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Uniform convexity of Banach spaces $l(\{p_i\})$

by

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The class of Banach sequence spaces $l(\{p_i\})$ studied originally by Nakano [4] has received attention in some of the recent papers. Klee [3] studied bounded summability property in the spaces $l(\{p_i\})$ while Waterman et al. [6] characterized reflexive $l(\{p_i\})$ spaces. In the present note we sharpen the main theorem in [6] by showing that the hypothesis in that theorem provides a characterization of uniformly convex $l(\{p_i\})$ spaces and that a reflexive $l(\{p_i\})$ space is uniformly convex. We accomplish the proofs of these results without appealing to the theorem in [6].

Let $\{p_i\}$ be a sequence of real numbers $1 \leqslant p_i < \infty$. Then $l(\{p_i\})$ is the set of all real sequences x such that

$$\sum_{i\geqslant 1} \frac{1}{p_i} |ax_i|^{p_i} < \infty$$

for some $a > 0$ depending on x. It is verified that with the usual definition of sum of two sequences and scalar multiple of a sequence the set $l(\{p_i\})$ is a real vector space. Further if for $x \in l(\{p_i\})$

$$(*) \qquad M(x) = \sum_{i\geqslant 1} \frac{1}{p_i} |x_i|^{p_i},$$

then M is a modular on $l(\{p_i\})$. For a detailed account of modulars on vector spaces we refer to Nakano [4]. If M is a modular on a vector space the norm induced by the modular M is given by the formula

$$\|x\| = \inf\left\{\frac{1}{\xi}\,\middle|\, \xi > 0,\, M(\xi x) \leqslant 1\right\}.$$

The space $l(\{p_i\})$ under the norm induced by the modular M defined in $(*)$ is a Banach space.

Before proceeding to the main result of this note we recall some terminology from Nakano [5] concerning modulars and state a theorem useful in the subsequent discussion.

Let M be a modular on a vector space E and let the norm induced by M be denoted by $\|\cdot\|$. A vector $x \in E$ is said to be *finite* if $M(\lambda x) < \infty$ for all real values of λ. The modular M is said to be *finite* if every vector $x \in E$ is finite. The modular M is said to be *uniformly finite* (*uniformly simple*) if

$$\sup_{M(x)\leqslant 1} M(\xi x) < \infty \quad \left(\operatorname*{Inf}_{M(x)\geqslant 1} M(\xi x) > 0 \right) \quad \text{for every real number } \xi.$$

The modular M is said to be *uniformly convex* if corresponding to any pair of positive real numbers r, ε there exists a $\delta > 0$ such that $M(x) \leqslant r$, $M(Y) \leqslant r$, $M(x-y) \geqslant \varepsilon \Rightarrow$,

$$M\left(\frac{x+y}{2}\right) \leqslant \tfrac{1}{2}[M(x)+M(Y)]-\delta.$$

For a definition of uniformly convex Banach spaces, see Day [2]. The theorem which is stated below relates the uniform convexity of the modular M with the uniform convexity of the norm induced by M.

THEOREM (Nakano). *If a modular M is uniformly convex, uniformly finite and uniformly simple, then the norm induced by M is uniformly convex.*

For a proof see Theorem 3 on p. 227 in Nakano [5].

We proceed next to the main theorems of this note. Let P be the set of positive integers. If $Q \subset P$ we denote by M_Q the function on $l(\{p_i\})$ defined by

$$M_Q(x) = \sum_{n\in Q} \frac{1}{p_n}|x_n|^{p_n}.$$

We note M_Q is a convex function. We further recall the following inequalities:

(i_1) If $p \geqslant 2$, then

$$|a+b|^p + |a-b|^p \leqslant 2^{p-1}[|a|^p+|b|^p]$$

for any two real numbers a, b.

(i_2) If $1 < p \leqslant 2$, then

$$\left|\frac{a+b}{2}\right|^p + \frac{p(p-1)}{2}\left|\frac{a-b}{|a|+|b|}\right|^{2-p}\left|\frac{a-b}{2}\right|^p \leqslant \frac{|a|^p+|b|^p}{2}$$

with a, b as in (i_1).

For a proof of (i_1) see Clarkson [1]. (i_2) follows from the Taylor expansion of $(1+t)^p$ for small t.

THEOREM 1. *The Banach space $l(\{p_i\})$ is uniformly convex if and only if*

(*) $$1 < \inf_{i\geqslant 1} p_i \leqslant \sup_{i\geqslant 1} p_i < \infty.$$

Proof. Let the sequence $\{p_i\}_{i\geqslant 1}$ satisfies the inequality stated in (*). Thus there exist real numbers A and B such that $1 < A \leqslant p_i \leqslant B < \infty$. We proceed to verify that the modular M is uniformly convex, uniformly finite and uniformly simple. Let r, ε be two positive numbers and $x, y \in l(\{p_i\})$ such that

$$M(x) \leqslant r, \quad M(y) \leqslant r \quad \text{and} \quad M(x-y) \geqslant \varepsilon.$$

Let us partition the set of positive integers into sets E, F defined by $n \in E$ if $p_n \geqslant 2$ and $n \in F$ if $p_n < 2$. We note that $M(x) = M_E(x)+M_F(x)$ for all $x \in l(\{p_i\})$. Thus $M(x-y) \geqslant \varepsilon$ implies either $M_E(x-y) \geqslant \varepsilon/2$ or $M_F(x-y) \geqslant \varepsilon/2$.

Case 1. Let $M_E(x-y) \geqslant \varepsilon/2$. Since $p_n \leqslant B$

$$M_E\left(\frac{x-y}{2}\right) \geqslant \frac{1}{2^B} M_E(x-y) \geqslant \frac{\varepsilon}{2^{B+1}}.$$

Further since, for $n \in E$, $p_n \geqslant 2$, it follows from the inequality (i_1) that

$$M_E\left(\frac{x+y}{2}\right) + M_E\left(\frac{x-y}{2}\right) \leqslant \tfrac{1}{2}[M_E(x)+M_E(y)].$$

Now noting that M_F is a convex function it is verified using the above inequalities that

$$\tfrac{1}{2}[M(x)+M(y)] \geqslant M_E\left(\frac{x+y}{2}\right) + M_E\left(\frac{x-y}{2}\right) + M_E\left(\frac{x+y}{2}\right)$$

$$\geqslant M\left(\frac{x+y}{2}\right) + \frac{\varepsilon}{2^{B+1}}.$$

Case 2. Let $M_F(x-y) \geqslant \varepsilon/2$. Let G be the subset of F consisting of the $n \in G$ such that

$$|x_n - y_n| \geqslant C(|x_n|+|y_n|),$$

where $C = \mathrm{Min}(\frac{1}{2}, \varepsilon/8r)$. With $G_0 = F \sim G$ it is verified that,

$$\sum_{n\in G_0} \frac{1}{p_n}|x_n-y_n|^{p_n} \leqslant \sum_{n\in G_0}\frac{1}{p_n}\{C^{p_n}(|x_n|+|y_n|)^{p_n}\}$$

$$\leqslant \sum_{n\in G_0} \frac{2^{p_n}}{p_n}\frac{[C(|x_n|+|y_n|)]^{p_n}}{2}$$

$$\leqslant \sum_{n\in G_0} \frac{2^{p_n}}{2p_n}[|Cx_n|^{p_n}+|Cy_n|^{p_n}]$$

$$\leqslant \tfrac{1}{2}[M(2cx)+M(2cy)].$$

Since $0 \leqslant 2c \leqslant 1$ and $M(x), M(y) \leqslant r$

$$M(2cx)+M(2cy) \leqslant 4cr.$$

Thus it is verified that

$$\sum_{n\in G_0} \frac{1}{p_n} |x_n - y_n|^{p_n} \leqslant 2cr \leqslant \frac{\varepsilon}{4} \quad \text{since } C \leqslant \frac{\varepsilon}{8r}.$$

Since $M_F(x-y) \geqslant \varepsilon/2$ it follows from the definition of G_0 that

$$(**) \qquad \sum_{n\in G} \frac{1}{p_n} |x_n - y_n|^{p_n} > \varepsilon/4.$$

Then from inequality (i_2) it follows that

$$(***) \quad \frac{1}{2}[M_G(x)+M_G(y)] \geqslant M_G\left(\frac{x+y}{2}\right)+M_G\left(\frac{x-y}{2}\right)\frac{(A-1)C}{2}.$$

Since for $n \in G$, $p_n < 2$ it is verified

$$M_G\left(\frac{x-y}{2}\right) \geqslant \frac{1}{4} M_G(x-y).$$

But from $(**)$ it follows that

$$M_G\left(\frac{x-y}{2}\right) \geqslant \frac{\varepsilon}{16}.$$

Thus inequality $(***)$ yields

$$\frac{1}{2}[M_G(x)+M_G(y)] \geqslant M_G\left(\frac{x+y}{2}\right)+\frac{(A-1)c\varepsilon}{32}.$$

Noting that the function M_Q is convex it is deduced from the above inequality that

$$\frac{1}{2}[M(x)+M(y)] \geqslant M_G\left(\frac{x+y}{2}\right)+M_{P\sim G}\left(\frac{x+y}{2}\right)+\frac{(A-1)c\varepsilon}{32}$$
$$= M\left(\frac{x+y}{2}\right)+\frac{(A-1)c\varepsilon}{32},$$

where P is the set of positive integers.

Thus choosing

$$\delta = \mathrm{Min}\left(\frac{\varepsilon}{2^{B+1}}, \frac{(A-1)c\varepsilon}{32}\right)$$

it is verified that the modular M is uniformly convex.

The modular M is uniformly finite for if S is the function defined on the real line by setting $S(\xi) = |\xi|^B$ if $|\xi| \geqslant 1$ and $S(\xi) = |\xi|^A$ if $|\xi| < 1$ it is verified that $M(\xi x) \leqslant S(\xi) M(x)$. Thus $\sup_{M(x)\leqslant 1} M(\xi x) \leqslant S(\xi)$.

Next we proceed to verify that M is uniformly simple. Let L be the function defined on the real line by setting $L(\xi) = |\xi|^A$ if $|\xi| \geqslant 1$ and $L(\xi) = |\xi|^B$ if $|\xi| < 1$. Then it follows that $M(\xi x) \geqslant L(\xi) M(x)$. Hence M is uniformly simple. Thus it follows from Nakano's theorem that the norm induced by M is uniformly convex.

We next proceed to the Converse of the above theorem.

THEOREM 2. *If $l(\{p_i\})$ is uniformly convex, then* $1 < \liminf p_i \leqslant \limsup p_i < \infty$.

Proof. If possible let $l(\{p_i\})$ be uniformly convex and $\liminf p_i = 1$. Thus there exists an infinite subsequence $\{p_{i_j}\}$ of $\{p_i\}$ such that $p_{i_j} \to 1$. By considering the vectors $x \in l(\{p_i\})$ such that $x_n = 0$ if $n \neq i_j$ for some j it is seen that the Banach space $l(\{p_{i_j}\})$ is isometrically isomorphic with a subspace of $l(\{p_i\})$. Thus $l(\{p_{i_j}\})$ is uniformly convex. Hence it is a reflexive Banach space. However, since $p_{i_j} \to 1$ by Theorem 2 in Nakano [4] the weak sequential convergence and norm convergence coincide in $l(\{p_{i_j}\})$. Since $l(\{p_{i_j}\})$ is reflexive the unit cell in $l(\{p_{i_j}\})$ is weakly compact. Thus it follows readily from Eberlein theorem (see [2], p. 51) that the unit cell in $l(\{p_{i_j}\})$ is compact in the norm topology. Hence $l(\{p_{i_j}\})$ is finite dimensional contradicting that $\{p_{i_j}\}$ is an infinite sequence. Hence $1 < \inf p_i$. If $\limsup p_i = \infty$ it is verified as in Lemma 1 in [6] that $l(\{p_i\})$ contains a subspace isomorphic to l^∞ contradicting that the space $l(\{p_i\})$ is reflexive. The proof of Theorem 2 is complete.

In conclusion we note that from Theorem 1 and proof of Theorem 2 in this note it is readily inferred that the Banach space $l(\{p_i\})$ is uniformly convex if and only if it is reflexive.

References

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