

## UNIFORM CONVEXITY AND ASSOCIATE SPACES

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*Abstract.* We prove that the associate space of a generalized Orlicz space  $L^{\varphi(\cdot)}$  is given by the conjugate modular  $\varphi^*$  even without the assumption that simple functions belong to the space. Second, we show that every weakly doubling  $\Phi$ -function is equivalent to a doubling  $\Phi$ -function. As a consequence, we conclude that  $L^{\varphi(\cdot)}$  is uniformly convex if  $\varphi$  and  $\varphi^*$  are weakly doubling.

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## 1. INTRODUCTION

Generalized Orlicz spaces  $L^{\varphi(\cdot)}$  have been studied since the 1940s. A major synthesis of functional analysis in these spaces, based on work, e.g. of Hudzik, Kamińska and Musielak, is given in the monograph [16]. Following ideas of Maeda, Mizuta, Ohno and Shimomura (e.g. [15]), we have studied these spaces from a point-of-view which emphasizes the possibility of choosing the  $\Phi$ -function generating the norm in the space appropriately [5], [9], [10], [12]. From this perspective, some classical concepts, like convexity of the  $\Phi$ -function, are too rigid.

Renewed interest in the topic has arisen recently from studies of PDE with non-standard growth, including the variable exponent case  $\varphi(x, t) = t^{p(x)}$  and the double phase case  $\varphi(x, t) = t^p + a(x)t^q$ . Such problems have been studied e.g. in [2], [3], [4], [8], [17]. For a detailed motivation of our context and additional references we refer to the introduction of [11].

In this note, we tie up some loose ends concerning the basic functional analysis of generalized Orlicz spaces in our monograph [6]. In the book we relied on the assumption that all simple functions belong to our space. This excludes for instance

the case  $\varphi(x, t) := |x|^{-n}t^2$ , where  $n$  is the dimension. We can now remove this assumption from the following result (cf. [6], Theorem 2.7.4). For simplicity, we consider only the Lebesgue measure on subsets of  $\mathbb{R}^n$ . See the next sections for definitions.

**Theorem 1.1.** *Let  $A \subset \mathbb{R}^n$  be measurable. If  $\varphi \in \Phi_w(A)$ , then  $(L^\varphi)' = L^{\varphi^*}$ , i.e. for all measurable  $f: A \rightarrow \mathbb{R}$*

$$\|f\|_{\varphi(\cdot)} \approx \sup_{\|g\|_{\varphi^*(\cdot)} \leq 1} \int_A |f(x)g(x)| \, dx.$$

The proof relies among other things on upgrading the weak  $\Phi$ -function to a strong  $\Phi$ -function based on our earlier work. The next result is of the same type, upgrading weak doubling to strong doubling.

**Theorem 1.2.** *Let  $A \subset \mathbb{R}^n$  be measurable. If  $\varphi \in \Phi_w(A)$  satisfies  $\Delta_2^w$  and  $\nabla_2^w$ , then there exists  $\psi \in \Phi_w(A)$  with  $\varphi \sim \psi$  satisfying  $\Delta_2$  and  $\nabla_2$ .*

Recall that a vector space  $X$  is *uniformly convex* if it has a norm  $\|\cdot\|$  such that for every  $\varepsilon > 0$  there exists  $\delta > 0$  with

$$\|x - y\| \geq \varepsilon \quad \text{or} \quad \|x + y\| \leq 2(1 - \delta)$$

for all unit vectors  $x$  and  $y$ . In the Orlicz case, it is well known that the space  $L^\varphi$  is reflexive and uniformly convex if and only if  $\varphi$  and  $\varphi^*$  are doubling [18], Theorem 2, page 297. Hudzik in [13] showed in 1983 that the same conditions are sufficient for uniform convexity (see also [7], [14]). With the equivalence technique, we are able to give a very simple proof of this result.

**Theorem 1.3.** *Let  $A \subset \mathbb{R}^n$  be measurable and  $\varphi \in \Phi_w(A)$ . If  $\varphi$  satisfies  $\Delta_2^w$  and  $\nabla_2^w$ , then  $L^{\varphi(\cdot)}$  is uniformly convex and reflexive.*

## 2. $\Phi$ -FUNCTIONS

By  $A \subset \mathbb{R}^n$  we denote a measurable set. The notation  $f \lesssim g$  means that there exists a constant  $C > 0$  such that  $f \leq Cg$ . The notation  $f \approx g$  means that  $f \lesssim g \lesssim f$ . By  $c$  we denote a generic constant whose value may change between appearances. A function  $f$  is *almost increasing* if there exists a constant  $L \geq 1$  such that  $f(s) \leq Lf(t)$  for all  $s \leq t$  (abbreviated  $L$ -almost increasing). *Almost decreasing* is defined analogously.

**Definition 2.1.** We say that  $\varphi: A \times [0, \infty) \rightarrow [0, \infty]$  is a *weak  $\Phi$ -function*, and write  $\varphi \in \Phi_w(A)$ , if the following conditions hold:

- ▷ For every  $t \in [0, \infty)$  the function  $x \mapsto \varphi(x, t)$  is measurable and for every  $x \in A$  the function  $t \mapsto \varphi(x, t)$  is non-decreasing and left-continuous.
- ▷  $\varphi(x, 0) = \lim_{t \rightarrow 0^+} \varphi(x, t) = 0$  and  $\lim_{t \rightarrow \infty} \varphi(x, t) = \infty$  for every  $x \in A$ .
- ▷ The function  $t \mapsto \varphi(x, t)/t$  is  $L$ -almost increasing for  $t > 0$  uniformly in  $A$ . “Uniformly” means that  $L$  is independent of  $x$ .

If  $\varphi \in \Phi_w(A)$  is convex, then it is called a  *$\Phi$ -function*, and we write  $\varphi \in \Phi(A)$ . If  $\varphi \in \Phi(A)$  is continuous as a function into the extended real line  $[0, \infty]$ , then it is a *strong  $\Phi$ -function*, and we write  $\varphi \in \Phi_s(A)$ .

We say that  $\varphi, \psi \in \Phi_w(A)$  are *weakly equivalent*,  $\varphi \sim \psi$ , if there exist  $D > 1$  and  $h \in L^1(A)$  such that

$$\varphi(x, t) \leq \psi(x, Dt) + h(x) \quad \text{and} \quad \psi(x, t) \leq \varphi(x, Dt) + h(x).$$

Two functions  $\varphi$  and  $\psi$  are *equivalent*,  $\varphi \simeq \psi$ , if the previous conditions hold with  $h \equiv 0$ . Note that  $\varphi \sim \psi$  if and only if  $L^{\varphi(\cdot)} = L^{\psi(\cdot)}$ . In the case  $\varphi, \psi \in \Phi$ , this has been proved in [6], Theorem 2.8.1. For the weak  $\Phi$ -functions the proof is the same.

We define the *doubling condition*  $\Delta_2$  and the *weak doubling condition*  $\Delta_2^w$  by

$$\varphi(x, 2t) \lesssim \varphi(x, t), \quad \varphi(x, 2t) \lesssim \varphi(x, t) + h(x),$$

respectively, where  $h \in L^1$  and the implicit constant are independent of  $x$ . If  $\varphi \in \Phi_w(A)$ , then we define a conjugate  $\Phi$ -function by

$$\varphi^*(x, t) := \sup_{s \geq 0} (st - \varphi(x, s)).$$

We say that  $\varphi$  satisfies  $\nabla_2$  or  $\nabla_2^w$  if  $\varphi^*$  satisfies  $\Delta_2$  or  $\Delta_2^w$ , respectively. All these assumptions are invariant under equivalence,  $\simeq$ , of  $\Phi$ -functions.

In some situations, it is useful to have a more quantitative version of the  $\Delta_2$  and  $\nabla_2$  conditions. It can be shown that (aDec) is equivalent to  $\Delta_2$  and (aInc) to  $\nabla_2$  (cf. [11], Lemma 2.6, and [5], Proposition 3.6), where (aInc) and (aDec) means the following:

- (aInc) There exist  $\gamma^- > 1$  and  $L \geq 1$  such that  $t \mapsto \varphi(x, t)/t^{\gamma^-}$  is  $L$ -almost increasing in  $(0, \infty)$ .
- (aDec) There exist  $\gamma^+ > 1$  and  $L \geq 1$  such that  $t \mapsto \varphi(x, t)/t^{\gamma^+}$  is  $L$ -almost decreasing in  $(0, \infty)$ .

Note that the optimal  $\gamma^-$  and  $\gamma^+$  correspond to the lower and upper Matuszewska-Orlicz indexes, respectively.

Let us start by showing that weak doubling can be upgraded to strong doubling via weak equivalence of  $\Phi$ -functions. For this we will use the *left-inverse* of a weak  $\Phi$ -function, defined by the formula

$$\varphi^{-1}(x, \tau) := \inf\{t > 0: \varphi(x, t) \geq \tau\}.$$

We point out that if  $\varphi \in \Phi_s(\Omega)$ , then by [9], page 4, we have for every  $t$  that

$$(2.1) \quad \varphi(x, \varphi^{-1}(x, t)) = t.$$

**P r o o f** of Theorem 1.2. By [10], Proposition 2.3, we may assume without loss of generality that  $\varphi \in \Phi_s(A)$ . By assumption,

$$\varphi(x, 2t) \leq D\varphi(x, t) + h(x), \quad \varphi^*(x, 2t) \leq D\varphi^*(x, t) + h(x)$$

for some  $D > 2$ ,  $h \in L^1$  and all  $x \in A$  and  $t \geq 0$ . Using  $\varphi = \varphi^{**}$  (see [6], Corollary 2.6.3), and the definition of the conjugate  $\Phi$ -function, we obtain from the second inequality that

$$\begin{aligned} \varphi(x, 2t) &= \sup_{u \geq 0} (2tu - \varphi^*(x, u)) \leq \sup_{u \geq 0} \left( 2tu - \frac{1}{D}(\varphi^*(x, 2u) - h(x)) \right) \\ &= \sup_{u \geq 0} \left( 2tu - \frac{1}{D}\varphi^*(x, 2u) \right) + \frac{1}{D}h(x) = \frac{1}{D} \sup_{u \geq 0} (Dt2u - \varphi^*(x, 2u)) + \frac{1}{D}h(x) \\ &= \frac{1}{D}\varphi(x, Dt) + \frac{1}{D}h(x). \end{aligned}$$

Define  $t_x := \varphi^{-1}(x, h(x))$  and suppose that  $t \geq t_x$  so that  $h(x) \leq \varphi(x, t)$ . By convexity, we conclude that  $Dh(x) \leq D\varphi(x, t) \leq \varphi(x, Dt)$ . Hence in the case  $t \geq t_x$  we have

$$\varphi(x, 2t) \leq (D+1)\varphi(x, t), \quad \varphi(x, 2t) \leq \frac{D+1}{D^2}\varphi(x, Dt).$$

Let  $p := \log_2(D+1)$  and

$$q := \frac{\log(D^2/(D+1))}{\log(D/2)}.$$

Note that  $q > 1$  since  $D^2/(D+1) > D/2$ . Divide the first inequality by  $(2t)^p$  and the second one by  $(2t)^q$ :

$$\begin{aligned} \frac{\varphi(x, 2t)}{(2t)^p} &\leq \frac{D+1}{2^p} \frac{\varphi(x, t)}{t^p} = \frac{\varphi(x, t)}{t^p}, \\ \frac{\varphi(x, 2t)}{(2t)^q} &\leq \frac{(D+1)D^q}{D^2 2^q} \frac{\varphi(x, Dt)}{(Dt)^q} = \frac{\varphi(x, Dt)}{(Dt)^q}. \end{aligned}$$

Let  $s > t \geq t_x$ . Then there exists  $k \in \mathbb{N}$  such that  $2^k t < s \leq 2^{k+1} t$ . Hence

$$\frac{\varphi(x, s)}{s^p} \leq \frac{\varphi(x, 2^{k+1}t)}{(2^{k+1}t)^p} = 2^p \frac{\varphi(x, 2^k t)}{(2^k t)^p} \leq 2^p \frac{\varphi(x, 2^k t)}{(2^k t)^p} \leq \dots \leq 2^p \frac{\varphi(x, t)}{t^p},$$

so  $\varphi$  satisfies (aDec) with  $\gamma^+ = p$  for  $t \geq t_x$ . Similarly, we find that  $\varphi$  satisfies (aInc) with  $\gamma^- = q$  for  $t \geq t_x$ .

Define

$$\psi(x, t) := \begin{cases} \varphi(x, t) & \text{for } t \geq t_x, \\ c_x t^2 & \text{otherwise,} \end{cases}$$

where  $c_x$  is chosen so that  $\psi$  is continuous at  $t_x$ . Then  $\psi$  satisfies (aDec) on  $[0, t_x]$  and  $[t_x, \infty)$ , hence on the whole real axis with  $\gamma^+ = \max\{p, 2\}$ , similarly for (aInc) with  $\gamma^- = \min\{q, 2\}$ .

Furthermore,  $\varphi(x, t) = \psi(x, t)$  when  $t \geq t_x$ , and so it follows that  $|\varphi(x, t) - \psi(x, t)| \leq \varphi(x, t_x) = h(x)$ , where (2.1) is used for the last step. Since  $h \in L^1$ , this means that  $\varphi \sim \psi$ , so  $\psi$  is the required function.  $\square$

**Remark 2.2.** From the proof of the previous theorem, we see that the two conditions are not interdependent, i.e. if  $\varphi \in \Phi_w(A)$  satisfies  $\Delta_2^w$ , then there exists  $\psi \in \Phi_w(A)$  with  $\varphi \sim \psi$  satisfying  $\Delta_2$ ; similarly for only  $\nabla_2^w$  and  $\nabla_2$ .

### 3. ASSOCIATE SPACES

We denote by  $L^0(A)$  the set of measurable functions in  $A$ .

**Definition 3.1.** Let  $\varphi \in \Phi_w(A)$  and define the *modular*  $\varrho_{\varphi(\cdot)}$  for  $f \in L^0(A)$  by

$$\varrho_{\varphi(\cdot)}(f) := \int_A \varphi(x, |f(x)|) dx.$$

The *generalized Orlicz space*, also called Musielak-Orlicz space, is defined as the set

$$L^{\varphi(\cdot)}(A) := \{f \in L^0(A) : \lim_{\lambda \rightarrow 0^+} \varrho_{\varphi(\cdot)}(\lambda f) = 0\}$$

equipped with the (Luxemburg) quasinorm

$$\|f\|_{\varphi(\cdot)} := \inf \left\{ \lambda > 0 : \varrho_{\varphi(\cdot)}\left(\frac{f}{\lambda}\right) \leq 1 \right\}.$$

Let us start with a lemma which shows that we can approximate the function 1 with a monotonically increasing sequence of functions in the generalized Orlicz space. Note that the next lemma is trivial if  $L^\infty \subset L^{\varphi(\cdot)}$ , as was assumed in [6] when dealing with associate spaces.

**Lemma 3.2.** *Let  $\varphi \in \Phi_w(A)$ . There exists positive  $h_k \in L^{\varphi(\cdot)}(A)$ ,  $k \in \mathbb{N}$ , such that  $h_k \nearrow 1$  and  $\{h_k = 1\} \nearrow A$ .*

Proof. For  $k \geq 1$  we define

$$E_k := \{x: \varphi(x, 2^{-k}) \leq 1\}.$$

Since  $\varphi(\cdot, t)$  is assumed to be measurable,  $E_k$  is a measurable set. Since  $\lim_{t \rightarrow 0^+} \varphi(x, t) = 0$ , there exists for every  $x \in A$  an index  $k_x$  such that  $x \in E_{k_x}$ . And since  $\varphi$  is non-decreasing, it follows that  $E_k \nearrow A$  as  $k \rightarrow \infty$ . We define

$$h(x) := \sum_{i=0}^{\infty} 2^{-i-1} \chi_{E_i}(x).$$

Then  $h(x) \in (0, 1]$  for every  $x$ , and  $h$  is measurable. Suppose that  $x \in E_{k+1} \setminus E_k$  for some  $k \in \mathbb{N}$ . Then

$$h(x) = \sum_{i=k+1}^{\infty} 2^{-i-1} = 2^{-(k+1)}.$$

Hence, by the definition of  $E_{k+1}$ , we find that  $\varphi(x, h(x)) \leq 1$ . Since  $A = \bigcup_k E_k$ , we have  $\varphi(x, h(x)) \leq 1$  in  $A$ . (The function  $h$  can alternatively be constructed using the left-inverse of  $\varphi$ , as in the previous section.)

Let us define  $h_k := \min\{kh\chi_{B(0,k)\cap A}, 1\}$ . Then

$$\varrho_{\varphi(\cdot)}(k^{-1}h_k) \leq \int_{B(0,k)\cap A} \varphi(x, h) \, dx \leq |B(0, k)| < \infty,$$

so that  $h_k \in L^{\varphi(\cdot)}(A)$ . Since  $h > 0$ , it follows that  $kh\chi_{B(0,k)\cap A} \nearrow \infty$  for every  $x$ , and so  $h_k \nearrow 1$ , as required.  $\square$

We define the *associate space* by  $(L^{\varphi(\cdot)})'(A) := \{f \in L^0(A): \|f\|_{(L^{\varphi(\cdot)})'} < \infty\}$ , where

$$\|f\|_{(L^{\varphi})'} := \sup_{\|g\|_{\varphi(\cdot)} \leq 1} \int_A fg \, dx.$$

If  $g \in (L^{\varphi})'$  and  $f \in L^{\varphi}$ , then  $fg \in L^1$  by the definition of the associate space. In particular, the integral  $\int_A fg \, dx$  is well defined and

$$\left| \int_A fg \, dx \right| \leq \|g\|_{(L^{\varphi})'} \|f\|_{\varphi(\cdot)}.$$

Hölder's inequality holds in generalized Orlicz spaces with constant 2, without restrictions on the  $\Phi_w$ -function ([6], Lemma 2.6.5):

$$(3.1) \quad \int_A |f| |g| \, dx \leq 2 \|f\|_{\varphi(\cdot)} \|g\|_{\varphi^*(\cdot)}.$$

Here  $\varphi^*$  is the conjugate  $\Phi$ -function defined in the previous section. Furthermore, we can define a conjugate modular on the dual space by the formula

$$(\varrho_{\varphi(\cdot)})^*(J) := \sup_{f \in L^{\varphi(\cdot)}} (J(f) - \varrho_{\varphi(\cdot)}(f))$$

for  $J \in (L^{\varphi(\cdot)})^*$ , i.e.  $J: L^{\varphi(\cdot)} \rightarrow \mathbb{R}$  is a bounded linear functional. By  $J_f$  we denote the functional  $g \mapsto \int fg \, dx$ .

*Proof of Theorem 1.1.* We follow the outlines of [6], Theorem 2.7.4, but use Lemma 3.2 to get rid of the extraneous assumption that simple functions belong to the space. The inequality  $\|f\|_{(L^{\varphi})'} \leq 2\|f\|_{\varphi^*(\cdot)}$  follows from (3.1).

Let then  $f \in (L^{\varphi})'$  and  $\varepsilon > 0$ . Let  $\{q_1, q_2, \dots\}$  be an enumeration of non-negative rational numbers with  $q_1 = 0$ . For  $k \in \mathbb{N}$  and  $x \in A$  define

$$r_k(x) := \max_{j=1, \dots, k} q_j |f(x)| - \varphi(x, q_j).$$

The special choice  $q_1 = 0$  implies  $r_k(x) \geq 0$  for all  $x \geq 0$ . Since  $\mathbb{Q}$  is dense in  $[0, \infty)$  and  $\varphi(x, \cdot)$  is left-continuous,  $r_k(x) \nearrow \varphi^*(x, |f(x)|)$  for every  $x \in A$  as  $k \rightarrow \infty$ .

Since  $f$  and  $\varphi(\cdot, t)$  are measurable functions, the sets

$$E_{i,k} := \{x \in A: q_i |f(x)| - \varphi(x, q_i) = \max_{j=1, \dots, k} (q_j |f(x)| - \varphi(x, q_j))\}$$

are measurable. Let  $F_{i,k} := E_{i,k} \setminus (E_{1,k} \cup \dots \cup E_{i-1,k})$ . Define

$$g_k := \sum_{i=1}^k q_i \chi_{F_{i,k}}.$$

Then  $g_k$  is measurable and bounded and

$$r_k(x) = g_k(x) |f(x)| - \varphi(x, g_k(x))$$

for all  $x \in A$ .

Let  $h_k \in L^{\varphi(\cdot)}(A)$  be as in Lemma 3.2, i.e.  $\{h_k = 1\} \nearrow A$  and  $0 < h_k \leq 1$ . Since  $g_k$  is bounded, it follows that  $w := \operatorname{sgn} f h_k g_k \in L^{\varphi(\cdot)}$ . Denote  $E := \{fw \geq \varphi(x, w)\}$ .

Since the conjugate modular is defined as a supremum over functions in  $L^{\varphi(\cdot)}$ , we get a lower bound by using the particular function  $w \chi_E$ . Thus

$$\begin{aligned} (\varrho_{\varphi(\cdot)})^*(J_f) &\geq J_f(w \chi_E) - \varrho_{\varphi(\cdot)}(w \chi_E) = \int_E fw - \varphi(x, w) \, dx \\ &\geq \int_{\{h_k=1\}} g_k |f| - \varphi(x, g_k) \, dx = \int_A r_k \chi_{\{h_k=1\}} \, dx. \end{aligned}$$

Since  $r_k \chi_{\{h_k=1\}} \nearrow \varphi^*(x, |f|)$ , it follows by monotone convergence that  $(\varrho_{\varphi(\cdot)})^*(J_f) \geq \varrho_{\varphi^*(\cdot)}(f)$ . From the definitions of  $(\varrho_{\varphi(\cdot)})^*$  and  $\varrho_{\varphi^*(\cdot)}$ ,

$$(\varrho_{\varphi(\cdot)})^*(J_f) = \sup_{g \in L^{\varphi(\cdot)}} \int_A fg - \varphi(x, g) \, dx \leq \int_A \varphi^*(x, f) \, dx = \varrho_{\varphi^*(\cdot)}(f).$$

Hence  $(\varrho_{\varphi(\cdot)})^*(J_f) = \varrho_{\varphi^*(\cdot)}(f)$ .

Since  $f \mapsto J_f$  is linear, it follows that  $(\varrho_{\varphi(\cdot)})^*(\lambda J_f) = \varrho_{\varphi^*(\cdot)}(\lambda f)$  for every  $\lambda > 0$  and therefore  $\|f\|_{\varphi^*(\cdot)} = \|J_f\|_{(\varrho_{\varphi(\cdot)})^*} \leq \|J_f\|_{(L^{\varphi(\cdot)})^*} = \|f\|_{(L^{\varphi(\cdot)})'}$ , where the second step follows from [6], Theorem 2.2.10.

Taking into account that  $\varphi^{**} \simeq \varphi$ , we have shown that  $L^{\varphi(\cdot)} = (L^{\varphi^*(\cdot)})'$ . By the definition of the associate space norm, this means that

$$\|f\|_{\varphi(\cdot)} \approx \sup_{\|g\|_{\varphi^*(\cdot)} \leq 1} \int |f| |g| \, dx$$

for  $f \in L^{\varphi(\cdot)}$ . In the case  $f \in L^0 \setminus L^{\varphi(\cdot)}$ , we can approximate  $h_k \min\{|f|, k\} \nearrow |f|$  with  $h_k$  as before. Since  $h_k \min\{|f|, k\} \in L^{\varphi(\cdot)}$ , the previous result implies that the formula holds, in the form  $\infty = \infty$ , when  $f \in L^0 \setminus L^{\varphi(\cdot)}$ .  $\square$

#### 4. UNIFORM CONVEXITY

The function  $\varphi \in \Phi_w(\mathbb{R}^n)$  is *uniformly convex* if for every  $\varepsilon > 0$  there exists  $\delta > 0$  such that

$$\varphi\left(x, \frac{s+t}{2}\right) \leq (1-\delta) \frac{\varphi(x, s) + \varphi(x, t)}{2}$$

for every  $x \in \mathbb{R}^n$  whenever  $|s-t| \geq \varepsilon \max\{|s|, |t|\}$ .

**Theorem 4.1.** *The function  $\varphi \in \Phi_w(A)$  is equivalent to a uniformly convex  $\Phi$ -function if and only if it satisfies (aInc).*

**Proof.** Assume first that  $\varphi$  satisfies (aInc) with  $\gamma^- = p > 1$ . By [10], Lemma 2.2, there exists  $\psi \in \Phi(A)$  such that  $\varphi \simeq \psi$  and  $\psi^{1/p}$  is convex for some  $p > 1$ . The claim follows once we show that  $\psi$  is uniformly convex. Let  $\varepsilon \in (0, 1)$  and  $s-t \geq \varepsilon s$  with  $s > t > 0$ . Since  $\psi^{1/p}$  is convex,

$$\psi\left(x, \frac{s+t}{2}\right)^{1/p} \leq \frac{\psi(x, s)^{1/p} + \psi(x, t)^{1/p}}{2}.$$

Since  $t \leq (1-\varepsilon)s$  and  $\psi$  is convex, we find that  $\psi(x, t) \leq \psi(x, (1-\varepsilon)s) \leq (1-\varepsilon)\psi(x, s)$ . Therefore  $\psi(x, t)^{1/p} \leq (1-\varepsilon')\psi(x, s)^{1/p}$  for some  $\varepsilon' > 0$ . Since  $t^p$  is uniformly convex,



we obtain that

$$\left(\frac{\psi(x, s)^{1/p} + \psi(x, t)^{1/p}}{2}\right)^p \leq (1 - \delta) \frac{\psi(x, s) + \psi(x, t)}{2}.$$

Combined with the previous estimate, this shows that  $\psi$  is uniformly convex.

Assume now conversely that  $\varphi \simeq \psi$  and  $\psi$  is uniformly convex. Choose  $\varepsilon = \frac{1}{2}$  and  $t = 0$  in the definition of uniform convexity:

$$\psi(x, s/2) \leq \frac{1}{2}(1 - \delta)\psi(x, s).$$

Divide this equation by  $(s/2)^p$ , where  $p$  is chosen so that  $2^{p-1}(1 - \delta) = 1$ :

$$\frac{\psi(x, s/2)}{(s/2)^p} \leq 2^{p-1}(1 - \delta) \frac{\psi(x, s)}{s^p} = \frac{\psi(x, s)}{s^p}.$$

The previous inequality holds for every  $s > 0$ . If  $0 < t < s$ , then we can choose  $k \in \mathbb{N}$  such that  $2^k t \leq s < 2^{k+1} t$ . Then by the previous inequality and monotonicity of  $\psi$ ,

$$\frac{\psi(x, t)}{t^p} \leq \frac{\psi(x, 2t)}{(2t)^p} \leq \dots \leq \frac{\psi(x, 2^k t)}{(2^k t)^p} \leq 2^p \frac{\psi(x, s)}{s^p}.$$

Hence,  $\psi$  satisfies (aInc) with  $\gamma^- = p$ . Since this property is invariant under equivalence, it holds for  $\varphi$  as well.  $\square$

We can now prove the uniform convexity of the space.

**Proof of Theorem 1.3.** By Theorem 1.2,  $\Delta_2^w$  and  $\nabla_2^w$  imply  $\Delta_2$  and  $\nabla_2$ . If  $\varphi$  satisfies (aInc), then it follows from Theorem 4.1 that it is equivalent to a uniformly convex  $\Phi$ -function  $\psi$ . By (aDec), also  $\psi$  is doubling. Hence by [16], Theorem 11.6 (see also [6], Theorem 2.4.14),  $L^{\psi(\cdot)}$  is uniformly convex. Since  $\varphi \simeq \psi$ ,  $L^{\varphi(\cdot)} = L^{\psi(\cdot)}$ , and hence we have proved  $L^{\varphi(\cdot)}$  is uniformly convex. Furthermore, every uniformly convex Banach space is reflexive [1], Chapter 1.  $\square$

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