

# UNIFORM EMBEDDABILITY OF RELATIVELY HYPERBOLIC GROUPS

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ABSTRACT. Let  $\Gamma$  be a finitely generated group which is hyperbolic relative to a finite family  $\{H_1, \dots, H_n\}$  of subgroups. We prove that  $\Gamma$  is uniformly embeddable in a Hilbert space if and only if each subgroup  $H_i$  is uniformly embeddable in a Hilbert space.

## 1. INTRODUCTION

Gromov introduced the notion of uniform embeddability (in Hilbert space), and suggested it should be relevant to the Novikov Conjecture [6, 5]. Subsequently Yu proved the Coarse Baum-Connes Conjecture for bounded geometry discrete metric spaces that are uniformly embeddable; applying a descent principle the Novikov Conjecture followed for groups that, when equipped with a word length metric, are uniformly embeddable [16]. (A condition on finiteness of the classifying space was later removed [14].)

The notion of  $C^*$ -exactness of discrete groups was introduced by Kirchberg. It has been extensively studied as a functional analytic property of groups, and developed by many authors. In particular, Ozawa gave a characterization of  $C^*$ -exact groups, from which it directly follows that a  $C^*$ -exact group is uniformly embeddable [11] (see [7] for the link to uniform embeddability). There is at present no known example of a group that is uniformly embeddable but not  $C^*$ -exact. Analogous statements for metric spaces involve Property A of Yu [16], which is equivalent to  $C^*$ -exactness for discrete groups.

The classes of metric spaces (and groups) that are uniformly embeddable, or have Property A (are  $C^*$ -exact) are the subject of intense study. In this note we introduce a ‘gluing’ technique for proving uniform embeddability: starting from the assumption that a space is covered in an appropriate way by uniformly embeddable sets we conclude that the space itself is uniformly embeddable. Thus, the individual uniform embeddings of the pieces are ‘glued’ to give a uniform embedding of the whole. A parallel technique is introduced for spaces with Property A; it applies to  $C^*$ -exact groups. The most primitive gluing result is summarized in the following theorem (compare Theorem 3.2):

**Theorem.** *Let  $X$  be a metric space. Assume that for all  $\lambda > 0$  there exists a partition of unity  $(\varphi_i)_{i \in I}$  on  $X$  such that*

- (i) *the associated  $\Phi : X \rightarrow l^1(I)$  (defined by  $\Phi(x)(i) = \varphi_i(x)$ ) is Lipschitz, and*
- (ii) *the subspaces  $(\text{supp}(\varphi_i))_{i \in I}$  are ‘equi’ uniformly embeddable.*

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Then  $X$  is uniformly embeddable.

Our gluing technique is inspired by the work of Bell and Dranishnikov on spaces of finite asymptotic dimension [2, 13], and by the work of Bell on Property A [1]. Our results differ from these works in several ways: first, we treat uniformly embeddable spaces, and allow spaces of unbounded geometry; second, we significantly relax the condition on the ‘parameter space’. With these refinements, the gluing technique is extremely versatile — it allows us to give a conceptual treatment of all known permanence properties of the classes of uniformly embeddable spaces and  $C^*$ -exact groups (see, for example, [3, 4, 9]).

In this note we concentrate on the basics of our gluing technique; we plan to further develop the technique and present the above mentioned applications elsewhere. We also describe one application of gluing, proving the following new permanence property of the class of uniformly embeddable groups:

**Theorem.** *Let  $\Gamma$  be a finitely generated group which is hyperbolic relative to a finite family  $\{H_1, \dots, H_n\}$  of subgroups. Then  $\Gamma$  is uniformly embeddable if and only if each subgroup  $H_i$  is uniformly embeddable.*

A parallel statement concerning  $C^*$ -exactness was recently obtained by Ozawa [12]. Our methods allow us to recover this result; indeed, the  $C^*$ -exactness result may be extracted from the combined work of Bell [1] and Osin [10]. Our approach to relative hyperbolicity, which is based on the work of Osin, is quite different from that taken by Ozawa however, who, given amenable  $H_i$ -spaces explicitly constructs an amenable  $\Gamma$ -space.

These results should be compared to a recent result of Osin [10]: in the situation described in the theorem,  $\Gamma$  has finite asymptotic dimension if and only if the  $H_i$  have finite asymptotic dimension.

## 2. PRELIMINARIES

Let  $X$  and  $Y$  be metric spaces, with metrics  $d_X$  and  $d_Y$ , respectively. A function  $F : X \rightarrow Y$  is a *uniform embedding* if there exist non-decreasing functions  $\rho_{\pm} : \mathbb{R}_+ \rightarrow \mathbb{R}_+$  such that  $\lim_{t \rightarrow \infty} \rho_{\pm}(t) = \infty$  and such that

$$(1) \quad \rho_-(d_X(x, x')) \leq d_Y(F(x), F(x')) \leq \rho_+(d_X(x, x')), \quad \text{for all } x, x' \in X.$$

The space  $X$  is (Hilbert space) *uniformly embeddable* if there exists a uniform embedding  $F$  of  $X$  into a (real) Hilbert space  $\mathcal{H}$ .

**Proposition 2.1.** *Let  $X$  be a metric space. Then  $X$  is uniformly embeddable if and only if for every  $R > 0$  and  $\varepsilon > 0$  there exists a Hilbert space valued map  $\xi : X \rightarrow \mathcal{H}$ ,  $(\xi_x)_{x \in X}$ , such that  $\|\xi_x\| = 1$ , for all  $x \in X$  and such that*

- (i)  $d(x, x') \leq R \Rightarrow \|\xi_x - \xi_{x'}\| \leq \varepsilon$
- (ii)  $\lim_{S \rightarrow \infty} \sup\{|\langle \xi_x, \xi_{x'} \rangle| : d(x, x') \geq S, x, x' \in X\} = 0.$  □

*Definition 2.2.* A family of metric spaces  $(X_i, d_i)$  is called equi-uniformly embeddable if there is a family of Hilbert space valued maps  $F_i : X_i \rightarrow H_i$  and non-decreasing functions  $\rho_{\pm} : \mathbb{R}_+ \rightarrow \mathbb{R}_+$  with  $\lim_{t \rightarrow \infty} \rho_{\pm}(t) = \infty$  such that

$$(2) \quad \rho_-(d_i(x, y)) \leq \|F_i(x) - F_i(y)\| \leq \rho_+(d_i(x, y)), \quad \text{for all } i \text{ and all } x, y \in X_i.$$

The proof of Proposition 2.1 given in [3] for arbitrary metric spaces also shows the following characterization of equi-uniform embeddability for families of metric spaces.

**Proposition 2.3.** *A family  $(X_i)_{i \in I}$  of metric spaces is equi-uniformly embeddable if and only if for every  $R > 0$  and  $\varepsilon > 0$  there exists a family  $(\xi_i)_{i \in I}$  of Hilbert space valued maps  $\xi_i : X_i \rightarrow \mathcal{H}$ , such that  $\|\xi_i(x)\| = 1$ , for all  $x \in X_i$ , and such that*

- (i)  $\forall i \in I \forall x, x' \in X_i \ d(x, x') \leq R \Rightarrow \|\xi_i(x) - \xi_i(x')\| \leq \varepsilon$
- (ii)  $\lim_{S \rightarrow \infty} \sup_{i \in I} \sup\{|\langle \xi_i(x), \xi_i(x') \rangle| : d(x, x') \geq S, x, x' \in X_i\} = 0.$  □

*Remark 2.4.* Let  $X$  be a metric space which is uniformly embeddable. Then any family  $(X_i)_{i \in I}$  of subspaces of  $X$  is equi-uniformly embeddable.

Property *A* is a condition on metric spaces introduced by Yu [16]. We do not recall the definition of Property *A* here; rather, we work with the following characterization of Property *A* obtained by Tu [15].

**Proposition 2.5** ([15]). *A discrete metric space  $X$  with bounded geometry has Property *A* if and only if for every  $R > 0$  and  $\varepsilon > 0$  there exist a function  $\xi : X \rightarrow \ell^1(X)$  and a number  $S > 0$  such that for all  $x, x' \in X$  we have  $\|\xi_x\| = 1$ , and*

- (i)  $d(x, x') \leq R \Rightarrow \|\xi_x - \xi_{x'}\| \leq \varepsilon$
- (ii)  $\text{supp } \xi_x \subset B(x, S).$

(Moreover one can arrange that  $\xi$  is nonnegative.)

*Equivalently, for every  $R > 0$  and  $\varepsilon > 0$  there exists an  $S > 0$  and a Hilbert space valued function  $\xi : X \rightarrow \mathcal{H}$  such that for all  $x, x' \in X$  we have  $\|\xi_x\| = 1$ , (i) as above and*

- (iii)  $\exists S > 0$  such that  $d(x, x') \geq S \Rightarrow \langle \xi_x, \xi_{x'} \rangle = 0.$  □

*Remark 2.6.* The existence of  $\xi$  satisfying the conditions (i), (ii) (respectively (iii)) of Proposition 2.5 is a consequence of Property *A* for *arbitrary* metric spaces as shown in [15]. The bounded geometry condition is needed only for the reverse implication.

Let  $X$  be a set. A *partition of unity* on  $X$  is a family of maps  $(\varphi_i)_{i \in I}$ , with  $\varphi_i : X \rightarrow [0, 1]$ , and such that  $\sum_{i \in I} \varphi_i(x) = 1$  for all  $x \in X$ . If  $x \in X$  we do not require that the set  $\{i \in I : \varphi_i(x) \neq 0\}$  be finite, although that will be the case in most of our examples. We say that  $(\varphi_i)_{i \in I}$  is *subordinated to a cover*  $\mathcal{U} = (U_i)_{i \in I}$  of  $X$  if each  $\varphi_i$  vanishes outside  $U_i$ . Sometimes a partition of unity subordinated to a cover  $\mathcal{U}$  will be denoted by  $(\varphi_{\mathcal{U}})_{\mathcal{U} \in \mathcal{U}}$ .

*Definition 2.7.* A metric space  $X$  is *exact* if for all  $R > 0$  and  $\varepsilon > 0$  there is a partition of unity  $(\varphi_i)_{i \in I}$  on  $X$  subordinated to a cover  $\mathcal{U} = (U_i)_{i \in I}$  and such that

- (i)  $\forall x, y \in X$  with  $d(x, y) \leq R$ ,  $\sum_{i \in I} |\varphi_i(x) - \varphi_i(y)| \leq \varepsilon.$
- (ii) the cover  $\mathcal{U} = (U_i)_{i \in I}$  is a uniformly bounded, i.e.  $\sup_{i \in I} \text{diam}(U_i) < \infty.$

*Definition 2.8.* A family of metric spaces  $(X_i)_{i \in I}$  is *equi-exact* if for all  $R > 0$  and  $\varepsilon > 0$  and for every  $i \in I$  there is a partition of unity  $(\psi_i^j)_{j \in J_i}$  on  $X_i$  subordinated to a cover  $\mathcal{U}_i = (U_i^j)_{j \in J_i}$  of  $X_i$  and such that

- (i)  $\forall i \in I, \forall x, y \in X_i$  with  $d(x, y) \leq R$ ,  $\sum_{j \in J_i} |\psi_i^j(x) - \psi_i^j(y)| \leq \varepsilon$ .
- (ii) the family  $(U_i^j)_{i \in I, j \in J_i}$  is uniformly bounded, i.e.  $\sup_{i \in I, j \in J_i} \text{diam}(U_i^j) < \infty$ .

*Remark 2.9.* Let  $X$  be an exact metric space. Then any family  $(X_i)_{i \in I}$  of subspaces of  $X$  is equi-exact.

**Proposition 2.10.** *Let  $X$  be a metric space.*

- (a) *If  $X$  has Property A then  $X$  is exact.*
- (b) *If  $X$  is discrete and has bounded geometry then  $X$  is exact if and only if it has Property A.*
- (c) *If  $X$  is exact then  $X$  is uniformly embeddable.*

*Proof.* For (a), assume that  $X$  has Property A. Let  $R > 0$  and  $\varepsilon > 0$  be given. Obtain  $\xi : X \rightarrow \ell^1(X)$  (which we assume to be non-negative) and  $S > 0$  as in Proposition 2.5. Define  $(\varphi_z)_{z \in X}$  by  $\varphi_z(x) = \xi_x(z)$ . If  $U_z = \{x \in X : \varphi_z(x) > 0\}$  then  $U_z \subset B(z, S)$  since  $\text{supp } \xi_x \subset B(x, S)$ . It is clear that  $\sum_{z \in X} \varphi_z(x) = \|\xi_x\| = 1$  and

$$\sum_{z \in X} |\varphi_z(x) - \varphi_z(y)| = \|\xi_x - \xi_y\| \leq \varepsilon$$

if  $d(x, y) \leq R$ .

For (b), assume that  $X$  is exact. It suffices to find a Hilbert space valued function on  $X$  satisfying the conditions from the second part of Proposition 2.5. Let  $R > 0$  and  $\varepsilon > 0$  be given. Let  $(\varphi_i)_{i \in I}$  be as in Definition 2.7. Define  $\xi : X \rightarrow \ell^2(I)$  by  $\xi_x(i) = \varphi_i(x)^{1/2}$ . Then  $\|\xi_x\|^2 = \sum_{i \in I} \varphi_i(x) = 1$ . Using the inequality  $|a^{1/2} - b^{1/2}|^2 \leq |a - b|$  we see that

$$\|\xi_x - \xi_y\|^2 = \sum_{i \in I} |\varphi_i(x)^{1/2} - \varphi_i(y)^{1/2}|^2 \leq \sum_{i \in I} |\varphi_i(x) - \varphi_i(y)| \leq \varepsilon$$

if  $d(x, y) \leq R$ . Finally we note that

$$\langle \xi_x, \xi_y \rangle = \sum_{i \in I} \varphi_i(x)^{1/2} \varphi_i(y)^{1/2} = 0$$

whenever  $d(x, y) > \sup_{i \in I} \text{diam}(U_i)$ .

The proof of (c) is similar to that of (b), but one applies Proposition 2.1. □

*Remark 2.11.* If  $f : X \rightarrow Y$  is a uniform embedding of metric spaces and  $Y$  is exact then  $X$  is exact. Therefore exactness of a metric space is a coarse invariant. This remark is generalized in Corollary 3.3.

## 3. GLUING SPACES USING PARTITIONS OF UNITY

Let  $X$  be a metric space and let  $\mathcal{U} = (U_i)_{i \in I}$  be a cover of  $X$ . Denote by  $\mathcal{U}_R$  the cover obtained by enlarging the sets in  $\mathcal{U}$  by taking their  $R$ -closed neighborhoods:

$$\mathcal{U}_R = \{U_i(R) : i \in I\}, \quad U_i(R) = \{x \in X : d(x, U_i) \leq R\}.$$

One verifies immediately that if the family  $\mathcal{U}$  (with the metric structure induced from  $X$ ) is equi-uniformly embeddable (or equi-exact) then so is  $\mathcal{U}_R$ .

**Theorem 3.1.** *Let  $X$  be a metric space. Suppose that for all  $R > 0$  and  $\varepsilon > 0$  there is a partition of unity  $(\varphi_i)_{i \in I}$  on  $X$  such that*

- (i)  $\forall x, y \in X$  with  $d(x, y) \leq R$ ,  $\sum_{i \in I} |\varphi_i(x) - \varphi_i(y)| \leq \varepsilon$ , and
- (ii)  $(\varphi_i)_{i \in I}$  is subordinated to an equi-exact cover  $(U_i)_{i \in I}$  of  $X$ .

*Then  $X$  is exact.*

*Proof.* Let  $R > 0$  and  $\varepsilon > 0$  be given. We are going to construct a Hilbert space valued function  $\eta : A \rightarrow \mathcal{H}$  satisfying the hypotheses of Proposition 2.1. By assumption there is a cover  $\mathcal{U} = (U_i)_{i \in I}$  of  $X$  which is equi-exact and there is a partition of unity  $(\varphi_i)_{i \in I}$  subordinated to  $\mathcal{U}$  such that  $\forall x, y \in X$  with  $d(x, y) \leq R$ ,

$$\sum_{i \in I} |\varphi_i(x) - \varphi_i(y)| \leq \varepsilon/2.$$

Since  $\mathcal{U}$  is equi-exact so is  $\mathcal{U}_R = \{U_i(R) : i \in I\}$ . Therefore for each  $U_i \in \mathcal{U}$  there is a cover  $\mathcal{V}_i = (V_i^j)_{j \in J_i}$  of  $U_i(R)$  such that the cover  $\{V_i^j : i \in I, j \in J_i\}$  of  $X$  is uniformly bounded. Moreover for each  $U_i \in \mathcal{U}$  there is a partition of unity  $(\psi_i^j)$  on  $U_i(R)$  subordinated to  $\mathcal{V}_i$  such that  $\forall x, y \in U_i(R)$  with  $d(x, y) \leq R$ ,

$$\sum_{j \in J_i} |\psi_i^j(x) - \psi_i^j(y)| \leq \varepsilon/2.$$

It is useful to extend  $\psi_i^j$  to  $X$  by setting it equal to zero outside  $U_i(R)$ . Define  $\theta_{i,j} = \varphi_i \psi_i^j$ . Then  $(\theta_{i,j})$  is a partition of unity on  $X$  subordinated to a uniformly bounded cover. Moreover,

$$\sum_{i,j} |\theta_{i,j}(x) - \theta_{i,j}(y)| \leq \sum_i \varphi_i(x) \sum_j |\psi_i^j(x) - \psi_i^j(y)| + \sum_i |\varphi_i(x) - \varphi_i(y)| \sum_j \psi_i^j(y).$$

Assume now that  $d(x, y) \leq R$ . If  $\varphi_i(x) \neq 0$  then  $x \in U_i$  hence  $y \in U_i(R)$  as  $d(x, y) \leq R$ . Therefore

$$\sum_i \varphi_i(x) \sum_j |\psi_i^j(x) - \psi_i^j(y)| \leq \varepsilon/2.$$

Since  $\sum_j \psi_i^j(y)$  equals 1 for  $y \in U_i(R)$  and 0 for  $y \notin U_i(R)$ ,

$$\sum_i |\varphi_i(x) - \varphi_i(y)| \sum_j \psi_i^j(y) \leq \sum_i |\varphi_i(x) - \varphi_i(y)| \leq \varepsilon/2.$$

Combining the above estimates we obtain that

$$\sum_{i,j} |\theta_{i,j}(x) - \theta_{i,j}(y)| \leq \varepsilon$$

whenever  $x, y \in X$  and  $d(x, y) \leq R$ . □

**Theorem 3.2.** *Let  $X$  be a metric space. Suppose that for all  $R > 0$  and  $\varepsilon > 0$  there is a partition of unity  $(\varphi_i)_{i \in I}$  on  $X$  such that*

- (i)  $\forall x, y \in X$  with  $d(x, y) \leq R$ ,  $\sum_{i \in I} |\varphi_i(x) - \varphi_i(y)| \leq \varepsilon$ .
- (ii)  $(\varphi_i)_{i \in I}$  is subordinated to an equi-uniformly embeddable cover  $(U_i)_{i \in I}$  of  $X$ .

Then  $X$  is uniformly embeddable.

*Proof.* Let  $R > 0$  and  $\varepsilon > 0$  be given. We construct a Hilbert space valued function  $\eta$  on  $X$  satisfying the conditions in Proposition 2.1. By assumption there is a cover  $\mathcal{U} = (U_i)_{i \in I}$  of  $X$  which is equi-uniformly embeddable and there is a partition of unity  $(\varphi_i)_{i \in I}$  subordinated to  $\mathcal{U}$  such that  $\forall x, y \in X$  with  $d(x, y) \leq R$ ,

$$\sum_{i \in I} |\varphi_i(x) - \varphi_i(y)| \leq \varepsilon^2/4.$$

Since  $\mathcal{U}$  is equi-uniformly embeddable so is  $\mathcal{U}_R = \{U_i(R) : i \in I\}$  where as above  $U_i(R) = \{x \in X : d(x, U_i) \leq R\}$ . Therefore there exist Hilbert space valued maps  $\xi_i : U_i(R) \rightarrow \mathcal{H}_i$ , with  $\|\xi_i(x)\| = 1$  for all  $x \in U_i(R)$  and such that

- (iv)  $\sup\{\|\xi_i(x) - \xi_i(y)\| : d(x, y) \leq R, x, y \in U_i(R)\} \leq \varepsilon/2$  for all  $i \in I$ .
- (v)  $\lim_{S \rightarrow \infty} \sup_{i \in I} \sup\{|\langle \xi_i(x), \xi_i(y) \rangle| : d(x, y) \geq S, x, y \in U_i(R)\} = 0$ .

We extend each  $\xi_i$  to  $X$  by setting  $\xi_i(x) = 0$  for  $x \in X \setminus U_i(R)$ . Define  $\eta : X \rightarrow \mathcal{H} = \bigoplus_{i \in I} \mathcal{H}_i$ ,  $\eta(x) = (\eta_i(x))_{i \in I}$  by

$$(3) \quad \eta_i(x) = \varphi_i(x)^{1/2} \xi_i(x).$$

One verifies that  $\|\eta(x)\| = 1$ ,  $\forall x \in X$ . Let  $x, y \in X$  with  $d(x, y) \leq R$ . Consider  $\alpha(x, y), \beta(x, y) \in \mathcal{H}$  with components

$$\alpha_i(x, y) = \varphi_i(x)^{1/2} (\xi_i(x) - \xi_i(y)),$$

$$\beta_i(x, y) = (\varphi_i(x)^{1/2} - \varphi_i(y)^{1/2}) \xi_i(y).$$

Note that  $\alpha(x, y)$  and  $\beta(x, y)$  are well-defined because of the following norm estimates.

$$\|\alpha(x, y)\|^2 = \sum_{i \in I} \varphi_i(x) \|\xi_i(x) - \xi_i(y)\|^2,$$

where the summation is done for those  $i$  with  $x \in U_i$ . If  $d(x, y) \leq R$  and  $x \in U_i$  then  $y \in U_i(R)$ , so that using (iv) we obtain  $\|\alpha(x, y)\| \leq \varepsilon/2$ . Since  $|a^{1/2} - b^{1/2}|^2 \leq |a - b|$  we

have

$$\begin{aligned} \|\beta(x, y)\|^2 &= \sum_{i \in I} \|(\varphi_i(x)^{1/2} - \varphi_i(y)^{1/2}) \xi_i(y)\|^2 \leq \sum_{i \in I} |\varphi_i(x)^{1/2} - \varphi_i(y)^{1/2}|^2 \\ &\leq \sum_{i \in I} |\varphi_i(x) - \varphi_i(y)| \leq \varepsilon^2/4, \end{aligned}$$

hence  $\|\beta(x, y)\| \leq \varepsilon/2$ . Therefore

$$\|\eta(x) - \eta(y)\| = \|\alpha(x, y) + \beta(x, y)\| \leq \|\alpha(x, y)\| + \|\beta(x, y)\| \leq \varepsilon$$

whenever  $d(x, y) \leq R$ . In order to prove the support condition (ii) of Proposition 2.1 we use that  $\varphi_i$  vanishes outside  $U_i$  and the Cauchy-Schwarz inequality. Thus for any  $x, y \in X$  with  $d(x, y) \geq S$  we have:

$$\begin{aligned} |\langle \eta(x), \eta(y) \rangle| &\leq \sum_{i \in I} \varphi_i(x)^{1/2} \varphi_i(y)^{1/2} |\langle \xi_i(x), \xi_i(y) \rangle| \leq \\ &\sup_{i \in I} \sup\{|\langle \xi_i(x'), \xi_i(y') \rangle| : d(x', y') \geq S, x', y' \in U_i\}. \end{aligned}$$

In view of (v), this concludes the proof.  $\square$

**Corollary 3.3.** *Let  $p : X \rightarrow Y$  be a map of metric spaces with the property that  $\forall R > 0 \exists S > 0$  such that  $d(p(x), p(x')) \leq S$  whenever  $d(x, x') \leq R$ . Suppose that  $Y$  is exact. If for each uniformly bounded cover  $(U_i)_{i \in I}$  of  $Y$ , the family  $(p^{-1}(U_i))_{i \in I}$  of subspaces of  $X$  is equi-uniformly embeddable (respectively, equi-exact), then  $X$  is uniformly embeddable (respectively, exact).*

*Proof.* Let  $R > 0$  and  $\varepsilon > 0$  be given and let  $S > 0$  be as in the statement. Since  $Y$  is exact, we find a uniformly bounded cover  $(U_i)_{i \in I}$  of  $Y$ , together with a partition of unity  $(\varphi_i)_{i \in I}$  as in Definition 2.7 with  $S$  playing the role of  $R$ . Then  $(\varphi_i \circ p)_{i \in I}$  is a partition of unity on  $X$  subordinated to  $(p^{-1}(U_i))_{i \in I}$  and satisfying the assumptions of Theorem 3.2 (respectively, 3.1).  $\square$

**Corollary 3.4.** *Let  $p : X \rightarrow Y$  be a Lipschitz map of metric spaces. Assume that a group  $\Gamma$  acts by isometries on both  $X$  and  $Y$ , that the action on  $Y$  is transitive and that  $p$  is  $\Gamma$ -equivariant. Assume  $Y$  is exact. If there exists  $y_0 \in Y$  such that for every  $n \in \mathbb{N}$  the inverse image  $p^{-1}(B(y_0, n))$  is uniformly embeddable (respectively, exact) then  $X$  is uniformly embeddable (respectively, exact).*

*Proof.* This follows immediately from Corollary 3.3. Indeed, let  $(U_i)_{i \in I}$  be a uniformly bounded cover of  $Y$ . Since  $\Gamma$  acts transitively and isometrically on  $Y$ , there are elements  $s_i \in \Gamma$  and  $n \in \mathbb{N}$  such that  $s_i U_i \subseteq B(y_0, n)$  for all  $i \in I$ . Since  $s_i p^{-1}(U_i) = p^{-1}(s_i U_i) \subset p^{-1}(B(y_0, n))$ , we see that the family  $(p^{-1}(U_i))_{i \in I}$  is isometric to a family of subspaces of  $p^{-1}(B(y_0, n))$ . Since the latter space is uniformly embeddable (exact), we conclude that the family  $(p^{-1}(U_i))_{i \in I}$  is equi-uniformly embeddable (equi-exact).  $\square$

4. PARTITIONS OF UNITY COMING FROM THE COMBINATORICS OF ASYMPTOTIC  
DIMENSION

Let  $\mathcal{U}$  be a cover of  $X$ . A *Lebesgue number* for  $\mathcal{U}$  is a number  $L > 0$  with the property that any subset  $B \subset X$  of diameter less than  $L$  is contained in some  $U \in \mathcal{U}$ . A cover  $\mathcal{U}$  of  $X$  has *multiplicity at most  $k$*  if any  $x \in X$  belongs to at most  $k$  members of  $\mathcal{U}$ . One way to construct partitions of unity with Lipschitz properties is given by the following proposition.

**Proposition 4.1.** *Let  $\mathcal{U}$  be a cover of a metric space  $X$  with multiplicity at most  $k + 1$ , ( $k \geq 0$ ) and Lebesgue number  $L > 0$ . For  $U \in \mathcal{U}$  define*

$$\varphi_U(x) = \frac{d(x, X \setminus U)}{\sum_{V \in \mathcal{U}} d(x, X \setminus V)}.$$

*Then  $(\varphi_U)_{U \in \mathcal{U}}$  is a partition of unity on  $X$  subordinated to the cover  $\mathcal{U}$ . Moreover each  $\varphi_U$  satisfies*

$$(4) \quad |\varphi_U(x) - \varphi_U(y)| \leq \frac{2k+3}{L} d(x, y), \quad \forall x, y \in X,$$

*and the family  $(\varphi_U)_{U \in \mathcal{U}}$  satisfies*

$$(5) \quad \sum_{U \in \mathcal{U}} |\varphi_U(x) - \varphi_U(y)| \leq \frac{(2k+2)(2k+3)}{L} d(x, y), \quad \forall x, y \in X.$$

*Proof.* This is folklore. See Bell's paper for a proof of (4). Note that (5) follows from (4) since any point in  $X$  belongs to at most  $k + 1$  distinct elements of the cover  $\mathcal{U}$ .  $\square$

A metric space  $X$  has *asymptotic dimension*  $\leq k$  if for every  $R > 0$  there exists a uniformly bounded cover  $\mathcal{U}$  of  $X$  such that every ball of radius  $R$  in  $X$  meets at most  $k + 1$  elements of  $\mathcal{U}$ .

In the context of non-uniformly bounded covers we require several closely related properties. Let  $\mathcal{V}$  be a family of nonempty subsets of the metric space  $X$ . The *multiplicity* of  $\mathcal{V}$  is the maximum number of elements of  $\mathcal{V}$  that contain a common point of  $X$ ; the  *$R$ -multiplicity* of  $\mathcal{V}$  is the maximum number of elements of  $\mathcal{V}$  that meet a common ball of radius  $R$  in  $X$ . If  $d(U, V) > L$  for all  $U, V \in \mathcal{V}$  with  $U \neq V$  then  $\mathcal{V}$  is  *$L$ -separated* ( $L > 0$ ). Note that if  $\mathcal{V}$  consists of just one set then  $\mathcal{V}$  is  $L$ -separated (vacuously) for every  $L > 0$ . A cover  $\mathcal{U}$  of  $X$  is  *$(k, L)$ -separated* ( $k \geq 0$  and  $L > 0$ ) if there is a partition of  $\mathcal{U}$  into  $k + 1$  families

$$\mathcal{U} = \mathcal{U}_0 \cup \dots \cup \mathcal{U}_k$$

such that each family  $\mathcal{U}_i$  is  $L$ -separated. In particular  $\mathcal{U}$  has multiplicity at most  $k + 1$ .

We make two observations, which we will apply to covers that are not necessarily uniformly bounded. First, a  $(k, 2R)$ -separated cover has  $R$ -multiplicity  $\leq k + 1$ . Second, if  $\mathcal{U}$  is a cover of  $X$  with  $L$ -multiplicity  $\leq k + 1$  then  $L$  is a Lebesgue number for the cover  $\mathcal{U}_L$  obtained by enlarging the sets in  $\mathcal{U}$  by taking their  $L$ -neighborhoods;

$$\mathcal{U}_L = \{U(L) : U \in \mathcal{U}\}, \quad U(L) = \{x \in X : d(x, U) \leq L\}.$$



Further, in this case, the cover  $\mathcal{U}_L$  has multiplicity  $\leq k + 1$ .

We summarize the previous discussion in the following form.

**Lemma 4.2.** *A  $(k, 2L)$ -separated cover of a metric space has  $L$ -multiplicity  $\leq k + 1$ . If a cover  $\mathcal{U}$  of a metric space has  $L$ -multiplicity  $\leq k + 1$  then the enlarged cover  $\mathcal{U}_L$  has multiplicity  $\leq k + 1$  and Lebesgue number  $L$ .  $\square$*

The following result was proven by Higson and Roe in the case of discrete bounded geometry metric spaces [8, Lemma 4.3]; in our more general setting it follows immediately from Proposition 4.1 and Definition 2.7.

**Proposition 4.3.** *A metric space of finite asymptotic dimension is exact.  $\square$*

We now prove a natural generalization of this result, where uniform boundedness of the cover is replaced by the appropriate uniform versions of uniform embeddability and Property A, defined earlier. We also provide the proper setting to generalize the ‘union theorems’ of Bell and Dranishnikov [1, 2].

**Theorem 4.4.** *Let  $X$  be a metric space. Assume that for every  $\delta > 0$  there is a  $(k, L)$ -separated cover  $\mathcal{U}$  of  $X$  with  $k^2 + 1 \leq L\delta$  and such that the family  $\mathcal{U}$  is equi-uniformly embeddable (where each  $U \in \mathcal{U}$  is given the induced metric from  $X$ ). Then  $X$  is uniformly embeddable. If instead we assume that the family  $\mathcal{U}$  is equi-exact then  $X$  is exact.*

*Proof.* The statement concerning uniform embeddability follows from Lemma 4.2, Proposition 4.1 and Theorem 3.2; for exactness use Theorem 3.1 instead of 3.2.

More precisely, given  $R > 0$  and  $\varepsilon > 0$  we fix a number  $\delta$ ,  $0 < \delta < 1/20R$ . Then

$$k^2 + 1 \geq 2(2k + 2)(2k + 3)R\delta,$$

for all integers  $k \geq 0$ . By assumption there is a  $(k, 2L)$ -separated cover  $\mathcal{U}$  of  $X$  such that  $\mathcal{U}$  is equi-uniformly embeddable and  $k^2 + 1 \leq 2L\delta\varepsilon$ . By Lemma 4.2 the cover  $\mathcal{U}_L$  has multiplicity  $\leq k + 1$  and Lebesgue number  $L$ . Proposition 4.1 provides a partition of unity subordinated to  $\mathcal{U}_L$  with the following property: for all  $x, y \in X$ , if  $d(x, y) \leq R$  then

$$\sum_{U(L) \in \mathcal{U}_L} |\varphi_{U(L)}(x) - \varphi_{U(L)}(y)| \leq \frac{(2k + 2)(2k + 3)R}{L} \leq \frac{k^2 + 1}{2L\delta} \leq \varepsilon.$$

Since the cover  $\mathcal{U}$  is equi-uniformly embeddable so is the cover  $\mathcal{U}_L$ . We conclude the proof by applying Theorem 3.2.  $\square$

**Corollary 4.5.** *If a metric space  $X$  is a union of finitely many uniformly embeddable subspaces then  $X$  is uniformly embeddable. A similar result is true for exact spaces.*

*Proof.* By assumption there is a finite cover  $\mathcal{U}$  of  $X$  with each  $U \in \mathcal{U}$  uniformly embeddable. Let  $\delta$  be given. Let  $k + 1$  denote the cardinality of  $\mathcal{U}$  and choose  $L$  such that  $k^2 + 1 \leq L\delta$ . Then  $\mathcal{U}$  is a  $(k, L)$ -separated cover of  $X$  and  $\mathcal{U}$  is equi-uniformly embeddable. The result follows now from Theorem 4.4.  $\square$

**Corollary 4.6.** *If a metric space  $X$  is the union of an equi-uniformly embeddable family of subspaces  $\mathcal{U}$  with the property that for every  $L > 0$  there is a uniformly embeddable subspace  $Y \subset X$  such that the family  $\{U \setminus Y : U \in \mathcal{U}\}$  is  $L$ -separated then  $X$  is uniformly embeddable. A similar result is true for exact spaces.*

*Proof.* Given  $\delta > 0$  we fix  $L \geq 2/\delta$ . Let  $\mathcal{U}$  and  $Y$  (depending on  $L$ ) be as in the statement. We apply Theorem 4.4 using the  $(1, L)$ -separated cover of  $X$  given by the families of sets  $\mathcal{U}_0 \cup \mathcal{U}_1$ , where  $\mathcal{U}_0 = \{Y\}$  and  $\mathcal{U}_1 = \{U \setminus Y : U \in \mathcal{U}\}$ .  $\square$

**Corollary 4.7.** *Let  $p : X \rightarrow Y$  be a Lipschitz map of metric spaces. Assume that a group  $\Gamma$  acts by isometries on both  $X$  and  $Y$ , that the action on  $Y$  is transitive and that  $p$  is  $\Gamma$ -equivariant. Assume  $Y$  has finite asymptotic dimension. If there exists  $y_0 \in Y$  such that for every  $n \in \mathbb{N}$  the inverse image  $p^{-1}(B_Y(y_0, n))$  is uniformly embeddable, then  $X$  is uniformly embeddable.*

*Proof.* This follows from Proposition 4.3 in conjunction with Corollary 3.4  $\square$

*Remark 4.8.* In the previous corollary, if we assume instead that  $p^{-1}(B_Y(y_0, n))$  is exact then we conclude that  $X$  is exact. The result so obtained is closely related to a result of Bell concerning Property A for groups acting on spaces of finite asymptotic dimension (compare [1, Theorem 1]).

## 5. RELATIVELY HYPERBOLIC GROUPS

Let  $\Gamma$  be a finitely generated group which is hyperbolic relative to a finite family  $\{H_1, \dots, H_n\}$  of subgroups. We prove that  $\Gamma$  is uniformly embeddable if and only if each subgroup  $H_i$  is uniformly embeddable. There are two analogous results in the literature: Osin proved an analogous statement for finite asymptotic dimension [10] and Ozawa proved an analogous statement for exactness [12]. We rely heavily on Osin's method (Ozawa's method is completely different), and are indebted to Ozawa for alerting us to Osin's paper.

If  $A$  is a symmetric set of generators of  $\Gamma$ , we denote by  $d_A$  the corresponding left-invariant metric on  $\Gamma$ . If  $B$  is another such set with  $A \subset B$  the identity map  $p : (\Gamma, d_A) \rightarrow (\Gamma, d_B)$  is equivariant and  $d_B(p(x), p(y)) \leq d_A(x, y)$ . Let  $S$  be a finite symmetric set generating  $\Gamma$ . Let

$$\mathcal{H} = \bigcup_k (H_k \setminus e)$$

Let  $d_S$  and  $d_{S \cup \mathcal{H}}$  be the left invariant metrics on  $\Gamma$  induced by  $S$  and  $S \cup \mathcal{H}$ , respectively. For  $n \geq 1$ , let

$$B(n) = \{g \in \Gamma : d_{S \cup \mathcal{H}}(g, e) \leq n\}.$$

We always view  $B(n)$  as a subspace of  $\Gamma$  equipped with the metric  $d_S$ . The following useful recursive decomposition of  $B(n)$  is contained in the proof of [10, Lemma 12]:

$$(6) \quad B(1) = S \cup \left( \bigcup_k H_k \right)$$

$$(7) \quad B(n) = \left( \bigcup_k B(n-1)H_k \right) \cup \left( \bigcup_{x \in S} B(n-1)x \right)$$

$$(8) \quad B(n-1)H_k = \bigsqcup_{g \in R(n-1)} gH_k,$$

where the final equality represents a partition of  $B(n-1)H_k$  into disjoint cosets according to a fixed set of coset representatives,  $R(n-1) \subset B(n-1)$ .

**Proposition 5.1** (Osin). *For every  $L > 0$  there exists  $\kappa(L) > 0$  such that if*

$$Y = \{x \in \Gamma : d_S(x, B(n-1)) \leq \kappa(L)\}$$

*then for each  $k$*

$$(9) \quad B(n-1)H_k \subset Y \cup \left( \bigcup_{g \in R(n-1)} gH_k \setminus Y \right)$$

*and the subspaces  $gH_k \setminus Y$ ,  $g \in R(n-1)$  of  $(\Gamma, d_S)$  are  $L$ -separated.*

*Proof.* The statement is implicit in the proof of [10, Lemma 12]. □

**Proposition 5.2.** *If each  $H_k$  is uniformly embeddable so is  $B(n)$ . A similar statement is true for exactness.*

*Proof.* The proof is by induction. For the basis, observe that  $B(1)$  is uniformly embeddable by (6) and the finite union theorem (Corollary 4.5). For the induction step, assume that  $B(n-1)$  is uniformly embeddable. Using again the finite union theorem and (7) we are reduced to verifying that each  $B(n-1)H_k$  is uniformly embeddable. This follows from the infinite union theorem (Corollary 4.6) and Proposition 5.1.

The proof for exactness is analogous (compare [1]). □

Osin also proves the following result [10] (although not explicitly stated, the result is the content of Lemmas 17, 18 and 19) :

**Proposition 5.3** (Osin). *The metric space  $(\Gamma, d_{S \cup \mathcal{H}})$  has finite asymptotic dimension.* □

**Theorem 5.4.** *Let  $\Gamma$  be a finitely generated group which is hyperbolic relative to a finite family  $\{H_1, \dots, H_n\}$  of subgroups. Then  $\Gamma$  is uniformly embeddable in a Hilbert space if and only if each subgroup  $H_i$  is uniformly embeddable in a Hilbert space.*

*Proof.* If  $\Gamma$  is uniformly embeddable, then so are its subgroups, the  $H_k$ . For the converse we apply Corollary 4.7 to the isometric actions of  $\Gamma$  on the metric spaces  $X = (\Gamma, d_S)$ ,  $Y = (\Gamma, d_{S \cup \mathcal{H}})$ , where  $p$  is the identity map and  $y_0 = e$ . Then  $B(n) = p^{-1}(B_Y(e, n))$  which is uniformly embeddable by Proposition 5.2. □

It is clear that the analogous result for  $C^*$ -exact groups, due to Ozawa [12], can be recovered arguing as above.

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