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Uniform Oriented Matroids Without the Isotopy Property*

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Abstract. We give an easy general construction for uniform oriented matroids with disconnected realization space. This disproves the longstanding isotopy conjecture for simple line arrangements or order types in the plane.

We write $\mathcal{R}(M)$ for the space of all vector realizations $(x_1, \ldots, x_n) \in (\mathbf{R}^3)^n$ of a rank 3 oriented matroid M on n points. (In other words, $\mathcal{R}(M)$ is the set of $3 \times n$-matrices whose maximal minors have signs given by the alternating map $M: \{1, 2, \ldots, n\}^3 \to \{-, 0, +\}$.) If M is uniform (i.e., all minors are nonzero) then $\mathcal{R}(M)$ is an open subset of $\mathbf{R}^{3n}$. White's earlier paper [8] gives a nonuniform oriented matroid M_W with $\mathcal{R}(M_W)$ disconnected and $n = 42$. The aim of the present paper is to present a uniform oriented matroid $\tilde{M}$ with $\mathcal{R}(\tilde{M})$ disconnected. That is, $\tilde{M}$ does not have the isotopy property.

A rank 3 oriented matroid M is said to be *constructible* if (x_1, x_2, x_3, x_4) is a projective basis and the point x_t is incident to at most two lines spanned by $\{x_1, x_2, \ldots, x_{t-1}\}$ for $t = 5, 6, \ldots, n$. Using the configuration $\lambda_1 = \Omega(17, 15, 13)[\lambda_0]$ in [4] or a similar modification of White's example [8], we easily get a constructible oriented matroid whose realization space has two connected components. For example, the space $\mathcal{R}(\lambda_1)$ modulo the connected group $\mathrm{PGL}(\mathbf{R}^3)$ equals the set of matrices

$$\begin{pmatrix} 1 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 1 & 1 & 1 & 1 & 5 & 5 & 0 & 1 & -1 \\ 0 & 1 & 0 & 1 & 1 & 0 & -1 & 2 & 2 & 2 & 4 & 6 & 0 & -1 & t & -t & t \\ 0 & 0 & 1 & 1 & 1 & 1 & 0 & 1 & 2 & 0 & 1 & 0 & 6 & 5 & 1 & 0 & t-1 \end{pmatrix}$$

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with $\frac{1}{5} < t < \frac{1}{2}(1 - 1/\sqrt{5})$ or $\frac{1}{2}(1 + 1/\sqrt{5}) < t < \frac{4}{5}$. Hence it suffices to prove the following:

Theorem. *Let M be a constructible rank* 3 *oriented matroid on n points. Then there exists a* uniform *rank* 3 *oriented matroid $\tilde{M}$ on at most* $4(n-3)$ *points and a continuous surjective map $\mathcal{R}(\tilde{M}) \to \mathcal{R}(M)$. Hence $\mathcal{R}(\tilde{M})$ is disconnected whenever $\mathcal{R}(M)$ is disconnected.*

Proof. We define a sequence $M =: M_n, M_{n-1}, M_{n-2}, \ldots, M_5, M_4 =: \tilde{M}$ of oriented matroids and maps between their realization spaces. Let $n \geq t \geq 5$. Then M_{t-1} is constructed from M_t as follows. First assume that x_t is incident to exactly two lines $x_i \vee x_j$ and $x_k \vee x_l$ with $1 \leq i, j, k, l < t$. Using the notation of Billera and Munson [1], we let M_t' be the oriented matroid obtained from M_t by the four successive *principal extensions*

$$x_{t,1} := [x_t^+, x_i^+, x_k^+], \qquad x_{t,2} := [x_t^+, x_i^+, x_k^-], \qquad x_{t,3} := [x_t^+, x_i^-, x_k^-], \qquad x_{t,4} := [x_t^+, x_i^-, x_k^+]. \tag{1}$$

These extensions can be carried out for *every* vector realization of M_t by setting

$$x_{t,1} := x_t + \varepsilon_1 x_i + \varepsilon_2 x_k, \qquad x_{t,2} := x_t + \varepsilon_3 x_i - \varepsilon_4 x_k,$$

$$x_{t,3} := x_t - \varepsilon_5 x_i - \varepsilon_6 x_k, \qquad x_{t,4} := x_t - \varepsilon_7 x_i + \varepsilon_8 x_k$$

where $1 \gg \varepsilon_1 \gg \varepsilon_2 \gg \cdots \gg \varepsilon_8 > 0$. This implies that the *deletion map* $\Pi: \mathcal{R}(M_t') \to \mathcal{R}(M_t)$ is surjective. Geometrically speaking, in every affine realization of M_t', the intersection point x_t is "caught" in the quadrangle $(x_{t,1}, x_{t,2}, x_{t,3}, x_{t,4})$. Define $M_{t-1} := M_t' \backslash x_t$ by deletion of that point, and let $\pi: \mathcal{R}(M_t') \to \mathcal{R}(M_{t-1})$ denote the corresponding map. (See Fig. 1.)

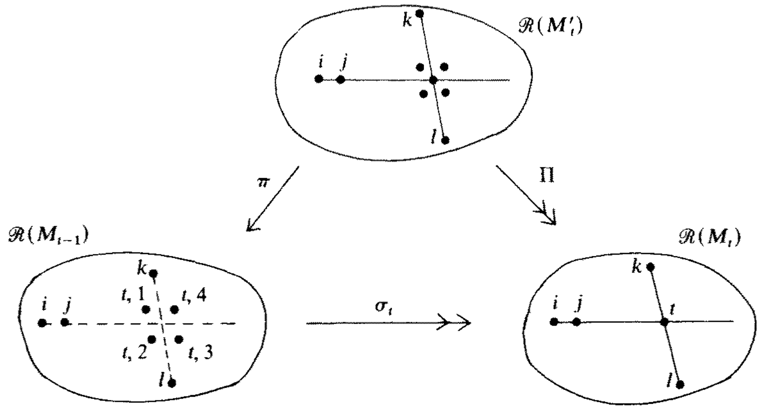

Fig. 1. Illustration of the oriented matroids M_t, M_t', and M_{t-1}.

Next consider an *arbitrary* realization $X := (x_1, \ldots, x_{t-1}, x_{t,1}, x_{t,2}, x_{t,3}, x_{t,4}, x_{t+1,1}, \ldots, x_{n,4})$ of M_{t-1}. As a consequence of the principal extension construction used in (1), $x_i \vee x_j$ and $x_k \vee x_l$ are the only lines spanned by $\{x_1, \ldots, x_{t-1}, x_{t+1,1}, \ldots, x_{n,4}\}$ which intersect the quadrangle $(x_{t,1}, x_{t,2}, x_{t,3}, x_{t,4})$. For any other such line the intersection point $x_t := (x_i \vee x_j) \wedge (x_k \vee x_l)$ is on the same side as $x_{t,1}, \ldots, x_{t,4}$. Therefore $\sigma_t(X) := (x_1, \ldots, x_t, x_{t+1,1}, \ldots, x_{n,4}) \in \mathcal{R}(M_t)$.

Hence we have a well-defined continuous map $\sigma_t\colon \mathcal{R}(M_{t-1}) \to \mathcal{R}(M_t)$, $X \mapsto \sigma_t(X)$. Moreover, σ_t is surjective because $\Pi = \sigma_t \circ \pi$ is surjective.

It remains to define M_{t-1} and σ_t when x_t is incident to less than two lines in M_t. If x_t is on no such line, then we define $M_{t-1} := M_t$ and σ_t as the identity map. Finally, suppose that x_t is on only one line $x_i \vee x_j$, $1 \le i, j < t$. In that case we replace (1) by setting $x_{t,1} := [x_i^+, x_k^+]$, $x_{t,2} := [x_i^+, x_j^+, x_k]$, $x_{t,3} := [x_i^-, x_j, x_k]$ for some $x_k \notin x_i \vee x_j$, and in the definition of the map σ_t we set $x_t := (x_i \vee x_j) \wedge (x_k \vee x_{t,1})$.

Iterating these constructions resolves all previous dependencies, and we obtain a uniform oriented matroid $\tilde{M} := M_4$ on $4(n-3)$ or fewer points. Moreover, we have a continuous surjection $\sigma := \sigma_n \circ \sigma_{n-1} \circ \cdots \circ \sigma_5$ from $\mathcal{R}(\tilde{M})$ onto $\mathcal{R}(M)$. □

Remarks. (a) Using a fairly straightforward procedure for doubling oriented matroids, we get the following corollary: *Given any integer C, there exists a uniform rank 3 oriented matroid $\tilde{M}_C$ with $4(n-3)C$ points such that $\mathcal{R}(\tilde{M}_C)$ has at least 2^C connected components.*

(b) After this paper had been accepted for publication we learned that the isotopy problem for uniform oriented matroids had been solved independently in 1985 by Mnev [9]. This result is part of a very general theory of configuration spaces developed by a group of Soviet topologists [10]. For further details see also [11].

(c) The construction presented here is an essentially simplified version of a construction originally found by the first two authors [4].

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