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Uniform boundary stabilization of the finite difference space discretization of the $1-d$ wave equation

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The energy of solutions of the wave equation with a suitable boundary dissipation decays exponentially to zero as time goes to infinity. We consider the finite-difference space semi-discretization scheme and we analyze whether the decay rate is independent of the mesh size. We focus on the one-dimensional case. First we show that the decay rate of the energy of the classical semi-discrete system in which the $1-d$ Laplacian is replaced by a three-point finite difference scheme is not uniform with respect to the net-spacing size h. Actually, the decay rate tends to zero as h goes to zero. Then we prove that adding a suitable vanishing numerical viscosity term leads to a uniform (with respect to the mesh size) exponential decay of the energy of solutions. This numerical viscosity term damps out the high frequency numerical spurious oscillations while the convergence of the scheme towards the original damped wave equation is kept. Our method of proof relies essentially on discrete multiplier techniques.

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Mathematics subject classifications (2000): 65M06, 35L05, 93D15.

1. Introduction and statement of the main results

Consider the $1-d$ damped wave equation

$$\begin{cases} y'' - y_{xx} = 0 & & \text{in } (0,1)\times(0,\infty), \\ y(0,t) = 0, & y_x(1,t) + \alpha y'(1,t) = 0 & \text{in } (0,\infty), \\ y(x,0) = y^0 & & \text{in } (0,1), \\ y'(0) = y^1 & & \text{in } (0,1), \end{cases} \tag{1.1}$$

where $\{y^0, y^1\} \in V \times L^2(0,1)$, $V = \{u \in H^1(0,1); u(0) = 0\}$, and α is a positive constant.

Here and in the sequel $'$ denotes partial differentiation with respect to time, i.e. $' = \partial \cdot / \partial t$.

System (1.1) arises in many applications in Engineering; in particular it may be viewed as a simplified model for a longitudinally vibrating bar with a viscous damper at the right end, and no load [8].

The energy of solutions of the damped wave equation (1.1)

$$E(t) = \frac{1}{2}\int_0^1 \left\{\left|y'(x,t)\right|^2 + \left|y_x(x,t)\right|^2\right\} \mathrm{d}x, \quad \forall t \geqslant 0, \tag{1.2}$$

obeys the following dissipation law:

$$\frac{\mathrm{d}E(t)}{\mathrm{d}t} = -\alpha\left|y'(1,t)\right|^2. \tag{1.3}$$

As (1.3) shows, the energy of each solution decreases as time increases.

By now it is well known (cf. [2,4,12,13,15,16,23,25,33]) that the energy of solutions of (1.1) satisfies, for some $M > 0$ and $\omega > 0$ independent of the solution but depending on the damping coefficient α, the estimate

$$E(t) \leqslant M \exp(-\omega t) E(0), \quad \forall t \geqslant 0. \tag{1.4}$$

As it will be proved in the sequel (see appendix B), the exponential decay property of solutions of (1.1) is equivalent to an observability inequality (O.I.) for the corresponding conservative system

$$\begin{cases} \varphi'' - \varphi_{xx} = 0 & & \text{in } (0,1) \times (0,\infty), \\ \varphi(0,t) = 0, & \varphi_x(1,t) = 0 & \text{in } (0,\infty), \\ \varphi(x,0) = \varphi^0 & & \text{in } (0,1), \\ \varphi'(0) = \varphi^1 & & \text{in } (0,1), \end{cases} \tag{C.S.}$$

where $\{\varphi^0, \varphi^1\} \in V \times L^2(0,1)$.

Note that the only difference between the damped system (1.1) and the conservative version above is that the dissipative boundary condition of (1.1) at the endpoint $x = 1$ has been replaced by the homogeneous Neumann boundary condition, that yields a conservative system. Indeed, it is easy to see that the energy E for solutions of (C.S.) is constant in time. Note also that (C.S.) can also be viewed as a degenerate particular case of (1.1) in which the damping constant α vanishes.

More precisely, by the energy dissipation law (1.3) and the semigroup property, it is easy to see that the uniform decay property (1.4) is equivalent to the existence of a positive time T and a positive constant C such that

$$E(y;0) \leqslant C\int_0^T \left|y'(1,t)\right|^2 \mathrm{d}t, \tag{O.I.D.}$$

for every solution y of (1.1). It is then easy to see, by means of a simple decomposition argument that (O.I.D.) holds for the solutions of the damped system (1.1) if and only if the same holds for the solutions of the conservative one (C.S.),

$$E(\varphi;0) \leqslant C\int_0^T \left|\varphi'(1,t)\right|^2 \mathrm{d}t. \tag{O.I.C.}$$

One can check that the time needed for (O.I.D.) and (O.I.C.) to be true is the same ($T > 2$ in this case), although the observability constant C may differ from one to the other.

Inequalities (O.I.D.) and (O.I.C.) guarantee that the energy of solutions is captured uniformly for all solutions at the endpoint $x = 1$ where the damping mechanism is being applied in time T.

There is by now an extensive literature on the subject and it is well known that, in the general multi-dimensional setting, observability inequalities of the form (O.I.D.) and (O.I.C.) are valid if and only if a suitable Geometric Control Condition (GCC) is satisfied (see [2]). Roughly speaking, the GCC requires that every ray of geometric optics reaches the region in which the damping mechanism is effective in a uniform time. Obviously this geometric control condition is trivially satisfied in the $1-d$ case under consideration. Indeed, for the $1-d$ wave equation, rays are straight lines of unit slope in the (x,t)-plane travelling from left to right and vice versa. It is clear that all rays reach the endpoint $x = 1$ where the damping is effective in time less than or equal to 2 after possibly being reflected at $x = 0$. Therefore for any value of the damping parameter $\alpha > 0$, the exponential decay property (1.4) holds for some $\omega = \omega(\alpha)$.

However as far as numerical approximation schemes are concerned, little is known about the uniform (w.r.t. the mesh size) exponential decay of the discretized energy. To our knowledge only the work by Banks et al. [1] addresses this issue. In [1] numerical simulations suggest that the exponential decay of the discretized energy might not be uniform, with respect to the step size, for the classical finite difference, and finite element schemes. To remedy this situation, the authors propose to use the mixed finite element method or other numerical schemes which produce a uniform decay rate. They do not provide any rigorous proof of the fact that classical finite difference, and finite element schemes do not keep the exponential decay of the discretized energy uniform (w.r.t. the mesh size). Our main purpose in this paper is twofold:

(i) to rigorously prove that for the classical finite difference scheme, the exponential decay of the discretized energy is not uniform,

(ii) to propose an alternative approach to the mixed finite element method of [1], that consists in adding a correcting numerical viscous term in the equation.

The study carried out in this paper is a natural complement to [29] where we proved similar results for the case where the damping term is effective on a subinterval.

Results similar to those we present in this paper are true in a much more general setting. For instance, the techniques we develop in this article allow to handle finite element discretization schemes (see [9,10] for the analysis of the conservative wave equation) and multi-dimensional models (see [34] for the conservative wave equation and [29] for the wave equation with internal damping) with similar results. But these subjects need to be addressed in detail.

We now introduce the finite difference scheme we will work on. For this purpose, let N be a nonnegative integer. Set $h = 1/(N+1)$ and consider the subdivision of (0, 1) given by

$$0 = x_0 < x_1 < \cdots < x_N < x_{N+1} = 1,$$

where $x_j = jh$.

The finite-difference space semi-discretization of system (1.1) that we consider is the following

$$\begin{cases} y_j'' - \dfrac{y_{j+1} - 2y_j + y_{j-1}}{h^2} = 0 & \text{in } (0,\infty),\ j = 1,2,\ldots,N, \\ y_0(t) = 0, \qquad \dfrac{y_{N+1}(t) - y_N(t)}{h} + \alpha y_{N+1}'(t) = 0 & \text{in } (0,\infty), \\ y_j(0) = y_j^0, \qquad y_j'(0) = y_j^1, & j = 1,2,\ldots,N, \end{cases} \tag{1.5}$$

where $y_j^0, y_j^1,\ j = 0,1,\ldots,N+1$, are approximations of the functions y^0 and y^1, respectively.

The energy of system (1.5) is given by

$$E_h(t) = \frac{h}{2}\sum_{j=0}^{N}\left\{\bigl(y_j'(t)\bigr)^2 + \left(\frac{y_{j+1} - y_j}{h}\right)^2\right\} \tag{1.6}$$

and it is a nonincreasing function of the time t. In fact its derivative is given by

$$E_h'(t) = -\alpha\bigl(y_{N+1}'(t)\bigr)^2. \tag{1.7}$$

Observe that E_h is a natural semi-discrete version of the energy E of system (1.1) and that (1.7) is the semi-discrete analogue of the energy dissipation law (1.3).

On the other hand, as we shall see below, this numerical approximation scheme converges in the classical sense to the continuous damped wave equation (1.1).

It is then reasonable to wonder whether the energies E_h decay exponentially and uniformly (with respect to $h \to 0$) to zero as the time t approaches infinity. For any $h > 0$ fixed it is easy to see that solutions of (1.5) tend exponentially to zero as time goes to infinity. But, as we previously mentioned, earlier results obtained by Banks et al. [1], and the authors [29] lead us to think that the decay rate degenerates as h tends to zero. Our first result confirms this fact:

Theorem 1.1. The exponential decay of E_h to zero is not uniform with respect to h. More precisely there do not exist positive constants M and ω which are independent of h such that for all $h > 0$ and $(y_j^0)_j$ and $(y_j^1)_j$ in $\mathbb{R}^N$,

$$E_h(t) \leqslant M\exp(-\omega t)E_h(0), \quad \forall t \geqslant 0. \tag{1.8}$$

Theorem 1.1 is in agreement with the negative observability results established in [9,10,21,34], and previously observed in [5,6] in the context of the control of the numerical approximations of the wave equation. Indeed, due to the existence of high frequency spurious solutions of the semi-discrete model, it is well known that there exist solutions that propagate very slowly (with group velocity of the order of the step size h) making the boundary observability property impossible to hold uniformly in h. To overcome this obstacle, several solutions were proposed: one consists in simply ruling out the high frequency spurious modes (cf. [5,9,10,34]), another one consists in the use of the Tychonoff regularization techniques as developed in [6], yet another one consists in using mixed finite elements [1], etc. We also refer to [30] for a deep qualitative analysis of the group velocity of solutions of numerical schemes for wave equations.

In [29] it was shown that adding a suitable numerical viscosity term is an efficient way to guarantee a uniform decay rate of the energy in the case where the damping term is locally distributed in the interior of the domain. Thus, in view of the negative result of theorem 1.1, it is reasonable to wonder whether such a numerical viscous damping mechanism could be used in the present setting to get a uniform decay rate. We now investigate this issue. To this end, introduce the new system with the extra numerical viscosity:

$$\begin{cases} y_j'' - \dfrac{y_{j+1} - 2y_j + y_{j-1}}{h^2} - \left(y_{j+1}' - 2y_j' + y_{j-1}'\right) = 0 & \text{in } (0,\infty),\ j = 1,2,\ldots,N, \\ y_0(t) = 0, \qquad \dfrac{y_{N+1}(t) - y_N(t)}{h} + \alpha y_{N+1}'(t) = 0 & \text{in } (0,\infty), \\ y_j(0) = y_j^0, \qquad y_j'(0) = y_j^1, & j = 1,2,\ldots,N. \end{cases} \tag{1.9}$$

The first equation of system (1.9) is the semi-discrete analogue of

$$y'' - y_{xx} - h^2 y_{xxt} = 0$$

which is a wave equation with viscous dissipation.

The energy of the new system (1.9) is

$$\mathcal{E}_h(t) = \frac{h}{2}\sum_{j=0}^{N}\left\{\left(y_j'(t)\right)^2 + \left(\frac{y_{j+1} - y_j}{h}\right)^2\right\} + \frac{\alpha h^2}{2}\left(y_{N+1}'(t)\right)^2. \tag{1.10}$$

The additional term $(\alpha h^2/2)(y_{N+1}'(t))^2$ appearing in (1.10) comes from the numerical viscosity term added in (1.9); this can be clearly seen by multiplying the first equation of (1.9) by hy_j' and taking the sums over j. The derivative of $\mathcal{E}_h$ is then given by

$$\mathcal{E}_h'(t) = -h^3\sum_{j=0}^{N}\left(\frac{y_{j+1}' - y_j'}{h}\right)^2 - \alpha\left(y_{N+1}'(t)\right)^2, \tag{1.11}$$

which shows that $\mathcal{E}_h$ is a nonincreasing function of the time variable t.

In the energy identity (1.11), we observe the extra dissipative effect that the numerical viscous damping term introduces in system (1.9).

For system (1.9), we prove:

(i) a decay rate of type (1.4) which is uniform with respect to the net-spacing h;

(ii) the convergence of its solutions towards those of the original wave equation (1.1) as $h \to 0$, in a suitable topology.

These two results show that the discretization (1.9) of system (1.1), in which a suitable artificial numerical viscosity term is introduced, is a good approximation scheme for (1.1) because not only it guarantees the convergence of solutions as $h \to 0$ (which is also true for the simpler approximation (1.5)) but because it also provides a uniform (with respect to $h \to 0$) decay rate of solutions as $t \to \infty$. This second fact shows that the viscous damping term in (1.9) correctly captures the long time asymptotic properties of system (1.1), and that it efficiently rules out the aforementioned high frequency numerical spurious oscillations. In particular, the choice of the h^2 multiplicative factor on the numerical viscous term is sharp and it is in fact the only one leading to uniform (as h goes to zero) decay properties.

The suitability of this numerical damping mechanism to restore the uniform exponential decay is closely connected to the efficiency of the Tychonoff regularization techniques developed in [6] when building up numerical schemes for the controllability of the wave equation. Indeed, in view of the multiplier techniques developed in [10] it can be easily seen that the following estimate holds for the solutions of the semi-discrete equation with Dirichlet boundary conditions: for any $T > 2$, there exists a positive constant C independent of h such that

$$E_h(0) \leqslant C \int_0^T \left| \frac{y_N}{h} \right|^2 \mathrm{d}t + Ch^3 \sum_{j=0}^{N} \int_0^T \left(\frac{y'_{j+1} - y'_j}{h} \right)^2 \mathrm{d}t. \tag{O.I$_h$}$$

As we shall show, a similar identity holds for the dissipative system (1.5). The last term in this inequality indicates the need of adding the numerical viscous term in order to gain uniformity (as h tends to zero) in the decay rate. Indeed, according to (O.I$_h$), in order to observe correctly the energy of solutions one has to measure both the discrete normal derivative (represented by y_N/h) and the internal viscosity. System (1.9) contains two damping mechanisms (an internal one and the other one located at the right endpoint) that take into account this fact.

We are now in the position to state our other results. The uniform stabilization result may be stated as follows:

Theorem 1.2. There exist positive constants M and ω independent of h such that for all $(y_j^0)_j$ and $(y_j^1)_j$ in $\mathbb{R}^N$, the energy $\mathcal{E}_h$ of system (1.9) satisfies

$$\mathcal{E}_h(t) \leqslant M \exp(-\omega t)\mathcal{E}_h(0), \quad \forall t \geqslant 0,\ 0 < h < 1. \tag{1.12}$$

Theorem 1.2 shows that the numerical viscosity term added in system (1.9) i.e.

$$-h^2\left(\frac{y'_{j+1}-2y'_j+y'_{j-1}}{h^2}\right)=-\left(y'_{j+1}-2y'_j+y'_{j-1}\right),$$

is enough to restore the uniform (with respect to $h\to 0$) exponential decay. This was already proved in [29] in the case where the damping term is locally distributed in the domain. Lately this approach was used in [24] to construct uniformly exponentially stable approximations for an abstract class of second order evolution equations with bounded feedback controls. This class is restricted essentially to $1-d$ models by the assumption that the continuous undamped system fulfills the classical spectral gap condition that is needed to apply Ingham type inequalities (see [31]). On the other hand, unbounded feedback operators, as it is typically the case for boundary stabilization problems, are not considered in [24]. Thus, theorem 1.2 may not be obtained as a particular case of the abstract result in [24]. It is also important to note that the multiplier techniques we employ in this article are not restricted to models in which the spectral gap condition is fulfilled, as mentioned above.

The main novelty of the present work is to extend the results in [29] to the case of the wave equation with boundary damping. We address only the $1-d$ case, although the discrete multiplier techniques we employ here can be easily extended to multi-dimensional problems as was shown in [29] in the case of locally distributed internal damping.

Before stating our convergence result, we need some additional notations. Set $\vec{y}_h=(y_j)_j$, $\vec{y}^0_h=(y^0_j)_j$ and $\vec{y}^1_h=(y^1_j)_j$, $j=1,2,\ldots,N$. Observe that, at the discrete level, we use arrows over vectors to distinguish them from their components; but we do not do this at the continuous level. Introduce the extension operators defined by (see [17]):

$$p_h\vec{v}_h=\begin{cases}\text{the continuous function, linear in each interval } \big[jh,(j+1)h\big],\\ \text{such that } p_h\vec{v}_h(jh)=v_j,\ \ j=0,1,\ldots,N+1,\end{cases}\tag{1.13}$$

$$q_h\vec{v}_h=\begin{cases}\text{the step function defined in each interval } \left(\left(j-\dfrac{1}{2}\right)h,\left(j+\dfrac{1}{2}\right)h\right)\cap(0,1)\\ \quad\text{by } q_h\vec{v}_h(x)=v_j,\ \ j=0,1,\ldots,N+1.\end{cases}\tag{1.14}$$

It is not hard to check that

$$\begin{aligned}&\int_0^1(p_h\vec{v}_h)_x(p_h\vec{w}_h)_x\,\mathrm{d}x=h\sum_{j=0}^{N}\left(\frac{v_{j+1}-v_j}{h}\right)\left(\frac{w_{j+1}-w_j}{h}\right),\\ &\int_0^1(q_h\vec{v}_h)_x(q_h\vec{w}_h)_x\,\mathrm{d}x=h\sum_{j=0}^{N}v_jw_j.\end{aligned}\tag{1.15}$$

We are now in the position to state our convergence result:

Theorem 1.3. Let $\vec{y}_h$ denote the solution of (1.9). Assume that $\vec{y}_h^0$ and $\vec{y}_h^1$ satisfy for $h \to 0$,

$$\begin{cases} p_h \vec{y}_h^0 \to y^0 & \text{weakly in } V, \\ q_h \vec{y}_h^1 \to y^1 & \text{weakly in } L^2(0,1). \end{cases} \tag{1.16}$$

Then as $h \to 0$, we have

$$\begin{aligned} p_h \vec{y}_h &\to y \quad \text{weakly* in } L^\infty(0,\infty; V), \\ q_h \vec{y}_h' &\to y' \quad \text{weakly* in } L^\infty\big(0,\infty; L^2(0,1)\big), \end{aligned} \tag{1.17}$$

where y is the solution of system (1.1).

Moreover if in addition to (1.16), $\mathcal{E}_h(0) \to E(0)$ as $h \to 0$, then we have

$$\begin{aligned} p_h \vec{y}_h &\to y \quad \text{strongly in } L^2(0,\infty; V), \\ q_h \vec{y}_h' &\to y' \quad \text{strongly in } L^2\big(0,\infty; L^2(0,1)\big), \\ p_h \vec{y}_h &\to y \quad \text{strongly in } C\big([0,\infty); L^2(0,1)\big), \end{aligned} \tag{1.18}$$

and

$$\lim_{h\to 0} \|\mathcal{E}_h - E\|_{C([0,\infty])} = 0. \tag{1.19}$$

Remark 1.4. The convergence hypotheses in (1.16) are natural in this context as the following comments indicate. We begin by observing that for every function u, continuous on $[0,1]$, if we set $\vec{u}_h = (u_j)_j = (u(jh))_j$, it follows that

$$q_h \vec{u}_h \to u \quad \text{strongly in } L^\infty(0,1). \tag{1.20}$$

Moreover, if $u \in H^1(0,1)$, then

$$\begin{aligned} \|p_h \vec{u}_h - q_h \vec{u}_h\|^2_{L^2(0,1)} &= h^2\left[\frac{h}{12}\sum_{j=0}^{N}\left(\frac{u_{j+1}-u_j}{h}\right)^2\right] = \mathrm{O}\big(h^2\big), \\ \big\|(p_h \vec{u}_h)_x\big\|^2_{L^2(0,1)} &= h\sum_{j=0}^{N}\left(\frac{u_{j+1}-u_j}{j}\right)^2 \leqslant \int_0^1 |u_x|^2\,\mathrm{d}x, \end{aligned} \tag{1.21}$$

so that $p_h \vec{u}_h$ is bounded in $H^1(0,1)$. Thus, according to (1.20), (1.21), and [3, proposition III.30], we have

$$p_h \vec{u}_h \to u \quad \text{strongly in } H^1(0,1). \tag{1.22}$$

Now let $v \in L^2(0,1)$. For all $j = 0, 1, \ldots, N$, set

$$v_j = \frac{1}{h}\int_{jh}^{(j+1)h} v(x)\,\mathrm{d}x, \quad \vec{v}_h = (v_j)_j. \tag{1.23}$$

Then

$$q_h \vec{v}_h \to v \quad \text{strongly in } L^2(0,1). \tag{1.24}$$

Indeed, let (φ_n) be a sequence of continuous functions on [0, 1] satisfying

$$\varphi_n \to v \quad \text{strongly in } L^2(0,1). \tag{1.25}$$

If we define (φ_{nh}) as we defined $\vec{v}_h$, it easily follows that

$$\|q_h\vec{v}_h - v\|^2_{L^2(0,1)} \leqslant 6\|\varphi_n - v\|^2_{L^2(0,1)} + 4\|q_h\varphi_{nh} - \varphi_n\|^2_{L^2(0,1)} \tag{1.26}$$

since, from the definition of q_h,

$$\|q_h\varphi_{nh} - q_n\vec{v}_h\|^2_{L^2(0,1)} \leqslant \|\varphi_n - v\|^2_{L^2(0,1)}. \tag{1.27}$$

Therefore (1.24) follows from (1.26), (1.25) and the uniform continuity of each φ_n.

According to (1.22) and (1.24), the convergence hypotheses (1.16) are satisfied if for any initial data $y^0 \in V$ and $y^1 \in L^2(0,1)$, we choose the approximations y_j^0 and y_j^1 by $y_j^0 = y^0(jh)$ (we can do so since $V \subset C([0,1])$), and

$$y_0^1 = 0, \qquad y_j^1 = \frac{1}{h}\int_{(j-1)h}^{jh} y^1(x)\,\mathrm{d}x, \quad j = 1, 2, \ldots, N+1.$$

Observe that these approximations lead to strong convergence results in (1.16).

The rest of the paper is organized as follows: section 2 is devoted to the proof of theorem 1.1. Section 3 deals with the proofs of theorem 1.2 and theorem 1.3. In section 4 we discuss some interesting open problems, while in appendix A we prove an important technical lemma used in the proof of theorem 1.2. Finally, in appendix B, we prove the equivalence between the exponential decay (1.4) and the observability inequality (O.I.C.).

2. Proof of theorem 1.1

This proof relies on the following lemmas:

Lemma 2.1. The following assertions are equivalent:

(i) There exist positive constants M and ω such that for all $(y_j^0)_j$ and $(y_j^1)_j$ in $\mathbb{R}^N$, one has

$$E_h(t) \leqslant M\exp(-\omega t)E_h(0), \quad \forall t \geqslant 0, \tag{2.1}$$

where E_h is the energy of system (1.5) given by (1.6).

(ii) There exist positive constant T_0 and C_0 such that for all $(u_j^0)_j$ and $(u_j^1)_j$ in $\mathbb{R}^N$, one has

$$h\sum_{j=0}^{N}\left\{\left|u_j^1\right|^2 + \left(\frac{u_{j+1}^0 - u_j^0}{h}\right)^2\right\} \leqslant C_0\int_0^{T_0}\left|u'_{N+1}(t)\right|^2\mathrm{d}t, \tag{2.2}$$

where $(u_j)_j$ solves

$$\begin{cases} u_j'' - \dfrac{u_{j+1} - 2u_j + u_{j-1}}{h^2} = 0 & \text{in } (0,\infty),\ j = 1,2,\ldots,N, \\ u_0(t) = 0, \qquad u_{N+1}(t) = u_N(t) & \text{in } (0,\infty), \\ u_j(0) = u_j^0, \qquad u_j'(0) = u_j^1, & j = 1,2,\ldots,N. \end{cases} \tag{2.3}$$

Moreover, there is a bicontinuous dependence between the constants (M,ω) in (2.1) and (C_0, T_0) in (2.2).

According to lemma 2.1, the problem of the uniform (w.r.t. h) exponential decay of the energy of system (1.5) is equivalent to that of finding a uniform observability inequality (2.2) for the undamped system (2.3). However, as pointed out in [9,10] and later further explained in [20,35], such an inequality may not hold for a uniform time T_0 and a uniform observability constant C_0 since there exist high frequency spurious solutions of (2.3) which are concentrated in the interior of the space interval and need a time of the order of $1/h$ to reach the boundary. They correspond to high frequency wave packets with group velocity of the order of the mesh-size h.

The following lemma provides a quantitative statement of this negative result.

Lemma 2.2. For all $T > 0$ there exist a positive constant $C(T)$ and initial data $(u_j^0)_j$ and $(u_j^1)_j$ in $\mathbb{R}^N$, such that the solution $(u_j)_j$ of (2.3) satisfies

$$E(\vec{u}_h;0) \geqslant \frac{C(T)}{h^2} \int_0^T \left|u_{N+1}'(t)\right|^2 \mathrm{d}t, \quad 0 < h < 1, \tag{2.4}$$

where

$$E(\vec{u}_h;0) = \frac{h}{2} \sum_{j=0}^{N} \left\{ \left|u_j^1\right|^2 + \left(\frac{u_{j+1}^0 - u_j^0}{h}\right)^2 \right\}.$$

Lemma 2.2 shows that whatever $T_0 > 0$ is, (2.2) may not be uniform with respect to h; the constant C_0 necessarily blows up as $h \to 0$. A more careful result in [21], based on a sharp analysis of the biorthogonal functions to the family of complex exponentials arising in the Fourier development of solutions, in the case of Dirichlet boundary conditions, shows that the constant C_0 blows-up exponentially in h. The same is certainly true for system (2.3), but this remains to be proved. Estimate (2.4) is sufficient for our purpose in this paper.

Proof of lemma 2.1. In the proof of theorem 1.1, we need only the implication (i) $\to$ (ii). So here we will prove only this implication. The proof of the other implication follows the same steps as that of lemma 3.2 provided in appendix A and will be omitted.

We now prove: (i) $\to$ (ii). Let us suppose that (2.1) holds. For all j, choose $u_j^0 = y_j^0$ and $u_j^1 = y_j^1$. It follows from the dissipation law (1.7) that for all $T > 0$,

$$E_h(0) - E_h(T) = \alpha \int_0^T \left|y'_{N+1}(t)\right|^2 \mathrm{d}t. \tag{2.5}$$

The combination of (2.1) and (2.5) shows that for T large enough (any $T \geqslant (\ln(4M/3))/\omega$ suffices), one has

$$\alpha \int_0^T \left|y'_{N+1}(t)\right|^2 \mathrm{d}t \geqslant \frac{1}{4} E_h(0) = \frac{1}{4} E(\vec{u}_h; 0). \tag{2.6}$$

Now set $\vec{y}_h = \vec{u}_h + \vec{v}_h$, where $\vec{y}_h = (y_j)_j$ is the solution of (1.5) and the difference $\vec{v}_h = (v_j)_j$ solves

$$\begin{cases} v''_j - \dfrac{v_{j+1} - 2v_j + v_{j-1}}{h^2} = 0 & \text{in } (0, \infty), \\ v_0(t) = 0, \qquad \dfrac{v_{N+1}(t) - v_N(t)}{h} + \alpha v'_{N+1}(t) = -\alpha u'_{N+1}(t) & \text{in } (0, \infty), \\ (v_j(0) = v'_j(0) = 0, & j = 1, 2, \ldots, N. \end{cases} \tag{2.7}$$

Note that $\vec{y}_h$ and $\vec{u}_h$ are respectively the solution of the dissipative system (1.5), and the solution of the conservative system (2.3) with the same initial data. The difference $\vec{v}_h$ has zero initial data but takes into account the difference of the boundary conditions of the systems that $\vec{y}_h$ and $\vec{u}_h$ solve.

For system (2.7), we have the energy equality

$$E(\vec{v}_h; t) + \alpha \int_0^T \left|v'_{N+1}(t)\right|^2 \mathrm{d}t = -\alpha \int_0^T v'_{N+1} u'_{N+1} \, \mathrm{d}s. \tag{2.8}$$

It easily follows from (2.8) that

$$\int_0^T \left|v'_{N+1}(t)\right|^2 \mathrm{d}t \leqslant \int_0^T \left|u'_{N+1}(t)\right|^2 \mathrm{d}t. \tag{2.9}$$

On the other hand, one has

$$\alpha \int_0^T \left|y'_{N+1}(t)\right|^2 \mathrm{d}t \leqslant 2\alpha \int_0^T \left|u'_{N+1}(t)\right|^2 \mathrm{d}t + 2\alpha \int_0^T \left|v'_{N+1}(t)\right|^2 \mathrm{d}t. \tag{2.10}$$

Using (2.9) and (2.10) in (2.6) we get

$$E(\vec{u}_h; 0) \leqslant 16\alpha \int_0^T \left|u'_{N+1}(t)\right|^2 \mathrm{d}t \tag{2.11}$$

which proves the claimed result. □

Proof of lemma 2.2. For this proof we observe that system (2.3) can be rewritten in the following simplified form:

$$\vec{U}_h'' + A_h \vec{U}_h = 0, \quad t > 0,$$

where $\vec{U}_h$ stands for the column vector

$$\begin{pmatrix} u_1 \\ \vdots \\ u_N \end{pmatrix},$$

A_h denotes the matrix

$$A_h = \frac{1}{h^2}\begin{pmatrix} 2 & -1 & 0 & \dots & \dots & 0 \\ -1 & 2 & -1 & 0 & \dots & 0 \\ 0 & \ddots & \ddots & \ddots & & \\ \vdots & & & & & \\ 0 & & & & 2 & -1 \\ 0 & \dots & \dots & \dots & -1 & 1 \end{pmatrix}$$

entering in the finite difference discretization of the Lapiacian with Dirichlet boundary condition at the left endpoint, and Neumann one at the right endpoint.

The eigenvectors of the matrix A_h satisfy the eigenvalue system

$$\begin{cases} -\dfrac{X_{j+1} - 2X_j + X_{j-1}}{h^2} = \lambda X_j, \\ X_0 = 0, \qquad \dfrac{X_{N+1} - X_N}{h} = 0, \quad j = 1, 2, \dots, N. \end{cases} \tag{2.12}$$

Proceeding as in [11, p. 456], one can show that

$$\begin{aligned} X_j^{k,h} &= \sin\left(\frac{(2k+1)\pi jh}{2-h}\right), & j = 0, 1, \dots, N, \\ \lambda_{k,h} &= \frac{4}{h^2}\sin^2\left(\frac{(2k+1)\pi h}{2(2-h)}\right), & k = 0, 1, \dots, N-1. \end{aligned} \tag{2.13}$$

Moreover, the eigenvectors $\vec{X}^{k,h}$, $k = 0, 1, 2, \dots, N-1$, are pairwise orthogonal, and we have

$$\begin{aligned} h\sum_{j=0}^{N} \left|X_j^{k,h}\right|^2 &= h\sum_{j=0}^{N}\left|\sin\left(\frac{(2k+1)\pi jh}{2-h}\right)\right|^2 \\ &= \frac{1}{2} - \frac{h}{2}\sum_{j=0}^{N}\cos\left(\frac{2(2k+1)\pi jh}{2-h}\right), \quad (\text{since } h(N+1) = 1) \\ &= \frac{1}{2} - \frac{h}{2}\frac{\cos((2k+1)\pi Nh/(2-h))\sin((2k+1)\pi(N+1)h/(2-h))}{\sin((2k+1)\pi h/(2-h))} \end{aligned}$$

$$= \frac{1}{2} - \frac{h}{2} \frac{\sin((2k+1)\pi(2N+1)h/(2-h)) + \sin((2k+1)\pi h/(2-h))}{\sin((2k+1)\pi h/(2-h))}$$
$$= \frac{2-h}{4}, \quad \forall k, h. \tag{2.14}$$

Choose $u_j^0 = X_j^{N-1,h}$ and $u_j^1 = \mathrm{i}\sqrt{\lambda_{N-1,h}} X_j^{N-1,h}$, where $\mathrm{i}^2 = -1$. It is easy to check that $\vec{u}_h = \exp(\mathrm{i}\sqrt{\lambda_{N-1,h}} t) \vec{X}^{N-1,h}$ solves (2.3). On the other hand, it is not difficult to verify that $E(\vec{u}_h; 0) = ((2-h)/4)\lambda_{N-1,h}$, while

$$\int_0^T \left|u'_{N+1}(t)\right|^2 \mathrm{d}t = \lambda_{N-1,h} T \left|X_{N+1}^{N-1,h}\right|^2 = \lambda_{N-1,h} T \sin^2\left(\frac{(2N-1)\pi}{2-h}\right)$$
$$= \lambda_{N-1,h} T \sin^2\left(\frac{\pi h}{2-h}\right). \tag{2.15}$$

Hence

$$E(\vec{u}_h; 0) = \frac{2-h}{4T \sin^2(\pi h/(2-h))} \int_0^T \left|u'_{N+1}(t)\right|^2 \mathrm{d}t$$
$$\geqslant \frac{2-h}{4T(\pi h/(2-h))^2} \int_0^T \left|u'_{N+1}(t)\right|^2 \mathrm{d}t \geqslant \frac{1}{4T\pi^2 h^2} \int_0^T \left|u'_{N+1}(t)\right|^2 \mathrm{d}t, \tag{2.16}$$

which proves (2.4) and the proof of lemma 2.2 is complete. □

The proof of lemma 2.2 is based on the fact that for monochromatic solutions of (2.3) constituted of a single eigenmode, the energy concentrated at the boundary (the right-hand side of (2.2)) is not strong enough to provide a uniform estimate of the solutions for high frequencies.

However, as pointed out in [9,10] and later further developed in [20,21,35], this is not the only reason for the observability constant C_0 to blow up as $h \to 0$ nor the worse one. Indeed, the gap between the square roots of two consecutive eigenvalues is of the order of h for high frequencies. This allows building high frequency wave packets for which the constant C_0 in (2.2) blows up at a polynomial rate (in $1/h$) of an arbitrarily large order. In fact, as shown in [21] in the context of the Dirichlet problem, the blow-up rate is exponential. The same is true for the problem under consideration since $\sqrt{\lambda_{N-1,h}} - \sqrt{\lambda_{N-2,h}} \sim ch$ for some suitable positive constant c.

Proof of theorem 1.1. We proceed by contradiction. Assume that (1.8) holds with M and ω independent of h. Then, as the proof of lemma 2.1 shows, estimate (2.2) would be uniform with respect to h; but this contradicts lemma 2.2, and we are done. □

3. Proofs of theorems 1.2 and 1.3

From now on, C will denote a generic positive constant independent of h and the initial data that may change from line to line.

3.1. Proof of theorem 1.2

The proof of theorem 1.2 will essentially rely on the following lemmas:

Lemma 3.1. For every $T>2$, we have

$$(T-2)E(\vec{u}_h;0) \leqslant \frac{h^3}{4}\sum_{j=0}^{N}\int_0^T\left(\frac{u'_{j+1}-u'_j}{h}\right)^2 \mathrm{d}t + \frac{2-h}{4}\int_0^T \left|u'_{N+1}\right|^2 \mathrm{d}t, \qquad (3.1.1)$$

for every solution $\vec{u}_h$ of (2.3) and all $h>0$.

Lemma 3.2. Let $(\vec{y}_h)$ be the solution of (1.9), and let $\vec{z}_h$ be the solution of

$$\begin{cases} z''_j - \dfrac{z_{j+1}-2z_j+z_{j-1}}{h^2} = h^2\left(\dfrac{y'_{j+1}-2y'_j+y'_{j-1}}{h^2}\right) & \text{in } (0,\infty), \\ z_0(t)=0, \qquad \dfrac{z_{N+1}(t)-z_N(t)}{h} = -\alpha y'_{N+1}(t) & \text{in } (0,\infty), \\ z_j(0)=z'_j(0)=0, & j=1,2,\dots,N. \end{cases} \qquad (3.1.2)$$

Let $T>0$. Then there exists a positive constant C depending on T, and a positive constant K independent of T such that for all $0<h<1$, we have the following estimate

$$\int_0^T\left[\left|z'_{N+1}\right|^2 + h^3\sum_{j=0}^{N}\left|\frac{z'_{j+1}-z'_j}{h}\right|^2\right]\mathrm{d}t$$
$$\leqslant C\int_0^T \alpha\left|y'_{N+1}\right|^2\mathrm{d}t + Ch^3\sum_{j=0}^{N}\int_0^T\left|\frac{y'_{j+1}-y'_j}{h}\right|^2\mathrm{d}t + KE_h(0), \qquad (3.1.3)$$

for every solution $\vec{z}_h$ of (3.1.2), where the energy E_h is given by (1.6).

The proof of lemma 3.1 will be given later. As for the proof of lemma 3.2 which is lengthy and technical, it is provided in appendix A. Let us use these lemmas to prove theorem 1.2 now. This theorem will be proved if we can find two constants $T>0$ and $0<\eta<1$ that are independent of h such that

$$\mathcal{E}_h(T) \leqslant \eta\mathcal{E}_h(0). \qquad (3.1.4)$$

In this case, applying the semigroup property, we get

$$\mathcal{E}_h(t) \leqslant M\exp(-\omega t)\mathcal{E}_h(0), \quad \forall t \geqslant T,\ 0<h<1, \qquad (3.1.5)$$

which is the claimed estimate with $M=1/\eta$, and $\omega=(\ln M)/T$.

We now prove (3.1.4). To this end, let $T>2$, and let $(\vec{y}_h)$ be the solution of (1.9). Set $\vec{y}_h=\vec{u}_h+\vec{z}_h$, where $\vec{u}_h$ solves (2.3) with $\vec{u}_h(0)=\vec{y}_h^{\,0}$ and $\vec{u}'_h(0)=\vec{y}_h^{\,1}$, and $\vec{z}_h$ is given by (3.1.2).

It follows from (3.1.1) that

$$\begin{aligned}(T-2)E_h(0) &= (T-2)E(\vec{u}_h;0)\\ &\leqslant \frac{h^3}{4}\sum_{j=0}^{N}\int_0^T\left(\frac{u'_{j+1}-u'_j}{h}\right)^2 \mathrm{d}t + \frac{2-h}{4}\int_0^T \left|u'_{N+1}\right|^2 \mathrm{d}t\\ &\leqslant \frac{h^3}{2}\sum_{j=0}^{N}\int_0^T\left(\frac{y'_{j+1}-y'_j}{h}\right)^2 \mathrm{d}t + \int_0^T \left|y'_{N+1}\right|^2 \mathrm{d}t\\ &\quad + \frac{h^3}{2}\sum_{j=0}^{N}\int_0^T\left(\frac{z'_{j+1}-z'_j}{h}\right)^2 \mathrm{d}t + \int_0^T \left|z'_{N+1}\right|^2 \mathrm{d}t. \end{aligned} \tag{3.1.6}$$

Thanks to lemma 3.2, from (3.1.6) we derive that

$$(T-2)E_h(0) \leqslant Ch^3\sum_{j=0}^{N}\int_0^T\left(\frac{y'_{j+1}-y'_j}{h}\right)^2 \mathrm{d}t + C\alpha\int_0^T \left|y'_{N+1}\right|^2 \mathrm{d}t + KE_h(0), \tag{3.1.7}$$

so that for $T > 2+K$, we get (see (1.11))

$$E_h(0) \leqslant C\int_0^T \left|\mathcal{E}'_h(t)\right| \mathrm{d}t. \tag{3.1.8}$$

On the other hand, one easily checks that

$$\mathcal{E}_h(0) \leqslant \left(1+\frac{h}{\alpha}\right)E_h(0) \leqslant \left(1+\frac{1}{\alpha}\right)E_h(0). \tag{3.1.9}$$

Consequently, $\mathcal{E}_h$ being nonincreasing,

$$\mathcal{E}_h(T) \leqslant \mathcal{E}_h(0) \leqslant C\int_0^T \left|\mathcal{E}'_h(t)\right| \mathrm{d}t, \tag{3.1.10}$$

from which we derive

$$\mathcal{E}_h(T) \leqslant \frac{C}{C+1}\mathcal{E}_h(0). \tag{3.1.11}$$

This establishes (3.1.4).

To complete the proof of theorem 1.2, it remains to prove lemmas 3.1 and 3.2. We now proceed with the proof of lemma 3.1. The proof of lemma 3.2 is provided in appendix A.

Proof of lemma 3.1. For this proof, we use the multiplier $j((u_{j+1}-u_{j-1})/2$ introduced in [9,10] which corresponds to a discrete version of the classical multiplier xu_x for the continuous wave equation.

Let $T > 2$. Multiply the first equation of (2.3) by $j((u_{j+1}-u_{j-1})/2$, integrate over $(0, T)$ and take the sum over j; this yields

$$h\sum_{j=1}^{N} u'_j j\left(\frac{u_{j+1}-u_{j-1}}{2}\right)\Big|_0^T - h\sum_{j=1}^{N} j\int_0^T u'_j\left(\frac{u_{j+1}-u_{j-1}}{2}\right)\mathrm{d}t$$
$$- h\sum_{j=1}^{N} j\int_0^T \left(\frac{u_{j+1}-2u_j+u_{j-1}}{h^2}\right)\left(\frac{u_{j+1}-u_{j-1}}{2}\right)\mathrm{d}t = 0. \quad (3.1.12)$$

Some elementary calculations show that

$$\begin{aligned}
&h\sum_{j=1}^{N} u'_j j\left(\frac{u_{j+1}-u_{j-1}}{2}\right)\Big|_0^T \\
&\quad= h^2\sum_{j=1}^{N} u'_j j\left(\frac{u_{j+1}-u_j}{2h}\right)\Big|_0^T + h^2\sum_{j=1}^{N} u'_j j\left(\frac{u_j-u_{j-1}}{2h}\right)\Big|_0^T \\
&\quad= h^2\sum_{j=0}^{N} u'_j j\left(\frac{u_{j+1}-u_j}{2h}\right)\Big|_0^T + h^2\sum_{j=0}^{N} u'_{j+1}(j+1)\left(\frac{u_{j+1}-u_j}{2h}\right)\Big|_0^T \\
&\quad= h^2\sum_{j=0}^{N}\left(ju'_j+(j+1)u'_{j+1}\right)\left(\frac{u_{j+1}-u_j}{2h}\right)\Big|_0^T, \qquad (3.1.13)
\end{aligned}$$

$$\begin{aligned}
&-h\sum_{j=1}^{N} j\int_0^T \left(\frac{u_{j+1}-2u_j+u_{j-1}}{h^2}\right)\left(\frac{u_{j+1}-u_{j-1}}{2}\right)\mathrm{d}t \\
&\quad= -h\sum_{j=1}^{N} j\int_0^T \left(\frac{(u_{j+1}-u_j)-(u_j-u_{j-1})}{h^2}\right)\left(\frac{(u_{j+1}-u_j)+(u_j-u_{j-1})}{2}\right)\mathrm{d}t \\
&\quad= \frac{h}{2}\sum_{j=0}^{N}\int_0^T \left(\frac{u_{j+1}-u_j}{h}\right)^2\mathrm{d}t \qquad (3.1.14)
\end{aligned}$$

and

$$\begin{aligned}
&-h\sum_{j=1}^{N}\int_0^T ju'_j\left(\frac{u'_{j+1}-u'_{j-1}}{2}\right)\mathrm{d}t \\
&\quad= \frac{h}{2}\sum_{j=0}^{N}\int_0^T u'_{j+1}u'_j\,\mathrm{d}t - \int_0^T \frac{|u'_{N+1}|^2}{2}\,\mathrm{d}t \\
&\quad= -\frac{h^3}{4}\sum_{j=0}^{N}\int_0^T \left(\frac{u'_{j+1}-u'_j}{h}\right)^2\mathrm{d}t + \frac{h}{2}\sum_{j=1}^{N}\int_0^T \left(v'_j\right)^2\mathrm{d}t + \frac{(h-2)}{4}\int_0^T \left|u'_{N+1}\right|^2\mathrm{d}t. \\
&\qquad (3.1.15)
\end{aligned}$$

Taking the sums in (3.1.13)–(3.1.15) side by side, using the fact that the energy of system (2.3) is conservative, and reporting the result in (3.1.12), we find

$$\begin{aligned} TE(\vec{u}_h;0) &= \int_0^T E(\vec{u}_h;t)\,\mathrm{d}t \\ &= -h^2 \sum_{j=0}^{N} \big(ju'_j + (j+1)u'_{j+1}\big)\left(\frac{u_{j+1}-u_j}{2h}\right)\bigg|_0^T \\ &\quad + \frac{h^3}{4}\sum_{j=0}^{N}\int_0^T \left(\frac{u'_{j+1}-u'_j}{h}\right)^2 \mathrm{d}t + \frac{2-h}{4}\int_0^T \left|u'_{N+1}\right|^2 \mathrm{d}t. \end{aligned} \tag{3.1.16}$$

Using Young inequality as well as the conservation of the energy, we obtain the inequality

$$\begin{aligned} &\left| -h^2 \sum_{j=0}^{N} \big(jv'_j + (j+1)u'_{j+1}\big)\left(\frac{u_{j+1}-u_j}{2h}\right)\bigg|_0^T \right| \\ &\quad\leqslant \left| -h^2 \sum_{j=0}^{N} ju'_j \left(\frac{u_{j+1}-u_j}{2h}\right)\bigg|_0^T \right| + \left| -h^2 \sum_{j=0}^{N-1} (j+1)u'_{j+1}\left(\frac{u_{j+1}-u_j}{2h}\right)\bigg|_0^T \right| \\ &\quad\leqslant 2E(\vec{u}_h;0). \end{aligned} \tag{3.1.17}$$

Reporting (3.1.17) in (3.1.16), we get

$$(T-2)E(\vec{u}_h;0) \leqslant \frac{h^3}{4}\sum_{j=0}^{N}\int_0^T \left(\frac{u'_{j+1}-u'_j}{h}\right)^2 \mathrm{d}t + \frac{2-h}{4}\int_0^T \left|u'_{N+1}\right|^2 \mathrm{d}t, \tag{3.1.18}$$

which establishes (3.1.1), and completes the proof of lemma 3.1. □

3.2. *Proof of theorem 1.3*

Observe that by the definitions of p_h and q_h, we have for every $t \geqslant 0$

$$\mathcal{E}_h(t) = \frac{1}{2}\big(\left\| p_h \vec{y}_h(t)\right\|_V^2 + \big(\left\| q_h y'_h(t)\right\|_{L^2(0,1)}^2\big) + \frac{\alpha h^2}{2}\big(y'_{N+1}(t)\big)^2. \tag{3.2.1}$$

Thanks to (1.16), (3.2.1) and the decreasing character of $\mathcal{E}_h$, we know that $p_h\vec{y}_h$ is bounded in $L^\infty(0,\infty;V) \cap W^{1,\infty}(0,\infty;L^2(0,1))$, while $q_h\vec{y}_h$ is bounded in $L^\infty(0,\infty;L^2(0,1))$. On the other hand, it follows from (1.11) that for all $t>0$

$$\mathcal{E}_h(0) = \mathcal{E}_h(t) + \int_0^t \left[\alpha\left|y'_{N+1}\right|^2 + h^3 \sum_{j=0}^{N}\left|\frac{y'_{j+1}-y'_j}{h}\right|^2\right]\mathrm{d}t, \tag{3.2.2}$$

from which we easily derive that $hp_h\vec{y}'_h$ is bounded in $L^2(0,\infty;V)$ while y'_{N+1} is bounded in $L^2(0,\infty)$.

Thus, up to the extraction of a subsequence, we have

$$\begin{cases} p_h\vec{y}_h \to y & \text{weakly}^* \text{ in } L^\infty(0,\infty;V), \\ p_h y_h' \to y' & \text{weakly}^* \text{ in } L^\infty\big(0,\infty;L^2(0,1)\big), \\ p_h,\vec{y}_h \to y & \text{strongly in } L^2_{\mathrm{loc}}\big(0,\infty;L^2(0,1)\big), \\ q_h\vec{y}_h \to y & \text{weakly}^* \text{ in } L^\infty\big(0,\infty;L^2(0,1)\big), \\ y_{N+1}' \to y'(1,t) & \text{weakly in } L^2(0,\infty), \\ q_h y_h' \to y' & \text{weakly}^* \text{ in } L^\infty\big(0,\infty;L^2(0,1)\big), \\ hp_h y_h' \to 0 & \text{weakly in } L^2(0,\infty;V). \end{cases} \tag{3.2.3}$$

The last convergence in (3.2.3) follows from the second one and the boundedness of the sequence $\{hp_h\vec{y}_h\}$ in the space $L^2(0,\infty;V)$. As for the fifth convergence, it follows from the first convergence and the boundedness of y_{N+1}' in $L^2(0;\infty)$.

Note that in (3.2.3) we implicitly claim that the limits of $p_h\vec{y}_h$ and $q_h\vec{y}_h$ are the same. To see this, it is sufficient to observe that thanks to (1.16), (3.2.3), and the definitions of p_h and q_h, we have, for every $t \geqslant 0$,

$$\int_0^1 \big|\big(p_h\vec{y}_h - q_h\vec{y}_h\big)(x,t)\big|^2\,\mathrm{d}x = \frac{h^3}{12}\sum_{j=0}^N\left(\frac{y_{j+1}-y_j}{h}\right)^2 \leqslant \frac{h^2}{6}\mathcal{E}_h(t) \leqslant ch^2. \tag{3.2.4}$$

We have to show now that the limit y is the solution of (1.1). To this end, let $w \in \mathcal{D}([0,1]\times(0,\infty))$ with $w(0,\cdot)\equiv 0$, and set $\vec{w}_h = (w_j)_j$, where $w_j = w(jh,\cdot)$. Multiplying the first equation of (1.9) by w_j, integrating by parts on $(0,\infty)$ and taking the sum over j, we find

$$\begin{aligned} &h\sum_{j=1}^N\int_0^\infty y_j w_j''\,\mathrm{d}t + h\sum_{j=0}^N\int_0^\infty\left(\frac{y_{j+1}-y_j}{h}\right)\left(\frac{w_{j+1}-w_j}{h}\right)\mathrm{d}t \\ &\quad + \alpha\int_0^\infty w_{N+1}y_{N+1}'\,\mathrm{d}t + h^3\sum_{j=0}^N\int_0^\infty\left(\frac{y_{j+1}'-y_j'}{h}\right)\left(\frac{w_{j+1}-w_j}{h}\right)\mathrm{d}t \\ &\quad - \alpha h^2\int_0^\infty w_{N+1}'y_{N+1}'\,\mathrm{d}t = 0. \end{aligned} \tag{3.2.5}$$

Using the definitions of p_h and q_h, it is easy to check that (3.2.5) is equivalent to

$$\begin{aligned} &\int_0^\infty\!\!\int_0^1 (q_h\vec{y}_h)\big(q_h\vec{w}_h''\big)\,\mathrm{d}x\,\mathrm{d}t + \int_0^\infty\!\!\int_0^1 (p_h\vec{y}_h)_x(p_h\vec{w}_h)_x\,\mathrm{d}x\,\mathrm{d}t \\ &\quad + \alpha\int_0^\infty w(1,t)y_{N+1}'\,\mathrm{d}t + h^2\int_0^\infty\!\!\int_0^1 \big(p_h y_h'\big)_x(p_h\vec{w}_h)_x\,\mathrm{d}x\,\mathrm{d}t \\ &\quad - \alpha h^2\int_0^\infty w'(1,t)y_{N+1}'\,\mathrm{d}t = 0. \end{aligned} \tag{3.2.6}$$

At this stage, we recall the elementary convergence results: for every $w \in \mathcal{D}([0,1] \times (0,\infty))$

$$\begin{aligned} p_h \vec{w}_h &\to w \quad \text{strongly in } L^2\big(0,\infty; H^1(0,1)\big), \\ q_h \vec{w}_h &\to w \quad \text{strongly in } L^2\big(0,\infty; L^2(0,1)\big). \end{aligned} \tag{3.2.7}$$

Thanks to (3.2.7) and (3.2.3), we can pass to the limit in all the terms in (3.2.6) getting

$$\int_0^\infty \int_0^1 y w'' \,\mathrm{d}x\,\mathrm{d}t + \int_0^\infty \int_0^1 y_x w_x \,\mathrm{d}x\,\mathrm{d}t + \alpha \int_0^\infty w(1,t) y'(1,t)\,\mathrm{d}t = 0. \tag{3.2.8}$$

Now, choose w such that we also have $w(1,\cdot) \equiv 0$, then we easily derive the first equation of (1.1) from (3.2.8). Then choose w with $w(1,\cdot) \not\equiv 0$, it follows that y satisfies the boundary condition at $x = 1$. Thus for y to solve (1.1), it remains to show that $y(0) = y^0$, and $y'(0) = y^1$. For this purpose, let $v \in \mathcal{D}((0,1))$ and $l \in \mathcal{D}([0,\infty))$, and set $\vec{v}_h = (v_j)_j$ where $v_j = v(jh)$. Multiplying the first equation of (1.9) by $v_j l$, integrating by parts on $[0,\infty)$, and taking the sum over j, we find

$$\begin{aligned} &-h \sum_{j=1}^N y_j^1 v_j l(0) + \sum_{j=1}^N y_j^0 v_j l'(0) + h \sum_{j=1}^N \int_0^\infty y_j v_j l'' \,\mathrm{d}t \\ &\quad + h \sum_{j=0}^N \int_0^\infty \left(\frac{y_{j+1}-y_j}{h}\right)\left(\frac{v_{j+1}-v_j}{h}\right) l \,\mathrm{d}t \\ &\quad + h^3 \sum_{j=0}^N \int_0^\infty \left(\frac{y'_{j+1}-y'_j}{h}\right)\left(\frac{v_{j+1}-v_j}{h}\right) l \,\mathrm{d}t = 0. \end{aligned} \tag{3.2.9}$$

Using the definitions of p_h and q_h, it is easy to check that (3.2.9) is equivalent to

$$\begin{aligned} &-l(0) \int_0^1 \big(q_h \vec{y}_h^1\big)(q_h \vec{v}_h)\,\mathrm{d}x + l'(0) \int_0^1 \big(q_h \vec{y}_h^0\big)(q_h \vec{v}_h)\,\mathrm{d}x \\ &\quad + \int_0^\infty \int_0^1 (q_h \vec{y}_h)(q_h \vec{v}_h) l'' \,\mathrm{d}x\,\mathrm{d}t + \int_0^\infty \int_0^1 (p_h \vec{y}_h)_x (p_h \vec{v}_h)_x l \,\mathrm{d}x\,\mathrm{d}t \\ &\quad + h^2 \int_0^\infty \int_0^1 \big(p_h y'_h\big)_x (p_h \vec{v}_h)_x l \,\mathrm{d}x\,\mathrm{d}t = 0. \end{aligned} \tag{3.2.10}$$

Passing to the limit as $h \to 0$ in (3.2.10), we get

$$-l(0) \int_0^1 y^1 v \,\mathrm{d}x + l'(0) \int_0^1 y^0 v \,\mathrm{d}x + \int_0^\infty \int_0^1 y v l'' \,\mathrm{d}x\,\mathrm{d}t + \int_0^\infty \int_0^1 y_x v_x l \,\mathrm{d}x\,\mathrm{d}t = 0 \tag{3.2.11}$$

from which we easily derive $y(0) = y^0$, and $y'(0) = y^1$. Since system (1.1) has a unique solution, we conclude that the convergence results in (3.2.3) hold for the whole sequence $\{h\}$, and not only for an extracted subsequence.

To complete the proof of theorem 1.3, it remains to show the strong convergence results in (1.18) and (1.19). First we show that (1.19) holds. We begin by noting that since $\mathcal{E}_h(t)$ decreases to zero as $t \to \infty$, it follows from (3.2.2) that,

$$\mathcal{E}_h(0) = h^3 \sum_{j=0}^{N} \int_0^\infty \left(\frac{y'_{j+1} - y'_j}{h}\right)^2 \mathrm{d}t + \alpha \int_0^\infty \left(y'_{N+1}(t)\right)^2 \mathrm{d}t. \tag{3.2.12}$$

Now, by our assumptions on the initial data,

$$\mathcal{E}_h(0) \to E(0), \quad \text{by hypothesis.} \tag{3.2.13}$$

On the other hand, $E(t) \to 0$ as $t \to \infty$, so that

$$E(0) = \alpha \int_0^\infty \left|y'(1,t)\right|^2 \mathrm{d}t. \tag{3.2.14}$$

It follows from (3.2.12)–(3.2.14) that

$$\limsup_{h\to 0} \int_0^\infty \left(y'_{N+1}(t)\right)^2 \mathrm{d}t \leqslant \int_0^\infty \left|y'(1,t)\right|^2 \mathrm{d}t, \tag{3.2.15}$$

which combined with the weak convergence in (3.2.3) yields

$$y'_{N+1} \to y'(1,\cdot) \quad \text{strongly in } L^2(0,\infty). \tag{3.2.16}$$

Consequently,

$$hp_h \vec{y}'_h \to 0 \quad \text{strongly in } L^2(0,\infty; V). \tag{3.2.17}$$

Further, thanks to (3.2.2) and

$$E(0) = E(t) + \alpha \int_0^\infty \left|y'(1,t)\right|^2 \mathrm{d}t, \quad \forall t > 0, \tag{3.2.18}$$

we have for all $t > 0$

$$\begin{aligned}
&\left|\mathcal{E}_h(t) - E(t)\right| \\
&\quad \leqslant \left|\mathcal{E}_h(0) - E(0)\right| + h^2 \int_0^t\!\!\int_0^1 \left|(p_h\vec{y}_h)_{xt}\right|^2 \mathrm{d}x\,\mathrm{d}t + \alpha \int_0^t \left|\left|y'_{N+1}\right|^2 - \left|y'(1,t)\right|^2\right| \mathrm{d}t \\
&\quad \leqslant \left|\mathcal{E}_h(0) - E(0)\right| + h^2 \int_0^\infty\!\!\int_0^1 \left|(p_h\vec{y}_h)_{xt}\right|^2 \mathrm{d}x\,\mathrm{d}t + \alpha \int_0^\infty \left|\left|y'_{N+1}\right|^2 - \left|y'(1,t)\right|^2\right| \mathrm{d}t,
\end{aligned} \tag{3.2.19}$$

so that, combining our assumption with (3.2.16) and (3.2.17), we get

$$\lim_{h\to 0} \|\mathcal{E}_h - E\|_{C([0,\infty))} = 0. \tag{3.2.20}$$

We will be done with the proof of theorem 1.3 if we establish the strong convergence results announced in (1.18). First, observe that for all $t \geqslant 0$

$$\begin{aligned}&\left\| p_h\vec{y}_h(t)-y(t)\right\|_V^2+\left\|q_h\vec{y}_h'(t)-y'(t)\right\|_{L^2(0,1)}^2+\alpha h^2\left|y_{N+1}'(t)\right|^2\\&\quad=2\mathcal{E}_h(t)+2E(t)-2\int_0^1 y_x(x,t)(p_h\vec{y}_h)_x(x,t)\,\mathrm{d}x-2\int_0^1 y'(x,t)q_h\vec{y}_h'(x,t)\,\mathrm{d}x.\end{aligned}\tag{3.2.21}$$

Since each term in (3.2.21) decays exponentially to zero as $t\to\infty$, it follows that all these terms are integrable over $(0,\infty)$. Therefore

$$\begin{aligned}&\int_0^\infty\left\| p_h\vec{y}_h(t)-y(t)\right\|_V^2\mathrm{d}t+\int_0^\infty\left\|q_h\vec{y}_h'(t)-y'(t)\right\|_{L^2(0,1)}^2\mathrm{d}t+\alpha h^2\int_0^\infty\left|y_{N+1}'(t)\right|^2\mathrm{d}t\\&\quad=2\int_0^\infty\mathcal{E}_h(t)\,\mathrm{d}t+2\int_0^\infty E(t)\,\mathrm{d}t-2\int_0^\infty\!\!\int_0^1 y_x(x,t)(p_h\vec{y}_h)_x(x,t)\,\mathrm{d}x\,\mathrm{d}t\\&\qquad-2\int_0^\infty\!\!\int_0^1 y_x'(x,t)q_h\vec{y}_h'(x,t)\,\mathrm{d}x\,\mathrm{d}t,\end{aligned}\tag{3.2.22}$$

from which we easily derive the first two convergence results in (1.18) by letting h go to zero, and using (3.2.20), the Lebesgue dominated convergence theorem, and (3.2.3). It remains to prove the last convergence result in (1.18). To this end, observe that one has for all $x\in[0,1]$ and $t>0$

$$\begin{aligned}&p_h\vec{y}_h(x,t)-y(x,t)\\&\quad=\int_0^t\big(p_h\vec{y}_h'(x,s)-y'(x,s)\big)\,\mathrm{d}s+p_h\vec{y}_h^0(x)-y^0(x)\\&\quad=\int_0^t\big(p_h\vec{y}_h'(x,s)-q_h\vec{y}_h'(x,s)\big)\,\mathrm{d}s+\int_0^t\big(q_h\vec{y}_h'(x,s)-y'(x,s)\big)\,\mathrm{d}s\\&\qquad+p_h\vec{y}_h^0(x)-y^0(x),\end{aligned}\tag{3.2.23}$$

from which one derives that for all $t>0$

$$\begin{aligned}&\left\|p_h\vec{y}_h(t)-y(t)\right\|_{L^2(0,1)}^2\\&\quad\leqslant 3\left\|\int_0^t\big(p_h\vec{y}_h'(s)-q_h\vec{y}_h'(s)\big)\,\mathrm{d}s\right\|_{L^2(0,1)}^2\\&\qquad+3\left\|\int_0^t\big(q_h\vec{y}_h'(s)-y'(s)\big)\,\mathrm{d}s\right\|_{L^2(0,1)}^2+3\left\|p_h\vec{y}_h^0-y^0\right\|_{L^2(0,1)}^2\\&\quad\leqslant 3\int_0^t\left\|\big(p_h\vec{y}_h'(s)-q_h\vec{y}_h'(s)\big)\right\|_{L^2(0,1)}^2\mathrm{d}s\\&\qquad+3\int_0^t\left\|\big(q_h\vec{y}_h'(s)-y'(s)\big)\right\|_{L^2(0,1)}^2\mathrm{d}s+3\left\|p_h\vec{y}_h^0-y^0\right\|_{L^2(0,1)}^2\\&\quad\leqslant 3\int_0^\infty\left\|\big(p_h\vec{y}_h'(s)-q_h\vec{y}_h'(s)\big)\right\|_{L^2(0,1)}^2\mathrm{d}s\end{aligned}$$

$$
\begin{aligned}
&+3\int_0^\infty \left\|\left(q_h\vec{y}_h'(s)-y'(s)\right)\right\|^2_{L^2(0,1)}\,\mathrm{d}s+3\left\|p_h\vec{y}_h^0-y^0\right\|^2_{L^2(0,1)}\\
&\leqslant \frac{h^2}{4}\int_0^\infty \left\|(p_h\vec{y}_h)_{xt}(s)\right\|^2_{L^2(0,1)}\,\mathrm{d}s+3\int_0^\infty \left\|\left(q_h\vec{y}_h'(s)-y'(s)\right)\right\|^2_{L^2(0,1)}\,\mathrm{d}s\\
&+3\left\|p_h\vec{y}_h^0-y^0\right\|^2_{L^2(0,1)}\quad \text{(see (3.2.4))}, \qquad (3.2.24)
\end{aligned}
$$

whence the desired convergence result thanks to (1.16), (3.2.17) and the second convergence result in (1.18). □

4. Some comments and open problems

4.1. Uniform polynomial decay rates

Theorem 1.1 shows that the exponential decay property of the energy of the $1-d$ damped wave equation (1.1) is not kept uniform for the classical finite difference space semi-discretization. However whether the polynomial decay of the discretized energy is uniform with respect to the mesh size h, for sufficiently smooth initial data, remains to be investigated.

4.2. Finite-element space discretizations

Theorem 1.2 shows that system (1.9), which is obtained from the classical finite difference space semi-discretization by the addition of an appropriate numerical viscosity term, is uniformly (w.r.t. the mesh size h) exponentially stable. This method may be applied to the classical finite element space semi-discretization with success, but the details are yet to be done.

4.3. Full discretizations

In [22] it was proved that the full centered finite difference discretization (also known as leapfrog scheme) of the $1-d$ wave equation is uniformly (with respect to the mesh-size) exactly controllable from the boundary with Dirichlet control provided that the time and space steps coincide. One would expect in this case the exponential stability property, under boundary feedback control, to be also uniform, but this is yet to be done. Note however that the results in [22] are non-generic since most fully discrete schemes fail to be controllable uniformly with respect to the mesh-size (see [35]). Thus, in general, one expects that adding extra numerical viscous terms will also be needed in the context of fully discrete schemes.

4.4. Multi-dimensional case

The extensions of the results of this paper to space dimensions greater than or equal to 2 are widely open problems. Note however that the main tools for doing that by means of discrete multiplier techniques have been already developed in [29,34].

Appendix A. Proof of lemma 3.2

Set

$$E(\vec{z}_h;t)=\frac{h}{2}\sum_{j=0}^{N}\left\{\left(z_j'(t)\right)^2+\left(\frac{z_{j+1}-z_j}{h}\right)^2\right\}. \tag{A.1}$$

Multiplying the first equation of (3.1.2) by hz_j', taking the sum over j and integrating over $[0,t]$, we find

$$\begin{aligned}E(\vec{z}_h;t)=&-\alpha\int_0^t z_{N+1}'y_{N+1}'\,\mathrm{d}s+h^2\sum_{j=0}^{N}\int_0^t\left(\frac{y_{j+1}'-y_j'}{h}\right)z_j'\,\mathrm{d}s\\&-h^2\sum_{j=0}^{N-1}\int_0^t\left(\frac{y_{j+1}'-y_j'}{h}\right)z_{j+1}'\,\mathrm{d}s.\end{aligned} \tag{A.2}$$

Using Gronwall lemma and Young inequality, we derive from (A.2) that for all $\varepsilon>0$

$$E(\vec{z}_h;t)\leqslant\varepsilon\int_0^T|z_{N+1}'|^2\,\mathrm{d}t+C_\varepsilon\int_0^T\left[\alpha|y_{N+1}'|^2\,\mathrm{d}t+h^3\sum_{j=0}^{N}\left|\frac{y_{j+1}'-y_j'}{h}\right|^2\right]\mathrm{d}t. \tag{A.3}$$

We will use estimate (A.3) with an appropriate choice of ε to obtain estimate (3.1.3) and complete the proof. To this end, multiply the first equation of (3.1.2) by $hj((z_{j+1}-z_{j-1})/2)$, integrate over $(0,T)$ and take the sum over j; this yields

$$\begin{aligned}&h\sum_{j=1}^{N}z_j'j\left(\frac{z_{j+1}-z_{j-1}}{2}\right)\Big|_0^T-h\sum_{j=1}^{N}j\int_0^T z_j'\left(\frac{z_{j+1}'-z_{j-1}'}{2}\right)\mathrm{d}t\\&\quad-h\sum_{j=1}^{N}j\int_0^T\left(\frac{z_{j+1}-2z_j+z_{j-1}}{h^2}\right)\left(\frac{z_{j+1}-z_{j-1}}{2}\right)\mathrm{d}t\\&=h^3\sum_{j=1}^{N}j\int_0^T\left(\frac{y_{j+1}'-2y_j'+y_{j-1}'}{h^2}\right)\left(\frac{z_{j+1}-z_{j-1}}{2}\right)\mathrm{d}t.\end{aligned} \tag{A.4}$$

Proceeding as in the proof of lemma 3.1, one can show that

$$\begin{aligned}&h\sum_{j=1}^{N}z_j'j\left(\frac{z_{j+1}-z_{j-1}}{2}\right)\Big|_0^T\\&=h^2\sum_{j=0}^{N}z_j'j\left(\frac{z_{j+1}-z_j}{2h}\right)\Big|_0^T+h^2\sum_{j=0}^{N-1}z_{j+1}'(j+1)\left(\frac{z_{j+1}-z_j}{2h}\right)\Big|_0^T,\end{aligned} \tag{A.5}$$

$$
\begin{aligned}
&-h\sum_{j=1}^{N} j\int_0^T\left(\frac{z_{j+1}-2z_j+z_{j-1}}{h^2}\right)\left(\frac{z_{j+1}-z_{j-1}}{2}\right)\mathrm{d}t\\
&\quad=\frac{h}{2}\sum_{j=0}^{N}\left(\frac{z_{j+1}-z_j}{h}\right)^2\mathrm{d}t-\frac{\alpha^2}{2}\int_0^T|y'_{N+1}|^2\,\mathrm{d}t
\end{aligned}\tag{A.6}
$$

and

$$
\begin{aligned}
&-h\sum_{j=1}^{N}\int_0^T jz'_j\left(\frac{z'_{j+1}-z'_{j-1}}{2}\right)\mathrm{d}t\\
&\quad=\frac{h}{2}\sum_{j=0}^{N}\int_0^T|z'_j|^2\,\mathrm{d}t+\frac{h}{4}\int_0^T|z'_{N+1}|^2\,\mathrm{d}t-\frac{h^3}{4}\sum_{j=0}^{N}\int_0^T\left|\frac{z'_{j+1}-z'_j}{h}\right|^2\mathrm{d}t\\
&\qquad+\frac{\alpha^2h^2}{4}\int_0^T\left|y''_{N+1}\right|^2\mathrm{d}t-\frac{1}{4}\int_0^T\left(|z'_{N+1}|^2+|z'_N|^2\right)\mathrm{d}t.
\end{aligned}\tag{A.7}
$$

It follows from (A.4)–(A.7) that

$$
\begin{aligned}
&\frac{h^3}{4}\sum_{j=0}^{N}\int_0^T\left|\frac{z'_{j+1}-z'_j}{h}\right|^2\mathrm{d}t+\frac{1-h}{4}\int_0^T|z'_{N+1}|^2\,\mathrm{d}t+\frac{1}{4}|z'_N|^2\,\mathrm{d}t\\
&\quad=h^2\sum_{j=0}^{N}z'_j j\left(\frac{z_{j+1}-z_j}{2h}\right)\Big|_0^T+h^2\sum_{j=0}^{N-1}z'_{j+1}(j+1)\left(\frac{z_{j+1}-z_j}{2h}\right)\Big|_0^T\\
&\qquad+\int_0^T E(\vec{z}_h;t)\,\mathrm{d}t-\frac{\alpha^2}{2}\int_0^T|y'_{N+1}|^2\,\mathrm{d}t+\frac{\alpha^2h^2}{4}\int_0^T|y''_{N+1}|^2\,\mathrm{d}t\\
&\qquad-h^3\sum_{j=1}^{N}j\int_0^T\left(\frac{y'_{j+1}-2y'_j+y'_{j-1}}{h^2}\right)\left(\frac{z_{j+1}-z_{j-1}}{2}\right)\mathrm{d}t.
\end{aligned}\tag{A.8}
$$

Some elementary calculations then show that

$$
\begin{aligned}
&\left|h^2\sum_{j=0}^{N}z'_j j\left(\frac{z_{j+1}-z_j}{2h}\right)\Big|_0^T+h^2\sum_{j=0}^{N-1}z'_{j+1}(j+1)\left(\frac{z_{j+1}-z_j}{2h}\right)\Big|_0^T\right|\\
&\quad\leqslant E(\vec{z}_h;T),\quad\text{since } E(\vec{z}_h;0)=0,
\end{aligned}\tag{A.9}
$$

$$
\begin{aligned}
&\left|h^3\sum_{j=1}^{N}j\int_0^T\frac{y'_{j+1}-2y'_j+y'_{j-1}}{h^2}\,\frac{z_{j+1}-z_{j-1}}{2}\,\mathrm{d}t\right|\\
&\quad\leqslant 2h^3\sum_{j=0}^{N}\int_0^T\left|\frac{y'_{j+1}-y'_j}{h}\right|^2\mathrm{d}t+\int_0^T E(\vec{z}_h;t)\,\mathrm{d}t,
\end{aligned}\tag{A.10}
$$

using Young inequality and (A.1).

Reporting (A.9) and (A. 10) in (A.8), we get

$$\begin{aligned}\frac{1}{4}\int_0^T |z_N'|^2\,\mathrm{d}t + \frac{h^3}{4}\sum_{j=0}^{N}\int_0^T \left|\frac{z_{j+1}'-z_j'}{h}\right|^2 \mathrm{d}t + \frac{1-h}{4}\int_0^T |z_{N+1}'|^2\,\mathrm{d}t \\ \leqslant E(\vec{z}_h;T) + 2\int_0^T E(\vec{z}_h;t)\,\mathrm{d}t + \frac{\alpha^2h^2}{4}\int_0^T |y_{N+1}''|^2\,\mathrm{d}t \\ + 2h^3\sum_{j=0}^{N}\int_0^T \left|\frac{y_{j+1}'-y_j'}{h}\right|^2 \mathrm{d}t.\end{aligned} \tag{A.11}$$

Using (A.3) in (A.11), we find, for all $\varepsilon > 0$

$$\begin{aligned}\frac{1}{4}\int_0^T |z_N'|^2\,\mathrm{d}t + \frac{h^3}{4}\sum_{j=0}^{N}\int_0^T \left|\frac{z_{j+1}'-z_j'}{h}\right|^2 \mathrm{d}t + \frac{1-h}{4}\int_0^T |z_{N+1}'|^2\,\mathrm{d}t \\ \leqslant \varepsilon(1+2T)\int_0^T \left|z_{N+1}'\right|^2 \mathrm{d}t + \frac{\alpha^2h^2}{4}\int_0^T \left|y_{N+1}''\right|^2 \mathrm{d}t \\ + C_\varepsilon\int_0^T\left[\alpha\left|y_{N+1}'\right|^2 \mathrm{d}t + h^3\sum_{j=0}^{N}\left|\frac{y_{j+1}'-y_j'}{h}\right|^2\right]\mathrm{d}t.\end{aligned} \tag{A.12}$$

Observing that for all real numbers a and b one has $(a+b)^2 \geqslant a^2/2 - b^2$, it follows that

$$\begin{aligned}\int_0^T |z_N'|^2\,\mathrm{d}t &= \int_0^T \left|z_{N+1}' + \alpha h y_{N+1}''\right|^2 \mathrm{d}t \\ &\geqslant \frac{1}{2}\int_0^T |z_{N+1}'|^2\,\mathrm{d}t - \alpha^2h^2\int_0^T \left|y_{N+1}''\right| \mathrm{d}t.\end{aligned} \tag{A.13}$$

Reporting (A.13) in (A.12), we get

$$\begin{aligned}\frac{h^3}{4}\sum_{j=0}^{N}\int_0^T \left|\frac{z_{j+1}'-z_j'}{h}\right|^2 \mathrm{d}t + \frac{3-2h}{8}\int_0^T \left|z_{N+1}'\right|^2 \mathrm{d}t \\ \leqslant \varepsilon(1+2T)\int_0^T \left|z_{N+1}'\right|^2 \mathrm{d}t + \frac{\alpha^2h^2}{2}\int_0^T \left|y_{N+1}''\right|^2 \mathrm{d}t \\ + C_\varepsilon\int_0^T\left[\alpha\left|y_{N+1}'\right|^2 \mathrm{d}t + h^3\sum_{j=0}^{N}\left|\frac{y_{j+1}'-y_j'}{h}\right|^2\right]\mathrm{d}t.\end{aligned} \tag{A.14}$$

At this stage, we observe that if we can show that there exists $C > 0$ independent of h, the initial data y_h^0 and y_h^1 and T, such that

$$\frac{\alpha^2h^2}{4}\int_0^T \left|y_{N+1}''\right|^2 \mathrm{d}t \leqslant C E_h(0), \tag{A.15}$$

then we will be done. In fact in this case, choosing $\varepsilon = (3-2h)/(16(1+2T))$ in (A.14), keeping in mind that $0 < h < 1$, and using (A.15), we get (3.1.3). To complete the proof of lemma 3.2 it remains to prove (A.15). First we observe that if $\vec{y}_h$ is a solution of (1.9), then so is $\vec{y}_h'$ since system (1.9) is linear. The energy F_h given by

$$F_h = \frac{h}{2}\sum_{j=0}^{N}\left\{\left(y_j''(t)\right)^2 + \left(\frac{y_{j+1}' - y_j'}{h}\right)^2\right\} + \frac{\alpha h^2}{2}\left(y_{N+1}''(t)\right)^2, \tag{A.16}$$

is obtained by substituting $\vec{y}_h'$ for $\vec{y}_h$ in the energy $\mathcal{E}_h$. It is easy to check that F_h is a non-increasing function of the time variable t with

$$F_h'(t) = -h^3\sum_{j=0}^{N}\left(\frac{y_{j+1}'' - y_j''}{h}\right)^2 - \alpha\left(y_{N+1}''(t)\right)^2. \tag{A.17}$$

Consequently,

$$\begin{aligned}\alpha^2 h^2\int_0^T \left|y_{N+1}''\right|^2 \mathrm{d}t \leqslant \alpha h^2\int_0^T \left|F_h'(t)\right|^2 \mathrm{d}t = \alpha h^2\left(F_h(0) - F_h(T)\right)\\ \leqslant \alpha h^2 F_h(0).\end{aligned} \tag{A.18}$$

It is not difficult to check that there exists $C > 0$ independent of h, the initial data y_h^0 and y_h^1, and T, such that $h^2F_h(0) \leqslant CE_h(0)$. □

Appendix B

In this section, we establish the equivalence between the exponential decay of the energy of solutions of system (1.1) and the boundary observability of the solutions of the associated conservative system (C.S.) in section 1.

More precisely, we will prove the following result:

Proposition B. The following assertions are equivalent:

(i) There exist $T_0 > 0$ and $C \geqslant 0$ such that for all $\{\varphi^0, \varphi^1\} \in V \times L^2(0,1)$

$$\|\varphi^0\|^2_{H^1(0,1)} + \|\varphi^1\|^2_{L^2(0,1)} \leqslant C\int_0^T \left|\varphi'(1,t)\right|^2 \mathrm{d}t, \quad \forall T > T_0. \tag{B.1}$$

(ii) There exist $M > 0$ and $\omega > 0$ such that for all $\{y^0, y^1\} \in V \times L^2(0,1)$

$$\begin{aligned}&\|y(\cdot,t)\|^2_{H^1(0,1)} + \|y'(.,t)\|^2_{L^2(0,1)}\\ &\quad\leqslant M\left[\exp(-\omega t)\right]\left[\|y^0\|^2_{H^1(0,1)} + \|y^1\|^2_{L^2(0,1)}\right], \quad \forall t \geqslant 0,\end{aligned} \tag{B.2}$$

where y solves (1.1), and $V = \{u \in H^1(0,1); u(0) = 0\}$.

Proof. First we prove that (i) implies (ii). To this end, choose $\varphi^0 = y^0$ and $\varphi^1 = y^1$. Set $y = \varphi + \psi$ where ψ is the solution of

$$\begin{cases} \psi'' - \psi_{xx} = 0 & \text{in } (0,1) \times (0,\infty), \\ \psi(0,t) = 0, \qquad \psi_x(1,t) = -\alpha y'(1,t) & \text{in } (0,\infty), \\ \psi(x,0) = 0 & \text{in } (0,1), \\ \psi'(0) = 0 & \text{in } (0,1). \end{cases} \tag{B.3}$$

It follows from (B.1) that for all $T > T_0$

$$\|y^0\|^2_{H^1(0,1)} + \|y^1\|^2_{L^2(0,1)} \leqslant C \int_0^T \left[|y'(1,t)|^2 + |\psi'(1,t)|^2 \right] \mathrm{d}t. \tag{B.4}$$

Thus if we can prove the existence of a positive constant C such that

$$\int_0^T |\psi'(1,t)|^2 \,\mathrm{d}t \leqslant C \int_0^T |y'(1,t)|^2 \,\mathrm{d}t, \tag{B.5}$$

then the combination of (B.4), (B.5) and the dissipation law (1.3), we will get

$$E(T) \leqslant \frac{C}{C+1} E(0). \tag{B.6}$$

Using the semi-group property, we derive (B.2) from (B.6). To complete the proof of the first implication, it remains to prove (B.5). To this end, let $T > T_0$, and $0 < t \leqslant T$. Multiply the first equation of (B.3) by ψ' and integrate by parts over $(0,1) \times (0,t)$; one can easily derive from this operation that for all $\varepsilon > 0$

$$\|\psi(\cdot,t)\|^2_{H^1(0,1)} + \|\psi'(\cdot,t)\|^2_{L^2(0,1)} \leqslant \frac{\alpha^2}{4\varepsilon} \int_0^T |y'(1,t)|^2 \,\mathrm{d}t + \varepsilon \int_0^T \|\psi'(1,t)\|^2 \,\mathrm{d}t. \tag{B.7}$$

Now multiplying the first equation of (B.3) by $2x\psi_x$ and integrating by parts over $(0,1) \times (0,T)$, and using (B.7) follows that

$$\int_0^T |\psi'(1,t)|^2 \,\mathrm{d}t \leqslant 2\alpha^2(1+T)^2 \int_0^T |y'(1,t)|^2 \,\mathrm{d}t,$$

which completes the proof of the first implication. We now turn to the proof of the statement “(ii) implies (i)”. Thanks to (B.2), we may find $T > 0$ such that

$$\alpha \int_0^T |y'(1,t)|^2 \,\mathrm{d}t \geqslant \frac{1}{4} \left[\|y^0\|^2_{H^1(0,1)} + \|y^1\|^2_{L^2(0,1)} \right]. \tag{B.8}$$

Let φ be an arbitrary solution of (C.S.). Let y be the solution of (1.1) such that $y^0 = \varphi^0$, and $y^1 = \varphi^1$. Multiply the first equation of (B.3) by ψ' and integrate by parts over $(0,1) \times (0,t)$; one can derive from this operation that

$$\frac{\alpha}{2} \int_0^T |\psi'(1,t)|^2 \,\mathrm{d}t \leqslant \frac{\alpha}{2} \int_0^T |\varphi'(1,t)|^2 \,\mathrm{d}t. \tag{B.9}$$

Now

$$\frac{\alpha}{2}\int_0^T \left|\psi'(1,t)\right|^2 \mathrm{d}t \geqslant \frac{\alpha}{4}\int_0^T \left|y'(1,t)\right|^2 \mathrm{d}t - \frac{\alpha}{2}\int_0^T \left|\varphi'(1,t)\right|^2 \mathrm{d}t. \tag{B.10}$$

Combining (B.8), (B.9) and (B.10), we find

$$\left\|\varphi^0\right\|^2_{H^1(0,1)} + \left\|\varphi^1\right\|^2_{L^2(0,1)} \leqslant 16\alpha \int_0^T \left\|\varphi'(1,t)\right\|^2 \mathrm{d}t,$$

which completes the proof of proposition B. □

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