqq \frac{3}{2}$ and therefore we make a change of variables to reduce P to a polynomial of degree $q + 1$. To do this, we introduce

$$(4.31)\qquad \Phi_q'(Z; \mathbf{A}^1) = \prod_{\nu=1}^{q} (Z - A_\nu^1); \qquad \mathbf{A}^1 = (A_1^1, \cdots, A_q^1)$$

and then set

$$(4.32)\qquad P(t; \tilde{\mathrm{x}}) = \Phi_q(Z; \mathbf{X}^1) = \int_0^Z \Phi_q'(Z'; \mathbf{A}^1)\, dZ' = P_q(Z; \mathbf{X}^1).$$

Here

$$(4.33)\qquad P_q(Z; \mathbf{X}^1) = \frac{Z^{q+1}}{q+1} + \sum_{\mu=1}^{q} \frac{X_\mu^1}{\mu} Z^\mu; \qquad \mathbf{X}^1 = (X_1^1, X_2^1, \cdots, X_q^1).$$

The elements of $\mathbf{A}^1$ or $\mathbf{X}^1$ are determined by the requirement that the change of variables be one-one analytic for $|t| \leqq \frac{3}{2}$ and in particular

$$(4.34)\qquad t = \tilde{\mathrm{a}}_\nu \leftrightarrow Z = A_\nu^1, \qquad \nu = 1, 2, \cdots, q.$$

Now $I^{(1)}$ becomes

$$(4.35)\qquad I^{(1)} = (k'/k)^{1/(p+1)} \int_{C_1} H_1(Z, \mathbf{A}^1) e^{-k'P_q(Z;\mathbf{X}^1)}\, dZ.$$

Here C_1 is the image of $C^{(1)}$ and

$$(4.36)\qquad H_1(Z; \mathbf{A}^1) = G_1(t; \tilde{\mathrm{a}})\, dt/dZ = \tilde{G}_1(t; \tilde{\mathrm{a}}) P_q'(Z; \mathbf{X}^1)/P'(t; \tilde{\mathrm{x}}).$$

For k' sufficiently large, we apply our induction hypothesis to $I^{(1)}$ and conclude that

$$(4.37)\qquad I^{(1)} = (k'/k)^{1/(p+1)} \Big\{ \sum_{\nu=0}^{q-1} \gamma_\nu [1 + O(1/k')] I_q^{(\nu)} - \sum_{C_1} F_1(Z; \mathbf{A}^1)[1/k' + O(1/k'^2)] e^{-kP_q(Z;\mathbf{X}^1)} \Big\}.$$

Here

$$(4.38)\qquad I_q^{(\nu)} = \int_{C_1} Z^\nu e^{-k'P_q(Z;\mathbf{X}^1)}\, dZ$$

and

$$H_1(Z; \mathbf{A}^1) = \sum_{\nu=0}^{q-1} \gamma_\nu^1 Z^\nu + P_q'(Z; \mathbf{X}^1) F_1(Z; \mathbf{A}^1). \tag{4.39}$$

In like manner we estimate $I^{(2)}$ by introducing the polynomial

$$\Phi_{p-q}'(Z; \mathbf{A}^2) = \prod_{\nu=1}^{p-q-1} (Z - A_\nu^2); \tag{4.40}$$

$$\Phi_{p-q}(Z; \mathbf{A}^2) = \int_0^Z \Phi_{p-q}'(Z'; \mathbf{A}^2)\, dZ' = P_{p-q}(Z; \mathbf{X}^2) \tag{4.41}$$

with

$$P_{p-q}(Z; \mathbf{X}^2) = \frac{Z^{p-q+1}}{p-q+1} + \sum_{\mu=1}^{p-q+1} \frac{X_\mu^2}{\mu} Z^\mu; \quad \mathbf{X}^2 = (X_1^2, X_2^2, \cdots, X_{p-q}^2). \tag{4.42}$$

The elements of $\mathbf{A}^2$ are determined by

$$t = \tilde{a}_\nu \leftrightarrow Z = A_{\nu-q}^2; \qquad \nu = q+1, \cdots, p. \tag{4.43}$$

It then follows that

$$I^{(2)} = (k'/k)^{1/(p+1)} \left\{ \sum_{\nu=0}^{p-q-1} \gamma_2^2 [1 + O(1/k')] I_{p-q}^{(\nu)} - \sum_{C_2} F_2(Z; \mathbf{A}^2)[1/k' + O(1/k'^2)] e^{-k' P_{p-q}(Z; \mathbf{X}^2)} \right\}. \tag{4.44}$$

Here

$$I_{p-q}^{(\nu)} = (k'/k)^{1/(p+1)} \int_{C_2} Z^\nu e^{-k' P_{p-q}(Z; \mathbf{X}^2)}\, dZ, \tag{4.45}$$

$$\begin{aligned} H_2(Z; \mathbf{A}) &= \tilde{G}_1(t; \tilde{\mathbf{a}}) P_{p-q}'(Z; \mathbf{X}^2) / P_{p-q}(t; \tilde{\mathbf{x}}) \\ &= \sum_{\nu=0}^{p-q-1} \gamma_\nu^2 Z^\nu + P_{p-q}'(Z; \mathbf{X}^2) F_2(Z; \mathbf{A}^2) \end{aligned} \tag{4.46}$$

and C_2 is the image of $C^{(2)}$.

By adding (4.37) and (4.44) we find that for k large enough

$$\begin{aligned} I_1 = I^{(1)} + I^{(2)} = (k'/k)^{1/p+1} \Big\{ &\sum_{\nu=0}^{q-1} \gamma_\nu^1 [1 + O(1/k')] I_q^{(\nu)} \\ &+ \sum_{\nu=0}^{p-q-1} \gamma^2 [1 + O(1/k')] I_{p-q} - \sum_{C_1} F_1 [1/k' + O(1/k'^2)] e^{-k' P_q(Z; \mathbf{X}^1)} \\ &- \sum_{C_2} F_2 [1/k' + O(1/k'^2)] e^{-k' P_{p-q}(Z; \mathbf{X}^2)} \Big\}. \end{aligned} \tag{4.47}$$

We must relate this estimate to a bound in terms of the functions $I_p^{(\nu)}$ or equivalently $U_p^{(\nu)}$. To do this we observe first that the functions $I_p^{(\nu)}$ also have estimates of the form (4.47). By performing the same operations on $I_p^{(\nu)}$ as above, we find that

$$I_p^{(\mu)} = (k'/k)^{(\mu+1)/(p+1)}\Big\{\sum_{\nu=0}^{q-1} C_\nu^\mu[1 + O(1/k)]I_q^{(\nu)} + \sum_{\nu=0}^{p-q-1} d_\nu^\mu[1 + O(1/k')]I_{p-q-1}^{(\nu)} - \sum_{C_1} F_1^\mu[1/k' + O(1/k'^2)]e^{-k'P_q(Z;X^1)} - \sum_{C_2} F_2^\mu[1/k' + O(1/k'^2)]e^{-k'P_{p-q}(Z;X^2)}\Big\}; \quad \mu = 0, 1, \cdots, p-1. \tag{4.48}$$

Here the explicit values of the coefficients are unimportant—only the form of (4.48) will enter into our further analysis. We note from the appendix, (I13), that

$$I_p^{(\mu)}k^{(\mu+1)/(p+1)} = U_p^{(\mu)}(\mathrm{y}; C') \tag{4.49}$$

where the function $U_p^{(0)}$ is a non-trivial solution of a homogeneous differential equation of degree $P + 1$. Therefore, $U_p^{(0)}$ and its derivatives to order $p - 1$ cannot all be zero simultaneously. Consequently not all of the expressions { } in (4.48) can be zero simultaneously and in fact we can find domains where each of the expressions is bounded below. As far as its order in k' is concerned, it must be directly identifiable with the order in k' of the expression { } in (4.47). Therefore we conclude that there exist constants c'_μ such that for k sufficiently large

$$|I_1| \leqq \sum_{\mu=0}^{p-1} (k/k')^{\mu/(p+1)}c'_\mu I_p^{(\mu)}. \tag{4.50}$$

When this result is substituted into (4.21) we see that again we obtain the desired estimate. If we apply our result to I_1 instead of I we find that

$$I = \sum_{\nu=0}^{p-1} (\gamma_\nu^{(1)} + \gamma_\nu^{(2)}k^{-1})I_p^{(\nu)} + k^{-2}I_2 . \tag{4.52}$$

Here the coefficients $\gamma_\nu^{(1)}$ are just the γ_ν's in (4.21), and the coefficients $\gamma_\nu^{(2)}$ are determined by setting

$$G_1(z; \mathrm{a}) = \sum_{\nu=0}^{p-1} \gamma_\nu^{(2)}z^\nu + P'(z; \mathrm{x})F_2(z; \mathrm{a}). \tag{4.53}$$

The integral I_2 is the same as I_1 with G replaced by $G_2 = F_2$. Therefore we obtain an estimate for I_2 of the form of (4.50) and conclude that

$$I = \sum_{\nu=0}^{p-1} \gamma_\nu^{(1)}[1 + O(k^{-1})]I_p^{(\nu)}. \tag{4.54}$$

Now let us suppose that C is of finite extent. We again introduce the contour C_E defined below (4.14) and the functions J, (4.15), and $J_p^{(\nu)}$, (4.16). If $C + C_E$ is a trivial contour, then we obtain an estimate completely in terms of endpoint contributions such as in (4.18). If $C + C_E$ is non-trivial we apply (4.54) to $I + J$ and obtain the result (4.19) in terms of $I_p^{(\nu)} + J_p^{(\nu)}$. Then (4.20) follows.

5. Example of a uniform asymptotic expansion. As an example, let us apply the methods of sections 2 and 3 to obtain the leading term of the uniform asymptotic expansion of

$$I(k; \alpha, \beta, r) = \frac{1}{\pi} \int_\Gamma (t - \beta)^r e^{-k\rho f(t;\alpha)} \, dt. \tag{5.1}$$

Here we take Γ to be some part of Γ_1, the Sommerfeld contour [3] for the Hankel function of the first kind (See Fig. 1.) and

$$f(t; \alpha) = -i\left[\cos t + \nu\left(t - \frac{\pi}{2}\right)\right]. \tag{5.2}$$

We seek a uniform asymptotic expansion for $k\rho \gg 1$, $\nu \approx 1$ and β free; α will be defined below as a function of ν.

By differentiating (5.2) we find that

$$f'(t; \alpha) = i[\sin t - \nu]; \qquad f''(t) = i \cos t, \qquad f'''(t) = -i \sin t \tag{5.3}$$

from which it follows that for $0 < \nu \leqq 1$ the stationary points of interest are

$$t = \alpha_\pm ; \qquad \sin \alpha_\pm = \nu; \qquad 0 < \alpha_+ \leqq \frac{\pi}{2} ; \qquad \frac{\pi}{2} \leqq \alpha_- = \pi - \alpha_+ < \pi. \tag{5.4}$$

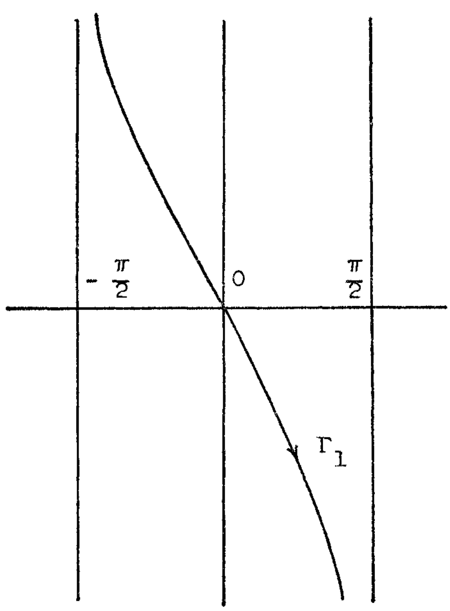

FIGURE 1

For this example the notation $\alpha_\pm$ will prove less cumbersome than the notation, α_1, α_2 of the previous sections and therefore we define

$$\boldsymbol{\alpha} = (\alpha_+, \alpha_-). \tag{5.5}$$

The integral (5.1) is therefore a special case of (3.1) with $p = 2$, $q = 1$. We note that for $\nu = 1$, $\alpha_+ = \alpha_- = \pi/2$, $f''(\pi/2; \boldsymbol{\alpha}) = 0$, but $f'''(\pi/2; \boldsymbol{\alpha}) = -i$.

We shall set f equal to a polynomial of degree three as in (2.6) and (2.7), but we shall also introduce the condition (2.11) so that the vector $\mathbf{a}$ in (2.4) is given by

$$\mathbf{a} = (a, -a); \tag{5.6}$$

then (2.7) becomes

$$f(t; \boldsymbol{\alpha}) = \frac{z^3}{3} - a^2 z + x_0 = \Phi(z; \mathbf{a}). \tag{5.7}$$

By using the requirement (2.9) with $\alpha_\pm \leftrightarrow \pm a$, we obtain a pair of equations for a and x_0;

$$\mp i\left[\cos\alpha_+ - \sin\alpha_+\left(\frac{\pi}{2} - \alpha_+\right)\right] = \mp 2\,\frac{a^3}{3} + x_0 . \tag{5.8}$$

It therefore follows that

$$x_0 = 0; \tag{5.9}$$

$$\tfrac{2}{3}a^3 = i\left[\cos\alpha_+ - \sin\alpha_+\left(\frac{\pi}{2} - \alpha_+\right)\right] = i[(1 - \nu^2)^{1/2} - \nu\cos^{-1}\nu]. \tag{5.10}$$

For $\nu < 1$, the expression [ ] is positive and therefore

$$\arg a^3 = \frac{\pi}{2} + 2n\pi; \qquad \arg a = \frac{\pi}{6} + \frac{2n\pi}{3} \qquad n = 0, 1, 2. \tag{5.11}$$

The value of n and the branch of the cubic $z = z(t; \boldsymbol{\alpha})$ in (5.7) must be determined. To do this, let us take $\nu < 1$ and consider the steepest descent contours $\Gamma_\pm$ through the stationary points $\alpha_\pm$. See Fig. 2. The images of these contours are steepest descent paths in the z plane. In Fig. 3 we show these paths for $n = 1$ in (5.11). For $n = 0$, this figure should be rotated through $2\pi/3$ clockwise, for $n = 2$, they should be rotated through $2\pi/3$ counter-clockwise.

We note that from (5.7)

$$\frac{dt}{dz} = \frac{z^2 - a^2}{f'(t; \boldsymbol{\alpha})} \tag{5.12}$$

for $t \neq \alpha_\pm$ and by L'Hospital's rule

$$\left(\frac{dt}{dz}\right)^2_{t=\alpha_\pm} = \frac{2a}{i\cos\alpha_+}; \qquad \left.\frac{dt}{dz}\right|_{t=\alpha_\pm} = \pm\left(\frac{2a}{i\cos\alpha_+}\right)^{1/2}. \tag{5.13}$$

The choice of sign in (5.13) is undetermined but we observe that it must be the same for $t = \alpha_\pm$ since, when $\alpha_\pm \to \pi/2$, we want to obtain the same limit. By analyzing $f(t; \alpha)$, we can determine arg Δt on the paths of descent at $t = \alpha_\pm$. By analyzing $\Phi(z; a)$ in (5.7) for each possible choice of arg a we can determine arg Δz on the paths of descent near $t = \pm a$. Analysis of arg $\Delta t/\Delta z$ leads us to conclude that the choice of sign in (5.13) will be the same (+) at $t = \alpha_\pm$ only when we choose $n = 1$. Therefore

$$a = e^{5\pi i/6}\{\tfrac{3}{2}[(1 - \nu^2)^{1/2} - \nu \cos^{-1} \nu]\}^{1/3} \tag{5.14}$$

and we take the positive cube root of { } for $\nu < 1$. Now we obtain an integral of the type (3.2) for (5.1);

$$I(k; \alpha; \beta, r) = \int_C (z - b)^r G(z) e^{-k\rho[z^3/3 - a^2 z]}\, dz. \tag{5.15}$$

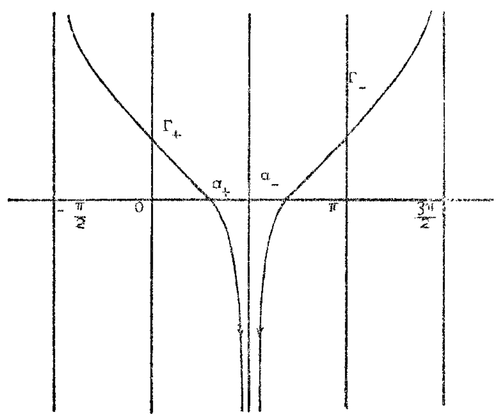

FIGURE 2

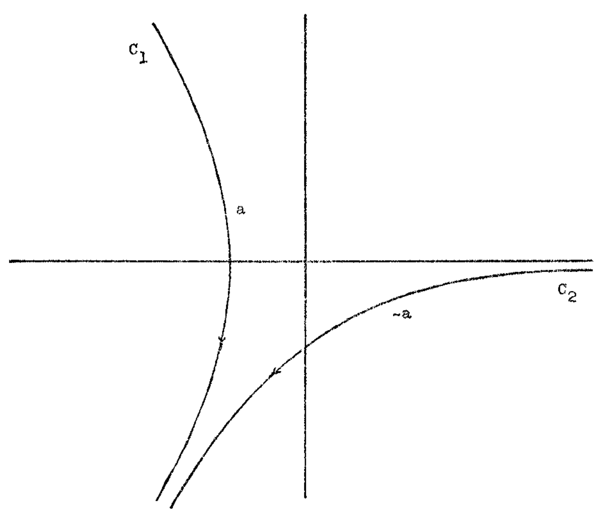

FIGURE 3

Here C is the image of Γ and

$$G(z) = \frac{1}{\pi}\left(\frac{t-\beta}{z-b}\right)^r \frac{dt}{dz}. \tag{5.16}$$

For brevity we have omitted the arguments $\mathbf{a}$, b in G as they appear in (3.2). The parameter b is determined by requiring that $t = \beta \leftrightarrow z = b$. Therefore from (5.7) b is one of the roots of the cubic equation

$$f(\beta; \boldsymbol{\alpha}) = \frac{b^3}{3} - a^2 b. \tag{5.17}$$

We choose that root for which $b = a$ when $\beta = \alpha_+$.

The next step is to carry out the expansion indicated in (3.5). That is, we set

$$G(z) = \gamma_0 + \gamma_1 z + \gamma_2 z^2 + (z-b)(z^2 - a^2)F_0(z). \tag{5.18}$$

The constants γ_0, γ_1, γ_2 are determined by three equations in which we set $z = \pm a$ and b; *i.e.*,

$$\begin{aligned} G(\pm a) &= \gamma_0 \pm \gamma_1 a + \gamma_2 a^2 \\ G(b) &= \gamma_0 + \gamma_1 b + \gamma_2 b^2. \end{aligned} \tag{5.19}$$

Here, from (5.16), (5.12) and (5.13) with (+),

$$\begin{aligned} G(\pm a) &= \frac{1}{\pi}\frac{(2\,|a|)^{1/2}}{(1-\nu^2)^{1/4}}\left[\frac{\alpha_\pm - \beta}{\pm a - b}\right]^r e^{i\pi/6}; \\ G(b) &= \frac{1}{\mu}\left[\frac{i(\sin\beta - \nu)}{b^2 - a^2}\right]^{r+1}. \end{aligned} \tag{5.20}$$

By solving the linear system (5.19) we find that

$$\begin{aligned} \gamma_0 &= \frac{2a^2 G(b) - b[G(a)(b+a) + G(-a)(b-a)]}{2\pi(a^2 - b^2)}; \\ \gamma_1 &= \frac{G(-a) - G(a)}{2\pi a}; \\ \gamma_2 &= \frac{G(a)(b+a) - G(-a)(b-a) - 2aG(b)}{2\pi a(a^2 - b^2)}. \end{aligned} \tag{5.21}$$

Now from (3.10) and (3.15) it follows that

$$\begin{aligned} I \sim \frac{\gamma_0}{(k\rho)^{(r+1)/3}} V_{21}^{(0)} &+ \frac{\gamma_1}{(k\rho)^{(r+2)/3}} V_{21}^{(1)} \\ &+ \frac{\gamma_2}{(k\rho)^{(r+3)/3}} V_{21}^{(2)} - \sum_C \frac{F_0}{k\rho} e^{-k\rho(z^3/3 - a^2 z)}. \end{aligned} \tag{5.22}$$

The functions $V_{21}^{(j)}$ are defined in the appendix. From (I14), (I20), (I21)

$$\begin{aligned} V_{pq}^{(j)} &= \int_{C'} \zeta^j (\zeta - (k\rho)^{1/3} b)^r e^{-[\zeta^3/3 - a^2 (k\rho)^{2/3}\zeta]}\, d\zeta \\ &= (k\rho)^{(r+j)/3} \int_C z^j (z-b)^r e^{-k\rho[z^3/3 - a^2 z]}\, dz; \qquad j = 0, 1, 2. \end{aligned} \tag{5.23}$$

Here C' is the image of C under the transformation

$$(k\rho)^{1/3}z = \zeta. \tag{5.24}$$

As a check, let us take $r = 0$ and $\Gamma = \Gamma_1$ in which case we should obtain a uniform asymptotic expansion of $H_{k\mu}^{(1)}(k\rho)$. Now the function $V_{21}^{(0)}$ reduces to an Airy function. Also in this special case

$$\sum_C = 0 \tag{5.25}$$

and integration by parts in (5.23), with $r = 0$, $j = 2$, yields

$$V_{21}^{(2)} = a^2(k\rho)^{2/3}V_{21}^{(0)}. \tag{5.26}$$

(This is simply Airy's differential equation.) It is then a straight forward calculation to show that

$$H_{k\mu}^{(1)}(k\rho) \sim -2\,\frac{[12]^{1/6}[(1-\nu^2)^{1/2} - \nu\cos^{-1}\nu]^{1/6}e^{2\pi i/3}}{(k\rho)^{1/3}[1-\nu^2]^{1/4}} \times Ai[-e^{2\pi i/3}\{\tfrac{3}{2}k\rho((1-\nu^2)^{1/2} - \nu\cos^{-1}\nu)\}^{2/3}], \qquad \nu = \frac{\mu}{\rho}. \tag{5.27}$$

Here

$$Ai(z) = -\frac{1}{2\pi i}\int_{C_1} e^{-[\tau^3/3 - z\tau]}\,d\tau \tag{5.28}$$

and (5.26) agrees with known results [9].

APPENDIX I

Special functions represented by integrals. Two classes of special functions are used in sections 2 and 3 to describe the uniform asymptotic expansions. Here we shall define the functions and, in particular, give the following information:

(i) an integral representation
(ii) an ordinary differential equation for each function
(iii) qualitative information about the classes of functions.

Under (iii) we shall indicate a relatively familiar element of each of the classes of functions and we shall write down representations of the functions and their derivatives for large values of the parameters involved.

Let us first consider the function

$$U_p(\mathrm{y}; C) = \int_C e^{-P(\zeta;\mathrm{y})}\,d\zeta. \tag{I1}$$

Here P is a polynomial of degree p,

$$P(\zeta; \mathrm{y}) = \frac{\zeta^{p+1}}{p+1} + \sum_{\mu=1}^{p} \frac{y_\mu}{\mu}\zeta^\mu + y_0\,; \qquad \mathrm{y} = (y_0, y_1, \cdots, y_p). \tag{I2}$$

(We could take $y_0 = 0$, but notation will be simplified in the application if we include y_0 in y.) We take the endpoints of C to be at ∞ in the valleys of $-P$. Modifications for finite endpoints will be indicated below. If one regards all of the elements of y except y_1 as parameters then differentiation with respect to y_1 is equivalent to multiplication by $-\zeta$ in the integrand and

$$\left(-\frac{d}{dy_1}\right)^j U_p(\mathbf{y}; C) = \int_C \zeta^j e^{-P(\zeta;\mathbf{y})}\, d\zeta. \tag{I3}$$

Consequently, U_p satisfies an ordinary differential equation with respect to y_1, namely

$$\begin{aligned} P'\left(-\frac{d}{dy_1}; \mathbf{y}\right) U_p(\mathbf{y}; C) &= \int_C P'(\zeta; \mathbf{y}) e^{-P(\zeta;\mathbf{y})}\, d\zeta \\ &= -\sum_C e^{-P(\zeta;\mathbf{y})} = 0. \end{aligned} \tag{I4}$$

This differential equation is of order p in which case we need only consider $U_p^{(j)}$ for $j \leqq p - 1$, all other derivatives being expressible in terms of these. In order to gain some insight into the nature of the functions U_p, let us consider the specific example where $p = 2$. In this case (I4) becomes

$$\frac{d^2 U_2}{dy_1^2} + y_2 \frac{dU_2}{dy_1} + y_1 U_2 = 0. \tag{I5}$$

If we set

$$U_2 = A(y_2^2/4 - y_1) e^{-(y_2/2) y_1} \tag{I6}$$

then $A(x)$ is a solution of Airy's differential equation [4]:

$$A''(x) - xA(x) = 0. \tag{I7}$$

Therefore we may look upon U_2 as a *modified Airy function* and $U_p(y; C')$ as a *generalized (modified) Airy function*. We may view the function

$$A(x) = \int_a^\infty e^{-[t^3/3 - xt]}\, dt \tag{I8}$$

as an incomplete Airy function. In like manner, when C has finite endpoints, we think of $U_p(\mathbf{y}; C')$ as an *incomplete* special function. In addition, the function

$$B(x) = \int_a^\infty -x e^{-[t^3/3 - xt]}\, dt \tag{I9}$$

is an incomplete derivative of the Airy function. We would have $B = A'$ only if a were independent of x. Therefore, in general we cannot say that the incomplete derivative is the derivative of the incomplete function. Nonetheless, we may always regard (I3) as an equation for the *incomplete derivatives* of U_p when C' has finite endpoint(s).

The special case U_2 also serves to cast some light on the asymptotic nature of U_2. In (I6) it is clear that one could replace the Airy function by its asymptotic expansion when

$$|y_1 - y_2^2/4| \gg 1. \tag{I10}$$

This quantity is just the discriminant of $P'(\zeta; \mathbf{y})$ and therefore measures the separation of the stationary points of P. The asymptotic expansion of the Airy function can be obtained by the method of steepest descents, with either one or both saddle points contributing to the result. In the context of the present analysis, this asymptotic expansion is just a sum of functions of the type U_1 (incomplete), each multiplied by an asymptotic series.

For the general function, U_p, let us suppose that the stationary points of P are separated into two groups, one with p' stationary points and the other with $p - p'$ stationary points. We claim that if these groups are well separated U_p has an asymptotic expansion in terms of $U_{p'}$, $U_{p'}^{(1)}$, $\cdots$ $U_{p'}^{(p-1)}$, $U_{p-p'}$, $\cdots$, $U_{p-p'}^{(p-p'-1)}$, of some as yet unspecified arguments. Each of these functions will be multiplied by an asymptotic series. Of course, the functions might be incomplete and there might also be an asymptotic series of endpoint contributions.

Let us now make a change of variables in (I1) in order to write U_p in the form in which it arises in this paper. We set

(I11)

$$\zeta = k^{1/(p+1)}z; \quad y_\mu = k^{1-\mu/(p+1)}x_\mu; \quad \mu = 0, 1, \cdots, p; \quad \mathbf{x} = (x_0, x_1, \cdots, x_p).$$

Now (I1) becomes

$$U_p(\mathbf{y}; C') = k^{1/(p+1)} \int_C e^{-kP(z;\mathbf{x})}\, dz. \tag{I12}$$

Here C is the image of C' under the stretching introduced in (I11). For the derivatives we have

$$\frac{1}{k^{(1+j)/(1+p)}} U_p^{(j)}(\mathbf{y}; C') = \int_C z^j e^{-kP(z;\mathbf{x})}\, dz; \qquad j = 0, 1, \cdots. \tag{I13}$$

The second class of functions to be considered is

$$V_{pq}(\mathbf{y}; \tilde{\mathbf{b}}, \mathbf{r}, C') = \int_{C'} \prod_{\nu=1}^{q} (\zeta - \tilde{b})^{r_\nu} e^{-P(\zeta;\mathbf{y})}\, d\zeta; \tag{I14}$$
$$\tilde{\mathbf{b}} = (\tilde{b}_1, \tilde{b}_2, \cdots, \tilde{b}_q), \qquad \mathbf{r} = (r_1, r_2, \cdots, r_q.)$$

Here P is as defined in (I1) and C' is again assumed to have endpoints at ∞ in the valleys of $-P$. We can obtain an ordinary differential equation in y_1 for v_{pq} by regarding $\tilde{\mathbf{b}}$ and all other elements of $\mathbf{y}$ as parameters. The equation is

(I15)

$$\left\{P'\left(-\frac{d}{dy_1}; \mathbf{y}\right) \prod_{\nu=1}^{q} \left(-\frac{d}{dy_1} - \tilde{b}_\nu\right) - \sum_{\mu=1}^{q} (r_\mu + 1) \prod_{\substack{\mu\neq\nu \\ \nu=1}}^{q} \left(-\frac{d}{dy_1} - \tilde{b}_\nu\right)\right\} V_{pq} = 0.$$

(For $q = 1$, interpret $\prod_{\mu \neq \nu} = 1$.) This equation is of order $p + q$ and therefore only the derivatives of V_{pq} up to order $p + q - 1$ are of interest. In any case,

$$\text{(I16)} \qquad \left(-\frac{d}{dy_1}\right)^j V_{pq} = V_{pq}^{(j)} = \int_{C'} \zeta^j \prod_{\nu=1}^{q} (\zeta - \tilde{b}_\nu)^{r_\nu} e^{-P(\zeta;y)} \, d\zeta.$$

As a special case, let us take $p = q = 1$, in which case (I15) becomes

$$\text{(I17)} \qquad \frac{d^2 V_{11}}{dy_1^2} - (y_1 - \tilde{b}_1) \frac{d}{dy_1} V_{11} - (\tilde{b}_1 y_1 + r_1 + 1) V_{11} = 0.$$

If one sets

$$\text{(I18)} \qquad V_{11} = e^{-\eta^2/4} W_{r_1}(y_1 + \tilde{b}_1); \qquad \eta = y_1 - \tilde{b}_1.$$

Then $W_{r_1}(x)$ is a solution of Weber's equation [17] of order $-(r_1 + 1)$:

$$\text{(I19)} \qquad W_{r_1}''(x) - [r_1 + \tfrac{1}{2} + x^2/4] W_{r_1}(x) = 0.$$

The function V_{11} is therefore a modified Weber function and the functions V_{pq} may be thought of as *generalized* (*modified*) *Weber* functions.

We could replace W_{r_1} by its asymptotic expansion in (I18) when

$$|y_1 + \tilde{b}_1| \gg 1.$$

This quantity measures the distance between the stationary point of P and the branch point. We conjecture that for the general function V_{pq} it is possible to obtain an asymptotic expansion in terms of $V_{p'q'}, V_{p'q'}^{(1)}, \cdots, V_{p'q'}^{(p'+q'-1)}, V_{p''q''}, V_{p''q''}^{(1)}, \cdots, V_{p''q''}^{(p''+q''-1)}; p' + p'' = p, q' + q'' = q$ when the stationary points and singular points are separated into two groups.

In order to express our results in the form needed in section 3 we use the change of variables (I11) and in addition we set

$$\tilde{b}_\nu = k^{(1/p+1)} b_\nu \, ; \qquad \nu = 1, 2, \cdots, q.$$

We then obtain the result

(I21)

$$\frac{1}{k^{([r]+j+1)/(p+1)}} V_{pq}^{(j)}(\mathbf{y}; \tilde{\mathbf{b}}; \mathbf{r}; C') = \int_C z^j \prod_{\nu=1}^{q} (z - \tilde{b}_\nu)^{r_\nu} e^{-kP(z;\mathbf{x})} \, dz; \quad j = 0, 1, \cdots.$$

Here

$$\text{(I22)} \qquad [r] = \sum_{\nu=1}^{q} r_\nu$$

and C is the image of C' under the change of variables in (I11).

The special case of (1.1) for $p = 2$ was discussed in a talk presented by Lawrence Levey at the URSI Meeting, Spring '66, Washington, D. C. The title of the talk was "On the Incomplete Airy Integral and its Application to Certain Diffraction Problems."

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