Differential and Integral Equations, Volume 6, Number 3, May 1993, pp. 507–533.

UNIFORM BOUNDARY STABILIZATION OF SEMILINEAR WAVE EQUATIONS WITH NONLINEAR BOUNDARY DAMPING

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Abstract. A semilinear model of the wave equation with nonlinear boundary conditions and nonlinear boundary velocity feedback is considered. Under the assumption that the velocity boundary feedback is dissipative and that the other nonlinear terms are conservative, uniform decay rates for the solutions are established.

1. Introduction.

Consider the semilinear equation

$$\begin{cases} y_{tt} = \Delta y - f_0(y) & \text{in } \Omega \times (0,\infty), \\ \frac{\partial y}{\partial \gamma} = -g(y_t \,|_{\Gamma_1}) - f_1(y\,|_{\Gamma_1}) & \text{on } \Gamma_1 \times (0,\infty), \\ y = 0 & \text{on } \Gamma_0 \times (0,\infty), \\ y(0) = y_0 \in H^1_{\Gamma_0}(\Omega), \quad y_t(0) = y_1 \in L_2(\Omega). \end{cases} \tag{1.1}$$

Here Ω is a bounded open region in $\mathbb{R}^n$, $n \geq 1$, with a smooth boundary $\Gamma \equiv \Gamma_0 \cup \Gamma_1$ and $H^1_{\Gamma_0}(\Omega) \equiv \{h \in H^1(\Omega) : h\,|_{\Gamma_0} = 0\}$, where Γ_0 and Γ_1 are closed and disjoint; γ is an outer unit vector normal to the boundary Γ_1. The following assumptions are made on the nonlinear functions f_i, $i = 0, 1$, and g:

(H–1) (i) $g(s)$ is a continuous, monotone, increasing function on $\mathbb{R}$;
(ii) $g(s)\,s > 0$ for $s \neq 0$;
(iii) $M_2 s^2 \leq g(s)\,s \leq M_1 s^2$ for $|s| \geq 1$, for some M_1, M_2, $0 < M_2 \leq M_1$;

(H–2) (i) $f_0(s)$ is a $W^{1,\infty}_{\text{loc}}(\mathbb{R})$, piecewise $C^1(\mathbb{R})$ function, differentiable at $s = 0$;
(ii) $f_0(s)\,s \geq 0$ for $s \in \mathbb{R}$;
(iii) $|f_0'(s)| \leq N(1 + |s|^{k_0 - 1})$, $1 < k_0 < \frac{n}{n-2}$ for $|s| > N$, N large enough, $n \geq 2$;

(H–3) (i) $f_1(s)$ is a continuous function, differentiable at $s = 0$;
(ii) $f_1(s)s \geq 0$ for $s \in \mathbb{R}$;
(iii) $|f_1(s)| \leq M|s|^{k_1} + A|s|$ for $s \in \mathbb{R}$, $k_1 < \frac{n-1}{n-2}$, M, A given constants.

Received for publication April 1992.
†Research partially supported by the National Science Foundation Grant NSF DMS 8902811 and the Air Force Office of Scientific Research Grant AFOSR-89-0511.
AMS Subject Classification: 35.

The main goal of this paper is to prove that under the above hypotheses, solutions to (1.1) exist in $C(0,\infty; H^1_{\Gamma_0}(\Omega)) \cap C^1(0,\infty; L_2(\Omega))$ and, moreover, that they decay to 0 with uniform rates when $t \to \infty$.

The problem of proving uniform decay rates for the solutions to the wave equation with a boundary dissipation has attracted a lot of attention in recent years. Indeed, the linear problem (i.e., when $g(y) = y$ and $f_i(y) \equiv 0$) has been treated by several authors; see for instance [6], [10], [16], [18], [9]. When the boundary conditions are nonlinear, the only cases considered so far in the literature ([1], [8], [7], [11], [13], [19], [20]) are marked by both of the following features:

(i) the nonlinearities give rise to a monotone problem (modulo Lipschitz perturbations);

(ii) the dissipative term on the boundary is of a preassigned polynomial growth at the origin.

These assumptions are critically invoked in the proofs in the literature in the following ways:

(a) monotonicity (modulo Lipschitz perturbations) plays an essential role in asserting well-posedness of the problem (existence, uniqueness, and regularity of the solutions);

(b) the polynomial growth of the origin of the boundary dissipation contributes (among other things) to confer a specific structure to the equation, that allows the construction of a standard Lyapunov-function, which is then used to yield desired decay rates.

With motivations coming from various physical applications, our goal in this paper is to dispense entirely with both of the above assumptions (i) and (ii). Indeed, in our formulation, the presence of the nonlinear functions f_0 and, particularly, f_1 entirely destroys monotonicity and, moreover, no growth assumption at the origin is imposed on the function g. As a consequence, the resulting problem is now faced with major technical difficulties, which require the development of new approaches and new techniques in successfully solving both (a) the problem of existence of solutions (no claim of uniqueness is, however, made) and (b) the problem of obtaining global, uniform decay rates.

Orientation. Existence and regularity. Our basic approach in proving existence of solutions (without claiming uniqueness) relies on a rather special construction of suitable approximating problems (see (2.11)) and on careful estimates of the approximating solutions (difficulties are primarily due to the presence of the term f_1). Passage to the limit then produces the desired claim of existence of solutions to the original problem. However, the absence of a claim of uniqueness (uniqueness can be asserted only if g is coercive, see Corollary 1) is a source of another difficulty. Given a solution (perhaps different from the one asserted through the aforementioned existence argument), p.d.e's estimates require that it be approximated by solutions with some regularity properties to carry out and justify computations. An important point to be stressed is that we cannot, in the present case, adopt the usual procedure of first restricting to smooth initial data, next obtain for these solutions the desired estimates, and finally extend them by density. No matter how smooth the initial data are, the corresponding solutions of the nonlinear problem need not be regular; besides, they do not necessarily depend continuously on the initial data. Nevertheless, a strategy is needed for approximating a given solution with regular functions. Since in the absence of uniqueness a given solution need not be the one produced by the existence approximating argument mentioned before, a second approximating scheme is then needed (see Lemma 2.2) which is defined in terms of a given solution of the original problem (1.1) and which produces regular/smooth approximations of this (given) solution. This scheme, by necessity, is then different from the one employed in the existence proof (notice the different boundary conditions in (2.11) and (2.39)). The need for approximating regular solutions appears already at the very preliminary step of the analysis when we derive the basic energy identity of Proposition 2.1. This proposition can be obtained *formally* by means of usual integration by part arguments, if one takes for granted that solutions are regular. A similar need arises in Proposition 3.1, which likewise requires p.d.e. estimates, hence regularity of approximations. A passage to the limit (on the final estimates) produces the desired estimates for the original problem.

Global, uniform estimates. In order to prove uniform decay rates for all solutions, we follow an approach which is based on the following two steps.

(i) We first obtain certain integral estimates for the energy functional (in place of the usual differential estimates as in the Lyapunov approach of prior literature). These integral estimates have the advantage of allowing application of certain nonlinear compactness-uniqueness arguments which in turn lead to a nonlinear functional (not differential) relation for the energy function; see Lemma 3.2 (this idea was first employed in [12]).

(ii) Next, we prove comparison theorems which relate the asymptotic behaviour of the energy and of the solutions to an appropriate nonlinear ordinary differential equation.

Below we state our main results.

Theorem 1.

(i) *Assume* (H-1)–(H-3). *Then, for each* $(y_0, y_1) \in H^1_{\Gamma_0}(\Omega) \times L_2(\Omega)$, *problem* (1.1) *has at least one solution* $y \in C_{\text{loc}}(0, \infty; H^1_{\Gamma_0}(\Omega)) \cap C^1_{\text{loc}}(0, \infty; L_2(\Omega))$ *such that*

$$y_t \in L_{2,\text{loc}}(0, \infty; \Gamma_1), \quad \frac{\partial y}{\partial \gamma} \in L_{2,\text{loc}}(0, \infty, \Gamma_1). \tag{1.2}$$

Remark 1.1. Actually, Theorem 1 can be proven under somewhat more general hypotheses assumed on the functions $f_i(s)$. Indeed, modification of some arguments in the proof of Theorem 1 allows us to replace conditions (H-2)–(H-3) (iii) by the requirement that f_0, f_1 are compact (as Nemytskii's operators) from $H^1(\Omega)$ to $L_2(\Omega)$, $L_2(\Gamma_1)$, resp.

Corollary 1. *In addition to the hypotheses of Theorem* 1, *we assume that either* $f_1 \equiv 0$ *or else* $[g(s_1) - g(s_2)][s_1 - s_2] \geq \alpha |s_1 - s_2|^2$ *and* f_1 *is locally Lipschitz from* $H^1(\Omega)$ *into* $L^2(\Gamma)$. *Then the solution* $(y,\ y_t)$ *is unique.*

In order to state our stability result, we introduce some notation. Let $h(s)$ be a real valued function which is defined for $s \geq 0$, it is concave, strictly increasing, $h(0) = 0$ and it satisfies

$$h(sg(s)) \geq s^2 + g^2(s) \text{ for } |s| \leq N, \text{ for some } N > 0. \tag{1.3}$$

Such a function can be always constructed by virtue of hypothesis (H-1). Indeed, define the increasing functions k_1, k_2 on $\mathbb{R}^1$ by

$$k_1(sg(s)) \geq s^2 + g^2(s) \quad \text{for } s \geq 0,$$
$$k_2(sg(s)) \geq s^2 + g^2(s) \quad \text{for } s \leq 0.$$

Then the function

$$h = \text{conc}(\max\{k_1, k_2\}) \quad \text{(concave envelope)}$$

has the desired properties. Let

$$\tilde{h}(x) \equiv h\Big[\frac{x}{\text{mes}\Sigma_1}\Big], \quad x \geq 0,$$

where $\sum_1 = \Gamma_1 \times (0,T)$ and T is a given constant. Since $\tilde{h}$ is monotone increasing, for every $c \geq 0$, $c + \tilde{h}$ is invertible. Define

$$p(x) \equiv (cI + \tilde{h})^{-1}(Kx), \tag{1.4}$$

where K is a positive constant. Then p is a positive, continuous, strictly increasing function with $p(0) = 0$. Let

$$q(x) \equiv x - (I + p)^{-1}(x), \quad x > 0. \tag{1.5}$$

Since $p(x)$ is positive, increasing, so is $q(x)$. Let $E(t)$ denotes the energy of the solution $(y,\, y_t)$; i.e.,

$$E(t) \cong \frac{1}{2}(|\nabla y(t)|^2_{L_2(\Omega)} + |y_t(t)|^2_{L_2(\Omega)}) + \int_{\Gamma_1} F_1(y)\, d\Gamma_1 + \int_\Omega F_0(y)\, d\Omega, \tag{1.6}$$

where

$$F_i(s) \equiv \int_0^s f_i(t)\, dt.$$

It will be shown that $E(t)$ remains bounded for the solutions in a bounded set of $H^1_{\Gamma_0}(\Omega) \times L_2(\Omega)$. We are ready to state our stabilization result. We shall need the following additional hypotheses.

(H-4) With $h \equiv x - x^0$, $x^0 \in \mathbb{R}^n$, the following geometric condition holds on the uncontrolled portion of the boundary Γ_0:

$$h\gamma \leq 0 \quad \text{on } \Gamma_0.$$

(H-5) At least one of the following conditions holds:

(i) f_0 is linear;

(ii) $\Gamma_0 = \emptyset$ and $f_0(u)u \geq \varepsilon u^2$ for some $\varepsilon > 0$ or $f_1(u)u \geq \varepsilon u^2$ for some $\varepsilon > 0$;

(iii) $\Gamma_0 = \partial\Omega_1 \neq \emptyset$, where Ω_1 is convex and $\Omega_1 \cap \Omega = \emptyset$.

We note that condition (H-5) will be necessary only in the compactness-uniqueness argument.

Theorem 2. *Assume hypotheses* (H-1)–(H-5). *Let* (y, y_t) *be a solution to* (1.1), *with the properties listed in Theorem* 1. *Then for some* $T_0 > 0$,

$$E(t) \leq S\big(\frac{t}{T_0} - 1\big)(E(0)) \quad \text{for } t > T_0, \tag{1.7}$$

where $S(t)$ *is the solution (contraction semigroup) of the differential equation*

$$\frac{d}{dt}S(t) + q(S(t)) = 0, \quad S(0) = E(0) \tag{1.8}$$

and $q(s)$ *is given by* (1.5), (1.4) *with the constant* K *in* (1.4) *depending in general (unless* $k = 1$, *where* $k = \max(k_0, k_1)$) *on* $E(0)$, *and the constant* $c = \frac{1}{\text{mes}\Sigma_1}(M_1 + M_2^{-1})$

Remark 1.2. Uniform decay rates for the wave equation with nonlinear monotone boundary feedback were obtained in [19]. However, the problem considered in [19] is fully monotone (f_0 and f_1 are zero), the nonlinear feedback $g(s)$ satisfies in addition to the hypothesis (H-1) a polynomial growth condition at the origin and, moreover, geometric restrictions on the domain Ω of "star-shaped" type are imposed. Thus, the result of [19] obtained by multipliers methods as in [6] or [10] combined with Liapunov technique of [21] is a very special case of Theorem 2.

If we additionally assume that the function $g(s)$ is of a polynomial growth at the origin, the following explicit decay rates are obtained.

Corollary 2. *Assume in addition to* (H-1)–(H-5) *that for some positive constants* a, b,

$$g(s)s \leq bs^2 \text{ for each real } s,$$
$$g(s)s \geq a|s|^{p+1} \text{ for } |s| \leq 1, \text{ for some } p \geq 1.$$

Then

$$E(t) \leq Ce^{-\alpha t} \quad \text{if } p = 1,$$
$$E(t) \leq Ct^{\frac{2}{1-p}} \quad \text{if } p > 1,$$

where both constants $C > 0$ *and* $\alpha > 0$ *depend in general on* $E(0)$ *(unless* $k = 1$).

Proof of Corollary 2. It is enough to construct a function h with the property (1.3). Indeed, we can take $h(s) = a^{-\frac{2}{p+1}}(1 + b^2)s^m$ where $m = \frac{2}{p+1} \leq 1$. Then $p(s) = (cI + \tilde{h})^{-1}(Ks)$; i.e., $cp + d(a, b)s^m = Ks$ where d is a suitable constant depending on a, b. Also, recall that

$$q(s) = s - (I + p)^{-1}(s).$$

Since asymptotically (for s small) we have, for some constant $\alpha > 0$ depending in general (unless $k = 1$) on $E(0)$,

$$p(s) \sim \alpha s^{\frac{1}{m}} \quad \text{and therefore} \quad q(s) \sim \alpha s^{\frac{1}{m}},$$

by solving equation (1.8) with q as above we obtain

$$S(t)x = \begin{cases} c_1\left(t + c_2 x^{\frac{1-p}{2}}\right)^{\frac{2}{1-p}} & \text{if } p > 1 \\ e^{-\alpha t}x & \text{if } p = 1, \end{cases}$$

where c_1, c_2 depend only on α, p. The conclusion now follows from Theorem 2.

Remark 1.3. Theorem 2 may be easily extended to the case when the function f_1 is not Lipschitz at "0". However, in this case, our proof does not provide a computable rate of energy decay.

Remark 1.4. Note that our results do not require any geometric conditions on the controlled portion of the boundary Γ_1. This is in contrast with most of the literature ([6], [10], [15], [18], [19], [9]), where the geometric restrictions on Γ_1 were imposed. In the linear case, stabilization results with $\Gamma_0 = Q$ and without any geometric conditions assumed on the boundary Γ_1 were obtained in [14]. The linear case when $\Gamma_0 \neq Q$ has been treated in [5], where sharp results are expressed in terms of geometric optics conditions.

2. Proof of Theorem 1. The proof of Theorem 1 follows from the following two step procedure. We first construct an auxiliary approximating problem for which the existence of the unique solution will be established by the arguments of nonlinear semigroup theory. In the second step, we obtain the solutions of problem (1.1) as the limits of the approximating equations.

To accomplish this, the following result will be needed.

Proposition 2.1. *Let u be a given function in $C[0,T;H^1(\Omega)] \cap C^1[0,T;L_2(\Omega)]$ such that*

$$\begin{cases} u_{tt} - \Delta u = f \in L_1[0,T;L_2(\Omega)], \\ u(0) = u_0 \in H^1(\Omega),\ u_t(0) = u_1 \in L_2(\Omega) \end{cases} \tag{2.1i}$$

and, moreover,

$$\begin{cases} u_t \text{ and } \left.\dfrac{\partial u}{\partial \gamma}\right|_\Gamma \text{ are in } L^2(0,T;L^2(\Gamma_1)), \\ u = 0 \text{ on } \Sigma_0. \end{cases} \tag{2.1ii}$$

Then the following energy identity holds for each $t > 0$:

$$E_1(t) - \int_0^t \int_{\Gamma_1} \left.\frac{\partial u}{\partial \gamma}\right|_\Gamma u_t \, d\Gamma_1 \, ds - \int_0^t \int_\Omega f u_t \, d\Omega \, ds = E_1(0),$$

where

$$E_1(t) = \frac{1}{2}(|\nabla u(t)|^2_{L_2(\Omega)} + |u_t(t)|^2_{L_2(\Omega)}).$$

Notice that the result of Proposition 2.1 can be *formally* obtained by using Green's formula and integration by parts in time. However, without the additional smoothness of the solutions u, such a procedure is only formal. Since we do not have any additional information on the smoothness of the solution (which is typical for nonlinear problems) and, moreover, the solution u may not depend continuously on the initial data, we must resort to a different approach which will be based on a certain approximation type of argument.

This argument, rather technical and independent from the main body of the proof of Theorem 1, is deferred to the end of this section.

Theorem 2.1. *Assume that*

$$\text{the functions } f_0(s) \text{ and } f_1(s) \text{ are Lipschitz continuous on } \mathbb{R}, \tag{2.2i}$$

$$g(s_1) - g(s_2) \geq \alpha(s_1 - s_2) \text{ for all } s_1 - s_2 \geq 0 \text{ and fixed } \alpha > 0. \tag{2.2ii}$$

Then, problem (1.1) *has an unique solution* $y \in C(0, \infty; H^1_{\Gamma_0}(\Omega)) \cap C^1(0, \infty; L_2(\Omega))$. *Moreover, if* g *satisfies hypothesis* (H-1) (iii), *then*

$$y_t \mid_{\Gamma} \in L_2(0, \infty; \Gamma_1), \quad \frac{\partial}{\partial \gamma} y \in L_2(0, \infty; \Gamma_1). \tag{2.3}$$

Proof of Theorem 2.1. This follows from nonlinear semigroup theory. Let $A : L_2(\Omega) \to L_2(\Omega)$ be the operator defined by

$$Au = -\Delta u \text{ with } \mathcal{D}(A) = \left\{ u \in H^2_{\Gamma_0}(\Omega) : \frac{\partial u}{\partial \gamma} \Big|_{\Gamma_1} = 0, u \mid_{\Gamma_0} = 0 \right\},$$

where $H^2_{\Gamma_0}(\Omega) \equiv H^2(\Omega) \cap H^1_{\Gamma_0}(\Omega)$. Let $N : L_2(\Omega) \to L_2(\Omega)$ be the Neumann map

$$\Delta(Ng) = 0, \; Ng \mid_{\Gamma_0} = 0, \; \frac{\partial}{\partial \gamma} Ng \mid_{\Gamma_1} = g.$$ [1]

It is well known that

$$N \in \mathcal{L}(L_2(\Gamma) \to H^{3/2}(\Omega) \subset \mathcal{D}(A^{3/4-\varepsilon})) \tag{2.4}$$

and

$$N^* A^* v = -v \mid_{\Gamma_1} \quad \text{for} \quad v \in \mathcal{D}(A^{1/2}). \tag{2.5}$$

Next define

$$\mathcal{A} \begin{bmatrix} u \\ v \end{bmatrix} \equiv \begin{bmatrix} -v \\ A(u - N[g(v) + f_1(u)]) + f_0(u) \end{bmatrix}$$

with

$$\mathcal{D}(\mathcal{A}) \equiv \{ u \in H^1_{\Gamma_0}(\Omega), \; v \in H^1_{\Gamma_0}(\Omega) : \; u - N[g(v) + f_1(u)] \in \mathcal{D}(A) \}.$$

Then (1.1) can be written as

$$\begin{bmatrix} y \\ y_t \end{bmatrix}_t = \mathcal{A} \begin{bmatrix} y \\ y_t \end{bmatrix}.$$

We shall prove that $\mathcal{A}$ is ω-accreative on the space $E = H^1_{\Gamma_0} \times L_2$ equipped with a norm

$$\begin{bmatrix} u \\ v \end{bmatrix}_E^2 \equiv |A^{1/2} u|^2_{L_2(\Omega)} + |v|^2_{L_2(\Omega)}.$$

[1] This definition is meaningful when $\Gamma_0 \neq \emptyset$. Otherwise, we consider $(\Delta - 1)Ng = 0$.

Indeed, with $\begin{bmatrix} u_1 \\ v_1 \end{bmatrix}, \begin{bmatrix} u_2 \\ v_2 \end{bmatrix} \in \mathcal{D}(\mathcal{A})$ and L_0, L_1 Lipschitz constants for f_0, f_1, resp., we have

$$\begin{aligned}
&\langle \mathcal{A} \begin{bmatrix} u_1 \\ v_1 \end{bmatrix} - \mathcal{A} \begin{bmatrix} u_2 \\ v_2 \end{bmatrix}, \begin{bmatrix} u_1 \\ v_1 \end{bmatrix} - \begin{bmatrix} u_2 \\ v_2 \end{bmatrix} \rangle_E \\
&\quad = -(A^{1/2}(v_1 - v_2), A^{1/2}(u_1 - u_2))_{L_2(\Omega)} + (A(u_1 - u_2), v_1 - v_2)_{L_2(\Omega)} \\
&\quad + (g(v_1) - g(v_2), v_1 - v_2)_{L_2(\Gamma)} + (f_1(u_1) - f_1(u_2), v_1 - v_2)_{L_2(\Gamma)} \\
&\quad + (f_0(u_1) - f_0(u_2), v_1 - v_2)_{L_2(\Omega)} \geq (\alpha - \varepsilon)|v_1 - v_2|^2_{L_2(\Gamma)} \\
&\quad - \frac{L_1}{4\varepsilon}|u_1 - u_2|^2_{L_2(\Gamma_1)} - \frac{L_0}{2}|u_1 - u_2|^2_{L_2(\Omega)} - |v_1 - v_2|^2_{L_2(\Omega)}.
\end{aligned}$$

Applying the Trace Theorem, taking $\varepsilon < \alpha$ and ω suitably large gives the desired conclusion.

We shall next prove that $\mathcal{A} + \omega\, I$ is maximal monotone. To this end, it suffices to prove (by Minty's Theorem) that for $\lambda > 0$ large enough the equation

$$\mathcal{A} \begin{bmatrix} u \\ v \end{bmatrix} + \lambda \begin{bmatrix} u \\ v \end{bmatrix} \equiv h$$

has a solution $\begin{bmatrix} u \\ v \end{bmatrix} \in E$ for any $h \in E$;

$$\begin{cases} -v + \lambda u = h_1, \\ A(u - N(g(v) + f_1(u\,|_\Gamma))) + f_0(u) + \lambda v = h_2. \end{cases} \tag{2.6}$$

Hence, $u = \frac{1}{\lambda}(h_1 + v)$ and

$$\begin{aligned}
&\lambda v + \frac{1}{\lambda}A(v) - ANg(v\,|_\Gamma) - ANf_1(\frac{1}{\lambda}(v\,|_\Gamma + h_1\,|_\Gamma) + f_0(\frac{1}{\lambda}(h_1 + v)) \\
&= -\frac{1}{\lambda}A(h_1) + h_2 \in \mathcal{D}(A^{1/2})'.
\end{aligned} \tag{2.7}$$

Let

$$B(v) = AN(g(v\,|_\Gamma) - f_1(\frac{1}{\lambda}(v\,|_\Gamma + h_1)), \tag{2.8}$$

$$C(v) = \frac{\lambda}{2}v + f_0(\frac{1}{\lambda}h_1 + v) + \frac{1}{\lambda}A. \tag{2.9}$$

In what follows, we consider the dual pair $\{H^1_{\Gamma_0}(\Omega), (H^1_{\Gamma_0}(\Omega))'\}$ with respect to the L^2 duality. For $\lambda\alpha > L_1$, $g(\cdot) - f_1(\frac{1}{\lambda}(\cdot + h_1))$ is increasing, therefore B is maximal monotone in $H^1_{\Gamma_0}(\Omega)$ (B may be written in the form $B = \partial\phi$, where ϕ is a convex integrand on $\partial\,\Omega$) (see [4] p.33). For $\frac{\lambda^2}{2} > L_2$, $\frac{\lambda}{2} \cdot + f_0(\frac{1}{\lambda}(\cdot + v))$ is also increasing, therefore C is lipschitz and monotone, continuous and coercive in $H^1_{\Gamma_0}(\Omega)$. Then, using standard perturbation results (see [3], Theorem 1.7, p.46), it follows that

$B + C$ is maximal monotone and coercive, therefore the left hand term in (2.7) is surjective (see [3]). Hence, $v \in H^1_{\Gamma_0}(\Omega)$ and from the first equation in (2.6) we infer that $u \in H^1_{\Gamma_0}(\Omega)$.

This completes the proof of the maximal monotonicity of $\mathcal{A}+\omega I$. From nonlinear semigroup theory and the density of $\mathcal{D}(\mathcal{A})$ in E, we obtain unique existence of the solution $y \in C(0,T; H^1_{\Gamma_0}(\Omega)) \cap C^1(0,T; L_2(\Omega))$ for any finite $T > 0$. To obtain (2.3), we first notice that with $\begin{bmatrix} u_0 \\ v_0 \end{bmatrix} \in \mathcal{D}(\mathcal{A})$, we have

$$v_0 \,|_{\Gamma} \in H^{1/2}(\Gamma)$$

and after using assumption (H-1) (iii) and (2.2i),

$$\frac{\partial u_0}{\partial \gamma} \in L_2(\Gamma_1), \quad g(v_0 \,|_{\Gamma}) \in L_2(\Gamma_1).$$

Let $(y(t), y_t(t))$ denote the solution of (1.1) corresponding to the initial state $(u_0, v_0) \in \mathcal{D}(\mathcal{A})$. Then, by the semigroup property, we have $(y(t), y_t(t)) \in \mathcal{D}(\mathcal{A})$ and consequently

$$y_t \,|_{\Gamma_1} \in L^\infty(0,T; L_2(\Gamma_1)), \ \frac{\partial y}{\partial \gamma}\Big|_{\Gamma_1} \in L^\infty(0,T; L_2(\Gamma_1)).$$

By the result of Proposition 2.1 and assumptions (2.2i) and (2.2ii), we obtain

$$|\nabla y(t)|^2_\Omega + |y_t(t)|^2_\Omega + \frac{\alpha}{2}\int_0^t |y_t|^2_{\Gamma_1}\, dt \le c\big[|\nabla u_0|^2_\Omega + |v_0|^2_\Omega + |u_0|^2_\Omega\big]. \tag{2.10}$$

Since $\mathcal{D}(\mathcal{A})$ is dense in E, the above inequality can be extended by density to all u_0, $v_0 \in H^1_{\Gamma_0} \times L_2(\Omega)$. Moreover, from hypothesis H-1 (iii) and from (2.10), it follows that $g(y_t \,|_{\Gamma_1}) \in L_2(0,\infty; L_2(\Gamma_1)$, and so from the equation we obtain $\frac{\partial y}{\partial \gamma} \in L_2(\sum_1)$ as desired for (2.3)

Remark 2.1. By using the technique from Proposition 2.9 [3], one could show that if g is also Lipschitz continuous then $\mathcal{D}(\mathcal{A}) \subset \{u \in H^2(\Omega) \cap H^1_{\Gamma_0}(\Omega),\ v \in H^1_{\Gamma_0}(\Omega)\}$; therefore, for all $(y_0, y_1) \in \mathcal{D}(\mathcal{A})$ the solution (y, y_t) satisfies $y \in L^\infty[0,T; H^2(\Omega)] \cap W^\infty_1[0,T; H^1_{\Gamma_0}(\Omega)]$.

We consider next the following approximation of equation (1.1). With $l \to \infty$ as the parameter of approximation,

$$\begin{cases} y_{ltt} = \Delta y_l - f_{0l}(y_l) & \text{in } \Omega \times (0,\infty), \\ \frac{\partial y_l}{\partial \gamma} = -g(y_{lt} \,|_{\Gamma}) - \frac{1}{l} y_{lt} \,|_{\Gamma} - (f_{1l}(y_l) \,|_{\Gamma}) & \text{on } \Gamma_1 \times (0,\infty), \\ y_l = 0 & \text{on } \Gamma_0 \times (0,\infty), \\ y_l(t=0) = y_0 \in H^1_{\Gamma_0}(\Omega), \quad y_{lt}(t=0) = y_1 \in L_2(\Omega), \end{cases} \tag{2.11}$$

where the f_{il} are defined by

$$f_{il}(s) \equiv \begin{cases} f_i(s), & |s| \le l \\ f_i(l), & s \ge l \\ f_i(-l), & s \le -l \end{cases} \qquad i = 1, 2. \tag{2.12}$$

Notice that for each value of the parameter l, the functions f_{il}, $i = 0, 1$, and $g_l(s) \equiv g(s) + \frac{1}{l}s$ satisfy the hypothesis of Theorem 2.1.[2] Thus, there exists a solution (y_l, y_{lt}) of (2.11) such that

$$y_l \in C(0,\infty; H^1_{\Gamma_0}(\Omega)) \cap C^1(0,\infty; L_2(\Omega))$$

and

$$\frac{\partial y_l}{\partial \gamma} \in L_2(0,\infty; H^1_{\Gamma_0}(\Omega)),\ y_{lt}\,\big|_{\Gamma_1} \in L_2(\Sigma_1),\ g_l(y_l)\,\big|_{\Gamma_1} \in L_2(0,\infty; H^1_{\Gamma_0}(\Omega)). \qquad (2.13)$$

We shall prove that the above sequence of solutions $y_l(t)$ has, on a subsequence, an appropriate limit which is a solution of the original problem (1.1). To accomplish this, we need the following.

Lemma 2.1. *Under the assumptions of Theorem* 1, *we have, as* $l \to \infty$ *and* $u_l \overset{\omega}{\rightharpoonup} u$ *in* $H^1(\Omega)$,

$$\int_\Omega F_{0l}(u)\,d\Omega + \int_\Gamma F_{1l}(u\,\big|_\Gamma)\,d\Gamma \le C(|u|_{H^1(\Omega)}), \qquad (2.14)$$

$$\begin{cases} \text{(i) } f_{1l}(u_l\,\big|_\Gamma) \to f_1(u\,\big|_\Gamma) \quad \textit{in } L_2(\Gamma) \\ \qquad \textit{and} \\ \text{(ii) } f_{0l}(u_l) \to f_0(u) \quad \textit{in } L_2(\Omega), \end{cases} \qquad (2.15)$$

where the constant $C(|u|_{H^1(\Omega)})$ *depends only on the* H^1 *norm of* u.

Proof of Lemma 2.1. Let $u \in H^1(\Omega)$. By Sobolev's Imbeddings,

$$\begin{cases} H^1(\Omega) \subset L_{\frac{2n}{n-2}}(\Omega),\ H^{1/2}(\Gamma) \subset L_{\frac{2n-2}{n-2}}(\Gamma), \quad n > 2, \\ H^1(\Omega) \subset L_p(\Omega),\ H^{1/2}(\Gamma) \subset L_p(\Gamma),\ 1 \le p < \infty, \quad n = 2, \end{cases} \qquad (2.16)$$

and the following injections are compact:

$$\begin{cases} H^1(\Omega) \subset L_{2k_0}(\Omega),\ H^{1/2}(\Omega) \subset L_{2k_1}(\Gamma), \quad n > 2; \\ H^1(\Omega) \subset L_p(\Omega),\ H^{1/2}(\Gamma) \subset L_p(\Gamma),\ p < \infty, \quad n = 2. \end{cases}$$

According to (H-2) (iii), we get $|f_0(s)| \le A + B\,|s|^{k_0}$, therefore also $|f_{0l}(s)| \le A + B|s|^{k_0}$. Hence, $F_{0l}(s) \le A_1 + B_1|s|^{k+1}$ and

$$\left|\int_\Omega F_{0l}(u(x))\,dx\right| \le \int_\Omega (A_1 + B_1|u(x)|^{k_0+1})\,dx \le c(|u|_{H^1(\Omega)}).$$

For the last inequality, we have used the Sobolev imbedding $L_{k_0+1}(\Omega) \subset H^1(\Omega)$. Applying the same argument to the term $F_{1l}(u\,\big|_\Gamma)$ (after using the Trace Theorem and injections (2.16)), we arrive at (2.14). As for (2.15) (i), we write

$$\int_\Gamma |f_{1l}(u_l\,\big|_\Gamma) - f_1(u_l\,\big|_\Gamma)|^2 d\Gamma \le 2\left[\int_{\Gamma_l} |f_1(u_l\,\big|_\Gamma)|^2 d\Gamma_l + \int_{\Gamma_l} |f_1(l)| + |f_1(-l)|^2 d\Gamma_l\right], \qquad (2.17)$$

[2]Here, without loss of generality, we have assumed that $f_1(s)$ is locally Lipschitz. Otherwise, it is enough to define $f_{1l}(s) \equiv \tilde{f}_i(s)$, $|s| \le l$, where $\tilde{f}_1(s)$ is a suitable Lipschitz approximation of $f_1(s)$.

where $\Gamma_l \equiv \{x \in \Gamma : \ |u_l(x)| > l\}$. Then, by Sobolev's Imbeddings, we have for $n > 2$,

$$\left(\int_{\Gamma_l} l^{\frac{2n-2}{n-2}}\right)^{\frac{n-2}{2n-2}} \leq \left(\int_{\Gamma} |u_l|^{\frac{2n-2}{n-2}}\right)^{\frac{n-2}{2n-2}} \leq C(|u_l|_{H^1(\Omega)}),$$

therefore

$$\operatorname{mes}\Gamma_l \leq C(|u_l|_{H^1(\Omega)}) l^{\frac{-2n+2}{n-2}}. \tag{2.18}$$

Analogously, for $n = 2$ the above inequality is valid with any exponent for l.

By assumption (H-3) and by (2.16), (2.18),

$$\begin{aligned} \int_{\Gamma_l} |f_1(u_l\,|_{\Gamma})|^2\, d\Gamma_l &\leq C \int_{\Gamma_l} |u_l|^{2k_1}\, d\Gamma_l \\ &\leq C\left[\int_{\Gamma_l} |u_l|^{\frac{2n-2}{n-2}}\right]^{\frac{k_1(n-2)}{n-1}} (\operatorname{mes}\Gamma_l)^{\frac{n-1-k_1(n-2)}{n-1}} \underset{l\to\infty}{\longrightarrow} 0 \end{aligned} \tag{2.19}$$

since $k_1(n-2) < n-1$.

$$\int_{\Gamma_l} |f_1(l)|^2\, d\Gamma_l \leq C l^{2k_1} \operatorname{mes}\Gamma_l \leq C(|u_k|_{H^1(\Omega)}) l^{2k_1 - \frac{2(n-1)}{n-2}} \underset{l\to\infty}{\longrightarrow} 0, \tag{2.20}$$

where $C(|u_k|_{H^1(\Omega)})$ is a constant depending only on $|u_k|_{H^1(\Omega)}$. Combining the results of (2.17), (2.19), (2.20) gives (2.15i). The proof of (2.15ii) is similar, hence omitted.

By using regularity properties (2.13), we are in a position to apply the energy equality (see Proposition 2.1); for each $t > 0$, we obtain

$$E_l(t) + 2\int_0^t \int_{\Gamma} y_{lt}(g(y_{lt}) + \frac{1}{l} y_{lt})\, d\Gamma\, ds = E_l(0), \tag{2.21}$$

where $E_l(t)$ is defined by (1.6) with y (respectively f) replaced by y_l (respectively f_l). By result (2.14) of Lemma 2.1, we obtain

$$E_l(0) \leq C(|y_0|_{H^1}, |y_1|_{L_2}) \text{ uniformly in } l \to \infty. \tag{2.22}$$

From hypothesis (H-1) and from (2.21), (2.22), we infer that

$$|y_{lt}|_{L_2(\Sigma)} \leq C(|y_0|_{H^1}, |y_1|_{L_2}), \tag{2.23}$$

$$|y_l|_{C(0,T;H^1_{\Gamma_0}(\Omega))} + |y_{lt}|_{C(0,T;L_2(\Omega))} \leq C. \tag{2.24}$$

If $\Gamma_0 \neq \emptyset$, then $|\nabla u|_{L_2(\Omega)}$ is an equivalent norm in $H^1_{\Gamma_0}(\Omega)$ and (2.23), (2.24) follow. Otherwise, we also need to obtain an estimate for $|y_1|_{L^2(\Omega)}$. But we have

$$\frac{d}{dt}\int_{\Omega} |y_l|^2 = 2\int_{\Omega} y_l\, y_{l,t} \leq \int_{\Omega} |y_l|^2 + |y_{l,t}|^2 \leq C + \int_{\Omega} |y_l|^2,$$

hence using Gronwall's Lemma we obtain the desired estimate.

Therefore, on a subsequence we have

$$y_l \to y \text{ weakly in } H^1(\Omega \times [0,T]) \tag{2.25}$$

and by [17] and the Trace Theorem,

$$y_l \big|_\Gamma \to y \big|_\Gamma \text{ strongly in } L_\infty(0,T;L_2(\Gamma)), \tag{2.26}$$

$$y_{lt} \big|_\Gamma \to y_t \big|_\Gamma \text{ weakly in } L_2(\Sigma). \tag{2.27}$$

Hypotheses (H-1)–(H-3) together with the compactness of the imbeddings in (2.16) and (2.26) also give

$$f_0(y_l) \to f_0(y) \text{ in } L_\infty(0,T;L_2(\Omega)), \tag{2.28}$$

$$f_1(y_l \big|_\Gamma) \to f_1(y \big|_\Gamma) \text{ in } L_\infty(0,T;L_2(\Gamma)), \tag{2.29}$$

$$g(y_{lt} \big|_\Gamma) \rightharpoonup g_0 \in L_2(\Sigma) \text{ weakly in } L_2(\Sigma) \text{ for some } g_0 \in L_2(\Sigma). \tag{2.30}$$

Let $y_l(y_m)$ be the solutions to (2.11) corresponding to the parameter l (resp. m). Then from the energy identity

$$\begin{aligned}
&|\nabla(y_l - y_m)(t)|^2_{L_2(\Omega)} + |(y_{lt} - y_{mt})(t)|^2_{L_2(\Omega)} \\
&+ \int_0^t \int_{\Gamma_1} (g(y_{lt} \big|_\Gamma) - g(y_{mt} \big|_\Gamma)][y_{lt} \big|_\Gamma - y_{mt} \big|_\Gamma]\, d\Gamma_1\, dt \\
&\le \left[\frac{1}{l} + \frac{1}{m}\right] \int_{\Sigma_1} |y_{lt}|^2\, d\Sigma_1 + \left[\frac{1}{l} + \frac{1}{m}\right] \int |y_{mt}|^2\, d\Sigma_1 \\
&+ \int_{\Sigma_1} |y_{lt} - y_{mt}|\, |f_{1l}(y_l) - f_{1m}(y_m)| d\Sigma_1 + \int_Q |y_{lt} - y_{mt}|\, |f_{0l}(y_l) - f_{0m}(y_m)| dQ.
\end{aligned} \tag{2.31}$$

The result (2.15) of Lemma 2.1 together with (2.25), (2.27) and (2.23) imply the convergence to zero (when l, $m \to \infty$) of the last two terms on the RHS of (2.31). Similarly, by (2.27) the first two terms on the RHS of (2.31) converge to zero as well. Thus, we have obtained

$$y_l \to y \text{ in } C[0,T;H^1_{\Gamma_0}(\Omega)] \cap C^1[0,T;L_2(\Omega)] \tag{2.32}$$

and

$$\lim_{l,m\to\infty} \int_{\Sigma_1} (g(y_{lt} \big|_\Gamma) - g(y_{mt} \big|_\Gamma))(y_{lt} \big|_\Gamma - y_{mt} \big|_\Gamma)\, d\Sigma_1 = 0. \tag{2.33}$$

From (2.27), (2.30) and (2.33), we also obtain

$$\lim_{l\to\infty} \left[\int_{\Sigma_1} g(y_{lt})y_{lt} - \int_{\Sigma_1} g(y_{lt})y_t - \int_{\Sigma_1} g_0 y_{lt}\right] + \lim_{m\to\infty} \int_{\Sigma_1} g(y_{mt})y_{mt} = 0.$$

Hence, again using (2.27), (2.30) and changing m to l, we obtain

$$2 \lim_{l\to\infty} \int_{\Sigma_1} g(y_{lt})y_{lt} = 2 \int_{\Sigma_1} g_0 y_t. \tag{2.34}$$

But (2.34) combined with (2.27), (2.30) and the monotonicity of g, by virtue of Lemma 13, p.42 [3], yields

$$g_0 = g(y_t \mid_\Gamma). \tag{2.35}$$

This (together with (2.25)-(2.30)) allows us to pass to the limit in equation (2.11) giving

$$\begin{cases} y_{tt} = \Delta y - f_0(y) \quad \text{in } \mathcal{D}'(Q), \\ \dfrac{\partial y}{\partial \gamma} = -g(y_t \mid_\Gamma) - f_1(y \mid_\Gamma) \quad \text{in } L_2(0,\infty;\Gamma_1), \\ y(0) = y_0, y_t(0) = y_1, \end{cases} \tag{2.36}$$

with the regularity

$$\frac{\partial y}{\partial \gamma},\ y_t \in L_2(0,T;\Gamma_1).$$

The proof of Theorem 1 is thus complete.

The uniqueness statement of Corollary 1 follows from a standard energy estimate which can be justified by virtue of Proposition 2.1. The reason why we can replace global Lipschitz continuity of f_1 with local Lipschitz continuity is that we have a priori bounds for the solutions, given by the energy estimates. Indeed, let $\tilde{y} \equiv y_1 - y_2$ where y_1, y_2 are two possible solutions satisfying regularity properties as in Theorem 1. By virtue of Proposition 2.1, we are in a position to apply the energy estimate;

$$\begin{aligned} &|\nabla\tilde{y}(t)|^2_{L_2(\Omega)} + |\tilde{y}_t(t)|^2_{L_2(\Omega)} + \alpha_0 \int_0^t |\tilde{y}_t(s)|^2_{L_2(\Gamma_1)}\, ds \\ &\le \int_0^t |f_0(y_1) - f_0(y_2)|_{L_2(\Omega)} |\tilde{y}_t(s)|_{L_2(\Omega)} ds \\ &\quad + \int_0^t |f_1(y_1) - f_1(y_2)|_{L_2(\Gamma)} |\tilde{y}_t|_{L_2(\Gamma)} ds \\ &\le C(E(0)) \left[\int_0^t (|\nabla\tilde{y}|^2_{L_2(\Omega)} + |\tilde{y}_t|^2_{L_2(\Omega)}) ds \right] \\ &\quad + C(E(0)) \left[\frac{1}{4\varepsilon} \int_0^t (|\nabla\tilde{y}|^2_{L_2(\Omega)} + \varepsilon |\tilde{y}_t|^2_{L_2(\Gamma_1)}) ds \right]. \end{aligned}$$

It remains to prove Proposition 2.1. To accomplish this, we prove the following approximation result.

Lemma 2.2. *Assume that a given function $u \in C[0,T;H^1(\Omega)] \cap C^1[0,T;L_2(\Omega)]$ satisfies* (2.1)*. Then, there exists a sequence of functions*

$$u_l \in C[0,T;H^2(\Omega)] \cap C^1[0,T;H^1_{\Gamma_0}(\Omega)],\ f_l \in C[0,T;H^1_{\Gamma_0}(\Omega)] \tag{2.37}$$

such that

$$\begin{cases} f_l \to f & \text{in } L_1[0,T;L_2(\Omega)], \\ u_{ltt} - \Delta u_l = f_l, & \\ u_l \to u & \text{in } C[0,T;H^1(\Omega)], \\ u_{lt} \to u_t & \text{in } C[0,T;L_2(\Omega)], \\ u_{lt} \to u_t & \text{in } L_2(\Sigma_1), \\ \frac{\partial u_l}{\partial \gamma} \to \frac{\partial u}{\partial \gamma} & \text{in } L_2(\Sigma_1). \end{cases} \tag{2.38}$$

Proof. Let f_l be any sequence in $C[0,T;H^1_{\Gamma_0}(\Omega)]$ such that $f_l \to f$ in $L_1(0,T;L_2(\Omega)]$. Consider the sequence of linear equations

$$\begin{cases} v_{ltt} = \Delta v_l + f_l & \text{in } Q, \\ v_l(0) = v_{l0},\ v_{lt}(0) = v_{l1} & \text{in } \Omega, \\ v_l = 0 & \text{on } \Sigma_0 \\ \frac{\partial}{\partial \gamma} v_l + v_{lt} = g_l & \text{on } \Sigma_1, \end{cases} \tag{2.39}$$

where $(v_{10}, v_{l1}) \in \mathcal{D}(A_F) \equiv \{(x,y) \in H^2(\Omega) \times H^1(\Omega) : \ x = 0 \text{ on } \Gamma_0, \frac{\partial}{\partial \gamma} x = -y \text{ on } \Gamma_1\}$ with $A_F \begin{bmatrix} u \\ v \end{bmatrix} \equiv \begin{bmatrix} -v \\ A[u - Nv] \end{bmatrix}$ and $(v_{10}, v_{l1}) \to (u_0, u_1)$ in $H^1_{\Gamma_0}(\Omega) \times L_2(\Omega)$. Moreover, we have selected a sequence g_l such that

$$\begin{cases} g_l \in H^1(0,T;L_2(\Gamma_1) \cap C(0,T;H^{1/2}(\Gamma_1)), \\ g_l(t=0) = 0, \\ g_l \to \dfrac{\partial u}{\partial \gamma} + u_t \mid_\Gamma \quad \text{in } L_2(\Sigma_1). \end{cases} \tag{2.40}$$

It is well known [6] that the operator A_F generates a strongly stable semigroup of contractions $e^{A_F t}$ on the space $E \equiv H^1_{\Gamma_0} \times L_2(\Omega)$. Moreover, the operator

$$(Lg)(t) \equiv \int_0^t e^{A_F(t-\tau)} \begin{bmatrix} 0 \\ ANg(\tau) \end{bmatrix} d\tau \tag{2.41}$$

is bounded from $L_2(\Sigma_1) \to C[0,T;E]$. We shall show that for all g such that $g(t=0)=0$ we have

$$|Lg|_{C[0,T;H^2(\Omega)]} + |\frac{d}{dt} Lg|_{C[0,T;H^1(\Omega)]} \le C[|g|_{H^1(0,T;L_2(\Gamma))}] + |g|_{C[0,T;H^{1/2}(\Gamma)]}. \tag{2.42}$$

Indeed, we can write

$$(Lg)(t) = A_F^{-1} \begin{bmatrix} 0 \\ ANg(t) \end{bmatrix} - e^{A_F t} A_F^{-1} \begin{bmatrix} 0 \\ ANg(0) \end{bmatrix} + \int_0^T e^{A_F(t-s)} A_F^{-1} \begin{bmatrix} 0 \\ AN\dot{g}(s) \end{bmatrix} ds.$$

Hence,

$$(Lg)(t) = \begin{bmatrix} Ng(t) \\ 0 \end{bmatrix} + A_F^{-1}(L\dot{g})(t).$$

Since $N \in \mathcal{L}(H^{1/2}(\Gamma_1) \to H^2(\Omega))$ and $A_F^{-1}L \in \mathcal{L}(L_2(\Sigma_1), C(0,T;\mathcal{D}(A_F))$, we obtain (with $g(t=0)=0$)

$$|Lg|_{C[0,T;H^2(\Omega))]} \le C[|g|_{H^1[0,T;L_2(\Gamma_1)]} + |g|_{C[0,T;H^{1/2}(\Gamma_1)]}]. \tag{2.43}$$

To establish the regularity of $\frac{d}{dt}Lg$, we notice that

$$A_F^{-1}\begin{bmatrix} 0 \\ ANu \end{bmatrix} = \begin{bmatrix} Nu \\ 0 \end{bmatrix},$$

$$\left[\frac{d}{dt}Lg\right](t) = -A_F^{-1}\begin{bmatrix} 0 \\ AN\dot{g}(t) \end{bmatrix} + \begin{bmatrix} N\dot{g}(t) \\ 0 \end{bmatrix} + L\dot{g}(t) = L\dot{g}(t),$$

and the conclusion in (2.42) follows from (2.43) and (2.41). On the other hand, the solution v_l to (2.39) can be written as (see [2])

$$\begin{bmatrix} v_l(t) \\ v_{lt}(t) \end{bmatrix} = e^{A_F t}\begin{bmatrix} v_{10} \\ v_{l1} \end{bmatrix} + \int_0^t e^{A_F(t-s)}\begin{bmatrix} 0 \\ f_l(s) \end{bmatrix} ds + Lg_l(t).$$

Since $\begin{bmatrix} v_{10} \\ v_{l1} \end{bmatrix} \in \mathcal{D}(A_F)$ and $\begin{bmatrix} 0 \\ f_l \end{bmatrix} \in \mathcal{D}(A_F)$, by standard semigroup arguments and by (2.42) we obtain

$$v_l \in C(0,T;H^2(\Omega)) \cap C^1(0,T;H^1(\Omega)).$$

Thus, we are in a position to apply the standard energy identity to equation (2.39). Let v_l and v_m be the solutions corresponding to (2.39). We take the difference of the two equations, we multiply by $v_{lt} - v_{mt}$ and we integrate by parts. This gives

$$\begin{aligned} &|\nabla(v_l - v_m)(t)|_\Omega^2 + |(v_{lt} - v_{mt})(t)|_\Omega^2 + \frac{1}{2}\int_0^t \int_{\Gamma_1} |v_{lt} - v_{mt}|^2 \, d\Gamma_1 \, ds \\ &\quad \le \int_0^t \int_\Omega (f_l - f_m)(v_{lt} - v_{mt}) \, d\Omega \, ds + \frac{1}{2}\int_0^t \int_{\Gamma_1} |g_l - g_m|^2 \, d\Gamma_1 \\ &\quad + |\nabla(v_{l0} - v_{m0})|_\Omega^2 + |v_{l1} - v_{m1}|_\Omega^2. \end{aligned}$$

Using Gronwall's inequality and passing to the limit, we obtain

$$\begin{cases} v_l \to v \quad \text{in } C[0,T;H^1(\Omega)] \cap C[0,T;L_2(\Omega)], \\ v_{lt}\,|_\Gamma \to v_t\,|_\Gamma \quad \text{in } L_2(\Sigma_1), \end{cases}$$

and from equation (2.39), $\frac{\partial}{\partial\gamma}v_l \to -v_t\,|_\Gamma + \frac{\partial u}{\partial\gamma} + u_t\,|_\Gamma$ in $L_2(\Sigma_1)$. Passing to the limit in equation (2.39), we obtain

$$\begin{cases} v_{tt} = \Delta v + f, \\ v(0) = u_0, \; v_t(0) = u_1, \\ \dfrac{\partial}{\partial\gamma}v = -v_t + \dfrac{\partial u}{\partial\gamma} + u_t\,|_\Gamma \quad \text{on } \Sigma_1, \\ v\,|_{\Gamma_0} = 0. \end{cases} \tag{2.44}$$

Since the function u satisfies the same equation (2.44), by the uniqueness of the solution to (2.44) v must be equal to u. Thus, the $u_l \equiv v_l$ constructed in (2.39) are the desired approximations of the function u.

Now we are ready to complete the proof of Proposition 2.1.

Proof. We first derive the energy identity for the approximating smooth solutions u_l,

$$\frac{1}{2}(|\nabla u_l(t)|^2_{L_2(\Omega)} + |u_{lt}(t)|^2_{L_2(\Omega)}) - \int_0^t \int_{\Gamma_1} \frac{\partial u_l}{\partial \gamma}\Big|_{\Gamma} u_{lt}\, d\Gamma_1 ds - \int_0^t \int_\Omega f_l u_{lt} d\Omega\, ds$$
$$= \frac{1}{2}(|\nabla u_l(0)|^2_{L_2(\Omega)} + |u_{lt}(0)|^2_{L_2(\Omega)}),$$

and then, by virtue of Lemma 2.2, we pass to the limit.

3. Proof of Theorem 2. We recall the notation $Q \equiv \Omega \times (0,T)$, $\Sigma_i \equiv \Gamma_i \times (0,T)$.

Proposition 3.1. *Assume that the hypothesis* (H-4) *is fulfilled. Let* $u \in C(0,T;H^1(\Omega)) \cap C^1[0,T;L_2(\Omega)]$ *be such that* (2.1) *holds. Then*

$$\int_\alpha^{T-\alpha} [|\nabla u(t)|^2_{L_2(\Omega)} + |u_t(t)|^2_{L_2(\Omega)}]dt \le C[|\nabla u|^2_{L_\infty[0,T;L_2(\Omega)]} + |u_t|^2_{L_\infty(0,T;L_2(\Omega))}]$$
$$+C\left[\int_{\Sigma_1} \left|\frac{\partial}{\partial\gamma}u\right|^2 d\Sigma_1 + \int_{\Sigma_1} |u_t|^2 d\Sigma_1 + \int_Q |f|^2 dQ\right] + C_T |u|^2_{L_2[0,T;H^{1/2+\rho}(\Omega)]}, \tag{3.1}$$

where the constant C does not depend on T, and α, $1/2 > \rho > 0$ are small enough, arbitrary but fixed.

Proof. By virtue of Lemma 2.2, it is enough to prove inequality (3.1) for smooth solutions $u \in C[0,T;H^2(\Omega)] \cap C^1[0,T;H^1_{\Gamma_0}(\Omega)]$ satisfying (2.1). We multiply (2.1) by $h \cdot \nabla u$ with $h \equiv x - x^0$ for some $x^0 \in \mathbb{R}^n$ and we integrate by parts (see [6], [10]). This gives

$$\int_Q u_{tt} h \nabla u\, dQ = \int_\Omega u_t h \nabla u\, d\Omega\Big|_0^T - 1/2 \int_{\Sigma_1} h\gamma |u_t|^2 d\Sigma_1 + \frac{n}{2}\int_Q |u_t|^2 dQ, \tag{3.2i}$$

$$\int_Q \Delta u\, h \nabla u\, dQ = 1/2 \int_\Sigma \left|\frac{\partial u}{\partial \gamma}\right|^2 h\cdot\gamma\, d\Sigma_1 - 1/2\int_{\Sigma_1} |\nabla_\tau u|^2 h \cdot \gamma\, d\Sigma_1$$
$$- \int_{\Sigma_1} h \cdot \nabla_\tau u \frac{\partial}{\partial\gamma} u\, d\Sigma_1 + \left(\frac{n}{2} - 1\right)\int_Q |\nabla u|^2 dQ, \tag{3.2ii}$$

where ∇_τ stands for tangential gradient.

From (3.2) and (H-4), we obtain

$$\frac{n}{2}\int_Q [|u_t|^2_\Omega - |\nabla u|^2_\Omega]\, dQ + \frac{1}{4}\int_Q |\nabla u|^2 dQ$$
$$\le C[|u_t|^2_{L_\infty(0,T;L_2(\Omega))} + |\nabla u|^2_{L_\infty(0,T;L_2(\Omega))} \tag{3.3i}$$
$$+ \int_{\Sigma_1} |u_t|^2 d\Sigma_1 + \int_{\Sigma_1} |\nabla u|^2 d\Sigma_1 + \int_Q |f|^2 dQ].$$

Multiplying (2.1) by u and integrating by parts gives

$$\begin{aligned}\int_Q [-|u_t|^2 + |\nabla u|]^2 dQ \le C\bigg[\int_{\Sigma_1}\left[\frac{1}{\varepsilon}\left|\frac{\partial u}{\partial\gamma}\right|^2 + \varepsilon|u|^2\right] d\Sigma_1 \\ +|u_t|^2_{L_\infty[0,T;L_2(\Omega)]} + |u|^2_{L_\infty(0,T;L_2(\Omega))} + \frac{1}{\varepsilon}\int_Q |f|^2 dQ + \varepsilon\int_Q |u|^2 dQ\bigg],\end{aligned} \tag{3.3ii}$$

where $\varepsilon > 0$ can be taken arbitrarily small. Combining (3.3i) with (3.3ii) and applying trace theory yields

$$\begin{aligned}\int_0^T [|\nabla u(t)|^2_{L_2(\Omega)} + |u_t(t)|^2_{L_2(\Omega)}] \le C[|\nabla u|^2_{L_\infty[0,T;L_2(\Omega)]} \\ + |u_t|^2_{L_\infty(0,T;L_2(\Omega))}] + C\bigg[\int_{\Sigma_1}\left|\frac{\partial}{\partial\gamma}u\right|^2 d\Sigma_1 \\ + \int_{\Sigma_1} |u_t|^2 + |\nabla u|\bigg]^2 dQ\, d\Sigma_1 + \int_Q |f|^2 dQ + \int_Q |u|^2 dQ].\end{aligned} \tag{3.4}$$

From Lemma 7.2, inequality 7.5 in [14], we have

$$\begin{aligned}\int_\alpha^{T-\alpha}\int_{\Gamma_1} |\nabla_\tau u|^2 d\Gamma_1 dt \le C_{\rho,\alpha}\bigg[\int_{\Sigma_1}\left|\frac{\partial u}{\partial\gamma}u\right|^2 + |u_t|^2 d\Sigma_1 \\ + C_T|u|^2_{L^2[0,T;H^{1/2+\rho}(\Omega)]} + \int_Q |f|^2 dQ\bigg],\end{aligned} \tag{3.5}$$

where α, ρ are as in the hypotheses. Applying (3.4) with $(0,T)$ replaced by $(\alpha, T-\alpha)$ and using the regularity result in (3.5) yields the final result in (3.1).

By virtue of Theorem 1, the solution (y, y_t) to (1.1) possesses the regularity properties listed in (2.1). Thus, we are in a position to apply the energy identity of Proposition 2.1. Hence, for all solutions (y, y_t) of (1.1) we have

$$E(t) + \int_0^t\int_{\Gamma_1} g(y_t)y_t\, d\Gamma_1\, ds = E(0). \tag{3.6}$$

Similarly, the result of Proposition 3.1 holds for these solutions as well. Hence, by (3.1) and (3.6),

$$\begin{aligned}\int_\alpha^{T-\alpha} [|\nabla y(t)|^2_{L_2(\Omega)} + |y_t(t)|^2_{L_2(\Omega)}]\, dt \le C[E(T) \\ + \int_{\Sigma_1} [g^2(y_t) + f_1^2(y) + |y_t|^2]\, d\Sigma_1 + \int_Q f_0^2(y)\, dQ + |y|^2_{L^2[0,T;H^{1/2+\rho}(\Omega)]}].\end{aligned}$$

On the other hand, for a fixed α,

$$\begin{aligned}\int_0^\alpha [|\nabla y(t)|^2_{L_2(\Omega)} + |y_t(t)|^2_{L_2(\Omega)}]\, dt + \int_{T-\alpha}^T [|\nabla y(t)|^2_{L_2(\Omega)} + |y_t(t)|^2_{L_2(\Omega)}]\, dt \\ \le 2\alpha E(0) \le 2\alpha[E(T) + \int_{\Sigma_1} (g^2(y_t) + y_t^2)\, d\Sigma_1.\end{aligned}$$

Hence,

$$\int_Q [|\nabla y|^2 + |y_t|^2]\, dQ \le C[E(T) + \int_{\Sigma_1} [g^2(y_t) + f_1^2(y) + |y_t|^2]\, d\Sigma_1 \\ + \int_Q f_0^2(y)\, dQ + |y|^2_{L^2[0,T;H^{1/2+\rho}(\Omega)]}]. \tag{3.7}$$

From hypotheses (H-2) and (H-3), we obtain

$$\int_\Omega F_0(y(t))\, d\Omega + \int_{\Gamma_1} F_1(y(t))\, d\Gamma_1 \le C\Big[\int_\Omega |y(t)|^2 d\Omega + \int_{\Gamma_1} |y(t)|^2 d\Gamma_1 \\ + \int_\Omega |y(t)|^{k_0+1} d\Omega + \int_{\Gamma_1} |y(t)|^{k_1+1}\, d\Gamma_1\Big],$$

and by Sobolev's Imbeddings, (2.16) and (3.6),

$$\int_\Omega F_0(y(t))\, d\Omega + \int_{\Gamma_1} F_1(y(t))\, d\Gamma_1 \le C(E(0)) \int_\Omega |\nabla y(t)|^2 d\Omega + C_1 \int_\Omega y^2 d\Omega, \tag{3.8}$$

where the function $C(E(0))$ remains bounded for bounded values of $E(0)$. Collecting (3.6), (3.7) and (3.8) and noticing that for any $\varepsilon > 0$,

$$|y|^2_{L^2[0,T;H^{1/2+\rho}(\Omega)]} \le \int_0^T \varepsilon |\nabla y|^2_{L^2(\Omega)} + C(\varepsilon)|y|^2_{L^2(\Omega)}\, dt,$$

we obtain the following.

Proposition 3.2. *Assume* (H-1)–(H-4). *Let* (y, y_t) *be a solution to* (1.1) *guaranteed by Theorem* 1. *Then*

$$\int_0^T E(t)\, dt \le C(E(0))\Big[\int_{\Sigma_1} [g^2(y_t) + |y_t|^2 + |f_1^2(y)]\, d\Sigma_1 + \int_Q (f_0^2(y) + y^2)\, dQ + E(T)\Big]. \tag{3.9}$$

Our next step is to estimate the nonlinear terms appearing in (3.9).

Proposition 3.3. *Assume* (H-1)–(H-5). *Let* y *be as above. Let* $\varepsilon > 0$ *be arbitrarily small and* $C(\varepsilon)$ *be a constant depending on* ε *and possibly on* $E(0)$. *Then*

$$\int_{\Sigma_1} f_1^2(y)\, d\Sigma_1 \le \varepsilon |E(0)|^{\frac{(k_1-1)}{1-\varepsilon k_1}} \int_0^T E(t)\, dt + C(\varepsilon) \int_{\Sigma_1} y^2 d\Sigma_1, \tag{3.10}$$

$$\int_Q f_0^2(y)\, dQ \le \varepsilon |E(0)|^{\frac{(k_0-1)}{1-\varepsilon k_0}} \int_0^T E(t) + C(\varepsilon) \int_Q y^2 d\Sigma_1. \tag{3.11}$$

If $k = 1$ *then the first terms on the RHS of* (3.10), (3.11) *can be omitted and* $C(\varepsilon)$ *is independent of* $E(0)$.

Proof. It is enough to prove this result for $k > 1$ (the case $k = 1$ is obvious).

Recalling hypothesis H-3 and applying interpolation inequalities for L_p spaces, i.e.,

$$|y|_{L_p} \le |y|_{L_2}^{1-q} |y|_{L_r}^q, \quad \frac{1}{p} = \frac{1-q}{2} + \frac{q}{r}$$

with $p = 2k_1$, $r = 2k_1 + s$ and $0 < s < 1/2$, we obtain

$$\int_{\Gamma_1} f_1^2(y) d\Gamma_1 \le C \int_{\Gamma_1} [y^2 + y^{2k_1}] \, d\Gamma_1 \le C\Big[\int_{\Gamma_1} y^2 d\Gamma_1 + |y|_{L_2(\Gamma_1)}^{(1-q)2k_1} |y|_{L_{2k_1+s}(\Gamma_1)}^{2qk_1} \Big], \quad (3.12)$$

where $0 < q < 1$ is now given by

$$q = 1 + \frac{s}{k_1(2 - 2k_1 - s)}.$$

Using the inequality

$$a\,b \le \frac{\varepsilon^{-p} a^p}{p} + \frac{b^{\bar{p}}}{\bar{p}} \varepsilon^{\bar{p}}, \quad \frac{1}{p} + \frac{1}{\bar{p}} = 1$$

with $p = \frac{1}{(1-q)k_1}$ and $\bar{p} = \frac{1}{1-k_1(1-q)}$, we obtain

$$\int_{\Gamma_1} f_1^2(y) d\Gamma_1 \le C\Big[\int_{\Gamma_1} y^2 \, d\Gamma_1 + \frac{1}{\varepsilon^{\frac{1}{(1-q)k_1}}} |y|_{L_2(\Gamma_1)}^2 + \varepsilon^{\frac{1}{1-k_1(1-q)}} |y|_{L_{2k_1+s}(\Gamma_1)}^{\frac{2k_1 q}{1-k_1(1-q)}} \Big]. \quad (3.13)$$

For $s < \frac{2n-2}{n-2} - 2k_1$ (which is positive by (H-3)), hypothesis (H-3) and Sobolev's Imbedding (2.16) combined with the Trace Theorem gives

$$|y|_{L_{2k_1+s}(\Gamma_1)} \le C |y|_{H^{1/2}(\Gamma_1)} \le C(|\nabla y|_{L_2(\Omega)} + |y|_{L_2(\Gamma_1)}).$$

Combining the above inequality with (3.13) yields

$$\begin{aligned} \int_{\Gamma_1} f_1^2(y) \, d\Gamma_1 \le C\big[(1 + \frac{1}{\varepsilon^{\frac{1}{(1-q)k_1}}}) |y|_{L_2(\Gamma_1)}^2 \\ + \varepsilon^{\frac{1}{1-k_1(1-q)}} \big[|\nabla y|_{L_2(\Omega)}^{\frac{2k_1 q}{1-k_1(1-q)}} + |y|_{L_2(\Gamma_1)}^{\frac{2k_1 q}{1-k_1(1-q)}} \big] \big]. \end{aligned} \quad (3.14)$$

Notice that

$$|y(t)|_{H^1(\Omega)} \le CE(0).$$

Indeed, if $\Gamma_0 \neq \emptyset$ then the above inequality follows from Poincare's Inequality. Otherwise, according to (H-5) (ii), we have either $F_0(x) \ge \varepsilon x^2$ or $F_1(x) \ge \varepsilon x^2$, hence

$$E(t) \ge |\nabla y(t)|_{L_2(\Omega)}^2 + \varepsilon |y(t)|_{L_2(\Omega)}^2 \ge c |y(t)|_{H^1(\Omega)}^2$$

or

$$E(t) \ge |\nabla y(t)|_{L_2(\Omega)}^2 + \varepsilon |y(t)|_{L_2(\Gamma)}^2 \ge c |y(t)|_{H^1(\Omega)}^2$$

which together with (3.6) gives the desired inequality.

Integrating the inequality (3.14) over $(0,T)$, using

$$|y|_{L_2(\Gamma_1)}^{\frac{2k_1 q}{1-k_1(1-q)}} \le CE(0)^{\frac{k_1-1}{1-k_1(1-q)}} \int_{\Gamma_1} y^2 d\Gamma_1$$

and rescaling $\varepsilon = c\varepsilon^{\frac{1}{1-k_1(1-q)}}$ gives

$$\int_{\Sigma_1} f_1^2(y)\, d\Sigma_1 \le \varepsilon |E(0)|^{\frac{(k_1-1)}{1-k_1(1-q)}} \int_0^T E(t)\, dt + C(\varepsilon) \int_{\Sigma_1} y^2 d\Sigma_1$$

as desired for (3.10).

The proof of (3.11) is similar, hence omitted.

Collecting the results of Propositions 3.2 and 3.3 gives that for any $\varepsilon > 0$ (suitably small) and $k = \max\{k_0, k_1\}$,

$$\begin{aligned}(1-\varepsilon[E(0)]^{2(k-1)}) \int_0^T E(t)\, dt \le C(E(0)) \Big\{ \int_{\Sigma_1} [g^2(y_t) + y_t^2(t)]\, d\Sigma_1 \\ + C(\varepsilon)\big[\int_{\Sigma_1} y^2 d\Sigma_1 + \int_Q y^2 dQ \big] + E(T) \Big\}.\end{aligned} \tag{3.15}$$

Our next step is to absorb the lower order terms on the RHS of (3.15). This will be accomplished by applying a suitable nonlinear version of a compactness argument.

Lemma 3.1. *Assume the hypotheses* (H-1)–(H-5). *Let* (y, y_t) *be a solution to* (1.1). *Then for* $T > T_0$*, where* T_0 *is sufficiently large, we have*

$$\int_{\Sigma_1} y^2 d\Sigma_1 + \int_Q y^2 dQ \le C(E(0)) \Big[\int_{\Sigma_1} (y_t^2 + g^2(y_t))\, d\Sigma_1 \Big]. \tag{3.16}$$

Proof. We shall argue by contradiction. Let $y_l(t)$ be a sequence of solutions to (1.1) such that

$$\lim_{l\to\infty} \frac{\int_{\Sigma_1} y_l^2\, d\Sigma_1 + \int_Q y_l^2\, dQ}{\int_{\Sigma_1} y_{lt}^2\, d\Sigma_1 + \int_{\Sigma_1} g^2(y_{lt})\, d\Sigma_1} = \infty, \tag{3.17}$$

while the energy of the initial data $(y_l(0), y_{lt}(0))$ denoted by $E_l(0)$ remains uniformly (in l) bounded by, say, $E_l(0) \le M$.

Since $E_l(0) \le M$, by the basic energy identity (3.6) we have $E_l(t) \le M$. Hence,

$$\begin{cases} y_l \to y \text{ weakly in } H^1(Q) \text{ and weakly* in } L^\infty(0,T; H^1(\Omega)), \\ y_l \to y \text{ strongly in } L_2(\Sigma) \cap L^2(Q). \end{cases} \tag{3.18}$$

Case A. Assume that $y \ne 0$. Then, we first notice that an Aubin's type compactness result (see [17]) gives

$$y_l \to y \text{ strongly in } L_\infty(0,T; H^{1-\varepsilon}(\Omega)) \text{ for } \varepsilon > 0.$$

Then, by hypotheses (H-2), (H-3), it follows easily that

$$f_0(y_l) \to f_0(y) \text{ strongly in } L_\infty(0,T;L_2(\Omega)),$$
$$f_1(y_l) \to f_1(y) \text{ strongly in } L_\infty(0,T;L_2(\Gamma)).$$

Also, by (3.17), y_{lt}, $g(y_{lt}) \to 0$ in $L_2(\Sigma)$. Then, passing to the limit in the equation, we get for y,

$$\begin{cases} y_{tt} - \Delta y = -f_0(y), \\ \dfrac{\partial y}{\partial \gamma} = f_1(y), \quad y_t = 0 \text{ on } \Gamma_1, \\ y = 0 \quad \text{on } \Gamma_0, \end{cases} \tag{3.19}$$

and for $y_t = v$,

$$\begin{cases} v_{tt} - \Delta v = -f_0'(y)v, \\ \dfrac{\partial v}{\partial \gamma} = v = 0 \text{ on } \Gamma_1, \\ v = 0 \quad \text{on } \Gamma_0. \end{cases}$$

Now, we consider the three possibilities in (H-5).

(i) If f_0 is linear, then we get $v = 0$ by standard uniqueness results for the wave equation.

(ii) If $\Gamma_0 = \emptyset$, first note that $y \in L^\infty(0,T;H^1(\Omega))$ implies $y \in L_{\frac{2n}{n-2}}(Q)$ and, according to (H-2) (iii), $f_0'(y) \in L_n(Q)$. Then for $T > 2$ diam Ω we may apply the uniqueness result of [15] (see Theorem 2) which yields

$$v = y_t = 0.$$

(iii) In this case, as in (ii), we get $v = 0$ using a straightforward adaptation of the results of [15]. The main idea is that Ω_1 is the intersection of a family of balls and outside each ball we can use the technique in [15].

Hence, we have proven that $y_t = 0$. Then, by (3.19), we get for y the elliptic equation

$$\begin{cases} -\Delta y = -f_0(y), \\ \dfrac{\partial y}{\partial v} = -f_1(y) \text{ on } \Gamma_1. \end{cases}$$

Multiplying by y, we obtain

$$\int_\Omega (|\nabla y|^2 + y f_0(y))\, d\Omega + \int_{\Gamma_1} y f_1(y)\, d\Gamma = 0. \tag{3.20}$$

Hence, $\nabla y = 0$. If $\Gamma_0 \neq \emptyset$, it follows that $y = 0$. If $\Gamma_0 = \emptyset$, by (H-5) (ii) and (3.20) we also obtain that either $y = 0$ in Ω or $y = 0$ in Γ, therefore $y = 0$. This contradicts our assumption that $y \neq 0$.

Case B. Assume that $y = 0$. Denote $C_l = (|y_l|^2_{L_2(\Sigma_1)} + |y_l|^2_{L_2(Q)})^{1/2}$, $\tilde{y}_l = \frac{1}{C_l} \cdot y_l$. Clearly,

$$|\tilde{y}|^2_{L_2(\Sigma_1)} + |\tilde{y}|^2_{L_2(Q)} = 1. \tag{3.21}$$

Also, because $y = 0$, we get $C_l \to 0$ as $l \to \infty$. By (3.17) we obtain

$$\tilde{y}_{lt} \to 0 \quad \text{in } L_2(\Sigma_1). \tag{3.22}$$

On the other hand, from (3.15) and (3.6) we obtain, after using the estimate

$$\int_0^T E(t)\,dt \geq TE(T) \geq TE(0) - T\int_{\Sigma_1} y_t g(y_t)\,d\Sigma_1$$

and taking ε suitable small,

$$[T - C(E(0))]E(0) \leq C_T(E(0))\left\{\int_{\Sigma_1} [g^2(y_t) + y_t^2 + y^2]\,d\Sigma_1 + \int_Q y^2 dQ\right\},$$

and again recalling (3.6),

$$E(t) \leq E(0) \leq C_T(E(0))\left\{\int_{\Sigma_1} [g^2(y_t) + y_t^2 + y^2]\,d\Sigma_1 + \int_Q y^2 dQ\right\}. \tag{3.23}$$

Dividing both sides of (3.23) (applied to the solution y_l) by $|y_l|^2_{L_2(\Sigma_1)} + |y_l|^2_{L_2(Q)}$ and invoking (3.17) yields

$$|\nabla \tilde{y}_l(t)|^2_{L_2(\Omega)} + |\tilde{y}_{lt}(t)|^2_{L_2(\Omega)} \leq C_T(E(0)), \quad 0 \leq t \leq T. \tag{3.24}$$

Therefore, if $\Gamma_0 \neq \mathit{emptyset}$, $\tilde{y}_l$ is bounded in $H^1(Q)$. If we are in case (ii) of (H.5) then we also obtain in the L.H.S. of (3.24) the term $\varepsilon|\tilde{y}_l|^2_{L_2(\Omega)}$ or $\varepsilon|\tilde{y}_l|^2_{L_2(\Gamma)}$, therefore we still get boundedness of $\tilde{y}_l$ in $H^1(Q)$. Thus, on a subsequence we have

$$\begin{cases} \tilde{y}_l \rightharpoonup \tilde{y} & \text{weakly in } H^1(Q), \\ \tilde{y}_l \to \tilde{y} & \text{strongly in } L_2(\Sigma) \cap L_2(Q). \end{cases} \tag{3.25}$$

Moreover, $\tilde{y}_l$ satisfies the equation

$$\begin{cases} \tilde{y}_{ltt} = \Delta \tilde{y}_l - \dfrac{f_0(y_l)}{C_l}, \\ \tilde{y}_l = 0 \quad \text{on } \Sigma_0, \\ \dfrac{\partial}{\partial \gamma}\tilde{y}_l = \dfrac{-g(y_{lt}) - f_1(y_l)}{C_l} \quad \text{on } \Sigma_1. \end{cases} \tag{3.26}$$

In order to pass to the limit in (3.26), we need to determine the limits of nonlinear terms.

Proposition 3.4.

$$\frac{g(y_{lt})}{C_l} \to 0 \quad \mathit{in}\ L_2(\Sigma_1) \quad \mathit{as}\ l \to \infty, \tag{3.27}$$

$$\frac{f_0(y_l)}{C_l} \to f_0'(0)\tilde{y} \quad \mathit{in}\ L^2(Q) \quad \mathit{as}\ l \to \infty, \tag{3.28}$$

$$\frac{f_1(y_l)}{C_l} \to f_1'(0)\tilde{y} \quad \mathit{in}\ L_2(\Sigma) \quad \mathit{as}\ l \to \infty. \tag{3.29}$$

Proof. (3.27) follows directly from (3.17). For (3.28), we estimate

$$\Delta_l = \left| f_0'(0)\tilde{y}_l - \frac{f_0(y_l)}{C_l} \right|^2_{L_2(Q)} \leq \int_{|y_l|\leq\varepsilon} \tilde{y}_l^2 \left| f_0'(0) - \frac{f_0(y_l)}{y_l} \right|^2 dQ$$

$$+ 2|f_0'(0)|^2 \int_{|y_l|>\varepsilon} |\tilde{y}_l|^2 dQ + 2\int_{|y_l|>\varepsilon} \frac{f_0^2(y_l)}{C_l^2}\, dQ.$$

Then, according to (H-2), we get

$$\Delta_l \leq |\tilde{y}_l|^2_{L^2(Q)} \rho_\varepsilon^2 + C\int_{|y_l|>\varepsilon} \left[\frac{y_l^2}{C_l^2} + \frac{y_l^{2k_0}}{C_l^2} \right] dQ, \tag{3.30}$$

where $\rho_\varepsilon = \sup_{|x|\leq\varepsilon} |f_0'(0) - \frac{f_0(x)}{x}|$, $\rho_\varepsilon \to 0$ as $\varepsilon \to 0$. It is enough to consider the case when $k_0 > 1$. Then, by (3.30), we get

$$\Delta_l \leq |\tilde{y}_l|^2_{L^2(Q)} \cdot \rho_\varepsilon^2 + C \cdot \int_{|y_l|>\varepsilon} \frac{|y_l|^{2k_0}}{C_l^2} \big(1 + \frac{1}{\varepsilon^{2k_0-2}}\big)\, dQ$$

$$\leq |\tilde{y}_l|^2_{L^2(Q)} \rho_\varepsilon^2 + C_\varepsilon \cdot \frac{1}{C_l^2} \cdot |y_l|^{2k_0}_{L_{2k_0}(Q)} = |\tilde{y}_l|^2_{L^2(Q)} \cdot \rho_\varepsilon^2 + C_\varepsilon \cdot |\tilde{y}_l|^{2k_0}_{L_{2k_0}(Q)} C_l^{2k_0-2}.$$

But $\tilde{y}_l$ is bounded in $L^\infty(H^1(\Omega))$; therefore, according to Sobolev's Imbeddings, $\tilde{y}_l$ is also bounded in $L_{2k_0}(Q)$. Then as $l \to 0$ we get

$$\limsup_{l\to\infty} \Delta_l \leq \sup_l |\tilde{y}_l|_{L^2(Q)} \rho_\varepsilon^2,$$

and as $\varepsilon \to 0$,

$$\lim_{l\to\infty} \Delta_l = 0;$$

that is, we get (3.28).

Also, (3.26) may be proven in the same way.

Applying the result of Proposition 3.4 to equation (3.24) and passing to the limit $l \to \infty$ gives

$$\begin{cases} \tilde{y}_{tt} = \Delta\tilde{y} - f_0'(0)\tilde{y}, \\ \dfrac{\partial}{\partial\gamma}\tilde{y} = -f_1'(0)\tilde{y} \quad \text{on } \Sigma_1, \end{cases} \tag{3.31}$$

$$\begin{cases} \tilde{y} = 0 \quad \text{on } \Sigma_0, \\ \tilde{y}_t = 0 \quad \text{on } \Sigma_1. \end{cases}$$

Thus, $v = \tilde{y}_t \in C[0,T; L_2(\Omega)]$ satisfies

$$\begin{cases} v_{tt} = \Delta v - f_0'(0)v, \\ \dfrac{\partial}{\partial\gamma} v = 0 \quad \text{on } \Sigma_1, \\ v\,\big|_\Sigma = 0. \end{cases} \tag{3.32}$$

We are in a position to apply standard uniqueness results for the wave equation, which yield for T large enough,

$$v = \tilde{y}_t \equiv 0. \tag{3.33}$$

Returning to (3.31) and exploiting (3.33) we obtain

$$\begin{cases} \Delta\tilde{y} - f_0'(0)\tilde{y} = 0 \quad \text{in } \Omega, \\ \dfrac{\partial}{\partial\gamma}\tilde{y} = -f_1'(0)\tilde{y} \quad \text{on } \Sigma_1, \\ \tilde{y} = 0 \quad \text{on } \Sigma_0. \end{cases} \tag{3.34}$$

As in case A, multiplying the first equation in (3.34) by $\tilde{y}$ we get $\tilde{y} = 0$, which contradicts (3.21).

Using inequality (3.15) with ε suitably small and (3.16), we obtain the following.

Proposition 3.5. *Let $T > 0$ be sufficiently large. Then*

$$E(T) \le C_T(E(0)) \int_{\Sigma_1} (y_t^2(t) + g^2(y_t))\, d\Sigma_1.$$

Our final estimate is the following.

Lemma 3.2. *With $p(s)$ defined by (1.6) and $T > 0$ sufficiently large, we have*

$$p(E(T)) + E(T) \le E(0).$$

Proof. Denote

$$\Sigma_A \equiv \{u \in L_2(\Sigma_1) : \ |u| \ge N \text{ a.e. } \},$$
$$\Sigma_B \equiv \Sigma_1 - \Sigma_A.$$

From hypothesis (H-1) (iii), we have

$$\int_{\Sigma_A} (g^2(y_t) + y_t^2)\, d\Sigma_A \le (M_1 + M_2^{-1}) \int_{\Sigma_A} y_t g(y_t)\, d\Sigma_A. \tag{3.35}$$

On the other hand, from (1.3) and (H-1) (ii),

$$\int_{\Sigma_B} (y_t^2 + g^2(y_t))\, d\Sigma_B \le \int_{\Sigma_B} h(y_t g(y_t))\, d\Sigma_A. \tag{3.36}$$

By Jensen's inequality,

$$\begin{aligned} \int_{\Sigma_B} h(y_t g(y_t))\, d\Sigma_B &\le \text{mes}\Sigma_1 h\Big(\frac{1}{\text{mes}\Sigma_1} \int_{\Sigma_1} y_t g(y_t)\, d\Sigma_1\Big) \\ &= \text{mes}\Sigma_1 \tilde{h}\Big(\int_{\Sigma_1} y_t g(y_t)\, d\Sigma_1\Big). \end{aligned} \tag{3.37}$$

Combining inequalities (3.35), (3.36), (3.37) with the result of Lemma 3.1 gives

$$\begin{aligned} E(T) \le & C_T(E(0))\Big((M_1 + M_2^{-1})\Big(\int_{\Sigma_1} y_t g(y_t)\, d\Sigma_1\Big) \\ & + \mathrm{mes}\Sigma_1 \tilde{h}\Big(\int_{\Sigma_1} y_t g(y_t)\, d\Sigma_1\Big)\Big). \end{aligned} \tag{3.38}$$

Setting

$$K = \frac{1}{C_T(E(0))\mathrm{mes}\Sigma_1} \quad \text{and} \quad c = \frac{M_1 + M_2^{-1}}{\mathrm{mes}\Sigma_1},$$

we obtain

$$p(E(T)) \le \int_{\Sigma_1} y_t\, g(y_t)\, d\Sigma_1 = E(0) - E(T),$$

which gives the result of Lemma 3.2

To conclude the proof of our theorem, we need the following.

Lemma 3.3. *Let p be a positive, increasing function such that $p(0) = 0$. Since p is increasing, we can define an increasing function q, $q(x) \equiv x - (I + p)^{-1}(x)$. Consider a sequence s_n of positive numbers which satisfies*

$$s_{m+1} + p(s_{m+1}) \le s_m. \tag{3.39}$$

Then $s_m \le S(m)$ where $S(t)$ is a solution of the differential equation

$$\frac{d}{dt}S(t) + q(S(t)) = 0, \quad S(0) = s_0. \tag{3.40}$$

Moreover, if $p(x) > 0$ for $x > 0$ then $\lim_{t\to\infty} S(t) = 0$.

Proof. Use induction. Assume $s_m \le S(m)$ and prove that $s_{m+1} \le S(m+1)$. Inequality (3.39) is equivalent to

$$(I + p)s_{m+1} \le s_m$$

and since $(I+p)^{-1}$ is monotone increasing, $s_{m+1} \le (I+p)^{-1}s_m$, hence $s_{m+1} - s_m \le -q(s_m)$ or

$$s_{m+1} \le s_m - q(s_m). \tag{3.41}$$

On the other hand, since q is an increasing function, the solution $S(t)$ of equation (3.40) is described by a nonlinear contraction. In particular,

$$S(t) \le S(\tau) \quad \text{for all} \quad t \ge \tau. \tag{3.42}$$

Integrating equation (3.40) from m to $m + 1$ yields

$$S(m+1) - S(m) + \int_m^{m+1} q(S(\tau))\, d\tau = 0.$$

Since q is increasing, by (3.42) we obtain

$$\begin{aligned} S(m+1) &\geq S(m) - q(S(m)) = (I-q)S(m) = (I+p)^{-1}S(m) \\ &\geq (I+p)^{-1}S_m = S_m - q(S_m), \end{aligned} \tag{3.43}$$

where we have used the inductive assumption $s_m \leq S(m)$ as well as the increasing property of $(I+p)^{-1}$. Comparing (3.41) with (3.43) yields the desired result.

Final step in the proof of Theorem 2. Applying the results of Lemma 3.2, we obtain

$$E(m(T+1)) + p(E(m(T+1)) \leq E(mT) \tag{3.44}$$

for $m = 0, 1, \ldots$ (note that the constant K in (1.4) is decreasing in $E(0)$ and that the energy $E(t)$ is also decreasing).

Thus, we are in a position to apply the result of Lemma 3.3 with

$$s_m \equiv E(mT), \quad s_0 \equiv E(0).$$

This yields

$$E(mT) \leq S(m), \quad m = 0, 1, 2 \ldots.$$

Setting $t = mT + \tau$ and recalling the evolution property gives

$$E(t) \leq E(mT) \leq S(m) \leq S\left(\frac{t-\tau}{T}\right) \leq s\left(\frac{t}{T} - 1\right) \text{ for } t > T,$$

which completes the proof of Theorem 2.

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