

Uniform convergence of double trigonometric series

by

CHANG-PAO CHEN and GWO-BIN CHEN (Hsinchu)

Abstract. It is shown that under certain conditions on $\{c_{jk}\}$, the rectangular partial sums $s_{mn}(x, y)$ converge uniformly on T^2 . These conditions include conditions of bounded variation of order $(1, 0)$, $(0, 1)$, and $(1, 1)$ with the weights $|j|$, $|k|$, $|jk|$, respectively. The convergence rate is also established. Corresponding to the mentioned conditions, an analogous condition for single trigonometric series is

$$\sum_{|k|=n}^{\infty} |\Delta c_k| = o(1/n) \quad (\text{as } n \rightarrow \infty).$$

For O -regularly varying quasimonotone sequences, we prove that it is equivalent to the condition: $nc_n = o(1)$ as $n \rightarrow \infty$. As a consequence, our result generalizes those of Chaundy–Jolliffe [CJ], Jolliffe [J], Nurcombe [N], and Xie–Zhou [XZ].

1. Introduction. Let $T^2 \equiv [-\pi, \pi] \times [-\pi, \pi]$. Consider the double trigonometric series

$$(1.1) \quad \sum_{j=-\infty}^{\infty} \sum_{k=-\infty}^{\infty} c_{jk} e^{i(jx+ky)},$$

where $\{c_{jk} : -\infty < j, k < \infty\}$ is a double sequence of complex numbers. The rectangular partial sums $s_{mn}(x, y)$ of (1.1) are defined as

$$s_{mn}(x, y) \equiv \sum_{|j| \leq m} \sum_{|k| \leq n} c_{jk} e^{i(jx+ky)} \quad (m, n \geq 0).$$

We are interested in finding conditions on $\{c_{jk}\}$ under which $s_{mn}(x, y)$ converges uniformly on T^2 . This problem for single trigonometric series has been discussed by Chaundy–Jolliffe [CJ], Jolliffe [J], Nurcombe [N], and

1991 *Mathematics Subject Classification*: Primary 42A32, 42B05; Secondary 42A20.
 Research of the first author is supported by National Science Council, Taipei, R.O.C. under Grant #NSC 85-2121-M-007-021.

Xie-Zhou [XZ]. For higher dimensional case, it is still open. Set

$$\begin{aligned}\Delta_{10}c_{jk} &= c_{jk} - c_{j+1,k}, \\ \Delta_{01}c_{jk} &= c_{jk} - c_{j,k+1}, \\ \Delta_{11}c_{jk} &= c_{jk} - c_{j+1,k} - c_{j,k+1} + c_{j+1,k+1}.\end{aligned}$$

It was proved in [M1] that if both of the conditions

$$(1.2) \quad c_{jk} \rightarrow 0 \quad \text{as } \max(|j|, |k|) \rightarrow \infty,$$

$$(1.3) \quad \sum_{j=-\infty}^{\infty} \sum_{k=-\infty}^{\infty} |\Delta_{11}c_{jk}| < \infty,$$

are satisfied, then $s_{mn}(x, y)$ converges pointwise in Pringsheim's sense to some measurable function $f(x, y)$ with $0 < |x|, |y| \leq \pi$. Moreover, $f \in L^p(T^2)$ for all $0 < p < 1$, and $s_{mn}(x, y)$ converges in $L^p(T^2)$ -metric to f as $\min(m, n) \rightarrow \infty$, where

$$\|f\|_p \equiv \left(\frac{1}{4\pi^2} \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} |f(x, y)|^p dx dy \right)^{1/p} \quad (p > 0).$$

In [CL, M2, M3], it was further proved that $s_{mn}(x, y)$ converges uniformly to $f(x, y)$ on the set $\{\alpha \leq |x| \leq \pi, \beta \leq |y| \leq \pi\}$ for all $0 < \alpha, \beta \leq \pi$. As indicated in [C1, C2, M3], f may not be Lebesgue integrable, and hence, condition (1.3) is insufficient for the uniform convergence of $s_{mn}(x, y)$ on T^2 . Instead of (1.3), we assume the following conditions:

$$(1.4) \quad \sup_{\mu, \nu \geq 0} \left(\sum_{|j| \leq \mu} \sum_{|k| \leq \nu} \chi_{mn}(|j|, |k|) c_{jk} \right) \rightarrow 0 \quad \text{as } \min(m, n) \rightarrow \infty,$$

$$(1.5) \quad \sup_{\mu \geq 1} \left(\mu \sum_{|j|=\mu}^{\infty} \sum_{k=-\infty}^{\infty} \chi_{mn}(|j|, |k|) |\Delta_{10}c_{jk}| \right) \rightarrow 0$$

as $\min(m, n) \rightarrow \infty$,

$$(1.6) \quad \sup_{\nu \geq 1} \left(\nu \sum_{j=-\infty}^{\infty} \sum_{|k|=\nu}^{\infty} \chi_{mn}(|j|, |k|) |\Delta_{01}c_{jk}| \right) \rightarrow 0$$

as $\min(m, n) \rightarrow \infty$,

$$(1.7) \quad \sup_{\mu, \nu \geq 1} \left(\mu \nu \sum_{|j|=\mu}^{\infty} \sum_{|k|=\nu}^{\infty} \chi_{mn}(|j|, |k|) |\Delta_{11}c_{jk}| \right) \rightarrow 0$$

as $\min(m, n) \rightarrow \infty$,

where χ_{mn} denotes the characteristic function:

$$(1.8) \quad \chi_{mn}(j, k) \equiv \begin{cases} 1 & \text{if } |j| > m \text{ or } |k| > n, \\ 0 & \text{otherwise.} \end{cases}$$

We shall prove in §2 that these conditions are sufficient for the uniform convergence of $s_{mn}(x, y)$ on T^2 . The convergence rate of s_{mn} is also established there.

Conditions (1.4)–(1.7) are growth conditions on $\{c_{jk}\}$ with (j, k) lying outside the rectangle $[-m, m] \times [-n, n]$. It is obvious that the u -convergence of series (1.1) with $x = y = 0$ implies condition (1.4), and (1.4) is stronger than the convergence of the same series in Pringsheim's sense (cf. [D]). Moreover, a double sine series automatically satisfies condition (1.4). For the one-dimensional case, (1.4) reduces to

$$\sup_{\nu > n} \left(\sum_{n < |k| \leq \nu} c_k \right) \rightarrow 0 \quad (\text{as } n \rightarrow \infty),$$

which is equivalent to

$$(1.9) \quad \sum_{k=1}^{\infty} (c_k + c_{-k}) \text{ converges.}$$

This is a necessary condition for the uniform convergence. Next, (1.5)–(1.7) are conditions of bounded variation of order (1, 0), (0, 1), and (1, 1) with the weights j, k, jk , respectively (cf. Corollary 2.3). Any of them implies (1.3). They are necessary for certain cases, in particular, for O -regularly varying quasimonotone sequences (cf. §4 for details). Corresponding to (1.5)–(1.7), an analogous condition for single trigonometric series is

$$(1.10) \quad \sum_{|k|=n}^{\infty} |\Delta c_k| = o(1/n) \quad (\text{as } n \rightarrow \infty),$$

where $\Delta c_k = c_k - c_{k+1}$. It takes the following form for the one-sided condition:

$$(1.11) \quad \sum_{k=n}^{\infty} |\Delta c_k| = o(1/n) \quad (\text{as } n \rightarrow \infty).$$

For O -regularly varying quasimonotone sequences, we shall prove in Theorem 4.2 that (1.11) is equivalent to

$$(1.12) \quad \lim_{n \rightarrow \infty} n c_n = 0.$$

This leads us to an alternative approach to the uniform convergence problem. Our result generalizes [CJ, J, N, XZ].

2. Double trigonometric series. As in [C1], define $\Delta_{10}^*c_{jk}, \Delta_{01}^*c_{jk}$, and $\Delta_{11}^*c_{jk}$ by

$$\begin{aligned}\Delta_{10}^*c_{jk} &= c_{jk} - c_{\tau(j),k}, & \Delta_{01}^*c_{jk} &= c_{jk} - c_{j,\tau(k)}, \\ \Delta_{11}^*c_{jk} &= \Delta_{10}^*\Delta_{01}^*c_{jk} = \Delta_{01}^*\Delta_{10}^*c_{jk}.\end{aligned}$$



Here $c_{0+,k} = c_{0-,k} = c_{0k}$, $c_{j,0+} = c_{j,0-} = c_{j0}$, and the function $\tau(j)$ is defined by $\tau(0+) = 1$, $\tau(0-) = -1$, $\tau(j) = j + 1$ for $j \geq 1$, and $\tau(j) = j - 1$ for $j \leq -1$. Obviously, $\Delta_{\alpha\beta}^* c_{jk}$ are the same as $\Delta_{\alpha\beta} c_{jk}$ for $j, k \geq 0+$. For other cases, they are related in the following way:

$$\begin{aligned} \Delta_{10}^* c_{jk} &= -\Delta_{10} c_{j-1,k} \quad (j \leq 0-), \\ \Delta_{01}^* c_{jk} &= -\Delta_{01} c_{j,k-1} \quad (k \leq 0-). \end{aligned}$$

LEMMA 2.1. Assume that (1.2) holds. Then for $m, n > 0$, we have

$$\begin{aligned} &\left\| \sum_{|j| \leq m} \sum_{|k| \leq n} c_{jk} e^{i(jx+ky)} \right\|_{\infty} \\ &\leq \sup_{\mu \leq m; \nu \leq n} \left| \sum_{|j| \leq \mu} \sum_{|k| \leq \nu} c_{jk} \right| + \sup_{\mu \geq 1} \left(9\pi\mu \sum_{|j|=\mu} \sum_{k=-\infty}^{\infty} |\Delta_{10}^* c_{jk}| \right) \\ &\quad + \sup_{\nu \geq 1} \left(9\pi\nu \sum_{j=-\infty}^{\infty} \sum_{|k|=\nu} |\Delta_{01}^* c_{jk}| \right) \\ &\quad + \sup_{\mu, \nu \geq 1} \left(81\pi^2 \mu\nu \sum_{|j|=\mu} \sum_{|k|=\nu} |\Delta_{11}^* c_{jk}| \right). \end{aligned}$$

Proof. Let $|x| \leq \pi$ and $|y| \leq \pi$. Define $M = \max([1/|x|], 1)$ and $N = \max([1/|y|], 1)$. Then we have

$$(2.1) \quad \left| \sum_{|j| \leq m} \sum_{|k| \leq n} c_{jk} e^{i(jx+ky)} \right| \leq \Sigma_{11} + \Sigma_{12} + \Sigma_{21} + \Sigma_{22},$$

where

$$\Sigma_{\alpha\beta} \equiv \left| \sum_{|j| \leq m} \sum_{|k| \leq n} \chi_M^{\alpha}(j) \chi_N^{\beta}(k) c_{jk} e^{i(jx+ky)} \right|$$

and χ_M^1 and χ_M^2 denote the characteristic functions $\chi_{[-M,M]}$ and $\chi_{\mathbb{R} \setminus [-M,M]}$, respectively. Set

$$\Sigma_{11}^{\gamma\delta} \equiv \left| \sum_{|j| \leq m} \sum_{|k| \leq n} \chi_M^1(j) \chi_N^1(k) c_{jk} (e^{ijx} - 1)^{\gamma} (e^{iky} - 1)^{\delta} \right|.$$

We have $\Sigma_{11} \leq \Sigma_{11}^{00} + \Sigma_{11}^{10} + \Sigma_{11}^{01} + \Sigma_{11}^{11}$. Obviously, $|e^{ijx} - 1| \leq \pi M^{-1}|j|$ and

$$\left| M^{-1} \sum_{|j| \leq m} \chi_M^1(j) d_j \right| \leq \sup_{1 \leq \mu \leq m} \sum_{|j|=\mu} |d_j|$$

for any sequence $\{d_j\}$ with $d_0 = 0$. The choice $d_j = \sum_{|k| \leq n} |j c_{jk}|$ gives

$$\Sigma_{11}^{10} \leq \frac{\pi}{M} \left(\sum_{|j| \leq m} \sum_{|k| \leq n} \chi_M^1(j) |j c_{jk}| \right) \leq \sup_{\mu \geq 1} \left(\pi\mu \sum_{|j|=\mu} \sum_{|k| \leq n} |\Delta_{10}^* c_{jk}| \right).$$

Similarly, we have

$$\begin{aligned} \Sigma_{11}^{01} &\leq \sup_{\nu \geq 1} \left(\pi\nu \sum_{|j| \leq m} \sum_{|k|=\nu} |\Delta_{01}^* c_{jk}| \right), \\ \Sigma_{11}^{11} &\leq \sup_{\mu, \nu \geq 1} \left(\pi^2 \mu\nu \sum_{|j|=\mu} \sum_{|k|=\nu} |\Delta_{11}^* c_{jk}| \right). \end{aligned}$$

The above discussion shows that

$$(2.2) \quad \begin{aligned} \Sigma_{11} &\leq \sup_{\mu \leq m; \nu \leq n} \left| \sum_{|j| \leq \mu} \sum_{|k| \leq \nu} c_{jk} \right| + \sup_{\mu \geq 1} \left(\pi\mu \sum_{|j|=\mu} \sum_{|k| \leq n} |\Delta_{10}^* c_{jk}| \right) \\ &\quad + \sup_{\nu \geq 1} \left(\pi\nu \sum_{|j| \leq m} \sum_{|k|=\nu} |\Delta_{01}^* c_{jk}| \right) \\ &\quad + \sup_{\mu, \nu \geq 1} \left(\pi^2 \mu\nu \sum_{|j|=\mu} \sum_{|k|=\nu} |\Delta_{11}^* c_{jk}| \right). \end{aligned}$$

To estimate Σ_{12} , we employ the functions $\Psi_k(t)$, which are defined by $\Psi_{0+}(t) = \Psi_{0-}(t) = 1/2$ and

$$\begin{aligned} \Psi_k(t) &= 1/2 + e^{it} + e^{i2t} + \dots + e^{ikt}, \\ \Psi_{-k}(t) &= 1/2 + e^{-it} + e^{-i2t} + \dots + e^{-ikt}, \end{aligned}$$

where $k \geq 1$. By [C1, Lemma 2], we get $\Sigma_{12} \leq \Sigma_{12}^{00} + \Sigma_{12}^{10} + \Sigma_{12}^{01} + \Sigma_{12}^{11}$, where

$$\begin{aligned} \Sigma_{12}^{\gamma 0} &\equiv \left| \sum_{|j| \leq m} \sum_{|k|=0 \pm} \chi_M^1(j) \Delta_{01}^* (\chi_N^2(k) c_{jk}) (e^{ijx} - 1)^{\gamma} \Psi_k(y) \right|, \\ \Sigma_{12}^{\gamma 1} &\equiv \left| \sum_{|j| \leq m} \sum_{|k|=n} \chi_M^1(j) \chi_N^2(\tau(k)) c_{j,\tau(k)} (e^{ijx} - 1)^{\gamma} \Psi_k(y) \right|. \end{aligned}$$

As indicated in [C1], we have $|\Psi_k(y)| \leq \pi/|y| \leq 2\pi N$. It is clear that $\Delta_{01}^* (\chi_N^2(k) c_{jk}) = 0$ for $|k| < N$, and

$$\Delta_{01}^* (\chi_N^2(k) c_{jk}) = \begin{cases} \Delta_{01}^* c_{jk} & \text{if } |k| > N, \\ -c_{j,\tau(k)} & \text{if } |k| = N. \end{cases}$$

Moreover, $|e^{ijx} - 1| \leq \pi M^{-1}|j|$. Based on these, we get

$$(2.3) \quad \Sigma_{12}^{0\delta} \leq \sup_{\nu \geq 1} \left(4\pi\nu \sum_{|j| \leq m} \sum_{|k|=\nu} |\Delta_{01}^* c_{jk}| \right) \quad (\delta = 0, 1),$$

$$\begin{aligned}
 (2.4) \quad \Sigma_{12}^{\delta} &\leq \sup_{\nu \geq 1} \left(\frac{4\pi^2 \nu}{M} \sum_{|j| \leq m} \sum_{|k|=\nu}^{\infty} \chi_M^1(j) |j \Delta_{01}^* c_{jk}| \right) \\
 &\leq \sup_{\mu, \nu \geq 1} \left(4\pi^2 \mu \nu \sum_{|j|=\mu}^{\infty} \sum_{|k|=\nu}^{\infty} |\Delta_{11}^* c_{jk}| \right) \quad (\delta = 0, 1).
 \end{aligned}$$

Putting (2.3)–(2.4) together gives

$$\begin{aligned}
 (2.5) \quad \Sigma_{12} &\leq \sup_{\nu \geq 1} \left(8\pi \nu \sum_{j=-\infty}^{\infty} \sum_{|k|=\nu}^{\infty} |\Delta_{01}^* c_{jk}| \right) \\
 &\quad + \sup_{\mu, \nu \geq 1} \left(8\pi^2 \mu \nu \sum_{|j|=\mu}^{\infty} \sum_{|k|=\nu}^{\infty} |\Delta_{11}^* c_{jk}| \right).
 \end{aligned}$$

Like Σ_{12} , the term Σ_{21} satisfies

$$\begin{aligned}
 (2.6) \quad \Sigma_{21} &\leq \sup_{\mu \geq 1} \left(8\pi \mu \sum_{|j|=\mu}^{\infty} \sum_{k=-\infty}^{\infty} |\Delta_{10}^* c_{jk}| \right) \\
 &\quad + \sup_{\mu, \nu \geq 1} \left(8\pi^2 \mu \nu \sum_{|j|=\mu}^{\infty} \sum_{|k|=\nu}^{\infty} |\Delta_{11}^* c_{jk}| \right).
 \end{aligned}$$

It remains to estimate Σ_{22} . With the help of [C1, Lemma 3], we get $\Sigma_{22} \leq \Sigma_{22}^{00} + \Sigma_{22}^{10} + \Sigma_{22}^{01} + \Sigma_{22}^{11}$, where

$$\begin{aligned}
 \Sigma_{22}^{00} &\equiv \left| \sum_{|j|=0 \pm}^m \sum_{|k|=0 \pm}^n \Delta_{11}^* (\chi_M^2(j) \chi_N^2(k) c_{jk}) \Psi_j(x) \Psi_k(y) \right|, \\
 \Sigma_{22}^{01} &\equiv \left| \sum_{|j|=0 \pm}^m \sum_{|k|=n} \Delta_{10}^* (\chi_M^2(j) \chi_N^2(\tau(k)) c_{j, \tau(k)}) \Psi_j(x) \Psi_k(y) \right|, \\
 \Sigma_{22}^{10} &\equiv \left| \sum_{|j|=m} \sum_{|k|=0 \pm}^n \Delta_{01}^* (\chi_M^2(\tau(j)) \chi_N^2(k) c_{\tau(j), k}) \Psi_j(x) \Psi_k(y) \right|, \\
 \Sigma_{22}^{11} &\equiv \left| \sum_{|j|=m} \sum_{|k|=n} \chi_M^2(\tau(j)) \chi_N^2(\tau(k)) c_{\tau(j), \tau(k)} \Psi_j(x) \Psi_k(y) \right|.
 \end{aligned}$$

A similar argument to (2.3)–(2.4) gives

$$\Sigma_{22}^{\gamma \delta} \leq \sup_{\mu, \nu \geq 1} \left(16\pi^2 \mu \nu \sum_{|j|=\mu}^{\infty} \sum_{|k|=\nu}^{\infty} |\Delta_{11}^* c_{jk}| \right) \quad (\gamma, \delta = 0, 1),$$

which implies

$$(2.7) \quad \Sigma_{22} \leq \sup_{\mu, \nu \geq 1} \left(64\pi^2 \mu \nu \sum_{|j|=\mu}^{\infty} \sum_{|k|=\nu}^{\infty} |\Delta_{11}^* c_{jk}| \right).$$

Putting (2.1)–(2.2) and (2.5)–(2.7) together, we obtain the desired result.

THEOREM 2.2. *If conditions (1.2) and (1.4)–(1.7) are satisfied, then s_{mn} converges uniformly on T^2 to some continuous function f as $\min(m, n) \rightarrow \infty$. Moreover,*

$$\begin{aligned}
 (2.8) \quad \|s_{mn} - f\|_{\infty} &\leq \sup_{\mu, \nu \geq 0} \left| \sum_{|j| \leq \mu} \sum_{|k| \leq \nu} \chi_{mn}(|j|, |k|) c_{jk} \right| \\
 &\quad + \sup_{\mu \geq 1} \left(18\pi \mu \sum_{|j|=\mu}^{\infty} \sum_{k=-\infty}^{\infty} \chi_{mn}(|j|, |k|) |\Delta_{10}^* c_{jk}| \right) \\
 &\quad + \sup_{\nu \geq 1} \left(18\pi \nu \sum_{j=-\infty}^{\infty} \sum_{|k|=\nu}^{\infty} \chi_{mn}(|j|, |k|) |\Delta_{01}^* c_{jk}| \right) \\
 &\quad + \sup_{\mu, \nu \geq 1} \left(324\pi^2 \mu \nu \sum_{|j|=\mu}^{\infty} \sum_{|k|=\nu}^{\infty} \chi_{mn}(|j|, |k|) |\Delta_{11}^* c_{jk}| \right),
 \end{aligned}$$

where χ_{mn} is defined by (1.8).

Proof. Let λ_{mn} be on the right of (2.8). Then (1.4)–(1.7) imply that $\lambda_{mn} \rightarrow 0$ as $\min(m, n) \rightarrow \infty$. Set $d_{jk} = \chi_{mn}(|j|, |k|) c_{jk}$. For $M > \max(m, n)$, it follows from Lemma 2.1 that

$$\begin{aligned}
 \|s_{mn} - s_{MM}\|_{\infty} &\leq \left\| \sum_{|j| \leq M} \sum_{|k| \leq M} d_{jk} e^{i(jx+ky)} \right\|_{\infty} \\
 &\leq \sup_{\mu, \nu \leq M} \left| \sum_{|j| \leq \mu} \sum_{|k| \leq \nu} d_{jk} \right| + \sup_{\mu \geq 1} \left(9\pi \mu \sum_{|j|=\mu}^{\infty} \sum_{k=-\infty}^{\infty} |\Delta_{10}^* d_{jk}| \right) \\
 &\quad + \sup_{\nu \geq 1} \left(9\pi \nu \sum_{j=-\infty}^{\infty} \sum_{|k|=\nu}^{\infty} |\Delta_{01}^* d_{jk}| \right) \\
 &\quad + \sup_{\mu, \nu \geq 1} \left(81\pi^2 \mu \nu \sum_{|j|=\mu}^{\infty} \sum_{|k|=\nu}^{\infty} |\Delta_{11}^* d_{jk}| \right) \leq \lambda_{mn}.
 \end{aligned}$$

Thus, $\{s_{MM}\}$ forms a Cauchy sequence in $C(T^2)$. Let f be its limit in $C(T^2)$. Then

$$\|s_{mn} - f\|_{\infty} = \lim_{M \rightarrow \infty} \|s_{mn} - s_{MM}\|_{\infty} \leq \lambda_{mn},$$

which is (2.8). The desired result follows from this.

It is clear that

$$\begin{aligned} \sup_{\mu \geq 1} \left(\mu \sum_{|j|=\mu} \sum_{k=-\infty}^{\infty} \dots |\Delta_{10}c_{jk}| \right) &\leq \sum_{|j|=1}^{\infty} \sum_{k=-\infty}^{\infty} \dots |j\Delta_{10}c_{jk}|, \\ \sup_{\nu \geq 1} \left(\nu \sum_{j=-\infty}^{\infty} \sum_{|k|=\nu} \dots |\Delta_{01}c_{jk}| \right) &\leq \sum_{j=-\infty}^{\infty} \sum_{|k|=1}^{\infty} \dots |k\Delta_{01}c_{jk}|, \\ \sup_{\mu, \nu \geq 1} \left(\mu\nu \sum_{|j|=\mu} \sum_{|k|=\nu} \dots |\Delta_{11}c_{jk}| \right) &\leq \sum_{|j|=1}^{\infty} \sum_{|k|=1}^{\infty} \dots |jk\Delta_{11}c_{jk}|. \end{aligned}$$

Hence, Theorem 2.2 has the following consequence.

COROLLARY 2.3. *Assume that conditions (1.2), (1.4), and the following conditions are satisfied:*

$$(2.9) \quad \sum_{j=-\infty}^{\infty} \sum_{k=-\infty}^{\infty} |j\Delta_{10}c_{jk}| < \infty,$$

$$(2.10) \quad \sum_{j=-\infty}^{\infty} \sum_{k=-\infty}^{\infty} |k\Delta_{01}c_{jk}| < \infty,$$

$$(2.11) \quad \sum_{j=-\infty}^{\infty} \sum_{k=-\infty}^{\infty} |jk\Delta_{11}c_{jk}| < \infty.$$

Then the conclusions of Theorem 2.2 hold.

3. Single trigonometric series. Denote by $s_n(t)$ the n th partial sum of the single trigonometric series $\sum_{k=-\infty}^{\infty} c_k e^{ikt}$. To modify the proofs of Lemma 2.1 and Theorem 2.2, we get the following one-dimensional analogue of Theorem 2.2, which corresponds to the case where $c_{j,k} = 0$ for $j \neq 0$. In this case, (1.4) is replaced by (1.9), and (1.10) takes the place of (1.5)–(1.7). We leave the proof to the reader.

THEOREM 3.1. *Let $\{c_k\}_{k=-\infty}^{\infty}$ be a null sequence of complex numbers. If conditions (1.9)–(1.10) are satisfied, then s_n converges uniformly on T to some continuous function f . Moreover,*

$$\|s_n - f\|_{\infty} \leq \sup_{\nu > n} \left| \sum_{n < k \leq \nu} (c_k + c_{-k}) \right| + \sup_{\nu > n} \left(18\pi\nu \sum_{|k|=\nu} |\Delta c_k| \right).$$

It is clear that condition (1.9) is weaker than the following condition:

$$(3.1) \quad \sum_{k=1}^{\infty} |c_k + c_{-k}| < \infty.$$

To replace (1.9) by (3.1), we get the following analogue of Theorem 3.1. In this case, condition (1.11) takes the place of (1.10). In §4, we shall relate this result to O -regularly varying quasimonotone sequences.

THEOREM 3.2. *Let $\{c_k\}_{k=-\infty}^{\infty}$ be a null sequence of complex numbers. If conditions (1.11) and (3.1) are satisfied, then s_n converges uniformly on T to some continuous function f . Moreover,*

$$\|s_n - f\|_{\infty} \leq \sum_{k > n} |c_k + c_{-k}| + \sup_{\nu > n} \left(18\pi\nu \sum_{k=\nu}^{\infty} |\Delta c_k| \right).$$

Proof. The proof is similar to that of Lemma 2.1 and Theorem 2.2. Let $|t| \leq \pi$. Define $N = \max([1/|t|], 1)$. For $m > n$, we have

$$(3.2) \quad |s_n(t) - s_m(t)| = \left| \sum_{n < |k| \leq m} c_k e^{ikt} \right| = \left| \sum_{n < k \leq m} (c_k + c_{-k}) e^{-ikt} + \sum_{n < k \leq m} c_k (e^{ikt} - e^{-ikt}) \right| \leq \sum_{n < k \leq m} |c_k + c_{-k}| + \Sigma_1 + \Sigma_2,$$

where

$$\Sigma_{\alpha} \equiv \left| \sum_{n < k \leq m} \chi_N^{\alpha}(k) c_k (e^{ikt} - e^{-ikt}) \right|$$

and χ_N^1 and χ_N^2 denote the characteristic functions $\chi_{[-N, N]}$ and $\chi_{\mathbb{R} \setminus [-N, N]}$, respectively. Since $|e^{ikt} - e^{-ikt}| \leq 2|kt| \leq 2\pi|k|N^{-1}$, we have

$$(3.3) \quad \Sigma_1 \leq \frac{2\pi}{N} \left(\sum_{n < k \leq m} \chi_N^1(k) |kc_k| \right) \leq \sup_{\nu > n} \left(2\pi\nu \sum_{k=\nu}^{\infty} |\Delta c_k| \right).$$

For Σ_2 , summation by parts yields

$$(3.4) \quad \begin{aligned} \Sigma_2 &\leq \sum_{n < k \leq m} |\Delta(\chi_N^2(k)c_k)| \cdot |\Psi_k(t) - \Psi_{-k}(t)| \\ &\quad + \left| \left\{ \sum_{k=n^*}^{\infty} - \sum_{k=n^*}^{\infty} \right\} \chi_N^2(\tau(k)) c_{\tau(k)} (\Psi_k(t) - \Psi_{-k}(t)) \right| \\ &\leq \sup_{\nu > n} \left(16\pi\nu \sum_{k=\nu}^{\infty} |\Delta c_k| \right), \end{aligned}$$

where $n^* = n$ for $n \geq 1$ and $n^* = 0 \pm$ for $n = 0$. Putting (3.2)–(3.4) together gives

$$\|s_n - s_m\|_{\infty} \leq \sum_{k > n} |c_k + c_{-k}| + \sup_{\nu > n} \left(18\pi\nu \sum_{k=\nu}^{\infty} |\Delta c_k| \right).$$

The desired result follows from this inequality, (1.11), and (3.1).

Write $\sum_{k=-\infty}^{\infty} c_k e^{ikt}$ in the form $c_0 + \sum_{k=1}^{\infty} (a_k \cos kt + b_k \sin kt)$. Then $a_k = c_k + c_{-k}$, $b_k = i(c_k - c_{-k})$, and

$$s_n(t) = c_0 + \sum_{k=1}^n (a_k \cos kt + b_k \sin kt) \quad (n \geq 1).$$

In this notation, condition (3.1) becomes

$$(3.5) \quad \sum_{k=1}^{\infty} |a_k| < \infty.$$

If a_k and b_k are real, then

$$\sum_{k=n}^{\infty} |\Delta c_k| = \frac{1}{2} \sum_{k=n}^{\infty} |\Delta a_k - i \Delta b_k| \geq \frac{1}{2} \max \left(\sum_{k=n}^{\infty} |\Delta a_k|, \sum_{k=n}^{\infty} |\Delta b_k| \right).$$

This indicates that condition (1.11) implies both of the following two conditions:

$$(3.6) \quad \sum_{k=n}^{\infty} |\Delta a_k| = o(1/n) \quad (\text{as } n \rightarrow \infty),$$

$$(3.7) \quad \sum_{k=n}^{\infty} |\Delta b_k| = o(1/n) \quad (\text{as } n \rightarrow \infty).$$

In the following, we shall extend Theorem 3.2 from (1.11) to (3.7).

THEOREM 3.3. *Let $\{a_k\}_{k=1}^{\infty}$ and $\{b_k\}_{k=1}^{\infty}$ be null sequences of complex numbers. If conditions (3.5) and (3.7) are satisfied, then s_n converges uniformly on T to some continuous function f . Moreover,*

$$\|s_n - f\|_{\infty} \leq \sum_{k>n} |a_k| + \sup_{\nu>n} \left(9\pi\nu \sum_{k=\nu}^{\infty} |\Delta b_k| \right).$$

Proof. Let $s_n^1(t)$ and $s_n^2(t)$ denote the n th partial sums of $c_0 + \sum_{k=1}^{\infty} a_k \cos kt$ and $\sum_{k=1}^{\infty} b_k \sin kt$, respectively. Condition (3.5) and the Weierstrass M-test ensure the existence of $f_1 \in C(T)$ such that $s_n^1 \rightarrow f_1$ uniformly on T . Moreover, $\|s_n^1 - f_1\|_{\infty} \leq \sum_{k>n} |a_k|$. Applying Theorem 3.2 to the sine series, we find $f_2 \in C(T)$ such that $s_n^2 \rightarrow f_2$ uniformly on T , and

$$\|s_n^2 - f_2\|_{\infty} \leq \sup_{\nu>n} \left(9\pi\nu \sum_{k=\nu}^{\infty} |\Delta b_k| \right).$$

Therefore, $f \equiv f_1 + f_2 \in C(T)$, $s_n = s_n^1 + s_n^2$ converges uniformly on T to

f , and

$$\begin{aligned} \|s_n - f\|_{\infty} &\leq \|s_n^1 - f_1\|_{\infty} + \|s_n^2 - f_2\|_{\infty} \\ &\leq \sum_{k>n} |a_k| + \sup_{\nu>n} \left(9\pi\nu \sum_{k=\nu}^{\infty} |\Delta b_k| \right). \end{aligned}$$

To end this section, we give an example to distinguish our results from the Weierstrass M-test. Let $c_0 = c_1 = c_{-1} = 0$, and for $k \geq 2$, $c_k = 1/(k \ln k)$, $c_{-k} = -1/(k \ln k)$. This example satisfies conditions (1.9)–(1.11) and (3.1). Hence, Theorems 3.1 and 3.2 apply for this case. However, $\sum_{k=-\infty}^{\infty} |c_k| = \infty$, so the Weierstrass M-test fails.

4. Application to O -regularly varying quasimonotone sequences. In this section, we shall relate Theorem 3.2 to O -regularly varying quasimonotone sequences. Our result generalizes [CJ, J, N, XZ].

A sequence $\{R(n)\}_{n=0}^{\infty}$ of positive numbers is said to be O -regularly varying if it is nondecreasing and for some $\lambda > 1$,

$$(4.1) \quad \limsup_{n \rightarrow \infty} \frac{R([\lambda n])}{R(n)} < \infty,$$

in other words,

$$\sup_n \frac{R([\lambda n])}{R(n)} < \infty.$$

This generalizes the concept of regularly varying sequences introduced in Karamata [K]. Set $\lambda_{1n} = [\lambda n]$, and define $\lambda_{\nu n} = [\lambda \lambda_{\nu-1, n}]$ for $\nu \geq 2$. For $\lambda^* > \lambda > 1$, choose ν so large that

$$\frac{\lambda^* n}{\lambda^{\nu}} + \frac{1}{\lambda^{\nu-1}} + \dots + \frac{1}{\lambda} \leq n \quad \left(\text{for all } n \geq \frac{2}{\lambda-1} \right).$$

Then for $n \geq 2/(\lambda-1)$, we have $[\lambda^* n] \leq \lambda_{\nu n}$, and so

$$\frac{R([\lambda n])}{R(n)} \leq \frac{R([\lambda^* n])}{R(n)} \leq \frac{R(\lambda_{1n})}{R(n)} \cdot \frac{R(\lambda_{2n})}{R(\lambda_{1n})} \cdots \frac{R(\lambda_{\nu n})}{R(\lambda_{\nu-1, n})}.$$

Based on this inequality, we see that if condition (4.1) holds for some $\lambda > 1$, then it is satisfied by all $\lambda > 1$.

As defined in [XZ], the sequence $\{c_n\}_{n=0}^{\infty}$ is said to be O -regularly varying quasimonotone if for some $\theta_0 \in [0, \pi/2)$ and some O -regularly varying sequence $\{R(n)\}_{n=0}^{\infty}$, the following relation holds:

$$(4.2) \quad \Delta(c_n/R(n)) \in K(\theta_0) \equiv \{z \in \mathbb{C} : |\arg z| \leq \theta_0\} \quad (\text{for all } n).$$

We have

$$\Delta c_n + \left(\frac{R(n+1)}{R(n)} - 1 \right) c_n = R(n+1) \left(\Delta \frac{c_n}{R(n)} \right).$$

Hence, condition (4.2) is equivalent to

$$(4.3) \quad \Delta c_n + \left(\frac{R(n+1)}{R(n)} - 1 \right) c_n \in K(\theta_0) \quad (\text{for all } n).$$

Notice that the quasimonotone sequences defined in Szász [S] correspond to the case of $R(n) = n^\alpha$.

LEMMA 4.1. *Let $\{c_n\}_{n=0}^\infty$ be an O -regularly varying quasimonotone null sequence. Assume that θ_0 and $\{R(n)\}_{n=0}^\infty$ are the corresponding angle and O -regularly varying sequence. Then the following assertions hold:*

- (i) $|c_n| \leq (\sec \theta_0) \operatorname{Re} c_n$ for all n ;
- (ii) $\{(\operatorname{Re} c_n)/R(n)\}_{n=0}^\infty$ is nonnegative and nonincreasing;
- (iii) $\left| \Delta c_n + \left(\frac{R(n+1)}{R(n)} - 1 \right) c_n \right| \leq (\sec \theta_0) \operatorname{Re} \left\{ \Delta c_n + \left(\frac{R(n+1)}{R(n)} - 1 \right) c_n \right\}$ for all n .

PROOF. (i) was proved in [XZ, Lemma 1]. For (ii), we have $\Delta(c_n/R(n)) \in K(\theta_0)$, so

$$\frac{\operatorname{Re} c_n}{R(n)} - \frac{\operatorname{Re} c_{n+1}}{R(n+1)} = \operatorname{Re} \left(\Delta \frac{c_n}{R(n)} \right) \geq 0.$$

This indicates that $\{(\operatorname{Re} c_n)/R(n)\}_{n=0}^\infty$ is nonincreasing. On the other hand,

$$|(\operatorname{Re} c_n)/R(n)| \leq |c_n|/R(n) \rightarrow 0 \quad \text{as } n \rightarrow \infty.$$

Hence, $(\operatorname{Re} c_n)/R(n) \geq 0$. This proves (ii). For (iii), we have

$$\Delta c_n + \left(\frac{R(n+1)}{R(n)} - 1 \right) c_n \in K(\theta_0),$$

and so for all n ,

$$\left| \Delta c_n + \left(\frac{R(n+1)}{R(n)} - 1 \right) c_n \right| \leq (\sec \theta_0) \operatorname{Re} \left\{ \Delta c_n + \left(\frac{R(n+1)}{R(n)} - 1 \right) c_n \right\}.$$

This completes the proof.

THEOREM 4.2. *Let $\{c_n\}_{n=0}^\infty$ be an O -regularly varying quasimonotone null sequence. Then condition (1.11) is equivalent to condition (1.12).*

PROOF. Obviously, (1.11) implies (1.12). This follows from the inequality

$$|nc_n| \leq n \left(\sum_{k=n}^\infty |\Delta c_k| \right).$$

Conversely, we assume that (1.12) holds. Choose a positive integer $\lambda > 1$ such that (4.1) holds. Set

$$(4.4) \quad M \equiv \max \left(\sec \theta_0, \sup_n \left| \frac{R(\lambda n)}{R(n)} - 1 \right| \right),$$

where θ_0 is the angle appearing in (4.2). By Lemma 4.1(i), (iii), we get

$$\begin{aligned} |\Delta c_k| &\leq \left| \Delta c_k + \left(\frac{R(k+1)}{R(k)} - 1 \right) c_k \right| + \left| \left(\frac{R(k+1)}{R(k)} - 1 \right) c_k \right| \\ &\leq M \operatorname{Re} \left\{ \Delta c_k + 2 \left(\frac{R(k+1)}{R(k)} - 1 \right) c_k \right\}. \end{aligned}$$

Summing up both sides with respect to k gives

$$(4.5) \quad \sum_{k=n}^\infty |\Delta c_k| \leq M \operatorname{Re} \left(\sum_{k=n}^\infty \Delta c_k \right) + 2M \sum_{k=n}^\infty \left(\frac{R(k+1)}{R(k)} - 1 \right) \operatorname{Re} c_k \leq M \operatorname{Re} c_n + I,$$

where

$$I \equiv 2M \left\{ \sum_{k=n}^\infty \left(\frac{R(k+1)}{R(k)} - 1 \right) \operatorname{Re} c_k \right\}.$$

Since $(\operatorname{Re} c_k)/R(k)$ is nonnegative and nonincreasing,

$$(4.6) \quad \begin{aligned} \frac{I}{2M} &= \sum_{\nu=0}^\infty \left\{ \sum_{\lambda^\nu n \leq k < \lambda^{\nu+1} n} (R(k+1) - R(k)) \frac{\operatorname{Re} c_k}{R(k)} \right\} \\ &\leq \sum_{\nu=0}^\infty \frac{\operatorname{Re} c_{\lambda^\nu n}}{R(\lambda^\nu n)} \left\{ \sum_{\lambda^\nu n \leq k < \lambda^{\nu+1} n} (R(k+1) - R(k)) \right\} \\ &= \sum_{\nu=0}^\infty \operatorname{Re} c_{\lambda^\nu n} \left(\frac{R(\lambda^{\nu+1} n)}{R(\lambda^\nu n)} - 1 \right) \leq M \sum_{\nu=0}^\infty |c_{\lambda^\nu n}| \\ &\leq M (\sup_{\nu \geq n} |c_{\lambda^\nu}|) \left(\sum_{\nu=0}^\infty \frac{1}{\lambda^\nu n} \right). \end{aligned}$$

Putting (1.12), (4.5) and (4.6) together yields

$$\begin{aligned} n \sum_{k=n}^\infty |\Delta c_k| &\leq M |nc_n| + 2M^2 (\sup_{\nu \geq n} |c_{\lambda^\nu}|) \left(\sum_{\nu=0}^\infty \frac{1}{\lambda^\nu} \right) \\ &\rightarrow 0 \quad \text{as } n \rightarrow \infty. \end{aligned}$$

This is (1.11) and the proof is complete.

Combining Theorem 4.2 with Theorem 3.2, we obtain the following result, which generalizes the result [XZ], and hence includes those of Chaundy-Jolliffe [CJ], Jolliffe [J], and Nurcombe [N] as special cases. As indicated in [XZ], condition (1.12) is a necessary condition for O -regularly varying quasimonotone sequences. Thus, (1.11) is also needed for such a case.

COROLLARY 4.3. *Let $\{c_n\}_{n=-\infty}^\infty$ be a null sequence of complex numbers, and suppose that $\{c_n\}_{n=0}^\infty$ is an O -regularly varying quasimonotone sequence.*

If conditions (1.12) and (3.1) are satisfied, then s_n converges uniformly on T to some continuous function f . Moreover,

$$\|s_n - f\|_\infty \leq \sum_{k>n} |c_k + c_{-k}| + 18\pi M^2 \frac{3\lambda - 1}{\lambda - 1} \sup_{\nu>n} |\nu c_\nu|,$$

where $\lambda > 1$ is a positive integer satisfying (4.1) and M is defined by (4.4).

Proof. The first conclusion follows from Theorems 3.2 and 4.2. For the second, Theorem 3.2 gives

$$(4.7) \quad \|s_n - f\|_\infty \leq \sum_{k>n} |c_k + c_{-k}| + \sup_{m>n} \left(18\pi m \sum_{k=m}^{\infty} |\Delta c_k| \right).$$

For $m > n$, the proof of Theorem 4.2 yields

$$(4.8) \quad m \sum_{k=m}^{\infty} |\Delta c_k| \leq M |m c_m| + 2M^2 \left(\sup_{\nu \geq m} |\nu c_\nu| \right) \left(\sum_{\nu=0}^{\infty} \frac{1}{\lambda^\nu} \right) \\ \leq M^2 \frac{3\lambda - 1}{\lambda - 1} \sup_{\nu \geq m} |\nu c_\nu|.$$

Combining (4.7) with (4.8), we get the desired estimate for $\|s_n - f\|_\infty$.

Acknowledgements. The authors express their gratitude to the referee for his valuable comments on the final version of this paper.

References

- [CJ] T. W. Chaundy and A. E. Jolliffe, *The uniform convergence of a certain class of trigonometric series*, Proc. London Math. Soc. (2) 15 (1916), 214–216.
- [C1] C.-P. Chen, *Weighted integrability and L^1 -convergence of multiple trigonometric series*, Studia Math. 108 (1994), 177–190.
- [C2] —, *Integrability of multiple trigonometric series and Parseval's formula*, J. Math. Anal. Appl. 186 (1994), 182–199.
- [CL] C.-P. Chen and C.-C. Lin, *Integrability, mean convergence, and Parseval's formula for double trigonometric series*, preprint.
- [D] M. I. Dyachenko, *The rate of u -convergence of multiple Fourier series*, Acta Math. Hungar. 68 (1995), 55–70.
- [J] A. E. Jolliffe, *On certain trigonometric series which have a necessary and sufficient condition for uniform convergence*, Math. Proc. Cambridge Philos. Soc. 19 (1921), 191–195.
- [K] J. Karamata, *Sur un mode de croissance régulière des fonctions*, Mathematica (Cluj) 4 (1930), 38–53.
- [M1] F. Móricz, *Convergence and integrability of double trigonometric series with coefficients of bounded variation*, Proc. Amer. Math. Soc. 102 (1988), 633–640.
- [M2] —, *On the integrability and L^1 -convergence of double trigonometric series*, Studia Math. 98 (1991), 203–225.

- [M3] F. Móricz, *On the integrability of double cosine and sine series I*, J. Math. Anal. Appl. 154 (1991), 452–465.
- [N] J. R. Nurcombe, *On the uniform convergence of sine series with quasimonotone coefficients*, *ibid.* 166 (1992), 577–581.
- [S] O. Szász, *Quasi-monotone series*, Amer. J. Math. 70 (1948), 203–206.
- [XZ] T. F. Xie and S. P. Zhou, *The uniform convergence of certain trigonometric series*, J. Math. Anal. Appl. 181 (1994), 171–180.
- [Z] A. Zygmund, *Trigonometric Series*, 2nd ed., Cambridge Univ. Press, Cambridge, 1968.

Department of Mathematics
National Tsing Hua University
Hsinchu, Taiwan 30043, Republic of China
E-mail: cpchen@math.nthu.edu.tw

Received August 7, 1995
Revised version December 11, 1995

(3513)