UNIFORM POINTWISE BOUNDS FOR MATRIX COEFFICIENTS OF UNITARY REPRESENTATIONS AND APPLICATIONS TO KAZHDAN CONSTANTS

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1. Introduction

Let k be a local field and G the group of k-rational points of a connected reductive linear algebraic group over k with k-semisimple rank$(G) \geq 2$. Let K be a good maximal compact subgroup of G. For a unitary representation ρ of G, a vector v in ρ is called K-finite if the subspace spanned by Kv is finite dimensional. We will use the term K-matrix coefficients (resp. K-finite matrix coefficients) of ρ to refer to its matrix coefficients with respect to K-invariant (resp. K-finite) unit vectors. Following [BT], we denote by G^+ the subgroup generated by the unipotent k-split subgroups of G. The main goal of the present paper is to construct a class of uniform pointwise bounds for the K-finite matrix coefficients of all infinite dimensional irreducible unitary representations of G, or more generally of all unitary representations of G without a non-zero G^+-invariant vector.

Let A be a maximal k-split torus and A^+ the closed positive Weyl chamber of A such that the Cartan decomposition $G = KA^+\Omega K$ holds where Ω is a finite subset of the

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centralizer of A (2.1). Denote by Φ the set of non-multipliable roots of A and by Φ^+ the set of positive roots in Φ. A subset $\mathcal{S}$ of Φ^+ is called a *strongly orthogonal* system of Φ if any two distinct elements α and β of $\mathcal{S}$ are strongly orthogonal, that is, neither of $\alpha \pm \beta$ belongs to Φ. The notation $\Xi_{PGL_2(k)}$ denotes the Harish-Chandra function of $PGL_2(k)$ (2.2). For simplicity, we use the notation:

$$\Xi_{PGL_2(k)}(x) := \Xi_{PGL_2(k)} \begin{pmatrix} x & 0 \\ 0 & 1 \end{pmatrix}.$$

We denote by $Z(G)$ the center of G and by *s.s rank* the semisimple rank of G.

1.1. Theorem. *Let k be any local field with char$(k) \neq 2$ and $k \neq \mathbb{C}$. Let G be the group of k-rational points of a connected reductive linear algebraic group over k with k-s.s.rank$(G) \geq 2$ and $G/Z(G)$ almost k-simple. Let $\mathcal{S}$ be a strongly orthogonal system of Φ. Then for any unitary representation ρ of G without a non-zero G^+-invariant vector and with K-finite unit vectors v and w,*

$$|\langle \rho(g)v, w\rangle| \leq \left([K : K \cap dKd^{-1}] \cdot dim\langle Kv\rangle dim\langle Kw\rangle\right)^{1/2} \prod_{\alpha\in\mathcal{S}} \Xi_{PGL_2(k)}(\alpha(a))$$

for any $g = k_1adk_2 \in KA^+\Omega K = G$.

When k is archimedean, we have $\Omega = \{e\}$, and for non-archimedean k, $K \cap gKg^{-1}$ is an open compact subgroup of K for any $g \in G$ and hence $[K : K \cap gKg^{-1}] < \infty$. For k non-archimedean, we fix a uniformizer q so that $|q| = p^{-1}$ where p is the cardinality of the residue field of k. The Harish-Chandra function $\Xi_{PGL_2(k)}$ has the following formula:

$$\Xi_{PGL_2(\mathbb{R})}(x) = \frac{2}{\pi\sqrt{x}} \int_0^{\pi/2} \left(\frac{\cos^2 t}{x^2} + \sin^2 t\right)^{-1/2} dt, \qquad \text{for } x \geq 1$$

and for k non-archimedean,

$$\Xi_{PGL_2(k)}(q^n) = \frac{1}{\sqrt{p}^n} \left(\frac{n(p-1) + (p+1))}{p+1}\right), \qquad \text{for } n \in \mathbb{N}.$$

Note that $\Xi_{PGL_2(k)}(q^n) = \Xi_{PGL_2(k)}(q^{-n})$ for any $n \in \mathbb{Z}$.

Theorem 1.2. *Let G be a connected reductive complex algebraic group with semisimple rank at least 2 and $G/Z(G)$ almost simple. Let $\mathcal{S}$ be a strongly orthogonal system of Φ. Let ρ be any unitary representation of G without a non-zero G^+-invariant vector.*

(1) *If $G/Z(G) \not\cong Sp_{2n}(\mathbb{C})$ (locally), then for any K-finite unit vectors v and w,*

$$|\langle \rho(g)v, w\rangle| \leq (dim\langle Kv\rangle dim\langle Kw\rangle)^{1/2} \prod_{\alpha\in\mathcal{S}} \Xi_{PGL_2(\mathbb{C})}(\alpha(a))$$

for any $g = k_1 a k_2 \in KA^+K = G$.

(2) *If* $G/Z(G) \cong Sp_{2n}(\mathbb{C})$ *(locally), let* $n_\alpha = \frac{1}{2}$ *if* α *is a long root and* $n_\alpha = 1$ *otherwise. Then for any* K*-invariant unit vectors* v *and* w*,*

$$|\langle \rho(g)v, w\rangle| \leq \prod_{\alpha\in\mathcal{S}} \Xi_{PGL_2(\mathbb{C})}^{n_\alpha}(\alpha(a))$$

for any $g = k_1 a k_2 \in KA^+K = G$.

The Harish-Chandra function $\Xi_{PGL_2(\mathbb{C})}$ is as follows:

$$\Xi_{PGL_2(\mathbb{C})}(x) = \frac{1}{\pi x}\int_0^{\pi/2}\left(\frac{\cos^2 t}{x^2} + \sin^2 t\right)^{-1}\sin(2t)\,dt, \qquad \text{for } x \geq 1.$$

Remark.

(1) Note in the above theorems that G/G^+ is finite mod center. It follows that any infinite dimensional irreducible unitary representation of G has no non-zero G^+-invariant vector.

(2) Note that $G = G^+$ in the case when G is almost k-simple and simply connected. If $k = \mathbb{R}$ and G is semisimple, then G^+ coincides with the connected component of the identity in G.

To simplify the explanation, we assume $G/Z(G) \ncong Sp_{2n}(\mathbb{C})$ in the rest of introduction,

Definition. For a strongly orthogonal system $\mathcal{S}$ of Φ, we set a bi-K-invariant function $\xi_{\mathcal{S}}$ of G as follows:

$$\xi_{\mathcal{S}}(k_1 a d k_2) = \prod_{\alpha\in\mathcal{S}} \Xi_{PGL_2(k)}(\alpha(a)) \text{ for any } k_1 a d k_2 \in KA^+\Omega K = G.$$

By the above two theorems, $\xi_{\mathcal{S}}$ presents a uniform pointwise bound for all K-matrix coefficients (resp. K-finite matrix coefficients) of G (resp. up to a constant). Here are additional properties of $\xi_{\mathcal{S}}$:

Properties of $\xi_{\mathcal{S}}$.

(1) $0 < \xi_{\mathcal{S}}(g) \leq 1$.

(2) For any $\epsilon > 0$, there are constants $d_1 > 0$ and $d_2(\epsilon) > 0$ such that

$$d_1\left(\prod_{\alpha\in\mathcal{S}}|\alpha(a)|\right)^{-\frac{1}{2}} \leq \xi_{\mathcal{S}}(g) \leq d_2(\epsilon)\left(\prod_{\alpha\in\mathcal{S}}|\alpha(a)|\right)^{-\frac{1}{2}+\epsilon} \quad \text{for any } g = k_1 a d k_2$$

where $|\cdot|$ denotes the absolute value on k in the sense of [We, Ch1].

(3) $\xi_{\mathcal{S}}(g) = 1$ if and only if $\alpha(g) = 1$ for all $\alpha \in \mathcal{S}$.

A strongly orthogonal system $\mathcal{S}$ is called *maximal* if the coefficient of each simple root in the formal sum $\sum_{\alpha\in\mathcal{S}}\alpha$ is not less than the one in $\sum_{\alpha\in\mathcal{O}}\alpha$ for any strongly orthogonal system $\mathcal{O}$ of Φ. A maximal strongly orthogonal system for each irreducible root system has been constructed in [Oh] (see the Appendix for the list). Let $\mathcal{Q}$ denote a maximal strongly orthogonal system of Φ. In view of the above inequality (2), the uniform pointwise bound function $\xi_{\mathcal{Q}}$ gives the sharpest bound in this construction. We remark that in general there exist more than one maximal strongly orthogonal systems in Φ. Note, however, that the formal sum $\eta(\Phi) := \frac{1}{2}\sum_{\alpha\in\mathcal{Q}}\alpha$, which determines the decay rate of $\xi_{\mathcal{Q}}$, does not depend on the choice of a maximal strongly orthogonal system.

Moreover, it turns out that for $G = SL_n(k)$ or $Sp_{2n}(k)$, $\xi_{\mathcal{Q}}$ is in fact the best possible uniform pointwise bound for K-finite matrix coefficients, more precisely, there exists an irreducible class one unitary representation of G whose K-matrix coefficient is bounded below by the function $\xi_{\mathcal{Q}}^{1+\epsilon}$ up to some constant. In the following theorem, the group $Sp_{2n}(k)$ is defined by the bi-linear form $\begin{pmatrix} 0 & \bar{I}_n \\ -\bar{I}_n & 0 \end{pmatrix}$ where $\bar{I}_n$ denotes the skew diagonal $n\times n$-identity matrix.

Theorem 1.3. *Let G be either $SL_n(k)$ $(n\geq 3)$ or $Sp_{2n}(k)$ $(n\geq 2$, char $k\neq 2, k\neq\mathbb{C})$. Let P be the maximal parabolic subgroup of G which stabilizes ke_1 and v be a unique K-invariant unit vector in $\mathrm{Ind}_P^G(1)$. Then for any $\epsilon > 0$, there exists a constant C depending on ϵ such that*

$$C\cdot\xi_{\mathcal{Q}}^{1+\epsilon}(g)\leq\langle\mathrm{Ind}_P^G(1)(g)v, v\rangle\leq\xi_{\mathcal{Q}}(g)$$

for any $g\in G$.

For $Sp_{2n}(\mathbb{C})$, see Theorem 6.6 below.

In our proofs of Theorems 1.1 and 1.2, a crucial notion is the following:

Definition. Let M be the group of the k-rational points of a connected reductive linear algebraic group over k with a good maximal compact subgroup K. A unitary representation ρ of M is said to be *tempered* if for any K-finite unit vectors v and w,

$$|\langle\rho(g)v, w\rangle|\leq(\dim\langle Kv\rangle\dim\langle Kw\rangle)^{1/2}\,\Xi_M(g)\text{ for any } g\in M$$

where Ξ_M denotes the Harish-Chandra function of M (2.2).

A unitary representation ρ being tempered is equivalent to the condition that ρ is weakly contained in the regular representation of M (see Theorem 2.4 for more equivalent definitions for temperedness).

Let $\tilde{G}$ be the underlying algebraic group of G, that is, $G = \tilde{G}(k)$. Also let $\tilde{A}$ be the maximal k-split torus of $\tilde{G}$ such that $A = \tilde{A}(k)$. Denote by $H_{\mathcal{S}}$ the group of k-rational points of the connected semisimple k-subgroup generated by the one-dimensional root subgroups $\tilde{U}_{\pm\alpha}$ corresponding to $\pm\alpha$, $\alpha \in \mathcal{S}$, and by $G_{\mathcal{S}}$ the group of k-rational points of the connected reductive k-subgroup generated by $\tilde{U}_{\pm\alpha}$, $\alpha \in \mathcal{S}$ and $\tilde{A}$ (see 5.1). The following theorem then plays a key role in the proof of Theorems 1.1 and 1.2.

Theorem 1.4. *Let S be any strongly orthogonal system of Φ. Then for any unitary representation ρ of G without a non-zero G^+-invariant vector, the restrictions $\rho|_{H_{\mathcal{S}}}$ and $\rho|_{G_{\mathcal{S}}}$ are tempered. In fact, $\xi_{\mathcal{S}}|_{H_{\mathcal{S}}} = \Xi_{H_{\mathcal{S}}}$ and $\xi_{\mathcal{S}}|_{G_{\mathcal{S}}} = \Xi_{G_{\mathcal{S}}}$.*

Note that for any $\alpha \in \Phi$, the singleton $\{\alpha\}$ is a strongly orthogonal system. We set $H_\alpha = H_{\{\alpha\}}$. In particular H_α is isomorphic to either $SL_2(k)$ or $PGL_2(k)$ (see 3.1). Hence here is a special case of Theorem 1.4:

Corollary 1.5. *Let $\alpha \in \Phi$ be any root. Then for any unitary representation ρ of G without a non-zero G^+-invariant vector, the restriction $\rho|_{H_\alpha}$ is tempered.*

Theorems 1.1 and 1.2 cover all the groups of k-rational points of a connected almost simple algebraic k-groups (char $k \neq 2$) with Kazhdan property (T), except for the two rank one real groups: $Sp(1,n)$, F_4^{-20}. In fact, Theorem 1.1 also holds for $Sp(1,n)$ (see Theorem 4.11). We remark that if one can provide an analogue of Theorem 4.8 for F_4^{-20}, the same theorem holds for this group as well.

The pointwise bound $\xi_{\mathcal{S}}$ provides us with a simple and general method of calculating Kazhdan constants (see 8.1 for definition) for various compact subsets of semisimple G, in particular for any compact subset properly containing K. For instance, in $SL_n(\mathbb{R})\,(n \geq 3)$, for any $m \in \mathbb{N}$, the number $\frac{0.08}{m}$ is a Kazhdan constant with respect to the Kazhdan set $\left\{SO(2), \mathrm{diag}(4^{1/m}, 4^{-1/m})\right\}$ (embedded in the left upper corner of $SL_n(\mathbb{R})$) (see Ex 8.7.1). We also have the following interesting example:

Theorem 1.7.

$$\inf_{p=\text{ prime}} \inf_{n\geq 3} \inf_{s\notin SL_n(\mathbb{Z}_p)} \kappa(SL_n(\mathbb{Q}_p), \{SL_n(\mathbb{Z}_p), s\}) > 0.10$$

where $\kappa(G,Q)$ is the best Kazhdan constant for Q, that is,

$$\kappa(G,Q) = \inf \max_{g\in Q} \|\rho(g)v - v\|$$

where the infimum is taken over all unitary representations ρ of G without a non-zero invariant vector and for all unit vectors v of ρ.

The problem of calculating Kazhdan constants was first raised by J. P. Serre (see [Bu], [HV]). Kazhdan constants for the group of k-rational points of a semisimple algebraic group over k and for its lattices have been obtained with special choices of Kazhdan sets (see [Bu], [CMS], [Sh], [Sh1], [Zu])

The paper is organized as follows: in section 2, after recalling Cartan decomposition and the definition of the Harish-Chandra functions, we study when a unitary representation of a reductive algebraic group is tempered and recall Howe's strategy; in section 3, we study strongly orthogonal systems, subgroups H_α and the Harish-Chandra function of $PGL_2(k)$; in section 4, we show the temperedness of $\rho|_{H_\alpha}$; in section 5, we prove Theorem 1.4 as well as our main results on uniform pointwise bounds; in section 6, we show that the bounds given in section 5 are optimal for $SL_n(k)$ and for $Sp_{2n}(k)$; in section 7, we give some upper bounds for the constant $p_K(G)$ as an application of our main theorems; finally in section 8, we discuss another application to computation of Kazhdan constants.

Some of the above results for $k = \mathbb{R}$ were announced in [Oh] where the application of these results to the classification of non-Riemannian homogeneous spaces not admitting compact quotients by discrete subgroups (cf. [Ma1], [Oh]) is discussed as well.

Let G be a connected almost simple simply connected $\mathbb{Q}$-group and $\Gamma \subset G(\mathbb{Q})$ a congruence subgroup of G. Combined with some results of Clozel and Ullmo in [CU], Theorem 1.1 in the case of p-adic fields yields an application in obtaining the equidistribution of Hecke points on $G(\mathbb{R})/\Gamma$ with the rate estimate [COU]. See also [GO] and [Oh1] for other applications.

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2. Temperedness and Howe's strategy

2.1. Cartan decomposition. Let k be a local field with the standard absolute value $|\cdot|$ in the sense of [We, Ch1]. Let $\tilde{G}$ be a connected linear reductive algebraic group defined over a local field k and let $G = \tilde{G}(k)$. Let $\tilde{A}$ be a maximal k-split torus and $\tilde{B}$ a minimal parabolic k-subgroup of $\tilde{G}$ containing $\tilde{A}$. Set $A = \tilde{A}(k)$ and $B = \tilde{B}(k)$. Denote by Φ' the set of roots of $\tilde{A}$ in $\tilde{G}$ and by Φ the set of non-multipliable roots in Φ' with the ordering given by $\tilde{B}$. Let $X(\tilde{A})$ denote the set of characters of $\tilde{A}$ defined over k whose ordering is induced from Φ. Denote by X^+ (resp. Φ^+) the set of positive characters (resp. roots) in $X(\tilde{A})$ with respect to that ordering.

If k is archimedean, i.e., isomorphic to $\mathbb{R}$ or $\mathbb{C}$, we set

$$k^0 = \{x \in \mathbb{R} \mid x \geq 0\} \text{ and } \hat{k} = \{x \in \mathbb{R} \mid x \geq 1\}.$$

When k is non-archimedean, we fix a uniformizer q of k such that $|q|^{-1}$ is the cardinality of the residue field of k, and set

$$k^0 = \{q^n \mid n \in \mathbb{Z}\} \text{ and } \hat{k} = \{q^{-n} \mid n \in \mathbb{N}\}.$$

We set

$$A^0 = \{a \in A \mid \alpha(a) \in k^0 \text{ for each } \alpha \in X(\tilde{A})\}; \text{ and}$$

$$A^+ = \{a \in A \mid \alpha(a) \in \hat{k} \text{ for each } \alpha \in \Phi^+\}.$$

Equivalently $A^+ = \{a \in A^0 \mid |\alpha(a)| \geq 1 \text{ for each } \alpha \in \Phi^+\}$. We call A^+ a positive Weyl chamber of G.

Let $\tilde{Z}$ denote the centralizer of $\tilde{A}$ in $\tilde{G}$ and $Z = \tilde{Z}(k)$. Since $X(\tilde{Z})$ can be considered as a subset of $X(\tilde{A})$ in a natural way, it has an induced ordering from this inclusion. Define

$$Z_+ = \{z \in Z \mid |\alpha(z)| \geq 1 \text{ for each } \alpha \in X(Z)^+\}; \text{ and}$$

$$Z_0 = \{z \in Z \mid |\alpha(z)| = 1 \text{ for each } \alpha \in X(Z)^+\}.$$

For any subgroup H of G, $N_G(H)$ denotes the normalizer of H, $C_G(H)$ denotes the centralizer of H and $Z(H)$ denotes the center of H.

Proposition. *There exists a maximal compact subgroup K of G such that*

(1) $N_G(A) \subset KA$,
(2) *the Cartan decomposition $G = K(Z_+/Z_0)K$ and the Iwasawa decomposition $G = K(Z/Z_0)R_u(B)$ hold, in the sense that for any $g \in G$, there are elements $a \in Z_+$ (unique up to mod Z_0) and $b \in Z$ (unique up to mod Z_0) such that $g \in KaK$ and $g \in KbR_u(B)$,*
(3) *for any subset Δ of the set of simple roots of Φ and for the subgroup $M := Z_G(\{a \in A \mid \alpha(a) = 1 \text{ for all } \alpha \in \Delta\})$, $M \cap K$ satisfies the above properties (1) and (2) with respect to $(M, M \cap A)$.*

See [GV, 2.2] for archimedean case and see [B-T], [Ti1], and [Si] for non-archimedean case. In general, the positive Weyl chamber A^+ has finite index in (Z_+/Z_0). Hence for some finite subset $\Omega \subset Z_+$, $G = K(A^+\Omega)K$, i.e., for any $g \in G$, there exist unique elements $a \in A^+$ and $d \in \Omega$ such that $g \in KadK$. A maximal compact subgroup K is called a *good* maximal compact subgroup of G if it satisfies the properties listed in the above proposition.

Remark. We have $G = KA^+K$ and $G = KA^0R_u(B)$

(1) if k is archimedean, that is, $k \cong \mathbb{R}$, or $\mathbb{C}$ or;
(2) if G is split over k or;

(3) if G is quasi-split and split over an unramified extension over a non-archimedean local field k, for example $G = SU(f)$ where f is a hermitian form of dimension $2n$ or $2n+1$ over an unramified quadratic extension over k with Witt index n, so that $G \cong SU(n,n)$ or $G \cong SU(n, n+1)$.

2.1.1. Example. For $G = SL_n(k)$, let A be the subgroup of all diagonal matrices and B the subgroup of all upper triangular matrices. If k is archimedean, set

$$K = \{g \in G \mid {}^t\bar{g}g = 1\},$$

$$A^+ = \left\{ \begin{pmatrix} a_1 & & 0 \\ & \ddots & \\ 0 & & a_n \end{pmatrix} \in G \mid a_i \in \mathbb{R},\, a_1 \geq \cdots \geq a_n > 0 \right\};$$

otherwise, set

$$K = SL_n(\mathcal{O}) \text{ where } \mathcal{O} \text{ is the ring of integers of } k, \text{ and}$$

$$A^+ = \left\{ \begin{pmatrix} q^{k_1} & & 0 \\ & \ddots & \\ 0 & & q^{k_n} \end{pmatrix} \in G \mid k_i \in \mathbb{Z},\, k_1 \leq \cdots \leq k_n \right\}.$$

Note that the condition on k_i's is equivalent to saying that the norms of q^{k_i}'s are decreasing as the index i increases. Then K is a good maximal compact subgroup of G and $G = KA^+K$.

2.1.2. Example. Let D be a central simple division algebra of degree m over a non-archimedean local field k, and let $G = SL_n(D)$ be the group of all $n \times n$ matrices with entries in D which have reduced norm one. We may choose A to be the group of all diagonal matrices in G with entries in k. Then Z is the full diagonal group of G. If $d \neq 1$, then $Z \neq A$. Denote by q_D a uniformizer of D, that is, any element x in D^* can be written as $q_D^k u$ for some integer k and a unit u in D. Then the representatives of Z_+/Z_0 can be taken as

$$\left\{ \begin{pmatrix} q_D^{k_1} & & 0 \\ & \ddots & \\ 0 & & q_D^{k_n} \end{pmatrix} \in G \mid k_i \in \mathbb{Z},\, k_1 \leq \cdots \leq k_n \right\}.$$

Here

$$A^+ = \left\{ \begin{pmatrix} q^{k_1} & & 0 \\ & \ddots & \\ 0 & & q^{k_n} \end{pmatrix} \in G \mid k_i \in \mathbb{Z},\, k_1 \leq \cdots \leq k_n \right\}$$

where q is a uniformizer of k and

$$\Omega = \left\{ \begin{pmatrix} q_D^{k_1} & & 0 \\ & \ddots & \\ 0 & & q_D^{k_n} \end{pmatrix} \in G \mid -m+1 \leq k_1 \leq \cdots \leq k_n \right\}.$$

2.2. The Harish-Chandra function Ξ_G. We denote by δ_B the modular function of B; in particular for $a \in A^0$,

$$\delta_B(a) = \prod_{\text{all positive } \alpha \in \Phi'} |\alpha(a)|^{m_\alpha}$$

where m_α denotes the multiplicity of α. The Harish-Chandra function Ξ_G is defined by

$$\Xi_G(g) = \int_K \delta_B(gk)^{-1/2}\, dk.$$

As is well known, Ξ_G is the diagonal matrix coefficient $g \mapsto \langle \mathrm{Ind}_B^G(1)(g) f_0, f_0 \rangle$ where $\mathrm{Ind}_B^G(1)$ is the representation which is unitarily induced from the trivial representation 1 and f_0 is its unique (up to scalar) K-invariant unit vector. In fact, f_0 is given by

$$f_0(kb) = \delta_B^{1/2}(b) \quad \text{for } k \in K,\, b \in B.$$

We list some well-known properties of Ξ_G (see [Wa, GV, Ha1]) which will be frequently used in this paper:

Proposition.

(1) Ξ_G *is a continuous bi-$KZ(G)$-invariant function of G with values in* $(0, 1]$.

(2) *For any $\epsilon > 0$, there exist constants c_1 and $c_2(\epsilon)$ such that*

$$c_1\, \delta_B^{-1/2}(b) \leq \Xi_G(b) \leq c_2(\epsilon)\, \delta_B^{-1/2+\epsilon}(b) \quad \textit{for all } b \in B.$$

(3) Ξ_G *is $L^{2+\epsilon}(G/Z(G))$-integrable for any $\epsilon > 0$.*

2.3. We now recall the definition of strongly L^p (cf. [Li]):

Definition. For a locally compact group M, a (continuous) unitary representation ρ of M is said to be *strongly L^p* if there is a dense subset V in the Hilbert space attached to ρ such that for any x and y in V, the matrix coefficient $g \mapsto \langle \rho(g)x, y\rangle$ lies in $L^p(M)$. We say ρ is strongly $L^{p+\epsilon}$ if it is strongly L^q for any $q > p$.

For an *irreducible* unitary representation ρ, the center $Z(M)$ acts by a character (Schur's lemma). Hence for any vectors v and w of ρ, $[g] \mapsto |\langle \rho(g)v, w\rangle|$ is a well-defined function on $M/Z(M)$. Therefore the notion of its matrix coefficient being in $L^p(M/Z(M))$ and that of ρ being strongly $L^p(M/Z(M))$ are well defined.

Since the matrix coefficients of a unitary representation with respect to unit vectors are bounded by 1, a strongly L^q representation is also strongly L^p for any $p \geq q$.

Any unitary representation ρ of G is decomposed into a direct integral $\int_X \rho_x d\mu(x)$ of irreducible unitary representations of G for some measure space (X, μ) (we refer to [Zi, 2.3] or [Ma] for more detailed account for the direct integral theory). If ρ has no invariant vector and $v = \int_X v_x d\mu(x)$ is a K-invariant unit vector of ρ, then for almost all $x \in X$, ρ_x is non-trivial and v_x is a K-invariant unit vector.

We say that ρ is *weakly contained* in a unitary representation σ of G if any diagonal matrix coefficients of ρ can be approximated, uniformly on compact sets, by convex combinations of diagonal matrix coefficients of σ ([Fe], also see [CHH], [Sh]). Note that ρ is weakly contained in a countable direct sum $\infty \cdot \rho$, and vice versa.

2.4. Temperedness.

Definition. A unitary representation ρ of G is said to be *tempered* if for any K-finite unit vectors v and w,

$$|\langle \rho(g)v, w\rangle| \leq (\dim\langle Kv\rangle \dim\langle Kw\rangle)^{1/2}\, \Xi_G(g) \text{ for any } g \in G$$

where $\langle Kv\rangle$ denotes the subspace spanned by Kv and similarly for $\langle Kw\rangle$.

The following establishes equivalent definitions of a tempered unitary representation of a reductive algebraic group over a local field, generalizing the results in [CHH] for the semisimple case.

Theorem. *For any unitary representation $\rho = \int_X \rho_x\, dx$ of G, the following are equivalent:*

(1) *For almost all $x \in X$, the (irreducible) representation ρ_x is strongly $L^{2+\epsilon}(G/Z(G))$.*
(2) *ρ is weakly contained in the regular representation $L^2(G)$.*
(3) *For almost all $x \in X$, ρ_x is weakly contained in the regular representation $L^2(G)$.*
(4) *ρ is tempered.*
(5) *For almost all $x \in X$, ρ_x is tempered.*

In the case when G is semisimple, the above is equivalent to saying that ρ is strongly $L^{2+\epsilon}$.

Proof. The equivalence of (2) and (3) is well known (for example, see [Zi, Proposition 7.3.8]). We will show (2) $\Rightarrow$ (4) $\Rightarrow$ (1) $\Rightarrow$ (2) and (3) $\Rightarrow$ (5) $\Rightarrow$ (1) $\Rightarrow$ (2). The directions (2) $\Rightarrow$ (4) and (3) $\Rightarrow$ (5) follow from [CHH, Theorem 2]. Even though it is assumed that G is semisimple in [CHH, Theorem 2], the proof works for any reductive group case as well without any change. To see the direction (4) $\Rightarrow$ (1), we may assume that ρ_x is weakly contained in ρ (up to equivalence) for each $x \in X$ since G is type I (see 2.6

below). Then by [Ho, Lemma 6.2], (4) implies that for almost all $x \in X$, the K-finite matrix coefficients of ρ_x are bounded by Ξ_G up to a constant multiple. Since Ξ_G is $L^{2+\epsilon}(G/Z(G))$-integrable for any $\epsilon > 0$ and the K-finite vectors of ρ_x are dense by Peter-Weyl theorem, this proves that (4) implies (1). The direction (5) $\Rightarrow$ (1) clearly follows from the above argument. It is now enough to show the direction (1) $\Rightarrow$ (2). Since (2) and (3) are equivalent, we may assume that ρ is irreducible. For the semisimple case, it is a direct consequence of [CHH, Theorem 1]. We now make some modification of the proof in [CHH] for our claim. For a unitary representation ρ and its attached Hilbert space H, define a operator $\rho(f) : H \to H$ for $f \in C_c(G)$ as follows:

$$\langle \rho(f)v, w\rangle = \int_G f(g)\langle \rho(g)v, w\rangle \, dg \quad \text{ for } v,\, w \in H.$$

By [Ey], (2) is equivalent to saying that for any $f \in C_c(G)$

$$\|\rho(f)\| \le \|\lambda(f)\|$$

where λ denotes the regular representation $L^2(G)$, and $\|\rho(f)\|$ denotes the operator norm of $\rho(f)$ and similarly for $\|\lambda(f)\|$. Let μ_1 and μ_2 be Haar measures on $Z(G)$ and $G/Z(G)$ respectively. Since $Z(G)$ is a normal subgroup of G, without loss of generality, we may assume that $G = Z(G) \times G/Z(G)$ and $dg = d\mu_1 \times d\mu_2$ (cf. [Pa, 1.11]). For $f \in C_c(G)$ such that supp $f = Y_1Y_2$ where $Y_1 \subset Z(G)$ and $Y_2 \subset G/Z(G)$ are compact subsets,

$$\int_G f(g)\, dg = \int_{Y_2}\int_{Y_1} f(zh)\, d\mu_1(z)\, d\mu_2(h).$$

Then the Haar measure μ_2 grows at most exponentially; hence, for some constants C and M,

$$\mu_2(Y_2^n) \le C \cdot M^n \quad \text{ for any } n \ge 1.$$

On the other hand, since $Z(G)$ is an abelian locally compact group, the Haar measure μ_1 grows polynomially (cf. [Pa, Theorem 6.17]), hence for some d and r,

$$\mu_1(Y_1^n) \le d \cdot n^r \quad \text{ for any } n \ge 1.$$

As is shown in [CHH, P. 102],

$$\|\lambda(f)\| = \lim_{n\to\infty} \|(f^* * f)^{*2n}\|_2^{\frac{1}{4n}}$$

and

$$\|\rho(f)\| = \sup_{\theta\in\rho} \lim_{n\to\infty} \left(\int_G (f^* * f)^{*2n}(x)\langle \rho(x)\theta, \theta\rangle d\mu(x)\right)^{\frac{1}{4n}}.$$

Let v be a (unit) vector in ρ such that the matrix coefficient $\langle \rho(x)v, v\rangle$ is $L^{2+\epsilon}(G/Z(G))$-integrable. Since Gv is a dense subset in ρ (ρ being irreducible), it suffices to consider vectors θ in $G.v$ in the above formula for $\|\rho(f)\|$. For $\theta \in Gv$, set $\psi(x) = \langle \rho(x)\theta, \theta\rangle$. Then we have

$$\begin{aligned}
\int_{(Y_1\times Y_2)^n} |\psi(x)|^2 d\mu(x) &\le \int_{Y_1^n} 1\, d\mu_1(x) \cdot \left(\int_{Y_2^n} |\psi(x)|^2 d\mu_2(x) \right) \\
&\le \mu_1(Y_1^n) \cdot \left(\int_{Y_2^n} |\psi(x)|^{2+\epsilon} d\mu_2(x) \right)^{\frac{2}{2+\epsilon}} \mu_2(Y_2^n)^{\frac{\epsilon}{2+\epsilon}} \\
&\le \mu_1(Y_1^n)\, \mu_2(Y_2^n)^{\frac{\epsilon}{2+\epsilon}} \left(\int_{Y_2^n} |\psi(x)|^{2+\epsilon} d\mu_2(x) \right)^{\frac{2}{2+\epsilon}} \\
&\le dn^r\, (C\, M^n)^{\frac{\epsilon}{2+\epsilon}}\, \|\psi\|^2_{L^{2+\epsilon}(G/Z(G))}
\end{aligned}$$

By plugging this into the formula for $\|\rho(f)\|$ and extracting roots, we obtain that

$$\|\rho(f)\| \le M^{\frac{\epsilon}{8(2+\epsilon)}} \lim \inf_{n\to\infty} \|(f^* * f)^{*2n}\|_2^{\frac{1}{4n}};$$

since ϵ is arbitrary,

$$\|\rho(f)\| \le \lim \inf_{n\to\infty} \|(f^* * f)^{*2n}\|_2^{\frac{1}{4n}}$$

Hence

$$\|\rho(f)\| \le \|\lambda(f)\|.$$

Since the subset of functions in $C_c(G)$ with support of the form Y_1Y_2 as above is dense in $C_c(G)$, this proves that ρ is weakly contained in $L^2(G)$. For semisimple G, the center $Z(G)$ is finite, and hence a matrix coefficient of an irreducible unitary representation ρ is $L^p(G)$-integrable if and only if $L^p(G/Z(G))$-integrable. Therefore ρ (not necessarily irreducible) being tempered is equivalent to saying that ρ is strongly $L^{2+\epsilon}(G)$ as was shown in [CHH]. $\square$

It is well known that the trivial representation (in fact, any unitary representation) of an amenable group is weakly contained in its regular representation (cf. [Zi, Proposition 7.3.6]). It follows that $\mathrm{Ind}_B^G(1)$ is weakly contained in the regular representation of G, because B is amenable and the induction map is continuous. The above theorem therefore implies that the Harish-Chandra function Ξ_G gives the sharpest point-wise bound for the K-finite matrix coefficients of unitary representations weakly contained in the regular representation of G.

2.5. The following follows from (the proof) of [CHH, Corollary in P. 108]:

Theorem. *For semisimple G and its unitary representation ρ without a non-zero invariant vector,*

(1) *if ρ is strongly $L^{2+\epsilon}$, then every non-zero matrix coefficients of ρ is $L^{2+\epsilon}$-integrable;*
(2) *if ρ is strongly $L^{2k+\epsilon}$ for some positive integer k, then for any K-finite unit vectors v and w,*

$$|\langle \rho(g)v, w\rangle| \leq (dim\langle Kv\rangle dim\langle Kw\rangle)^{1/2}\, \Xi_G^{1/k}(g) \text{ for any } g \in G$$

2.6. The group of k-rational points of a connected reductive algebraic group over k is known to be of *Type I* (see [Wa], [Be]). Hence we have:

Proposition (cf. [Ho, Proposition 6.3]**).** *Let k be a local field. Let I be a finite set, and for each $i \in I$, let G_i be the group of k-rational points of a connected reductive algebraic group over k and K_i a good maximal compact subgroup of G_i. Then for any irreducible unitary representation ρ of $\prod_{i\in I} G_i$ having $\prod_{i\in I} K_i$-invariant unit vector v, we have $\rho = \otimes_{i\in I}\rho_i$ and $v = \otimes_{i\in I} v_i$ where ρ_i is an irreducible unitary representation of G_i and v_i is a K_i-invariant unit vector of ρ_i for each $i \in I$.*

2.7. Howe's strategy. In the spirit of Howe's strategy (cf. [Ho, Proposition 6.3], [LZ, Theorem 3.1]), we have the following:

Proposition. *For $1 \leq i \leq k$, let H_i be the group of k-rational points of a connected reductive k-subgroup of G such that $H_i \cap B$, $H_i \cap A$ and $H_i \cap K$ are a minimal parabolic subgroup, a maximal split torus, and a good maximal compact subgroup of H_i respectively.*

(1) *Suppose that for all $1 \leq i \neq j \leq k$, $x_i x_j = x_j x_i$ for $x_i \in H_i$ and $x_j \in H_j$, and $H_i \cap H_j$ is a finite subset of $K \cap H_i$.*
(2) *Suppose that for each $1 \leq i \leq k$, there exists a bi-$(H_i \cap K)$-invariant positive function ϕ_i of H_i such that for any non-trivial irreducible unitary representation σ of G, and the $H_i \cap K$-matrix coefficients of $\sigma|_{H_i}$ are bounded by ϕ_i.*

Then for any unitary representation ρ of G without a non-zero invariant vector and with K-invariant unit vectors v and w,

$$|\langle \rho(\prod_{i=1}^{k} h_i c)v, w\rangle| \leq \prod_{i=1}^{k} \phi_i(h_i)$$

where $h_i \in H_i$ for each $1 \leq i \leq k$ and $c \in \cap_{i=1}^{k} C_G(H_i)$.

Proof. We denote by H_I the formal direct product of H_i's, that is, $H_I = \prod_{i=1}^{k} H_i$. Let H be the subgroup of G generated by H_i's. Then we have a natural homomorphism f from

H_I onto H, whose kernel is in $\prod_{i=1}^k (H_i \cap K)$. Let $\tilde{\rho} = \rho \circ f$ and $\tilde{\rho}|_{H_I} = \int_X \rho_x d\mu(x)$. Since $c \in \cap_{i=1}^k C_G(H_i)$, the vector $\rho(c)v$ is obviously $\prod_{i=1}^k (H_i \cap K)$-invariant. Write $\rho(c)v$ and w as $\int v_x d\mu(x)$ and $\int w_x d\mu(x)$ respectively where v_x and w_x are vectors in ρ_x. Without loss of generality, we may assume that for all $x \in X$, ρ_x is a non-trivial unitary representation of H_I and v_x and w_x are $\prod_{i=1}^k (H_i \cap K)$- invariant unit vectors. Hence by the assumption (2),

$$|\langle \rho_x(h)v_x, w_x \rangle| \le \phi_i(h) = \phi_i(f(h)) \quad \text{for all } h \in H_i.$$

The last equality holds since $\ker f|_{H_i} \subset K \cap H_i$ and ϕ_i is bi-$K \cap H_i$-invariant. Fixing $x \in X$, by Proposition 2.6, we have $\rho_x|_{H_I} = \otimes_{i=1}^k \rho_{xi}$, $v_i = \otimes_{i=1}^k v_{xi}$, and $w_x = \otimes_{i=1}^k w_{xi}$, where ρ_{xi} is an irreducible class-one representation of H_i, and v_{xi} and w_{xi} are $K \cap H_i$-invariant unit vectors for each $1 \le i \le k$.

If $h_i \in H_i$, then

$$|\langle \rho_x(\prod_{i=1}^k h_i)v_x, w_x \rangle| = \prod_{i=1}^k |\langle \rho_{xi}(h_i)v_{xi}, w_{xi} \rangle| \le \prod_{i=1}^k \phi_i(h_i).$$

Hence for any $h_i' \in H_i$ such that $f(h_i') = h_i$,

$$\begin{aligned} |\langle \rho(\prod_{i=1}^k h_i c)v, w \rangle| &= |\langle \tilde{\rho}(\prod_{i=1}^k h_i')(cv), w \rangle| \le \int_x |\langle \rho_x(\prod_{i=1}^k h_i')v_x, w_x \rangle| \, d\mu(x) \\ &\le \prod_{i=1}^k \phi_i(h_i') = \prod_{i=1}^k \phi_i(h_i). \end{aligned}$$

This proves our claim. □

Remark. In fact, the proof of [Ho, Proposition 6.3] shows that if for any $(K \cap H_i)$-finite vectors v and w, the matrix coefficient $\langle \sigma|_{H_i}(h)v, w \rangle$ is bounded by $C_{vw}\phi_i(h)$ where C_{vw} is some constant depending only on $\dim\langle Kv \rangle$ and $\dim\langle Kw \rangle$, the above proposition holds also for K-finite unit vectors v and w provided we multiply the function $\prod_{i=1}^k \phi_i$ by some constant depending on $\dim\langle Kv \rangle$ and $\dim\langle Kw \rangle$.

3. The subgroups H_α, strongly orthogonal systems and the Harish-Chandra function $\Xi_{PGL_2(k)}$

3.1. We keep the notation from section 2.1. For each $\alpha \in \Phi$, set $\tilde{U}_{(\alpha)}$ to be the unipotent subgroup of $\tilde{G}$ attached to the root α so that Lie$(\tilde{U}_{(\alpha)})$ coincides with the root subspace $\{x \in \text{Lie}(\tilde{G}) \mid \text{Ad } a(x) = \alpha(a)x \text{ for each } x \in \tilde{A}\}$. The group $\tilde{G}$ contains a connected reductive k-split k-subgroup $\tilde{G}_0$ such that we have $\tilde{A} \subset \tilde{G}_0$, $\Phi = \Phi(\tilde{A}, \tilde{G}_0)$, and for each $\alpha \in \Phi$, the subgroup $\tilde{U}_{(\alpha)} \cap \tilde{G}_0$ coincides with the one dimensional k-split root subgroup $\tilde{U}_\alpha$ in $\tilde{G}_0$ (cf. [BT, Theorem 7.2]). As usual, we set $G_0 = \tilde{G}_0(k)$ and $U_\alpha = \tilde{U}_\alpha \cap G_0$.

Proposition [Ti]**.** *For any $\alpha \in \Phi$, there is a homomorphism $\phi_\alpha : SL_2 \to \tilde{G}_0$ defined over k such that the kernel of ϕ_α is contained in the center of $SL_2(k)$ and*

$$\phi_\alpha \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \in U_\alpha,\ \phi_\alpha \begin{pmatrix} 1 & 0 \\ y & 1 \end{pmatrix} \in U_{-\alpha}, \quad \textit{and}\ \phi_\alpha \begin{pmatrix} z & 0 \\ 0 & z^{-1} \end{pmatrix} \in A$$

for any $x, y \in k$ and $z \in k^*$.

For each $\alpha \in \Phi$, we set $\tilde{H}_\alpha = \phi_\alpha(SL_2)$ and $H_\alpha = \tilde{H}_\alpha(k)$. Denote by U^+ and U^- the upper and lower triangular subgroup of SL_2. Since SL_2 is generated by $U^\pm$ and $\phi_\alpha(U^\pm) = \tilde{U}_{\pm\alpha}$, $\tilde{H}_\alpha$ is the closed subgroup of $\tilde{G}_0$ generated by $\tilde{U}_{\pm\alpha}$. Note that H_α is isomorphic to either $SL_2(k)$ (when $\ker\phi_\alpha = 1$) or $PGL_2(k)$ (when $\ker\phi_\alpha = \pm 1$).

Denote by $\bar{A}$ the diagonal subgroup of SL_2 and by $\bar{B}$ the upper triangular subgroup of SL_2. Consider the simple root $\bar{\alpha}$ of SL_2 defined by

$$\bar{\alpha} \begin{pmatrix} z & 0 \\ 0 & z^{-1} \end{pmatrix} = z^2.$$

Denote by $\bar{K}$ the good maximal compact subgroup of $SL_2(k)$ as defined in Example 2.1.1 so that the Cartan decomposition $SL_2(k) = \bar{K}\bar{A}^+\bar{K}$ and the Iwasawa decomposition $SL_2(k) = \bar{K}\bar{A}^0U^+$ hold. Here $\bar{A}^+$ and $\bar{A}^0$ are defined as in 2.1 with respect to $\bar{\alpha}$.

Lemma.

(1) $\bar{\alpha} = \alpha \circ \phi_\alpha$
(2) $\phi_\alpha(\bar{A}) = \tilde{A} \cap \tilde{H}_\alpha$
(3) $\phi_\alpha(\bar{A}^{+,0}) \subset \{a \in A \cap H_\alpha \mid \alpha(a) \in k^{+,0}\}$
(4) $\phi_\alpha(\bar{K}) \subset K \cap H_\alpha$.
(5) *The set $\{\alpha, -\alpha\}$ is a root system of H_α with respect to $A \cap H_\alpha$ and the Cartan decomposition $H_\alpha = (K \cap H_\alpha)A^+(H_\alpha)(K \cap H_\alpha)$ holds where*

$$A^+(H_\alpha) = \{a \in A \cap H_\alpha \mid \alpha(a) \in k^{+,0}\}.$$

Proof. It suffices to verify the claim (1) by Proposition 2.1. Let $a \in \bar{A}$ and $u \in U^+$. Then $aua^{-1} = \bar{\alpha}(a)u$. Hence, applying ϕ_α on both sides, we get

$$\phi_\alpha(a)\phi_\alpha(u)\phi_\alpha(a^{-1}) = \bar{\alpha}(a)\phi_\alpha(u)$$

(note that $\phi_\alpha|_{U^+}$ is a k-linear map). On the other hand, since $\phi_\alpha(a) \in A$ and $\phi_\alpha(u) \in U_\alpha$, it follows that

$$\phi_\alpha(a)\phi_\alpha(u)\phi_\alpha(a^{-1}) = \alpha(\phi_\alpha(a))(\phi_\alpha(u)).$$

Hence we have $\alpha(\phi_\alpha(a)) = \bar{\alpha}(a)$ for any $a \in \bar{A}$, proving the claim (1).$\square$

3.2. Since PGL_2 is the adjoint group of type A_1, there exists a unique k-rational isogeny $\psi_\alpha : \tilde{H}_\alpha \to PGL_2$ such that $\psi_\alpha \circ \phi_\alpha = j$ where j denotes the natural projection map $j : SL_2 \to PGL_2$. The group $PGL_2(k)$ has the Cartan decomposition and Iwasawa decomposition which are compatible with those of $SL_2(k)$ described in 3.1.

Lemma. *For any $a \in A \cap H_\alpha$,*

$$\Xi_{H_\alpha}(a) = \Xi_{PGL_2(k)} \begin{pmatrix} \alpha(a) & 0 \\ 0 & 1 \end{pmatrix}.$$

Proof. Since $\Xi_{H_\alpha}(a) = \Xi_{PGL_2(k)}(\psi_\alpha(a))$ for any $a \in A \cap H_\alpha$, it is enough show that $\psi_\alpha(a) = \begin{pmatrix} \alpha(a) & 0 \\ 0 & 1 \end{pmatrix}$. Since

$$a = \phi_\alpha \begin{pmatrix} y & 0 \\ 0 & y^{-1} \end{pmatrix} \quad \text{for some } y \in k^*,$$

we have

$$\psi_\alpha(a) = \psi_\alpha \circ \phi_\alpha \begin{pmatrix} y & 0 \\ 0 & y^{-1} \end{pmatrix} = j \begin{pmatrix} y & 0 \\ 0 & y^{-1} \end{pmatrix}.$$

Since

$$\alpha(a) = \alpha \circ \phi_\alpha \begin{pmatrix} y & 0 \\ 0 & y^{-1} \end{pmatrix} = \bar{\alpha} \begin{pmatrix} y & 0 \\ 0 & y^{-1} \end{pmatrix} = y^2,$$

we have

$$\psi_\alpha(a) = j \begin{pmatrix} y & 0 \\ 0 & y^{-1} \end{pmatrix} = \begin{pmatrix} y^2 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} \alpha(a) & 0 \\ 0 & 1 \end{pmatrix}.$$

proving the lemma. □

3.3. We recall that two distinct roots α and β in Φ^+ are said to be *strongly orthogonal* if neither of $\alpha \pm \beta$ is a root.

Definition (cf. [Oh]**).**

(1) A subset $\mathcal{S}$ of Φ^+ is called a *strongly orthogonal system* of Φ if any two elements of $\mathcal{S}$ are strongly orthogonal to each other.

(2) A strongly orthogonal system $\mathcal{S}$ is called *large* if every simple root of Φ^+ has a non-zero coefficient in the formal sum $\sum_{\alpha \in \mathcal{S}} \alpha$.

(3) A strongly orthogonal system $\mathcal{S}$ is called *maximal* if the coefficient of each simple root in the formal sum $\sum_{\alpha \in \mathcal{S}} \alpha$ is not less than the one in $\sum_{\alpha \in \mathcal{O}} \alpha$ for any strongly orthogonal system $\mathcal{O}$ of Φ.

Example. Let $G = SL_3(k)$ and α_1, α_2 the simple roots. Then there are three strongly orthogonal systems: $\{\alpha_1\}$, $\{\alpha_2\}$, $\{\alpha_1 + \alpha_2\}$.

For $G = SL_4(k)$, let α_1, α_2, α_3 be the simple roots. Then the following is a complete list of strongly orthogonal systems: {a positive root}, $\{\alpha_1, \alpha_3\}$, $\{\alpha_1 + \alpha_2, \alpha_2 + \alpha_3\}$ and $\{\alpha_1 + \alpha_2 + \alpha_3, \alpha_2\}$.

Clearly for any $\alpha \in \Phi^+$, a singleton $\{\alpha\}$ is a strongly orthogonal system. If γ is the highest root of Φ^+, then the singleton $\{\gamma\}$ is a large strongly orthogonal system. See the Appendix for a list of a maximal strongly orthogonal system for each irreducible root system constructed in [Oh]. We remark that a maximal strongly orthogonal system is not unique in general (see Remark in [Oh, 2.3]).

We note that a priori it is not clear from the definition whether a maximal strongly orthogonal system always exists. It will be interesting to give an intrinsic explanation for its existence.

3.4. Lemma. *Let $\mathcal{S}$ be a subset of Φ^+. Then $\mathcal{S}$ is a strongly orthogonal system if and only if for any $\alpha \neq \beta$ in $\mathcal{S}$, $x_\alpha x_\beta = x_\beta x_\alpha$ for any $x_\alpha \in H_\alpha$ and $x_\beta \in H_\beta$.*

Proof. Since $\tilde{H}_\alpha$ is generated by $U_{\pm\alpha}$ for any $\alpha \in \Phi$, the claim follows from the Chevalley's commutator relations [St]. □

Recall that for a group H, the notation $Z(H)$ denotes the center of H.

3.5. Corollary. *If $\mathcal{S}$ is a strongly orthogonal system of Φ,*

$$H_\alpha \cap H_\beta \subset Z(H_\alpha) \subset K \cap Z(H_\alpha)$$

for any $\alpha \neq \beta$ in $\mathcal{S}$.

Proof. If $x \in H_\alpha \cap H_\beta$, x centralizes H_α by Lemma 3.4, and hence $x \in Z(H_\alpha)$. Note that $Z(H_\alpha)$ is non-trivial only when ϕ_α is injective, and hence when H_α is isomorphic to $SL_2(k)$. Since $Z(SL_2(k)) \subset \bar{K}$, we have $\phi_\alpha(Z(SL_2(k)) = Z(H_\alpha) \subset K \cap H_\alpha$ by Lemma 3.2. □

3.6. We will now show that for any root $\alpha \in \Phi^+$, the subgroup H_α is embedded in G in one of the following four ways as the lemma below describes. We denote by $k^2 \ltimes k$ (resp. $k^4 \ltimes k$) the Heisenberg group of dimension 3 (resp. 5).

Lemma. *Assume that $G/Z(G)$ is almost k-simple with $k-$s.s. rank $(G) \geq 2$. Then for any $\alpha \in \Phi^+$, there exist a connected almost simple k-split subgroup $\tilde{G}_\alpha$ of rank 2 and a unipotent algebraic k-subgroup $\tilde{N}_\alpha$ of $\tilde{G}$ such that $C_{G_\alpha}(\tilde{H}_\alpha)\tilde{N}_\alpha$ is a Levi-decomposition of a parabolic subgroup of $\tilde{G}_\alpha$. Moreover if we set $G_\alpha = \tilde{G}_\alpha(k)$ and $N_\alpha = \tilde{N}_\alpha(k)$, one* *of the following holds for (type of G_α, N_α, the action of H_α on $N_\alpha/[N_\alpha, N_\alpha]$) up to local isomorphism:*

(1) A_2, $N_\alpha \cong k^2$, *the standard representation of* $SL_2(k)$ *on* k^2.
(2) C_2, $N_\alpha \cong k^3$, *the adjoint representation of* $SL_2(k)$ *on* k^3.
(3) C_2, $N_\alpha \cong k^2 \ltimes k$, *the standard representation of* $SL_2(k)$ *on* k^2.
(4) G_2, $N_\alpha \cong k^4 \ltimes k$, *the symplectic representation of* $SL_2(k)$ *on* k^4.

Moreover in the root system of G_α, α is a short root in the cases (2) and (4), and a long root in the case (3).

Proof. Since $G/Z(G)$ is almost k-simple, the group $\tilde{G}_0/Z(\tilde{G}_0)$ (see 3.1 for notation) is almost simple and split over k. Since Φ is a reduced irreducible root system, there exists a root $\beta \in \Phi$ such that one of $\alpha \pm \beta$ belongs to Φ. Consider $\Psi = \{i\alpha + j\beta \in \Phi \mid i, j \in \mathbb{Z}\}$. Then Ψ is a reduced irreducible root system of rank 2. Denote by $\tilde{G}_0'$ the closed subgroup of $\tilde{G}_0$ generated by the one-dimensional root sub-subgroups $\tilde{U}_\gamma$ of $\tilde{G}_0$, $\gamma \in \Psi$. Then $\tilde{G}_0'$ is of type A_2, C_2 or G_2. Denote by $\{\alpha_1, \alpha_2\}$ the set of simple roots of $\tilde{G}_0'$. Let α_1 be the short simple root if the lengths of α_1 and α_2 are different. Since any root is conjugate to a simple root by a Weyl element, we may assume that $\alpha = \alpha_i$ for $i = 1$ or 2. In the following case by case proof according to the type of $(\tilde{G}_\alpha, \alpha)$, except in the last case, we will set $\tilde{G}_\alpha = \tilde{G}_0'$. For simplicity, we omit the notation $\tilde{}$ in the rest of proof.

(1) For $(A_2, \alpha_1$ or $\alpha_2)$: since α_1 and α_2 are conjugate in A_2, it suffices to consider $\alpha = \alpha_1$. Then it is enough to consider the unipotent subgroup N_α generated by U_{α_2} and $U_{\alpha_1+\alpha_2}$. Note that N_α is a 2-dimensional abelian subgroup.
(2) For (C_2, α_1): it suffices to consider the unipotent subgroup N_α generated by U_{α_2}, $U_{\alpha_1+\alpha_2}$ and $U_{2\alpha_1+\alpha_2}$. Note that N_α is a 3-dimensional abelian subgroup.
(3) For (C_2, α_2): consider the unipotent subgroup N_α generated by U_{α_1}, $U_{\alpha_1+\alpha_2}$ and $U_{2\alpha_1+\alpha_2}$. Hence N_α is a Heisenberg subgroup with the center $U_{2\alpha_1+\alpha_2}$.
(4) For (G_2, α_1): consider the unipotent subgroup N_α generated by U_{α_2}, $U_{\alpha_1+\alpha_2}$, $U_{2\alpha_1+\alpha_2}$, $U_{3\alpha_1+\alpha_2}$, and $U_{3\alpha_1+2\alpha_2}$. Hence N_α is a Heisenberg subgroup with the center $U_{3\alpha_1+2\alpha_2}$.
(5) For (G_2, α_2): since the root system generated by long roots of G_2 is A_2, it suffices to set G_α to be the corresponding subgroup of type A_2 and to apply the case of A_2.

Now the proof of lemma is straightforward from the above list. □

Remark. Unless $\Phi = C_n$ $(n \geq 2)$ and α is a long root (simultaneously), we can always take β in the proof so that the case (3) in the above lemma does not occur. This can be seen as follows: first we may assume that α is a simple root up to conjugation. If $\Phi = G_2$, the case (4) or (5) happens as explained in the above proof. In all other cases except when α is a short (resp. long) root in $\Phi = B_n\,(n \geq 3)$ (resp. $C_n\,(n \geq 2)$), it is clear from the Dynkin diagram that we can take β so that Ψ is A_2 (here we regard B_2 as C_2). Finally if α is a short root in B_n, the case (2) arises.

3.7. The Harish-Chandra function of $PGL_2(k)$. We explicitly calculate the Harish-Chandra function of $PGL_2(k)$. Let A (resp. B) be the image of the diagonal (resp. the upper triangular) subgroup of $GL_2(k)$ under the natural projection $GL_2(k) \to PGL_2(k)$. Then the set of representatives of A can be taken to be $\{\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix} \mid a \in k^*\}$. We set $\tilde{a} = \begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix} \in A^+$. Denote by α the simple root of $PGL_2(k)$ defined by $\alpha(\tilde{a}) = a$. In the Iwasawa decomposition $PGL_2(k) = KA^0R_u(B)$, the A^0-part of an element of $PGL_2(k)$ is uniquely determined. The modular function δ_B satisfies:

$$\delta_B(k\tilde{b}n) = |\frac{b_1}{b_2}| \qquad \text{for } k \in K,\ \tilde{b} = \text{diag}\,(b_1, b_2) \in A^0 \text{ and } n \in R_u(B)$$

Recall from 2.2 that

$$\Xi_{PGL_2(k)}(\tilde{a}) = \int_K \delta_B(\tilde{a}k)^{-1/2}\, dk$$

where dk is a normalized Haar measure on K.

3.7.1. For $k = \mathbb{R}$: since

$$\Xi_{PGL_2(\mathbb{R})} \begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix} = \Xi_{SL_2(\mathbb{R})} \begin{pmatrix} \sqrt{a} & 0 \\ 0 & \sqrt{a}^{-1} \end{pmatrix}$$

we may calculate $\Xi_{SL_2(\mathbb{R})}$. Let $G = SL_2(\mathbb{R})$ and

$$K = SO_2 = \left\{ \begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix} \mid 0 \leq t < 2\pi \right\},\ dk = \frac{1}{2\pi} dt$$

where dt is the Lebesgue measure on $[0, 2\pi)$. Let $\tilde{a} = \begin{pmatrix} \sqrt{a} & 0 \\ 0 & \sqrt{a}^{-1} \end{pmatrix}$. To compute A^0-component of $\tilde{a}k$, let

$$\tilde{a}k = k'\tilde{b}n \qquad \text{where } \tilde{b} = \begin{pmatrix} b & 0 \\ 0 & b^{-1} \end{pmatrix},\ b > 0$$

and apply the standard vector e_1 on both sides. Then $\tilde{a}k.e_1 = k'b.e_1$. Hence

$$b^2 = a\cos^2 t + a^{-1}\sin^2 t \qquad \text{where } k = \begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix}$$

since $k_{11}^2 + k_{21}^2 = 1$ for any $k = (k_{ij}) \in K$.

Since $\delta_B(\tilde{a}k)^{-1/2} = b^{-1}$,

$$\Xi_{PGL_2(\mathbb{R})} \begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix} = \Xi_{SL_2(\mathbb{R})} \begin{pmatrix} \sqrt{a} & 0 \\ 0 & \sqrt{a}^{-1} \end{pmatrix} = \frac{2}{\pi\sqrt{a}} \int_0^{\pi/2} \left(\frac{\cos^2 t}{a^2} + \sin^2 t \right)^{-1/2} dt \text{ for } a \geq 1.$$

3.7.2. For $k = \mathbb{C}$: we can parameterize $K = SU_2$ by

$$\{k(\psi, t, \phi) = \begin{pmatrix} \cos t\, e^{(i(\phi+\psi)/2)} & i \sin t\, e^{(i(\phi-\psi)/2)} \\ i \sin t\, e^{(i(\psi-\phi)/2)} & \cos t\, e^{(i(\phi+\psi)/2)} \end{pmatrix} \mid \\ 0 \le t < \pi/2,\ 0 \le \psi < 2\pi,\ -2\pi \le \psi < 2\pi\}$$

and

$$dk = \frac{1}{8\pi^2} \sin(2t) d\psi\, dt\, d\phi$$

where each of $d\psi$, dt and $d\phi$ is the Lebesgue measure on the corresponding domain of each variable [KV].

By the same argument as in 3.7.1, now using the fact that the equation $k_{11}\overline{k_{11}} + k_{21}\overline{k_{21}} = 1$ for every $k = (k_{ij}) \in K$ (where k_{ij} denotes the (i, j)-th entry of k), we deduce from the Iwasawa decomposition $\tilde{a}k = k'\tilde{b}n$ that $b_1 b_2 = a$ and

$$b_1^2 = a^2 \cos^2 t + a^{-2} \sin^2 t \quad \text{where } k = k(\psi, t, \phi)$$

(here recall that a, b_1 and b_2 are positive reals).

Since

$$\delta_B(\tilde{a}k)^{-1/2} = |\left(\frac{b_1}{b_2}\right)|^{-1/2} = \left(\frac{b_1}{b_2}\right)^{-1}$$

(recall that $|z| = z\bar{z}$),

$$\int_K \delta_B(\tilde{a}k)^{-1/2}\, dk = \frac{1}{8\pi^2} \int_{-2\pi}^{2\pi} \int_0^{\pi/2} \int_0^{2\pi} \left(\frac{b_1}{b_2}\right)^{-1} \sin(2t) d\psi\, dt\, d\phi.$$

Hence

$$\Xi_{PGL_2(\mathbb{C})} \begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix} = \frac{1}{\pi a} \int_0^{\pi/2} \left(\frac{\cos^2 t}{a^2} + \sin^2 t \right)^{-1} \sin(2t)\, dt, \text{ for } a \ge 1.$$

3.7.3. For k non-archimedean with a uniformizer q such that $|q|^{-1} = p$: in the realization of $\mathrm{Ind}_B^G(1)$ on $L^2(k)$ (see 6.2), the K-invariant unit vector f_0 is given by

$$f_0(x) = \sqrt{\frac{p}{p+1}} \cdot \max(1, |x|)^{-1} \text{ for any } x \in k.$$

Hence

$$\Xi_{PGL_2(k)}(\tilde{a}) = \langle \mathrm{Ind}_B^G(1)(\tilde{a}) f_0, f_0 \rangle = \frac{p}{p+1} \int_k |a|^{-1/2} \max\left(|a|^{-1}, |ax|\right)^{-1} \max(1, |x|)^{-1}\, dx$$

where dx is a normalized Haar measure on k. For $a = q^{-n}$, we split this integral into three parts-the integral over $|x| < p^{-n}$, the integral over $p^{-n} < |x| \leq 1$ and the integral over $|x| > 1$. Then the integral becomes

$$\frac{p}{\sqrt{p}^n (p+1)} \left(\int_{|x|>1} p^{-n} |x|^{-2} \, dx + \int_{p^{-n}<|x|\leq 1} p^{-n} |x|^{-1} \, dx + \int_{|x|<p^{-n}} p^n \, dx \right).$$

Since (cf. [GGP, Ch 3.10])

$$\int_{|x|>1} |x|^{-2} \, dx = p^{-1}, \int_{p^{-n}<|x|\leq 1} |x|^{-1} \, dx = \frac{n(p-1)}{p} \text{ and } \int_{|x|<p^{-n}} dx = p^{-n},$$

we obtain

$$\Xi_{PGL_2(k)} \begin{pmatrix} q^{-n} & 0 \\ 0 & 1 \end{pmatrix} = \frac{n(p-1) + (p+1)}{\sqrt{p}^n (p+1)}.$$

Since $\begin{pmatrix} q^n & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & q^{-n} \end{pmatrix}$ in $PGL_2(\mathbb{Q}_p)$ and the latter belongs to $K \begin{pmatrix} q^{-n} & 0 \\ 0 & 1 \end{pmatrix} K$, we have

$$\Xi_{PGL_2(k)} \begin{pmatrix} q^n & 0 \\ 0 & 1 \end{pmatrix} = \Xi_{PGL_2(k)} \begin{pmatrix} q^{-n} & 0 \\ 0 & 1 \end{pmatrix}.$$

3.8. In summary, the function $\Xi_{PGL_2(k)}$ is a bi-K-invariant function of $PGL_2(k)$ such that

$$\Xi_{PGL_2(k)} \begin{pmatrix} x & 0 \\ 0 & 1 \end{pmatrix} = \begin{cases} \frac{2}{\pi\sqrt{x}} \int_0^{\pi/2} \left(\frac{\cos^2 t}{x^2} + \sin^2 t \right)^{-1/2} dt, & \text{for } x \geq 1\text{: for } k = \mathbb{R} \\ \frac{1}{\pi x} \int_0^{\pi/2} \left(\frac{\cos^2 t}{x^2} + \sin^2 t \right)^{-1} \sin(2t) \, dt, & \text{for } x \geq 1\text{: for } k = \mathbb{C} \\ \frac{n(p-1)+(p+1)}{\sqrt{p}^n (p+1)}, & \text{for } x = q^{\pm n} \text{ and } n \in \mathbb{N}\text{: for } k \text{ non-archimedean} \end{cases}$$

(recall that $|q| = p^{-1}$ where p is the cardinality of the residue field of k).

4. Temperedness of $\rho|_{H_\alpha}$

4.1. We continue the notation from 2.1 and 3.1. We also denote by G^+ the subgroup generated by the subgroups $U(k)$ where U runs through the set of unipotent k-split subgroups of $\tilde{G}$ (cf. [BT]). The goal of section 4 is to show the following two theorems 4.1 (for $k \neq \mathbb{C}$) and 4.2 (for $k = \mathbb{C}$) which play key roles in the construction of our uniform pointwise bounds for K-matrix coefficients.

Theorem. *($k \neq \mathbb{C}$) Let k be a local field not of characteristic 2. Assume that $G/Z(G)$ is almost k-simple with k-s.s. rank $G \geq 2$. For any $\alpha \in \Phi$ and for any unitary representation ρ of G without a non-zero G^+-invariant vector, the restriction $\rho|_{H_\alpha}$ is tempered.*

Moreover if $G/Z(G) \cong SL_n(k)$ (locally), k can be any local field.

4.2. When k is the complex field, there appears different phenomenon, mainly due to the fact that the oscillator representation occurs as a representation of $SL_2(\mathbb{C})$, whereas, if $k \neq \mathbb{C}$ it is a representation of a double covering of $SL_2(k)$ but not of $SL_2(k)$. More precisely, when $H_\alpha \subset G = Sp_{2n}(\mathbb{C})$ and α is a long root, the restrictions to H_α of the $K \cap H_\alpha$-matrix coefficients of a unitary representation of G without a non-zero G^+-invariant vector are in general strongly $L^{4+\epsilon}$ but not strongly $L^{2+\epsilon}$ (see Theorem 6.6 below for an example of such a representation).

Theorem. *($k = \mathbb{C}$) Let G be a connected reductive complex algebraic group with s.s. rank $G \geq 2$ and $G/Z(G)$ almost simple. For any $\alpha \in \Phi$ and for any unitary representation ρ of G without a non-zero G^+-invariant vector, the restriction $\rho|_{H_\alpha}$ is tempered, except the case when $G/Z(G) \cong Sp_{2n}(\mathbb{C})\,(n \geq 2)$ (locally) and α is a long root. In the latter case, $\rho|_{H_\alpha}$ is strongly $L^{4+\epsilon}$.*

4.3. Recall the following direct consequence of the well known Howe-Moore theorem on vanishing of the matrix coefficients at infinity [HM]: if G is the group of k-rational points of a connected reductive algebraic group over any local field k with $G/Z(G)$ almost k-simple and ρ is a unitary representation of G without a non-zero G^+-invariant vector and M is a closed subgroup of G with $M/(M \cap Z(G))$ non-compact, then ρ has no M-invariant vector. In view of this, we may assume that $G = G_\alpha$ (see Lemma 3.6 for notation) in the proofs of Theorems 4.1 and 4.2. We will then carry out case by case analysis based on the position of H_α in G_α along with the type of G_α given in Lemma 3.6. By the following lemma, we may also assume that G_α is of simply connected type and hence H_α is isomorphic to $SL_2(k)$ without loss of generality.

Lemma. *Let $\mathcal{G}_\alpha$ be the simply connected covering of $\tilde{G}_\alpha$ and $\pi : \mathcal{G}_\alpha \to \tilde{G}_\alpha$ be the k-isogeny. Set $\mathcal{H}_\alpha := \pi^{-1}(\tilde{H}_\alpha)$. Suppose that for any unitary representation ρ of $\mathcal{G}_\alpha(k)$ with no non-zero invariant vector, $\rho|_{\mathcal{H}_\alpha(k)}$ is tempered. Then for any unitary representation ρ of G_α with no non-zero G_α^+-invariant vector, $\rho|_{H_\alpha}$ is tempered.*

Proof. Let ρ be a unitary representation of G_α with no non-zero G_α^+-invariant vector. Note that $\mathcal{G}_\alpha(k) = \mathcal{G}_\alpha(k)^+$ and $\pi(\mathcal{G}_\alpha(k)) = G_\alpha(k)^+$. Hence $\rho \circ \pi|_{\mathcal{G}_\alpha(k)}$ is a unitary representation of $\mathcal{G}_\alpha(k)$ with no non-zero $\mathcal{G}_\alpha(k)$-invariant vector. Note that $\pi(\mathcal{H}_\alpha(k))$ is a subgroup of finite index in H_α. Let $h_1, \cdots, h_k$ be a set of representatives in $H_\alpha/\pi(\mathcal{H}_\alpha(k))$. Let v and w be non-zero vectors in ρ and set $w_i = \rho(h_i^{-1})w$ for each $1 \leq i \leq k$. Denote by dh a Haar measure on H_α. Since $\pi(\mathcal{H}_\alpha(k))$ is an open subgroup of H_α, the restriction of dh defines a Haar measure on $\pi(\mathcal{H}_\alpha(k))$, which will also be denoted by dh. Let dm denote the Haar measure on $\mathcal{H}_\alpha(k)$ which is the pull back of dh under the covering map

$\pi : \mathcal{H}_\alpha(k) \to \pi(\mathcal{H}_\alpha(k))$. Fix $\epsilon > 0$. Then

$$\begin{aligned}\int_{H_\alpha} |\langle \rho(h)v, w\rangle|^{2+\epsilon}\, dh &= \sum_{i=1}^{k} \int_{\pi(\mathcal{H}_\alpha(k))} |\langle \rho(h_i h)v, w\rangle|^{2+\epsilon}\, dh \\ &= \frac{1}{c}\sum_{i=1}^{k} \int_{\mathcal{H}_\alpha(k)} |\langle \rho\circ\pi(m)v, w_i\rangle|^{2+\epsilon}\, dm\end{aligned}$$

where c is the cardinality of $\ker(\pi)\cap\mathcal{H}_\alpha(k)$. Since *any* non-zero matrix coefficients of $\rho\circ\pi|_{\mathcal{H}_\alpha(k)}$ is $L^{2+\epsilon}$-integrable for any $\epsilon > 0$ by the assumption (Theorem 2.5), it follows that $\rho|_{H_\alpha}$ is tempered by Theorem 2.4. □

Our main tool is the theory of Mackey on representations of a semi-direct product of groups (cf. [Zi, Ex 7.3.4], [Ma, III.4.7], [LZ]). The first two cases of Lemma 3.6 will be handled by the following proposition:

4.4. Proposition. *Let k be any local field and let $H = SL_2(k)$. Let G be the group $H \ltimes k^2$ (resp. $H \ltimes k^3$) where H acts on k^2 (resp. k^3) as the standard (resp. adjoint) representation. Let K be a good maximal compact subgroup of H. Then for any unitary representation ρ of without any k^2 (resp. k^3)-invariant vector, $\rho|_H$ is tempered.*

Proof. Set $N = k^2$ or k^3 accordingly. Conisder the action of H on the character group $\hat{N}$ of N defined by

$$h.\chi(n) := \chi(h^{-1}nh)$$

for any $h \in H$, $\chi \in \hat{N}$ and $n \in N$.

Let N' denote the space of k-linear forms on N, and fix a non-trivial additive character λ of k. Then the map $\phi : N' \to \hat{N}$ defined by $\phi(n')(x) = \lambda(n'(x))$ for any $x \in n$ is a bijection (cf. [We, Ch II-5, Theorem 3]). Through this identification of $\hat{N}$ with $N' \cong N$, the action of H on $\hat{N}$ is equivalent to the standard $SL_2(k)$-action on k^2 if $N = k^2$, and to the standard $SL_2(k)$-action on the symmetric power $\mathrm{Sym}^2(k^2)$ of k^2 if $N = k^3$.

Therefore the actions of H in $\hat{N}$ are algebraic and hence the H-orbits on $\hat{N}$ are locally closed (see [BZ, 6.15 and 6.8]). We can easily check that the zero element in $\hat{N}$ is the only H-fixed point in $\hat{N}$. Hence the stabilizer in H of any non-zero element in $\hat{N}$ is amenable, since any proper algebraic subgroup of $SL_2(k)$ is amenable.

Now assume that ρ is irreducible. Applying Mackey's theory, we conclude that ρ is induced from an irreducible unitary representation σ of the stabilizer in G of an element, say χ, of $\hat{N}$ and if χ is trivial, then $\rho|_N$ contains the trivial representation (cf. [Zi, Theorem 7.3.1]). It then follows from the assumption that χ must be non-trivial and hence the stabilizer of χ in G, which is the semi-direct product of the stabilizer of χ in H with N, is amenable. Recall the well known fact that any irreducible unitary representation of an amenable group is weakly contained in its regular representation (cf. [Ma, Ch1, 5.5.3]). Hence ρ is weakly contained in the regular representation of G, since the induced representation of a regular representation is the regular representation and the induction map is continuous (cf. [Zi, Proposition 7.3.7]). It now follows that $\rho|_H$ is weakly contained in a multiple of the regular representation $L^2(H)$. In general, in the direct integral decomposition $\rho = \int_X \rho_x \, d\mu(x)$ where ρ_x is irreducible, for almost all $x \in X$, $\rho_x|_N$ has no non-zero invariant vector. Hence $\rho_x|_H$ is weakly contained in the regular representation $L^2(H)$ for almost all $x \in X$, by the above argument. By Theorem 2.4, this finishes the proof. □

4.5. Now the last two two cases of Lemma 3.6 will be based on Propositions 4.5 and 4.6. In both cases, H_α is a subgroup (in fact, the derived subgroup of a Levi subgroup) of a parabolic subgroup of G_α whose unipotent radical N_α is a Heisenberg subgroup. Letting $W = N_\alpha/[N_\alpha, N_\alpha]$, H_α acts on W as a sympletic representation and hence $H_\alpha \hookrightarrow Sp(W)$. Fix a subset Δ of Φ^+ such that W admits a polarization

$$W = \left(\sum_{\beta \in \Delta} U_\beta \right) \oplus \left(\sum_{\beta \in \Delta} U_{\beta^*} \right).$$

Denote by $\hat{H}_\alpha$ the double cover of H_α for $k \neq \mathbb{C}$, and set $\hat{H}_\alpha = H_\alpha$ for $k = \mathbb{C}$. Applying Mackey's theory and the theory of oscillator representation, we can conclude that for any unitary representation of G_α without a non-zero invariant vector, there exist unitary representations μ_t of $\hat{H}_\alpha$, $t \in k^*/(k^*)^2$, such that

$$\rho|_{H_\alpha} = \sum_{t \in k^*/(k^*)^2} \omega_t|_{\hat{H}_\alpha} \otimes \mu_t$$

(up to equivalence) where ω_t is the oscillator representation of the symplectic group $Sp(W)$ corresponding to $t \in k^*/(k^*)^2$ (cf. [LZ, Proposition 2.1]). Here the representation $\omega_t|_{\hat{H}_\alpha} \otimes \mu_t$ factors through $\hat{H}_\alpha \to H_\alpha$. Therefore there exists a dense subset V of $\rho|_{H_\alpha}$ such that for any $x, y \in V$, the matrix coefficient $\langle \rho|_{H_\alpha}(a)x, y \rangle$ is bounded by a constant multiple of

$$\phi(a) = \prod_{\beta \in \Delta} \left(\max(|\beta(a)|, |\beta(a)|^{-1}) \right)^{-1/2}$$

for any $a \in A(H_\alpha)$ (cf. [LZ, Proposition 2.2]). Here ϕ is essentially a matrix coefficient of $\omega_t|_{\hat{H}_\alpha}$. We keep these notation in the proofs of the following two propositions.

Proposition. *Let k be a local field not of characteristic 2. Let G be the group of k-rational points of a connected simply connected almost simple k-split group of type G_2*

and α a short root in Φ. For any unitary representation ρ of G without a non-zero invariant vector, the restriction $\rho|_{H_\alpha}$ is tempered.

Proof. Let α_1 and α_2 be as in the the proof of Lemma 3.6 (in particular α_1 is short). Then we may assume thar $\alpha = \alpha_1$ (up to conjugation by a Weyl element). We may assume that ρ is irreducible without loss of generality. The maximal split torus A of G_2 can be identified as

$$A = \{a = \mathrm{diag}(a_1, a_2, a_3) \in SL_3(k) \mid a_i \in k^*\},$$

and

$$\alpha_1(a) = a_2 \text{ and } \alpha_2(a) = \frac{a_1}{a_2}.$$

Then

$$A^+ \cap H_{\alpha_1} = \{b = \mathrm{diag}(b_1^{-1}, b_1^2, b_1^{-1}) \mid b_1 \in \hat{k}\}.$$

Since Δ can be taken as $\{\alpha_2, \alpha_1 + \alpha_2\}$,

$$\phi(b) = |\alpha_2(b)|^{-1/2} |\alpha_1 \alpha_2(b)|^{-1/2} = |b_1|^{-2}.$$

Therefore $\rho|_{H_\alpha}$ is strongly $L^{1+\epsilon}$; hence $\rho|_{H_\alpha}$ is tempered by Theorem 2.4. □

4.6. Proposition. *Let k be a local field not of characteristic 2. Let G be the group of k-rational points of a connected simply connected almost simple k-split group of type C_2 and α a long root in Φ. Let ρ be a unitary representation of G without a non-zero invariant vector.*

1. *For $k \neq \mathbb{C}$, $\rho|_{H_\alpha}$ is tempered.*
2. *For $k = \mathbb{C}$, $\rho|_{H_\alpha}$ is strongly $L^{4+\epsilon}$.*

Proof. We may realize G by $Sp_4(k)$ so that

$$A = \{a = \mathrm{diag}(a_1, a_2, a_2^{-1}, a_1^{-1}) \mid a_1, a_2 \in k^*\}$$

and

$$\alpha_1(a) = \frac{a_1}{a_2}, \quad \alpha_2(a) = a_2^2$$

are simple roots. We may assume that $\alpha = \alpha_2$. and

$$A^+(H_\alpha) = \{b = \mathrm{diag}(1, b_1, b_1^{-1}, 1) \mid b_1 \in \hat{k}\}.$$

Then

$$\phi(b) = |\alpha_1(b)|^{-1/2} = |b_1|^{-1/2}$$

for any $b \in A(H_\alpha)$. Hence $\rho|_{H_\alpha}$ is strongly $L^{4+\epsilon}$. Now consider the case when $k \neq \mathbb{C}$. By the remark in 4.1, we may assume that ρ is irreducible. By [Ho, Corollary 2.15], any non-trivial irreducible unitary representation of $Sp_4(k)$ has pure rank 2. By [Ho, Corollary 2.12], the representations ω_t are representations of $\hat{H}_\alpha$ of pure rank 1. Since $\rho|_{H_\alpha}$ is a representation of H_α and ω_t acts on the kernel of $\hat{H}_\alpha \to H_\alpha$ non-trivially, it follows that μ_t is a genuine representation of $\hat{H}_\alpha$. It is well known that the genuine irreducible unitary representation of $\hat{H}_\alpha = \hat{S}L_2(k)$ is strongly $L^{4+\epsilon}$. Hence so is μ_t, being the direct integral of irreducible. Since both ω_t and μ_t are strongly $L^{4+\epsilon}$, $\rho|_{H_\alpha}$ is strongly $L^{2+\epsilon}$. By Theorem 2.4, $\rho|_{H_\alpha}$ is tempered. The above argument for $k \neq \mathbb{C}$ is borrowed from the proof of [DHL, Proposition 4.4.]. □

4.7. Proof of Theorems 4.1 and 4.2. Let G_α and N_α be as in Lemma 3.6. By Howe-Moore, $\rho|_{G_\alpha}$ has no G_α^+-invariant vectors. Hence we may assume $G = G_\alpha$. Suppose that G_α contains $H_\alpha \ltimes k^2$, or $H_\alpha \ltimes k^3$ as the first two cases of Lemma 3.6. Since $\rho|_{k^i}$ ($i = 2, 3$ respectively) has no non-zero invariant vector again by Howe-Moore, we apply Proposition 4.4 to prove the claim. For the last two cases of Lemma 3.6, Propositions 4.5 and 4.6 imply the claim. Since the case (3) in Lemma 3.6 can be avoided unless $\Phi = C_n$ ($n \geq 2$) (i.e., $G/Z(G) \cong Sp_{2n}(\mathbb{C})$) and α is a long root (see the remark following Lemma 3.5), we need to set $n_\alpha = 1/2$ only in that case with $k = \mathbb{C}$ as in Theorem 4.2. The condition that the characteristic of k is not 2 is required only in using oscillator representations of the double cover of a symplectic group. For example, if $\Phi = A_{n-1}$, i.e.,$G/Z(G) \cong SL_n(k)$, this case does not happen. Hence Theorem 4.1 is valid in this case for any local field k.

Remark. The proof of Proposition 3.4 in [Oh] is incomplete since only the cases when N_α can be taken abelian were treated. The above proof fills up its missing part.

4.8. $Sp(1, n)$ **case.** We also show a theorem analogous to Theorem 4.1 for the real rank one group $G = Sp(1, n)$, ($n \geq 2$). In this case, we have $\Phi^+ = \{\alpha\}$.

Theorem. *Let ρ be a unitary representation of $G = Sp(1, n)$ without a non-zero invariant vector. Then $\rho|_{H_\alpha}$ is tempered.*

Proof. Consider the natural embedding $Sp(1, 2) \hookrightarrow G$. Thanks to Howe-Moore again, we may assume that $G = Sp(1, 2)$. If we denote Φ by $\{\alpha, -\alpha\}$, we have that $G_0 = H_\alpha$. It is known that any non-trivial irreducible unitary representation of G is strongly $L^{5+\epsilon}$ from the classification [Ko]. By Theorem 2.5, any K-finite matrix coefficients of ρ is bounded by $\Xi_G^{1/3}$ up to a constant multiple. But $\Xi_G^{1/3} \approx \alpha^{-5/6}$ (see 5.6 for notation). Hence $\Xi_G^{1/3}|_{H_\alpha}$ is L^2-integrable and $\rho|_{H_\alpha}$ is strongly L^2. This proves the claim. □

4.9. We now obtain the following from Theorems 4.1, 4.2 and Proposition 2.7:

Theorem. *Let k be a local field with char $(k) \neq 2$. Let G be the group of k-rational points of a connected reductive linear algebraic group over k with k-s.s. rank $G \geq 2$ and $G/Z(G)$ almost k-simple, or $G = Sp(1,n)$ (over $\mathbb{R}$). Let $\mathcal{S}$ be a strongly orthogonal system of Φ. Let $n_\alpha = \frac{1}{2}$ if $k = \mathbb{C}$, $\Phi = C_n$ and α is a long root, and $n_\alpha = 1$ otherwise. If ρ is a unitary representation of G without a non-zero G^+-invariant vector and v and w are $K \cap H_{\mathcal{S}}$-invariant unit vectors, then for any $h_\alpha \in H_\alpha$ and $c \in \cap_{\alpha\in\mathcal{S}} C_G(H_\alpha)$, we have*

$$|\langle \rho(\prod_{\alpha\in\mathcal{S}} h_\alpha c)v, w\rangle| \leq \prod_{\alpha\in\mathcal{S}} \Xi_{H_\alpha}^{n_\alpha}(h_\alpha).$$

4.10. Definition. For a strongly orthogonal system $\mathcal{S}$, define a bi-K-invariant function $\xi_{\mathcal{S}}$ of $G = KA^+\Omega K$ as follows:

$$\xi_{\mathcal{S}}(k_1 a d k_2) = \prod_{\alpha\in\mathcal{S}} \Xi_{PGL_2(k)}^{n_\alpha}\begin{pmatrix} \alpha(a) & 0 \\ 0 & 1 \end{pmatrix} \qquad \text{for } a \in A^+, d \in \Omega,\ k_1, k_2 \in K$$

where $n_\alpha = \frac{1}{2}$ if $k = \mathbb{C}$, $\Phi = C_n$ and α is a long root (all at the same time), and $n_\alpha = 1$ in all other cases.

Lemma. *For any element $a = \prod_{\alpha\in\mathcal{S}} a_\alpha c$ where $a_\alpha \in A^+(H_\alpha)$ and $c \in \cap_{\alpha\in\mathcal{S}} C_{A^0} H_\alpha$, we have*

$$\xi_{\mathcal{S}}(g) = \prod_{\alpha\in\mathcal{S}} \Xi_{H_\alpha}^{n_\alpha}(a_\alpha).$$

Proof. It suffices to show that

$$\Xi_{PGL_2(k)}\begin{pmatrix} \alpha(a) & 0 \\ 0 & 1 \end{pmatrix} = \Xi_{H_\alpha}(a_\alpha).$$

Since

$$\Xi_{H_\alpha}(a_\alpha) = \Xi_{PGL_2(k)}\begin{pmatrix} \alpha(a_\alpha) & 0 \\ 0 & 1 \end{pmatrix} \text{ for } a_\alpha \in A \cap H_\alpha$$

by Lemma 3.2, it suffices to show that $\alpha(a) = \alpha(a_\alpha)$. Since the element $\prod_{\beta\in\mathcal{S},\beta\neq\alpha} a_\beta c$ lies in $C_G(H_\alpha)$, we only need to show that $\alpha(b) = 1$ for any $b \in A \cap C_G(H_\alpha)$. Note that for any $u \in U_\alpha$, $bub^{-1} = \alpha(b)u$. Therefore if b commutes with H_α, and hence with u, we have $\alpha(b)u = u$, yielding $\alpha(b) = 1$. □

4.11. Recall from 2.1 that if k is archimedean, we have the Cartan decomposition $G = KA^+K$. Therefore the following theorem presents a uniform pointwise bound on G for all K-matrix coefficients of unitary representations of G without non-zero G^+-invariant vectors in the archimedean field case.

Theorem. *Let $k = \mathbb{R}$ or $\mathbb{C}$, and let G be the k-points of a connected reductive linear algebraic group over k with k-s.s. rank$(G) \geq 2$ and $G/Z(G)$ almost k-simple or $G = Sp(1,n)$. Let $\mathcal{S}$ be a strongly orthogonal system of Φ. Then for any unitary representation ρ of G without a non-zero G^+-invariant vector and with K-invariant unit vectors v and w, we have*

$$|\langle \rho(g)v, w\rangle| \leq \xi_{\mathcal{S}}(g) \quad \text{for } g \in G.$$

Proof. Since both functions are bi-K-invariant, it suffices to consider the case when $g \in A^+$. In fact, we have

$$A^+ \subset \prod_{\alpha\in\mathcal{S}} A^+(H_\alpha) \cdot (\cap_{\alpha\in\mathcal{S}} C_{A^0}(H_\alpha))$$

for k-archimedean (see Lemma 5.2 below). Hence it only remains to apply Theorem 4.9 with Lemma 4.10. □

Theorem 1.2 (2) is a special case of the above theorem.

5. Uniform pointwise bound $\xi_{\mathcal{S}}$

5.1. Let k be any local field with char $(k) \neq 2$. Unless stated otherwise, G denotes the group of k-rational points of a connected reductive linear algebraic group over k with k-s.s. rank$(G) \geq 2$ and $G/Z(G)$ almost k-simple. We also assume that $G/Z(G) \not\cong Sp_{2n}(\mathbb{C})$ (locally).

We state some more structure theory of algebraic groups. We continue the same notation from 2.1, 3.1 and 4.1. In particular, recall that for each $\alpha \in \Phi$, $\tilde{U}_\alpha$ (resp. $\tilde{A}$) denotes the one dimensional root subgroup (resp. the maximal k-split torus) of $\tilde{G}_0$ such that $\tilde{U}_\alpha \cap G_0 = U_\alpha$ (resp. $\tilde{A} \cap G_0 = A$). In the following discussion we freely use some facts about algebraic groups from [BT, 3.8-3.11]. Let $\mathcal{S}$ be a strongly orthogonal system of Φ. Then the set $\pm\mathcal{S} = \{\alpha, -\alpha \mid \alpha \in \mathcal{S}\}$ is a *closed* subset of Φ. Denote by $\tilde{G}_{\mathcal{S}}$ the subgroup of $\tilde{G}_0$ generated by $\tilde{A}$ and $\tilde{U}_\alpha$, $\alpha \in \pm\mathcal{S}$. Then $\tilde{G}_{\mathcal{S}}$ is a connected reductive algebraic subgroup of $\tilde{G}_0$ defined over k. We also denote by $\tilde{H}_{\mathcal{S}}$ the subgroup of $\tilde{G}_0$ generated by $\tilde{U}_\alpha$, $\alpha \in \pm\mathcal{S}$. Then $\tilde{H}_{\mathcal{S}}$ is a connected semisimple algebraic subgroup of $\tilde{G}_0$ defined over k. We set $G_{\mathcal{S}} = \tilde{G}_{\mathcal{S}}(k)$ and $H_{\mathcal{S}} = \tilde{H}_{\mathcal{S}}(k)$. Then $G_{\mathcal{S}} = H_{\mathcal{S}}A$. It follows from Proposition 2.1 that $H_{\mathcal{S}}$ and $G_{\mathcal{S}}$ admit Cartan decomposition and Iwasawa decomposition compatible with those of G:

$$H_{\mathcal{S}} = (K \cap H_{\mathcal{S}})A^+(H_{\mathcal{S}})(K \cap H_{\mathcal{S}}) = (K \cap H_{\mathcal{S}})A^0(H_{\mathcal{S}})(R_u(B) \cap H_{\mathcal{S}});$$

$$G_{\mathcal{S}} = (K \cap G_{\mathcal{S}})A^+(K \cap G_{\mathcal{S}}) = (K \cap G_{\mathcal{S}})A^0(R_u(B) \cap G_{\mathcal{S}}).$$

5.2. The subgroup $\tilde{H}_{\mathcal{S}}$ is an almost direct product of $\tilde{H}_\alpha$, $\alpha \in \mathcal{S}$, and hence the subgroup generated by H_α, $\alpha \in \mathcal{S}$ has a finite index in $H_{\mathcal{S}}$. Since the centralizer of any subset is algebraic, $C_{A^0}(H_{\mathcal{S}}) = \cap_{\alpha\in\mathcal{S}} C_{A^0}(H_\alpha)$ and $A^+(H_\alpha) \subset A^+(H_{\mathcal{S}})$ for each $\alpha \in \mathcal{S}$.

Lemma. *Let $\mathcal{S}$ be a strongly orthogonal system of Φ. Then we have*

$$\begin{cases} A^+ \subset A^+(H_{\mathcal{S}}) \cdot (\cap_{\alpha\in\mathcal{S}} C_{A^0}(H_\alpha)) & \textit{for } k \textit{ archimedean} \\ 2A^+ \subset A^+(H_{\mathcal{S}}) \cdot (\cap_{\alpha\in\mathcal{S}} C_{A^0}(H_\alpha)) & \textit{for } k \textit{ non-archimedean} \end{cases}$$

where $2A^+ = \{a^2 \mid a \in A^+\}$. In fact, for any $a \in A^+$ (resp. for $a \in 2A^+$ for the latter case), there exist elements $a_\alpha \in A^+(H_\alpha)$ (unique up to mod $K \cap H_\alpha$), $\alpha \in \mathcal{S}$ and $c \in C_{A^0}(H_{\mathcal{S}})$ such that $a = \left(\prod_{\alpha\in\mathcal{S}} a_\alpha\right) \cdot c$.

Proof. Consider the character map $\alpha : A^0 \to k^0$ for each $\alpha \in \mathcal{S}$. For k archimedean, α must be surjective, since both $\alpha(A^0)$ and k^0 are one-dimensional connected groups. Hence $A^0 = (A^0 \cap H_\alpha)\ker(\alpha)$. When k is non-archimedean, we claim that $2A^0 \subset (A^0 \cap H_\alpha)2(\ker(\alpha))$. Denote by π the natural projection $A^0 \to A^0/\ker\alpha$. Consider the map

$$\tilde{\alpha} : A^0/\ker(\alpha) \to k^0$$

which is induced by α. Then $\tilde{\alpha}(\pi(a)) = \alpha(a)$ for all $a \in A^0 \cap H_\alpha$.

Denote by a (resp. b) the generator of $A^0 \cap H_\alpha$ (resp. $A^0/\ker(\alpha)$) as a $\mathbb{Z}$-module. We may assume that $\pi(a) = b^m$ for some positive integer m. Define a character $\beta : A^0 \cap H_\alpha \to k^0$ by setting $\beta(a) = \tilde{\alpha}(b)$. Then $\beta^m(a) = \tilde{\alpha}(b^m) = \tilde{\alpha}\pi(a) = \alpha(a)$. Hence $\beta^m = \alpha$ on $H_\alpha \cap A^0$. Since H_α is locally isomorphic to $SL_2(k)$, the $\mathbb{Z}$-module generated by the root α of H_α is of index 2 in the $\mathbb{Z}$-module generated by all characters of $A(H_\alpha)$ defined over k. Hence $m = 1$ or $m = 2$.

Therefore

$$2A^0/\ker(\alpha) \subset A^0 \cap H_\alpha;$$

hence

$$(2A^0/\ker(\alpha))(2\ker(\alpha)) \subset (A^0 \cap H_\alpha)(2\ker(\alpha)).$$

This shows that $2A^0 \subset (A^0 \cap H_\alpha)2(\ker(\alpha))$ for k non-archimedean. For the rest of the proof, let $r = 1$ or 2 depending on whether k is archimedean or not. Note that $\ker(\alpha) = C_{A^0}(H_\alpha)$ and hence $A^0 \cap H_\beta \subset \ker(\alpha)$ for all $\beta \in \mathcal{S}\backslash\{\alpha\}$. If $\beta \in \mathcal{S}\backslash\{\alpha\}$, by the same argument as before, we have

$$r\ker(\alpha)/(\ker(\alpha) \cap \ker(\beta)) \subset (A^0 \cap H_\beta).$$

By an inductive argument, we obtain that

$$rA^0 \subset \left(\prod_{\alpha\in\mathcal{S}} A^0(H_\alpha) \cdot (\cap_{\alpha\in\mathcal{S}} \ker(\alpha))\right).$$

Since $\prod_{\alpha\in\mathcal{S}} A^0(H_\alpha \subset A^0(H_\alpha)$, $\cap_{\alpha\in\mathcal{S}}\ker(\alpha) = C_{A^0}(H_{\mathcal{S}})$ and $\alpha(a) \in k^+$ for any $a \in A^+$ and $\alpha \in \Phi^+$, this proves the inclusion relation. To show the uniqueness, assume that $\prod_{\alpha\in\mathcal{S}} a_\alpha c = \prod_{\alpha\in\mathcal{S}} a'_\alpha c'$ where $a_\alpha, a'_\alpha \in H_\alpha$ and $c, c' \in \cap_{\alpha\in\mathcal{S}} C_{A^0}(H_\alpha)$. Then for each $\alpha \in \mathcal{S}$, $a_\alpha^{-1}a'_\alpha \in C_{A^0}(H_\alpha)$, since $H_\beta \subset C_{A^0}(H_\alpha)$ for any $\beta \neq \alpha$ in $\mathcal{S}$. Therefore $a_\alpha^{-1}a'_\alpha \in Z(H_\alpha) \subset K \cap H_\alpha$. This finishes the proof. $\square$

Example. For $G = SL_4(k)$, let $a = \mathrm{diag}(a_1, a_2, a_3, a_4) \in A^+$. Consider the simple roots α_1, α_2 and α_3 such that $\alpha_i(a) = \frac{a_i}{a_{i+1}}$ for each $1 \le i \le 3$. Set

$$\gamma_1(a) = \frac{a_1}{a_4} \text{ and } \gamma_2(a) = \frac{a_2}{a_3}.$$

Then $\{\gamma_1, \gamma_2\}$ is a (maximal) strongly orthogonal system of Φ. Observe that a is decomposed into

$$\begin{pmatrix} \sqrt{\frac{a_1}{a_4}} & & & \\ & 1 & & \\ & & 1 & \\ & & & \sqrt{\frac{a_4}{a_1}} \end{pmatrix} \begin{pmatrix} 1 & & & \\ & \sqrt{\frac{a_2}{a_3}} & & \\ & & \sqrt{\frac{a_3}{a_2}} & \\ & & & 1 \end{pmatrix} \begin{pmatrix} \sqrt{a_1a_4} & & & \\ & \sqrt{a_2a_3} & & \\ & & \sqrt{a_2a_3} & \\ & & & \sqrt{a_1a_4} \end{pmatrix}.$$

Therefore the decomposition of a into $A^+(H_{\gamma_1})A^+(H_{\gamma_2})C_{A^0}(H_{\gamma_1}H_{\gamma_2})$ can be achieved provided $\gamma_i(a) \in \hat{k}^2$. For k archimedean, this is always the case. For k non-archimedean, $\gamma_i(a) = q^n$ for some positive integer n. Hence $\gamma_i(a) \in \hat{k}^2$ if and only if n is even. Therefore A^+ cannot be contained in $A^+(H_{\gamma_1})A^+(H_{\gamma_2})C_{A^0}(H_{\gamma_1}H_{\gamma_2})$.

5.3. We also denote by $M_{\mathcal{S}}$ the subgroup generated by $H_{\mathcal{S}}$ and $C_A(H_{\mathcal{S}})$. Then $M_{\mathcal{S}}$ has a finite index in $G_{\mathcal{S}}$, since $2A^+ \subset M_{\mathcal{S}}$ by Lemma 5.2 and A^+ has a finite index in A. Observe that $H_{\mathcal{S}}/Z(H_{\mathcal{S}}) = M_{\mathcal{S}}/Z(M_{\mathcal{S}})$. Since $\tilde{H}_{\mathcal{S}}/Z(\tilde{H}_{\mathcal{S}})$ has finite index in $\tilde{G}_{\mathcal{S}}/Z(\tilde{G}_{\mathcal{S}})$ and PGL_2 is of adjoint type, the map $\prod_{\alpha\in\mathcal{S}} \psi_\alpha$ (see 3.2) factors through $\tilde{G}_{\mathcal{S}}/Z(\tilde{G}_{\mathcal{S}})$ and the following diagram is commutative:

$$\begin{array}{ccc} H_{\mathcal{S}}/Z(H_{\mathcal{S}}) & \xleftarrow{\prod_{\alpha\in\mathcal{S}}\phi_\alpha} & \prod_{\alpha\in\mathcal{S}} SL_2(k) \\ {\scriptstyle i}\big\downarrow & & {\scriptstyle j}\big\downarrow \\ G_{\mathcal{S}}/Z(G_{\mathcal{S}}) & \xrightarrow{\prod_{\alpha\in\mathcal{S}}\psi_\alpha} & \prod_{\alpha\in\mathcal{S}} PGL_2(k) \end{array}$$

where i and j are canonical maps. We mention that the above map $\prod_{\alpha\in\mathcal{S}} \phi_\alpha$ is not in general surjective and $H_{\mathcal{S}}/Z(H_{\mathcal{S}})$ and $G_{\mathcal{S}}/Z(G_{\mathcal{S}})$ are not isomorphic.

Example. Let $G = SL_4(k)$ and $\mathcal{S} = \{\gamma_1, \gamma_2\}$ where

$$\gamma_1 \begin{pmatrix} a_1 & & & \\ & a_2 & & \\ & & a_3 & \\ & & & a_4 \end{pmatrix} = \frac{a_1}{a_2} \quad \gamma_1 \begin{pmatrix} a_1 & & & \\ & a_2 & & \\ & & a_3 & \\ & & & a_4 \end{pmatrix} = \frac{a_3}{a_4}.$$

Then

$$H_{\mathcal{S}} = \left\{ \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} \mid A, B \in SL_2(k) \right\},$$

$$M_{\mathcal{S}} = \left\{ \begin{pmatrix} xA & 0 \\ 0 & x^{-1}B \end{pmatrix} \mid A, B \in SL_2(k), x \in k^* \right\}$$

and

$$G_{\mathcal{S}} = \left\{ \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} \mid A, B \in GL_2(k), \det(AB) = 1 \right\}.$$

Hence

$$H_{\mathcal{S}}/Z(H_{\mathcal{S}}) \cong M_{\mathcal{S}}/Z(M_{\mathcal{S}}) \cong PSL_2(k) \times PSL_2(k)$$

and

$$G_{\mathcal{S}}/Z(G_{\mathcal{S}}) \cong PGL_2(k) \times PGL_2(k).$$

5.4. Recall the Cartan decomposition $G = KA^+\Omega K$. Here Ω is a finite subset in the centralizer of A. The bi-K-invariant function $\xi_{\mathcal{S}}$ of $G = KA^+\Omega K$ is defined as follows (4.10):

$$\xi_{\mathcal{S}}(k_1 a d k_2) = \prod_{\alpha \in \mathcal{S}} \Xi_{PGL_2(k)} \begin{pmatrix} \alpha(a) & 0 \\ 0 & 1 \end{pmatrix} \qquad \text{for } k_1 a d k_2 \in KA^+\Omega K.$$

Lemma. *For a strongly orthogonal system $\mathcal{S}$ of Φ, we have*

$$\xi_{\mathcal{S}}|_{H_{\mathcal{S}}} = \Xi_{H_{\mathcal{S}}} \text{ and } \xi_{\mathcal{S}}|_{G_{\mathcal{S}}} = \Xi_{G_{\mathcal{S}}}.$$

Proof. If $a \in A \cap H_{\mathcal{S}}$, then $\Xi_{H_{\mathcal{S}}}(a) = \Xi_{G_{\mathcal{S}}}(a)$. Since for any $a \in A$,

$$\Xi_{G_{\mathcal{S}}}(a) = \prod_{\alpha \in \mathcal{S}} \Xi_{PGL_2(k)}(\psi_\alpha(a)),$$

it suffices to see that $\psi_\alpha(a) = \begin{pmatrix} \alpha(a) & 0 \\ 0 & 1 \end{pmatrix}$, which can be found in the proofs of Lemma 3.2 and Lemma 4.10. $\square$

5.5. Theorem. *For any unitary representation ρ of G without a non-zero G^+-invariant vector, the restrictions $\rho|_{H_{\mathcal{S}}}$ and $\rho|_{G_{\mathcal{S}}}$ are tempered.*

Proof. Denote by H the subgroup generated by H_α, $\alpha \in \mathcal{S}$. By Theorem 4.9 and the remark following Proposition 2.7 imply that the $K \cap H$-finite matrix coefficients of $\rho|_H$ are bounded by a constant multiple of $\xi_{\mathcal{S}}|_H$. Since $\Xi_H = \xi_{\mathcal{S}}|_H$, $\rho|_H$ is tempered. Since H has finite index in $H_{\mathcal{S}}$, a similar argument to the proof of Lemma 4.3 shows that $\rho|_{H_{\mathcal{S}}}$ is tempered. Now to show that $\rho|_{G_{\mathcal{S}}}$ is tempered, write $\rho|_{G_{\mathcal{S}}}$ as a direct integral $\int_Y \sigma_y$ of irreducible representations of $G_{\mathcal{S}}$. Without loss of generality, we may assume that for all $y \in Y$, σ_y has no non-zero $G_{\mathcal{S}}^+$-invariant vector and hence $\sigma_y|_{H_{\mathcal{S}}}$ is tempered. Consider the matrix coefficient $g \mapsto \langle \sigma_y(g)v, w\rangle$ where v and w are $K \cap G_{\mathcal{S}}$-finite vectors of σ_y. Recall that the subgroup $M_{\mathcal{S}}$ (5.3) has a finite index in $G_{\mathcal{S}}$. Let $h_1, \cdots, h_k$ be a set of representatives in $G_{\mathcal{S}}/M_{\mathcal{S}}$. We denote by $d[g]$ the G-invariant measure on $G_{\mathcal{S}}/Z(G_{\mathcal{S}})$. We also denote by $d[m]$ and $d[h]$ the restrictions of $d[g]$ to $M_{\mathcal{S}}/Z(M_{\mathcal{S}})$ and $H_{\mathcal{S}}/Z(H_{\mathcal{S}})$ respectively. Fix $\epsilon > 0$. Then

$$\begin{aligned}\int_{G_{\mathcal{S}}/Z(G_{\mathcal{S}})} |\langle \sigma_y([g])v, w\rangle|^{2+\epsilon}\, d[g] &= \sum_{i=1}^{k} \int_{M_{\mathcal{S}}/Z(M_{\mathcal{S}})} |\langle \sigma_y(h_i[m])v, w\rangle|^{2+\epsilon}\, d[m] \\ &= \sum_{i=1}^{k} \int_{H_{\mathcal{S}}/Z(H_{\mathcal{S}})} |\langle \sigma_y([h])v, h_i^{-1}w\rangle|^{2+\epsilon}\, d[h]\end{aligned}$$

Since *any* non-zero matrix coefficients of $\sigma_y|_{H_{\mathcal{S}}}$ is $L^{2+\epsilon}$-integrable (Theorem 2.5) and hence $L^{2+\epsilon}(H_{\mathcal{S}}/Z(H_{\mathcal{S}}))$-integrable (since $Z(H_{\mathcal{S}})$ is finite), we have shown that

$$\int_{G_{\mathcal{S}}/Z(G_{\mathcal{S}})} |\langle \sigma_y([g])v, w\rangle|^{2+\epsilon}\, d[g] < \infty.$$

Hence σ_y is strongly $L^{2+\epsilon}\,(G_{\mathcal{S}}/Z(G_{\mathcal{S}}))$-integrable. By Theorem 2.4, this implies that $\rho|_{G_{\mathcal{S}}}$ is tempered. □

5.6. For k archimedean, a translation of a K-finite vector is not necessarily K-finite. However that is the case for the non-archimedean field case.

Lemma. *Let k be non-archimedean and ρ any unitary representation of G. Then for any K-finite vector v and for any $g \in G$, the vector gv is K-finite. Furthermore*

$$dim\langle K(gv)\rangle \le [K : gKg^{-1} \cap K] \cdot dim\langle Kv\rangle.$$

Proof. Since $(gKg^{-1})(gv) = gKv$, the subspace spanned by $(gKg^{-1})(gv)$ has the same dimension as $\langle Kv\rangle$. Now $gKg^{-1} \cap K$ is an open compact subgroup of K, and hence has finite index in K. Therefore

$$\begin{aligned}\dim\langle K(gv)\rangle &\le [K : gKg^{-1} \cap K] \cdot \dim\langle (gKg^{-1} \cap K)(gv)\rangle \\ &\le [K : gKg^{-1} \cap K] \cdot \dim\langle (gKg^{-1})(gv)\rangle \\ &= [K : gKg^{-1} \cap K] \cdot \dim\langle Kv\rangle.\end{aligned}$$

□

5.7. Main Theorem. We are now ready for our main theorem:

Theorem. *Let k be a local field with char $(k) \neq 2$. Let G be the k-rational points of a connected linear reductive algebraic group with k-s.s. rank $(G) \geq 2$, $G/Z(G)$ almost k-simple and $G/Z(G) \ncong Sp_{2n}(\mathbb{C})$ (locally). Let $\mathcal{S}$ be a strongly orthogonal system of Φ. Then for any unitary representation ρ of G without a non-zero G^+-invariant vector and with K-finite unit vectors v and w, we have*

$$|\langle \rho(g)v, w\rangle| \leq \left([K : K \cap dKd^{-1}] \cdot dim\langle Kv\rangle dim\langle Kw\rangle\right)^{1/2} \xi_{\mathcal{S}}(g).$$

for all $g = k_1 a d k_2 \in KA^+\Omega K = G$.

Proof. By Theorem 5.5, $\rho|_{G_{\mathcal{S}}}$ is tempered. By the definition of $G_{\mathcal{S}}$, $A^+ \subset G_{\mathcal{S}}$. For $g = k_1 a d k_2 \in KA^+\Omega K$ (recall $\Omega = \{e\}$ for $k = \mathbb{R}, \mathbb{C}$), we have

$$\begin{aligned}
|\langle \rho(k_1 a d k_2)v, w\rangle| &= |\langle \rho(a)(dk_2 v), (k_1^{-1}w)\rangle| \\
&\leq \left(\dim\langle K(d(k_2 v))\rangle \dim\langle K(k_1^{-1}w)\rangle\right)^{1/2} \Xi_{G_{\mathcal{S}}}(a) \\
&\leq \left([K : K \cap dKd^{-1}] \cdot \dim\langle K(k_2 v)\rangle \dim\langle K(k_1^{-1}w)\rangle\right)^{1/2} \Xi_{G_{\mathcal{S}}}(a) \\
&= \left([K : K \cap dKd^{-1}] \cdot \dim\langle Kv\rangle \dim\langle Kw\rangle\right)^{1/2} \Xi_{G_{\mathcal{S}}}(a).
\end{aligned}$$

Since $\Xi_{G_{\mathcal{S}}}(a) = \xi_{\mathcal{S}}(a) = \xi_{\mathcal{S}}(g)$ by Lemma 5.4, this finishes the proof.□

As an immediate corollary, we have:

5.8. Corollary. *Keeping the same notation as above, assume furthermore that $G = KA^+K$ holds. With the same conditions on ρ, v and w as above, we have*

$$|\langle \rho(g)v, w\rangle| \leq \left(dim\langle Kv\rangle dim\langle Kw\rangle\right)^{1/2} \xi_{\mathcal{S}}(g) \quad \text{for all } g \in G.$$

5.9. We now summarize:

Theorem (Properties of $\xi_{\mathcal{S}}$).

(1) $0 < \xi_{\mathcal{S}}(k_1 a d k_2) = \xi_{\mathcal{S}}(a) \leq 1$ *for any $a \in A^+$, $d \in \Omega$ and $k_1, k_2 \in K$.*
(2) *For any $a \in A^+$, $\xi_{\mathcal{S}}(a) = 1$ if and only if $\alpha(a) = 1$ for all $\alpha \in \mathcal{S}$.*
(3) *For any $\epsilon > 0$, there are positive constants c_1 and c_2 such that*

$$c_1 \left(\prod_{\alpha \in \mathcal{S}} |\alpha(a)|\right)^{-1/2} \leq \xi_{\mathcal{S}}(a) \leq c_2 \left(\prod_{\alpha \in \mathcal{S}} |\alpha(a)|\right)^{-1/2+\epsilon} \quad \text{for any } a \in A^+.$$

For instance, in the case when $k = \mathbb{Q}_p$, if we set

$$n_{\mathcal{S}}(g) = \frac{1}{2} \sum_{\alpha \in \mathcal{S}} \log_p |\alpha(g)|,$$

then for any $\epsilon > 0$, there exists constants c_1 and $c_2(\epsilon)$ such that for any $g \in G(\mathbb{Q}_p)$,

$$C_1 \, p^{-n_{\mathcal{S}}(g)} \leq \xi_{\mathcal{S}}(g) \leq C_2(\epsilon) \, p^{-n_{\mathcal{S}}(g)(1-\epsilon)}.$$

In the inequality (3), using the well-known estimate of $\Xi_{PGL_2(k)}$, we may replace $\prod_{\alpha \in \mathcal{S}} |\alpha(a)|^{\epsilon}$ by some polynomial of variables $|\alpha(a)|$, more precisely, for any sufficiently large integer r, there are positive constants c_1 and c_2 such that

$$c_1 \leq \left(\prod_{\alpha \in \mathcal{S}} |\alpha(a)| \right)^{1/2} \xi_{\mathcal{S}}(a) \leq c_2 \prod_{\alpha \in \mathcal{S}} \left(1 + |\alpha(a)|\right)^r$$

(see [GV] for k archimedean, and see [Si] for k non-archimedean). For two non-negative bi-K-invariant functions f_1 and f_2 of KA^+K such that $f_1(g)$, $f_2(g) \leq 1$ for all $g \in A^+$, we will write $f_1 \approx f_2$ if for any $\epsilon > 0$, there are constants d_1 and d_2 such that $d_1 f_1(a) \leq f_2(a) \leq d_2 f_1(a)^{1-\epsilon}$ for all $a \in A^+$. Since $\xi_{\mathcal{S}} \approx \prod_{\alpha \in \mathcal{S}} |\alpha|^{-1/2}$ clearly $\xi_{\mathcal{S}}$ decays fastest when $\mathcal{S}$ is a maximal strongly orthogonal system of Φ, that is, $\sum_{\alpha \in \mathcal{S}} \alpha$ is the largest among all strongly orthogonal systems of Φ (see 3.3). Set $\eta(\Phi) := \frac{1}{2} \sum_{\alpha \in \mathcal{Q}} \alpha$ where $\mathcal{Q}$ is a maximal strongly orthogonal system of Φ. Note that $\eta(\Phi)$ does not depend on the choice of maximal strongly orthogonal systems. In the appendix we present the list of $\eta(\Phi)$.

6. Representations with the slowest decay

In this section we show that for $G = SL_n(k)$ or $G = Sp_{2n}(k)$, the pointwise bound function $\xi_{\mathcal{Q}}$ is an optimal bound for the K-matrix coefficients of the class one part of the unitary dual of G for a maximal strongly orthogonal system $\mathcal{Q}$. We remark that a priori it is not clear whether there should exist one representation whose K-matrix coefficients behave essentially like $\xi_{\mathcal{Q}}^{1+\epsilon}$ in *every* direction of A^+. This is indeed the case for $G = SL_n(k)$ or $Sp_{2n}(k)$.

6.1. Consider the case when $G = SL_n(k)$. Let A^+ and K be as in Example 2.1.1 so that $G = KA^+K$ holds. Define the characters γ_i's by

$$\begin{cases} \gamma_i = \sum_{k=i}^{n-i} \alpha_k & \text{for } i \leq \lfloor n/2 \rfloor - 1 \\ \gamma_{\lfloor n/2 \rfloor} = \begin{cases} \alpha_{\lfloor n/2 \rfloor} & \text{for } n \text{ even} \\ \alpha_{\lfloor n/2 \rfloor} + \alpha_{\lfloor n/2 \rfloor + 1} & \text{for } n \text{ odd} \end{cases} \end{cases}$$

where $\alpha_i(a) = \frac{a_i}{ai+1}$ for each $1 \le i \le n-1$ and for $a = \mathrm{diag}(a_1, \cdots, a_n) \in A^+$. That is,

$$\gamma_i(a) = \frac{a_i}{a_{n+1-i}}.$$

Then $\mathcal{Q} = \{\gamma_i \mid 1 \le i \le \lfloor n/2 \rfloor\}$ is a maximal strongly orthogonal system of Φ [Oh, Proposition 2.3]. The function $\xi_{\mathcal{Q}}$ (5.4) is a bi-K-invariant function of G defined by

$$\xi_{\mathcal{Q}}(a) = \prod_{i=1}^{\lfloor n/2 \rfloor} \Xi_{PGL_2(k)} \begin{pmatrix} \frac{a_i}{a_{n+1-i}} & 0 \\ 0 & 1 \end{pmatrix}.$$

Then by 5.9, for any $\epsilon > 0$, there exist constants d_1 and d_2 (depending only on ϵ) such that

$$d_1 \cdot \xi_{\mathcal{Q}}(a)^{1+\epsilon} \le F(a) \le d_2 \cdot \xi_{\mathcal{Q}}(a) \text{ for all } a \in A^+$$

where F is a bi-K-invariant function of G defined by

$$F(a) = \begin{cases} \prod_{i=1}^{n/2} |a_i|^{-1} & \text{for n even} \\ \left(\prod_{i=1}^{(n-1)/2} |a_i|^{-1}\right) |a_{(n+1)/2}|^{-1/2} & \text{for n odd .} \end{cases}$$

6.2. We recall the formula for the matrix coefficients of the induced representation $\mathrm{Ind}_P^G(1)$ (cf. [Kn]) where P is a parabolic subgroup of G. Consider the Langlands decomposition of P: $P = MA_PN$. Denote by $\bar{N}$ the unipotent radical of the opposite parabolic subgroup to P with the common Levi subgroup MA_P. If g decomposes under the decomposition $G = KMA_PN$, we denote by $\exp H(g)$ the A_P-component of g. If g decomposes under $\bar{N}MA_PN$ as

$$g = \bar{n}(g)m(g)\exp a(g)n(g),$$

then the action is given by

$$\mathrm{Ind}_B^G(1)(g)f(x) = e^{-\delta_0(a(g^{-1}x))} f(\bar{n}(g^{-1}x)) \text{ for any } f \in L^2(\bar{N}, dx) \text{ and } x \in \bar{N}$$

where δ_0 is the half sum of positive N-roots.

Define the vector f_0 of $\mathrm{Ind}_P^G(1)$ as follows:

$$f_0(x) = e^{-\delta_0(H(x))}.$$

It is not difficult to see that f_0 is K-invariant and the matrix coefficient of $\mathrm{Ind}_P^G(1)$ with respect to f_0 is as follows:

$$\langle \mathrm{Ind}_P^G(1)(g)f_0, f_0 \rangle = \int_{\bar{N}} e^{-\delta_0(a(g^{-1}x))} e^{-\delta_0(H(\bar{n}(g^{-1}x)))} e^{-\delta_0(H(x))} dx.$$

Let $G = SL_n(k)$, and P the maximal parabolic subgroup of G which stabilizes the line $k.e_1$. We write an element of $\bar{N}$ as $x = (x_1, x_2, \cdots, x_n)^T$ where $x_1 = 1$. The realization of the representation $\mathrm{Ind}_P^G(1)$ on $L^2(\bar{N}, dx)$ can be formulated as follows (see [Oh, 4.4]): for $a \in A^+$, for $f \in L^2(\bar{N}, dx)$ and $x = (x_1 = 1, x_2, \cdots, x_n)^T \in \bar{N}$

$$\left(\mathrm{Ind}_P^G(1)(a)f\right)(x) = |a_1|^{n/2} f(a_1^{-1}x_1, a_2^{-2}x_2 \cdots a_n^{-1}x_n)$$

and the K-invariant vector f_0 is defined by

$$f_0(x) = |x|^{-n/2}.$$

Here the absolute value $|\cdot|$ in k^n is defined as follows:

$$|x| = \begin{cases} \sqrt{\sum_{i=1}^n x_i^2} \text{ for } k = \mathbb{R} \\ \sum_{i=1}^n x_i \bar{x}_i \text{ for } k = \mathbb{C} \\ \max_{1 \le i \le n} |x_i| \text{ for } k \text{ non-archimedean.} \end{cases}$$

6.3. Theorem. *Let k be any local field and $G = SL_n(k)$ $(n \ge 3)$. Then for any $\epsilon > 0$, there is a constant $c > 0$ (depending on ϵ) such that for any $g \in G$,*

$$c \cdot \xi_{\mathcal{Q}}(g)^{1+\epsilon} \cdot \|f_0\| \le \langle \mathrm{Ind}_P^G(1)(g) f_0, f_0 \rangle \le \xi_{\mathcal{Q}}(g) \cdot \|f_0\|.$$

Proof. This theorem is shown in Proposition 4.4 in [Oh] for $k = \mathbb{R}$. For $k = \mathbb{C}$, essentially the same proof goes through. We will briefly go over the case when k is a non-archimedean local field. The proof given here follows line by line the proof of Proposition 4.4 in [Oh]. For $a \in A^+$, we have

$$\langle \mathrm{Ind}_P^G(1)(a) f_0, f_0 \rangle = \int_{k^{n-1}} \max_{1 \le i \le n} \frac{|x_i|}{|a_i|}^{-n/2} \max_{1 \le i \le n} |x_i|^{-n/2}\, dm$$

where dm is a normalized Haar measure on k^{n-1}.

Set $r = \lfloor \frac{n+1}{2} \rfloor$, and let T be the following set:

$$\{x = (x_2, \cdots, x_n) \in k^{n-1} \mid 0 \le |x_i| \le 1 \text{ for } 2 \le i \le r-1,$$

$$1 \le |x_r| \le 2, |x_i| \le \frac{|a_i|}{|a_r|}|x_r| \text{ for } r+1 \le i \le n\}.$$

Note that if $x = (x_2, \cdots, x_n) \in T$, then for each $1 \le i \le n$, we have

$$|x_i| \le |q| \quad \text{and} \quad \frac{|x_i|}{|a_i|} \le \frac{|x_r|}{|a_r|}.$$

Therefore

$$\langle \mathrm{Ind}_P^G(1)(\tilde{a}) f_0, f_0 \rangle \ge C \int_T |a_r|^{n/2} dm \ge C\, |a_r|^{n/2} \prod_{i=r+1}^n \left(\frac{|a_i|}{|a_r|}\right) = C \prod_{\alpha \in \mathcal{Q}} |\alpha(a)|^{-1/2}.$$

Hence we have

$$\langle \mathrm{Ind}_P^G(1)(a) f_0, f_0 \rangle \ge d \cdot \xi_{\mathcal{Q}}(a)^{1+\epsilon}$$

for some constant $d > 0$. $\square$

6.4. We now consider the case when $G = Sp_{2n}(k)$. The group $Sp_{2n}(k)$ is defined by the bi-linear form $\begin{pmatrix} 0 & \bar{I}_n \\ -\bar{I}_n & 0 \end{pmatrix}$ where $\bar{I}_n$ denotes the skew diagonal $n \times n$-identity matrix. We may take the positive Weyl chamber A^+ and the maximal compact subgroup K to be the intersections of those of $SL_n(k)$ with G. Then an element a of A^+ is of the form

$$a = \mathrm{diag}(a_1, \cdots, a_n, a_n^{-1}, \cdots, a_1^{-1}).$$

Define the characters γ_i's by

$$\begin{cases} \gamma_i = \left(\sum_{k=i}^{n-1} 2\alpha_k\right) + \alpha_n \text{ for } i \leq n-1 \\ \gamma_n = \alpha_n \end{cases}$$

where $\alpha_i(a) = \frac{a_i}{a_{i+1}}$, $1 \leq i \leq n-1$ and $\alpha_n(a) = a_n^2$ are the simple roots of Φ. That is,

$$\gamma_i(a) = a_i^2 \quad \text{for each } i = 1, \cdots, n.$$

Then $\mathcal{Q} = \{\gamma_1, \cdots, \gamma_n\}$ is a maximal strongly orthogonal system of Φ [Oh, Proposition 2.3]

By the definition of $\xi_{\mathcal{Q}}$, we have

$$\xi_{\mathcal{Q}}(a) = \begin{cases} \prod_{i=1}^n \Xi_{PGL_2(k)} \begin{pmatrix} a_i^2 & 0 \\ 0 & 1 \end{pmatrix} \text{ for } k \neq \mathbb{C} \\ \prod_{i=1}^n \left(\Xi_{PGL_2(\mathbb{C})} \begin{pmatrix} a_i^2 & 0 \\ 0 & 1 \end{pmatrix}\right)^{1/2} \text{ for } k = \mathbb{C} \end{cases}$$

(note that γ_i is a long root in C_n for each $1 \leq i \leq n$). Hence for any $\epsilon > 0$, there exist constants d_1 and d_2 (depending on ϵ) such that

$$d_1 \cdot \xi_{\mathcal{Q}}(a)^{1+\epsilon} \leq F(a) \leq d_2 \cdot \xi_{\mathcal{Q}}(a) \text{ for all } a \in A^+$$

where

$$F(a) = \begin{cases} \prod_{i=1}^n |a_i|^{-1} & \text{for } k \neq \mathbb{C} \\ \prod_{i=1}^n |a_i|^{-1/2} = \prod_{i=1}^n {a_i}^{-1} & \text{for } k = \mathbb{C} \text{ (all } a_i \text{ positive).} \end{cases}$$

6.5. Let P be the maximal parabolic subgroup of G which stabilizes the line $k.e_1$. Let $P = MA_PN$ denote the Lang lands decomposition of P and $\bar{N} \cong k^{2n-1}$ the unipotent radical of the opposite parabolic subgroup to P with the common Levi subgroup MA_P. Then an element x of $\bar{N}$ can be identified with

$$x = (x_1 = 1, x_2, \cdots, x_n, y_n, \cdots, y_1)^T.$$

In the realization of the representation $\mathrm{Ind}_P^G(1)$ on $L^2(\bar{N}, dx)$, we have that for $a \in A^+$, for $f \in L^2(\bar{N}, dx)$ and $x \in \bar{N}$,

$$\left(\mathrm{Ind}_P^G(1)(a)f\right)(x) = |a_1|^n f\left((a_1^{-1}x_1, a_2^{-2}x_2, \cdots, a_n^{-1}x_n, a_n^{-1}y_n, \cdots, a_1^{-1}y_1)^T\right).$$

Then vector f_0 given by

$$f_0(x) = |x|^{-n}$$

is K-invariant, where the absolute value $|\cdot|$ of k^{2n} is defined similarly as in 6.2.

Theorem ($k \neq \mathbb{C}$ and char $k \neq 2$). *Let $G = Sp_{2n}(k)$ $(n \geq 2)$. Then for any $\epsilon > 0$, there is a constant $c > 0$ (depending on ϵ) such that for any $g \in G$,*

$$c \cdot \xi_{\mathcal{Q}}(g)^{1+\epsilon} \cdot \|f_0\| \leq \langle \mathrm{Ind}_P^G(1)(g) f_0, f_0 \rangle \leq \xi_{\mathcal{Q}}(g) \cdot \|f_0\|.$$

Proof. As the case of Theorem 6.3, the other inequality is also shown in Proposition 4.4 in [Oh] for $k = \mathbb{R}$. Let k be a non-archimedean local field. Then for $a \in A^+$,

$$\langle \mathrm{Ind}_P^G(1)(a) f_0, f_0 \rangle = \int_{k^{2n-1}} \max_{1 \leq i \leq n} \{\frac{|x_i|}{|a_i|}, |a_i y_i|\}^{-n} \cdot \max_{1 \leq i \leq n} \{|x_i|, |y_i|\}^{-n} \, dm$$

where $x_1 = 1$. Let T be the following set:

$$\{(x_1, \cdots, x_n, y_n, \cdots, y_1) \mid 1 \leq |y_n| \leq 2,$$
$$|y_i| \leq \frac{|a_n|}{|a_i|}|y_n|, 0 \leq |x_i| \leq 1, 0 \leq |y_i| \leq 2 \text{ for } 1 \leq i \leq n\}.$$

Note that if $(x_1, \cdots, x_n, y_n, \cdots, y_1) \in T$, then

$$|x_i| \leq |a_i a_n y_n|$$

since $|a_i| \geq 1$ for all $1 \leq i \leq n$. Therefore

$$\langle \mathrm{Ind}_P^G(1)(a) f_0, f_0 \rangle \geq C \int_T (|a_n y_n|)^{-n} dm \geq C \frac{1}{|a_1 \cdots a_n|} = d \cdot \xi_{\mathcal{Q}}^{1+\epsilon}(a)$$

where C and d are some positive constant.□

6.6. The inequality in Theorem 6.2 is not true for $G = Sp_{2n}(\mathbb{C})$, in fact, the K-matrix coefficient of the representation $\mathrm{Ind}_P^G(1)$ satisfies

$$\xi_{\mathcal{Q}}(a) \approx \left(\langle \mathrm{Ind}_P^G(1)(a) f_0, f_0 \rangle\right)^{1/2}.$$

In this case, the minimal pointwise decay can be achieved by the oscillator representation ω of $Sp_{2n}(\mathbb{C})$ (we refer the reader to [Ho] for a detailed description of ω). In the realization of ω in $L^2(\mathbb{C}^n)$, we have the following formula: for $a \in A^+$, for $f \in L^2(\mathbb{C}^n)$ and $(z_1, \cdots, z_n) \in C^n$,

$$\omega(a) f(z_1, \cdots, z_n) = \prod_{i=1}^n |a_i|^{-1/2} f(a_1^{-1} z_1, \cdots, a_n^{-1} z_n)$$

where $a =$ diag $(a_1, \cdots, a_n, a_n^{-1}, \cdots, a_1^{-1}) \in A^+$ (recall that $|a_i| = a_i \bar{a}_i = a_i^2$). The representation ω decomposes into two irreducible components, the even part ω^+ and the odd part ω^-. In the realization of ω in $L^2(\mathbb{C}^n)$, the space ω^+ can be taken as the even functions: functions such that $f(-x) = f(x)$ and the space ω^- consists of the odd functions: functions such that $f(-x) = -f(x)$. And only the even part ω^+ is class one. In fact, the function

$$f_0(z_1, \cdots, z_n) = \exp\left(\pi^{-2}(\sum_{i=1}^n z_i \bar{z}_i)\right)$$

is a K-invariant unit vector in ω^+ where $K = SU(2n) \cap Sp_{2n}(\mathbb{C})$. Then

$$\langle \omega^+(a) f_0, f_0 \rangle = \prod_{i=1}^n (a_i)^{-1} \int_{\mathbb{R}} \int_{\mathbb{R}} \exp\left(-\pi^2(1 + a_i^{-2})(x_i^2 + y_i^2)\right) dx_i\, dy_i.$$

Since $\int_{\mathbb{R}^2} \exp\left(-(x^2 + y^2)\right) dx\, dy = \pi$,

$$\langle \omega^+(a) f_0, f_0 \rangle = \prod_{i=1}^n (a_i^2 + 1)^{-1/2}.$$

Since $F_{Sp_{2n}(\mathbb{C})}(a) = \prod_{i=1}^n |a_i|^{-1/2} = \prod_{i=1}^n a_i{}^{-1}$, we have shown:

Theorem. *Let $G = Sp_{2n}(\mathbb{C})\,(n \geq 2)$. For any $\epsilon > 0$, there exists a constant $C > 0$ (depending on ϵ) such that*

$$C \cdot \xi_{\mathcal{Q}}^{1+\epsilon}(g) \leq \langle \omega^+(g) f_0, f_0 \rangle \leq \xi_{\mathcal{Q}}(g)$$

for any $g \in G$.

6.7. Corollary. *Let $G = Sp_{2n}(\mathbb{C})\,(n \geq 2)$ and α a long root in its root system C_n. Then $\omega^+|_{H_\alpha}$ is strongly $L^{4+\epsilon}$ but not strongly $L^{2+\epsilon}$.*

Proof. By conjugation, we may assume $\alpha = \gamma_1$ where γ_1 is defined as in 6.4. It then follows from the above theorem that for any $\epsilon > 0$, there exists a constant $C > 0$ such that

$$C \cdot \Xi_{H_{\gamma_1}}^{1/2+\epsilon}(h) \leq \langle \omega^+(h) f_0, f_0 \rangle \leq \Xi_{H_{\gamma_1}}^{1/2}(h) \text{ for any } h \in H_{\gamma_1}.$$

Note that $\Xi_{H_{\gamma_1}}^{1/2}$ is $L^{4+\epsilon}$-integrable for any $\epsilon > 0$. Hence the restriction to H_{γ_1} of the matrix coefficient $\langle \omega^+(h) f, f \rangle$ is $L^{4+\epsilon}$-integrable for any $f \in G f_0$. Since ω^+ is irreducible, $G f_0$ spans a dense subset in the Hilbert space attached to ω^+. Hence $\omega^+|_{H_{\gamma_1}}$ is strongly $L^{4+\epsilon}$. Now suppose that $\omega^+|_{H_{\gamma_1}}$ is strongly $L^{2+\epsilon}$. Then by Theorem 2.4, its $K \cap H_{\gamma_1}$-matrix coefficient $\langle \omega^+(h) f_0, f_0 \rangle$ must be $L^{2+\epsilon}$-integrable. This is contradiction since it is bounded from below by the function $C \cdot \Xi_{H_{\gamma_1}}^{1/2+\epsilon}$ which is not $L^{2+\epsilon}$-integrable. $\square$

7. Uniform L^p-bound for matrix coefficients: $p_K(G)$

7.1. In this section, let G be the group of k-rational points of a connected almost k-simple algebraic group over k where k is any local field. As before, let K be a good maximal compact subgroup of G. Denote by $\hat{G}$ the set of equivalence classes of infinite dimensional irreducible unitary representations of G. Recall that $p(G)$ denotes the smallest real number such that for any non-trivial $\rho \in \hat{G}$, ρ is strongly L^q for any $q > p(G)$ (cf. [Li], [LZ]). Similarly we denote by $p_K(G)$ the smallest real number such that for any non-trivial $\rho \in \hat{G}$, the K-finite matrix coefficients of ρ are L^q-integrable for any $q > p_K(G)$. By Peter-Weyl theorem, we have $p(G) \le p_K(G)$ (we point out that in the literature the number $p(G)$ has been implicitly identified with $p_K(G)$). Cowling showed that $p(G) < \infty$ if and only if G property (T) [Co]. For $G = Sp_{2n}(k)\,(n \ge 2)$, Howe showed that $p_K(Sp_{2n}(\mathbb{C})) = 4n$ and $p_K(G) = 2n$ for other local field $k \neq \mathbb{C}$ with char $(k) \neq 2$ [Ho]. For other real (or complex) classical simple Lie groups, the exact number $p_K(G)$ is obtained by combining the known cases of a classification of the unitary dual by Vogan and Barbasch, and the results of Li (see [Li] for references). The precise values of $p_K(G)$ are not known in general but upper bounds have been given in many cases (see [Ho], [Li], [LZ], [Oh]).

7.2. Let $\mathcal{Q}$ be a maximal strongly orthogonal system of Φ. Then it follows from Theorem 5.7 that

$$p_K(G) \le \inf\{q \in \mathbb{R} \mid \xi_{\mathcal{Q}} \in L^q(G)\}.$$

7.3. Lemma. *Let f be a continuous function on G such that $f(k_1adk_2) = f(a)$ for any $k_1adk_2 \in KA^+\Omega K$. If $\int_{A^+} |f(a)|^p \delta_B(a)\, da < \infty$ for some $p > 0$, then $f \in L^p(G)$.*

Proof. For $K = \mathbb{R}$ or $\mathbb{C}$, this can be seen using the decomposition of the Haar measure dg on $G = KA^+K$: $dg = \Delta(a)dk_1\, dadk_2$ and the well known fact (cf. [Kn, Proposition 5.2.8]) that for any $t > 1$, there exist constants $d_1(t)$ and d_2 such that

$$d_1(t)\, \delta_B(a) \le \Delta(a) \le d_2\, \delta_B(a)$$

for all

$$a \in A_t^+ = \{g \in A^+ \mid |\alpha(g)| \ge t \text{ for all } \alpha \in \Phi^+\}.$$

For k non-archimedean, we have

$$\int_G |f(g)|^p\, dg = \sum_{ad \in A^+\Omega} \mathrm{Vol}(KadK)|f(a)|^p.$$

In fact there exist positive constants c_1 and c_2 such that $c_1\, \delta_B(ad) \le \mathrm{Vol}(KadK) \le c_2\, \delta_B(ad)$ for all $ad \in A^+\Omega$ [Si, Lemma 4.1.1]. Since the modular function δ_B is a homomorphism on B and Ω is finite, it follows that for some positive constant c_1' and c_2', $c_1'\,\delta_B(a) \le \mathrm{Vol}(KadK) \le c_2'\,\delta_B(a)$ for any $ad \in A^+\Omega$. Hence the above claim follows. $\square$

7.4. Hence by the above lemma together with the inequality (3) in 5.9, getting an upper bound for $p_K(G)$ boils down to a matter of comparing the coefficients of each simple root in $\prod_{\alpha\in\mathcal{Q}}\alpha^{-1/2}$ with those in the modular function δ_B.

Theorem. *Let k be a local field with char $(k) \neq 2$. Let G be the k-rational points of a connected linear almost k-simple algebraic group over k with k-rank $(G) \geq 2$ and $G/Z(G) \not\cong Sp_{2n}(\mathbb{C})$ (locally). Let δ_B be the modular function of B and set $\eta(\Phi) = \frac{1}{2}\sum_{\alpha\in\mathcal{Q}}\alpha$ for a maximal strongly orthogonal system $\mathcal{Q}$ of Φ (for example, one in the appendix). Then*

$$p_K(G) \le \max\left\{ \frac{\text{the coefficient of } \alpha_i \text{ in } \ \delta_B}{\text{the coefficient of } \ \alpha_i \text{ in } \eta(\Phi)} \,\middle|\, i = 1, \cdots, n \right\}$$

where $\{\alpha_1, \cdots, \alpha_n\}$ is the set of simple roots in Φ.

For example, if G is split over k with rank ≥ 2, $p_K(G)$ is bounded above by below:

Φ:	A_n	B_n	C_n	$D_{n:\text{even}}$	$D_{n:\text{odd}}$	E_6	E_7	E_8	F_4	G_2
$p_K(G) \le$:	$2n$	$2n$	$2n$	$2(n-1)$	$2n$	16	18	29	11	6

We remark that for $k = \mathbb{R}$, if we let $F(k_1ak_2) = \prod_{\alpha\in\mathcal{Q}}\alpha^{-1/2}(a)$ for any $k_1ak_2 \in KA^+K = G$, this function F coincides with F_G in [Oh] except for D_n, n odd, in which case F was improved by replacing two of the H_γ's by $SO(3.3)$ (see the remark in 3.5 in [Oh]) and one obtains a stronger estimate $2n-2$ for $p_K(G)$. Since we believe that the novelty of the above corollary lies in the simplicity of our method giving an upper bound for $p_K(G)$ rather than in improving the bound, we do not elaborate on here.

7.5. Moreover the results in section 6 yields the precise number $p_K(G)$ in the following cases (in which the classification of the unitary dual is also known):

Theorem.

(1) *Let $n \geq 3$. Then $p_K(SL_n(k)) = 2(n-1)$ for any local field k.*
(2) *Let $n \geq 2$. Then $p_K(Sp_{2n}(k)) = 2n$ for any non-archimedean local field k (char $k \neq 2$) or $k = \mathbb{R}$.*
(3) *Let $n \geq 2$. Then $p_K(Sp_{2n}(\mathbb{C})) = 4n$*

7.6. By the work of Cowling, Haggerup and Howe [CHH] (see Theorem 2.5), we have a passage from a uniform L^p-bound to a uniform pointwise bound, that is, let m be any integer such that $2m \geq p_K(G)$. Then any K-finite matrix coefficients $\langle\rho(g)v, w\rangle$

of an infinite dimensional $\rho \in \hat{G}$ is bounded by $(\dim\langle Kv\rangle \dim\langle Kw\rangle)^{1/2}\, \Xi_G(g)^{1/m}$. We remark that even in the case when the number $p_K(G)$ is precisely known, Theorem 5.7 (for a maximal strongly orthogonal system) provides a much sharper pointwise bound in general.

8. Kazhdan constants

In this section, we discuss some applications of the above results in terms of a quantitative estimate of Kazhdan property (T) of the group G, namely, Kazhdan constant.

8.1. For a locally compact group G, we say that a unitary representation ρ of G almost has an invariant vector if for any $\epsilon > 0$ and any compact subset Q of G, there exists a unit vector v which is (Q, ϵ)-invariant, that is, $\|\rho(g)v - v\| \le \epsilon$ for all $g \in Q$. Recall that G is said to have *Kazhdan property* (T) if any unitary representation of G which almost has an invariant vector actually has a non-zero invariant vector.

Definition (cf. [HV], [Bu]**).** For a locally compact group G with a compact subset Q, a positive number ϵ is said to be a *Kazhdan constant for* (G, Q) if for any unitary representation ρ without a non-zero G^+-invariant vector and for any unit vector v of ρ,

$$\max_{s \in Q} \|\rho(s)v - v\| \ge \epsilon.$$

If there exists such an ϵ, we call Q a *Kazhdan set* for G.

In other words, if ϵ is a Kazhdan constant for (G, Q), then any unitary representation of G which has a (Q, ϵ)-invariant vector actually has a non-zero invariant vector.

8.2. In what follows, we keep the notation etc. from section 2.1.

Proposition. *Let H be any subset of G such that $K \cap H$ is a closed subgroup of K. Let Φ be a bi-$K \cap H$-invariant function of H with the following properties:*

(1) $0 < \Phi(h) \le 1$ *for any* $h \in H$;
(2) *For* $h \in H$, $\Phi(h) = 1$ *if and only if* $h \in K \cap H$;
(3) *For any unitary representation ρ of G and any $K \cap H$-fixed unit vector v,*

$$|\langle \rho(h)v, v\rangle| \le \Phi(h) \quad \textit{for any } h \in H.$$

Set

$$\chi(h) = \frac{\sqrt{2(1 - \Phi(h))}}{\sqrt{2(1 - \Phi(h))} + 3} \quad \textit{for any } h \in H.$$

Then χ is a bi-$K \cap H$-invariant function of H satisfying

(1) $0 \le \chi(h) < \frac{\sqrt{2}}{\sqrt{2}+3}$ *for any* $h \in H$;

(2) *For* $h \in H$, $\chi(h) = 0$ *if and only if* $h \in K \cap H$;
(3) *We have*

$$\inf \max_{s \in \{K \cap H, h\}} \|\rho(s)v - v\| \geq \chi(h) \quad \textit{for any } h \in H$$

where the infimum is taken over all unitary representations ρ *of* G *without a non-zero invariant vector and for all unit vectors* v *of* ρ.

Proof. (1) and (2) are obvious from the definition of χ. Let h be a non-trivial element of H such that $h \notin K \cap H$. Fix a unit vector v of ρ. Suppose that for all $k \in K \cap H$, we have $\|\rho(k)v - v\| \leq \chi(h)$. We will show that $\|\rho(h)v - v\| \geq \chi(h)$. Let v_1 be the average of the $K \cap H$-transform of v:

$$v_1 = \int_{K \cap H} kv dk$$

where dk is the normalized Haar measure on $K \cap H$. Note that v_1 is $K \cap H$-fixed. We compute

$$\|v - v_1\| \leq \chi(h), \text{ so that } \|v_1\| \geq 1 - \chi(h).$$

Since $\chi(h) < 1$, the inequality implies that v_1 is non-zero. Recall that for any unit vector w,

$$\|\rho(h)w - w\|^2 = 2 - 2\mathrm{Re}\langle \rho(h)w, w\rangle.$$

Hence

$$\left\|\rho(h)\left(\frac{v_1}{\|v_1\|}\right) - \frac{v_1}{\|v_1\|}\right\| \geq \sqrt{2(1 - \Phi(h))}$$

and

$$\|\rho(h)v_1 - v_1\| \geq \sqrt{2(1 - \Phi(h))}\|v_1\| \geq \sqrt{2(1 - \Phi(h))}(1 - \chi(h)).$$

Therefore

$$\|\rho(h)v - v\| = \|\rho(h)v_1 - v_1 + (\rho(h)v - \rho(h)v_1) + (v_1 - v)\|$$

is greater than or equal to

$$\sqrt{2(1 - \Phi(h))}(1 - \chi(h)) - 2\chi(h) = \chi(h).$$

Hence

$$\|\rho(h)v - v\| \geq \chi(h).$$

This shows that for any $h \in H$,

$$\max_{\{s \in K \cap H, h\}} \|\rho(s)v - v\| \geq \chi(h),$$

proving the proposition. □

The above proposition can be considered as a generalization of a quantitative statement of the fact that the unitary representations of G which are not class one are uniformly bounded away from the trivial representation in the Fell topology: if ρ has an (K,ϵ)-invariant vector, say v, for some $0<\epsilon<1$, then ρ does have a K-invariant vector. This can be seen by considering the average $\int_K \rho(k)v$ of v along its K-transform. If $\epsilon<1$, then the average is non-zero, which is K-invariant. This is the reason that a pointwise bound for the matrix coefficients (only) with respect to K-invariant vectors yields Kazhdan constants.

8.3. In what follows, let k be a local field with char $(k)\neq 2$ and let G be the group of k-rational points of a connected simply connected almost k-simple linear group over k with k-rank at least 2 with the Cartan decomposition $G=KA^+K$ (see Remark in 2.1). In particular $G=G^+$.

Recall the bi-K-invariant function $\xi_{\mathcal{S}}$ on G for a strongly orthogonal system $\mathcal{S}$ defined in 4.10 (see 5.9 as well).

Notation. Define a bi-K-invariant function $\chi_{\mathcal{S}}$ on G by

$$\chi_{\mathcal{S}}(a)=\frac{\sqrt{2(1-\xi_{\mathcal{S}}(a))}}{\sqrt{2(1-\xi_{\mathcal{S}}(a))}+3}\quad\text{for any } a\in G.$$

Since $H_{\mathcal{S}}$ (see 5.1) and $\xi_{\mathcal{S}}$ satisfy Proposition 8.2 by Corollary 5.8 and Theorem 5.9, we have:

Theorem. *Let $\mathcal{S}$ be a strongly orthogonal system of Φ. Then for any $s\in H_{\mathcal{S}}\backslash K$, $\chi_{\mathcal{S}}(s)$ is a Kazhdan constant for $(\{K\cap H_{\mathcal{S}},s\},G)$.*

8.4. Note that if $\mathcal{S}$ is a *large* strongly orthogonal system $\mathcal{S}$ (3.3), then

$$K=\{g\in KA^+K \mid \xi_{\mathcal{S}}(g)=1\};$$

hence it follows that $\chi_{\mathcal{S}}(g)>0$ for any $g\notin K$.

Theorem. *Let $\mathcal{S}$ be a large strongly orthogonal system of Φ.*

(1) *For any $s\notin K$, $\chi_{\mathcal{S}}(s)$ is a Kazhdan constant for $(\{K,s\},G)$.*
(2) *For any compact subset Q properly containing K, $\max_{g\in Q}\chi_{\mathcal{S}}(g)$ is a Kazhdan constant for (Q,G).*

Proof. (1) follows from Corollary 5.8 and Proposition 8.2. To deduce (2) from (1), it suffices to observe that $\max_{g\in Q}\chi_{\mathcal{S}}(g)=\chi_{\mathcal{S}}(x)$ for some $x\in Q$, since $\chi_{\mathcal{S}}$ is a continuous function on G and Q is compact. □

Using an explicit form of $\xi_{\mathcal{S}}$ and hence of $\chi_{\mathcal{S}}$ in the above theorem provides us a simple machine which produces a Kazhdan constant for any set $\{K, s\}$, $s \notin K$, and hence for any compact subset properly containing K. One of the simplest methods will be to use $\mathcal{S} = \{\text{the highest root}\}$, but to get an optimal Kazhdan constant from this method, we can use a maximal strongly orthogonal system as in the Appendix.

Remark. The assumption that Q contains K can be loosened in many generic cases when k is archimedean. For instance, if Q contains some neighborhood of identity of G, then there is a positive integer n such that Q^n contains K properly, where Q^n denotes the set of all elements whose word lengths with respect to Q are at most n. Then $\max_{g\in Q^n} \chi_{\mathcal{S}}(g)$ is a Kazhdan constant for (Q^n, G) by Theorem 8.4. Observe that

$$n \cdot \max_{g\in Q} \|\rho(g)v - v\| \geq \max_{g\in Q^n} \|\rho(g)v - v\|$$

for any unitary representation ρ and a unit vector v of ρ. Therefore $\frac{1}{n}\max_{g\in Q^n} \chi_{\mathcal{S}}$ is a Kazhdan constant for (Q, G).

Remark. For the real group $Sp(1, n)\,(n \geq 2)$, Theorem 8.4 remains valid in view of Theorem 4.9.

8.5. Corollary. *For any non-trivial irreducible class one unitary representation ρ of G and a K-invariant unit vector v,*

$$\|\rho(g)v - v\| \geq \chi_{\mathcal{S}}(g) \quad \textit{for all } g \in G.$$

Proof. Since v is K-invariant, $\rho(k)v - v = 0$ for any $k \in K$. If $g \in K$, then the both sides are 0. For $g \notin K$, by the above theorem,

$$\max_{s\in\{K,g\}} \|\rho(s)v - v\| \geq \chi_{\mathcal{S}}(g).$$

But $\max_{s\in\{K,g\}} \|\rho(s)v - v\| = \|\rho(g)v - v\|$, proving the claim. □

8.6. While the function $\xi_{\mathcal{S}}$ gives an exact value for k non-archimedean, it is not the case for $k = \mathbb{R}$ or $\mathbb{C}$. In the following we discuss some estimates of $\xi_{\mathcal{S}}$ for $k = \mathbb{R}$ (the process is similar for $k = \mathbb{C}$).

We have that (cf. [HT, Ch V, 3.1])

$$\Xi_{\mathbb{R}}(x) = \frac{2}{\pi\sqrt{x}} \int_0^{\pi/2} \left(\frac{\cos^2 t}{x^2} + \sin^2 t\right)^{-1/2} dt \leq \frac{1.09 + \log x}{\sqrt{x}} \qquad \text{for any } x \geq 16.$$

Hence

$$\xi_{\mathcal{S}}(s) \leq \prod_{\gamma\in I(s)} h_{\mathbb{R}}(\gamma(s)) \text{ for any } s \notin K$$

where $h_{\mathbb{R}}(x) = \frac{1.09+\log x}{\sqrt{x}}$ and $I(s) = \{\gamma \in \mathcal{S} \mid \gamma(s) \geq 16\}$.

Lemma. *For any $s \in H_{\mathcal{S}} \backslash K$, the set $\{K \cap H_{\mathcal{S}}, s\}$ is a Kazhdan set with a Kazhdan constant*

$$\frac{1}{N_s}\chi_{\mathcal{S}}(s^{N_s}) \geq \frac{1}{N_s} f(h_{\mathbb{R}}(16)^{|I(s^{N_s})|})$$

where N_s denotes the minimum positive integer n such that $I(s^n) \neq \emptyset$ and $f(x) = \frac{\sqrt{2(1-x)}}{\sqrt{2(1-x)}+3}$.

Proof. Observe that for any positive integer n,

$$\frac{1}{n} \max_{g\in\{K,s^n\}} \|\rho(g)v - v\| \leq \max_{g\in\{K,s\}} \|\rho(g)v - v\|$$

for any unitary representation ρ and a unit vector v of ρ. On the other hand, since $f(x)$ is a strictly decreasing function on $[0,1]$, we have $\chi_{\mathcal{S}}(s^n) \geq f\left(h_{\mathbb{R}}(16)^{|I(s^n)|}\right)$. Since $\max_{g\in\{K,s^n\}} \|\rho(g)v - v\| \geq \frac{1}{n}\chi_{\mathcal{S}}(s^n)$, the lemma is proved. □

8.7. Examples of Kazhdan constants. We note that any compact subset of G, which contains K properly, generates the group G in the topological sense. Any compact generating subset of a Kazhdan property (T) group is a Kazhdan set (see [HV, Ch 1, Proposition 15]). Furthermore, any compact subset of G which generates a non-amenable subgroup is in fact a Kazhdan set (see [Sh]). The following theorem yields examples of Kazhdan sets which are contained in a proper closed semisimple subgroup of G. We give examples of Kazhdan constants where Kazhdan sets are taken from $SL_2(k)$ or $SL_4(k)$ embedded into the upper left corner of $SL_n(k)$.

Example 8.7.1. Let m be any non-zero integer.

(1) For any $n \geq 3$, the group $SL_n(\mathbb{R})$ has a Kazhdan constant

$$\left(\left\{SO(2), \mathrm{diag}(4^{1/m}, 4^{-1/m})\right\}, \frac{0.08}{|m|}\right).$$

(2) For any $n \geq 4$, the group $SL_n(\mathbb{R})$ has a Kazhdan constant

$$\left(\left\{SO(4), \mathrm{diag}(4^{1/m}, 4^{1/m}, 4^{-1/m}, 4^{-1/m})\right\}, \frac{0.109}{|m|}\right)$$

Proof. Let $m = 1$. Denote by s_1 and s_2 the diagonal elements in (1) and (2) respectively. Set

$$\gamma_1(a) = \frac{a_1}{a_n} \text{ and } \gamma_2(a) = \frac{a_2}{a_{n-1}},$$

where $a = \mathrm{diag}(a_1, a_2, \cdots, a_n) \in A^+$, that is, $a_i \geq a_{i+1} > 0$ for all $1 \leq i \leq n-1$. Note that $\mathcal{S}_1 = \{\gamma_1\}$ and $\mathcal{S}_2 = \{\gamma_1, \gamma_2\}$ are large strongly orthogonal systems (since γ_1 is the highest root). Then $\gamma_1(s_1) = 16$, $\gamma_1(s_2) = 16$ and $\gamma_2(s_2) = 16$. Therefore by Lemma 8.6, $\chi_{\mathcal{S}_1}(s_1) \geq f(h_{\mathbb{R}}(16)) \geq 0.08$ and $\chi_{\mathcal{S}_2}(s_2) \geq f(h_{\mathbb{R}}(16)^2) \geq 0.109$. Now the claim for an arbitrary m follows from Lemma 8.6. □

Example 8.7.2. *Let m be any non-zero integer.*

(1) *For any $n \geq 3$, the group $SL_n(\mathbb{R})$ has a Kazhdan constant*

$$\left(\left\{SO(2), \begin{pmatrix} 1 & \frac{4}{m} \\ 0 & 1 \end{pmatrix}\right\}, \frac{0.104}{|m|}\right)$$

(2) *For any $n \geq 4$, the group $SL_n(\mathbb{R})$ has a Kazhdan constant*

$$\left(\left\{SO(4), \begin{pmatrix} 1 & \frac{4}{m} & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & \frac{4}{m} \\ 0 & 0 & 0 & 1 \end{pmatrix}\right\}, \frac{0.139}{|m|}\right)$$

Proof. By Lemma 8.6, it suffices to consider the case of $m = 1$. Let

$$s_1 = \begin{pmatrix} 1 & 4 \\ 0 & 1 \end{pmatrix}, \quad s_2 = \begin{pmatrix} 1 & 4 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

For $g \in G$, denote by $A^+(g)$ the A^+-part of the decomposition of g in the Cartan decomposition $G = KA^+K$. It is not difficult to see that

$$A^+(s_1) = \operatorname{diag}((9 + 4\sqrt{5})^{1/2}, 1, \cdots, 1, (9 + 4\sqrt{5})^{-1/2}),$$

and that for $n \geq 4$,

$$A^+(s_2) = \operatorname{diag}((9 + 4\sqrt{5})^{1/2}, (9 + 4\sqrt{5})^{1/2}, 1, \cdots, 1, (9 + 4\sqrt{5})^{-1/2}, (9 + 4\sqrt{5})^{-1/2}).$$

Let γ_1, γ_2, $\mathcal{S}_1$ and $\mathcal{S}_2$ be as in the proof of Example 8.7.1. Therefore $\gamma_1(s_1) = \gamma_1(s_2) = \gamma_2(s_2) = 9 + 4\sqrt{5} \geq 16$. By Lemma 8.6,

$$\chi_{S_1}(s_1) \geq f(h_{\mathbb{R}}(\gamma_1(s_1))) \geq 0.104$$

and

$$\chi_{S_2}(s_2) \geq f(h_{\mathbb{R}}(\gamma_1(s_2) h_{\mathbb{R}}(\gamma_2(s_2))) \geq 0.139.$$

□

The following two examples are direct application of Theorem 8.4. Let

$$c_n = \frac{n(p-1) + (p+1)}{\sqrt{p}^n (p+1)} \text{ and } f(x) = \frac{\sqrt{2(1-x)}}{\sqrt{2(1-x)} + 3}.$$

Example 8.7.3. *For $n \geq 3$, $SL_n(\mathbb{Q}_p)$ has Kazhdan constants:*

$$\left(\left\{SL_2(\mathbb{Z}_p), diag\,(p^m, p^{-m})\right\}, f\left(c_{2|m|}\right)\right)$$

for any $m \in \mathbb{Z}$.

Example 8.7.4. *For $n \geq 3$, the group $SL_n(\mathbb{Q}_p)$ has Kazhdan constants:*

$$\left(\{SL_3(\mathbb{Z}_p), diag\,(p^{n_1}, p^{n_2}, p^{n_3})\}, \max_{1\leq i,j\leq 3} f(c_{|n_i-n_j|})\right)$$

for any $n_1, n_2, n_3 \in \mathbb{Z}$ such that $\sum_{i=1}^{3} n_i = 0$.

So for $SL_n(\mathbb{Q}_2)$, $SL_n(\mathbb{Q}_3)$, and $SL_n(\mathbb{Q}_5)$, the following are Kazhdan constants respectively:

$$\left(\left\{SL_4(\mathbb{Z}_2), \mathrm{diag}\,(2, 2^{-1}, 2^2, 2^{-2})\right\}, 0.25)\right);$$

$$\left(\left\{SL_4(\mathbb{Z}_3), \mathrm{diag}\,(3, 3^{-1}, 3^2, 3^{-2})\right\}, 0.29)\right); \text{ and}$$

$$\left(\left\{SL_4(\mathbb{Z}_5), \mathrm{diag}(5, 5^{-1}, 5^2, 5^{-2})\right\}, 0.31)\right)$$

8.8. Recall the definition of $\kappa(G, Q)$ from the introduction.

Proposition. *Let k be any local field with $char(k) \neq 2$ and G the group of k-rational points of a connected simply connected almost k-simple algebraic group over k with k-rank$(G) \geq 2$.*

(1) *Let k be non-archimedean and G be k-split. Then*

$$\inf_{s\in G\setminus K} \kappa(G, \{K, s\}) \geq f\left(\frac{2\sqrt{p}}{p+1}\right)$$

where p is the cardinality of the residue field of k.

(2) *We have*

$$\inf_{p=\text{ prime}} \inf_{n\geq 3} \inf_{s\notin SL_n(\mathbb{Z}_p)} \kappa(SL_n(\mathbb{Q}_p), \{SL_n(\mathbb{Z}_p), s\}) \geq f\left(\frac{2\sqrt{2}}{3}\right) > 0.10.$$

(3) *Let $k = \mathbb{R}$ or $\mathbb{C}$. Then*

$$\inf_{s\in G\setminus K} \kappa(G, \{K, s\}) = 0.$$

(4) *For any sequence $g_i \in G$ going to the infinity,*

$$\liminf_{i\to\infty} \kappa(G, \{K, g_i\}) \geq \frac{\sqrt{2}}{(\sqrt{2}+3)}.$$

Proof. Claim (1): Denote by γ the highest root in Φ. Then $\mathcal{S} = \{\gamma\}$ forms a large strongly orthogonal system of Φ. Let $s \in G \backslash K$. By replacing s by its A^+-component in the Cartan decomposition $G = KA^+K$, we may assume that $s \in A^+$. Hence $|\gamma(s)| = p^m$ for some positive integer m. By the definition of $\xi_{\mathcal{S}}$,

$$\xi_{\mathcal{S}}(s) = \Xi_{PGL_2\mathbb{Q}_p}\begin{pmatrix} \gamma(s) & 0 \\ 0 & 1 \end{pmatrix} \leq \Xi_{PGL_2\mathbb{Q}_p}\begin{pmatrix} p^{-1} & 0 \\ 0 & 1 \end{pmatrix}.$$

Hence we have

$$\kappa(G, \{K, s\}) \geq f(\xi_{\mathcal{S}}(s)) \geq f\left(\Xi_{PGL_2\mathbb{Q}_p}\begin{pmatrix} p^{-1} & 0 \\ 0 & 1 \end{pmatrix}\right) \geq f\left(\frac{2\sqrt{p}}{p+1}\right).$$

Claim (2): Since $p \geq 2$ for $k = \mathbb{Q}_p$, we have

$$\frac{2\sqrt{p}}{p+1} \geq \frac{2\sqrt{2}}{3}.$$

Hence by Claim (1), we have

$$\kappa(SL_n(\mathbb{Q}_p), \{SL_n(\mathbb{Z}_p), s\}) \geq f\left(\frac{2\sqrt{2}}{3}\right) > 0.10.$$

Since this is independent of n and p, Claim (2) follows.

Claim (3): Let ρ be any class one unitary representation of G without any invariant vector (of which we know existence), and let v be a K-invariant unit vector of ρ. Then the matrix coefficient $\langle \rho(g)v, v\rangle$ is a continuous function which has value 1 for all $g \in K$. Hence for a sequence g_i tending to an element of K with $g_i \notin K$, we have $\lim_{i\to\infty}\langle \rho(g_i)v, v\rangle = 1$. By the well known equality that states $\|\rho(g)v - v\|^2 = 2 - 2Re\langle \rho(g)v, v\rangle$, we obtain $\lim_{i\to\infty} \|\rho(g_i)v - v\| = 0$.

Claim (4): Without loss of generality, we may assume $g_i \in A^+$. Then $f(\xi_{\mathcal{S}}(g_i))$ tends to $\frac{\sqrt{2}}{\sqrt{2}+3}$. Thus the claim follows from Theorem 8.4 □

8.9. Let Γ be a lattice in G. If (Q, ϵ) is a Kazhdan constant for G, then in principle one can find some positive real number R (at least bigger than the radius of Q) such that $\Gamma \cap B_R$ yields a Kazhdan set for Γ where B_R denotes a ball of radius R of the identity in a suitable metric in G/K (see [Sh, Theorem B], also [HV, Lemma 3.3]). For instance, our results imply that if Γ is a co-compact lattice in the group $G = PGL_3(\mathbb{Q}_p)$ such that Γ acts simply transitively on $G/PGL_3(\mathbb{Z}_p)$, then 0.10 is a Kazhdan constant for $(\Gamma, B_1 \cap \Gamma)$ where $B_1 = \{g \in PGL_3(\mathbb{Q}_p) \mid g \in PGL_3(\mathbb{Z}_p)\mathrm{diag}(p,1,1)PGL_3(\mathbb{Z}_p)\}$.

However obtaining Kazhdan constants for a lattice Γ of G using this method involves understanding the size of the fundamental domain of Γ in G, which seems highly nontrivial in general.

Appendix: maximal strongly orthogonal systems

Let Φ be a reduced irreducible root system with a basis $\alpha_1, ..., \alpha_n$.

The subscripts of α_i's are determined by the following choice of the highest root [Bo].

Φ	the highest root
A_n	$\alpha_1 + \alpha_2 + \cdots + \alpha_n$
B_n	$\alpha_1 + 2\alpha_2 + \cdots + 2\alpha_n$
C_n	$2\alpha_1 + 2\alpha_2 + \cdots + 2\alpha_{n-1} + \alpha_n$
D_n	$\alpha_1 + 2\alpha_2 + \cdots + 2\alpha_{n-2} + \alpha_{n-1} + \alpha_n$
E_6	$\alpha_1 + 2\alpha_2 + 2\alpha_3 + 3\alpha_4 + 2\alpha_5 + \alpha_6$
E_7	$2\alpha_1 + 2\alpha_2 + 3\alpha_3 + 4\alpha_4 + 3\alpha_5 + 2\alpha_6 + \alpha_7$
E_8	$2\alpha_1 + 3\alpha_2 + 4\alpha_3 + 6\alpha_4 + 5\alpha_5 + 4\alpha_6 + 3\alpha_7 + 2\alpha_8$
F_4	$2\alpha_1 + 3\alpha_2 + 4\alpha_3 + 2\alpha_4$
G_2	$3\alpha_1 + 2\alpha_2$

The notation $\lfloor x \rfloor$ denotes the largest integer not bigger than x. Set

$$N(\Phi) = N = \begin{cases} [\frac{n+1}{2}] & \text{for} \Phi = A_n \\ 2[\frac{n}{2}] & \text{for } \Phi = D_n \\ 4 & \text{for } \Phi = E_6 \\ \text{rank}(\Phi) & \text{for } \Phi = B_n, C_n, F_4, G_2, E_7, E_8 \end{cases}$$

In the following we list maximal strongly orthogonal systems $\mathcal{Q}$ constructed in [Oh]. We correct a typo for γ_3 in E_8 from [Oh].

Φ	$\mathcal{Q}(\Phi)$: a maximal strongly orthogonal system of Φ
A_n	$\begin{cases} \gamma_i = \alpha_i + \cdots + \alpha_{n-i+1} \text{ for } 1 \le i \le N-1 \\ \gamma_N = \begin{cases} \alpha_N & \text{for } n \text{ odd} \\ \alpha_N + \alpha_{N+1} & \text{for } n \text{ even} \end{cases} \end{cases}$
B_n, $(n \ge 2)$	$\begin{cases} \gamma_{2i-1} = \alpha_i + \cdots + \alpha_{n-i} + 2\alpha_{n-i+1} + \cdots + 2\alpha_n \\ \gamma_{2i} = \alpha_i + \cdots + \alpha_{n-i} \text{ for } 1 \le i \le [\frac{n}{2}] \\ \gamma_n = \alpha_{(n+1)/2} + \cdots + \alpha_n \text{ for } n \text{ odd} \end{cases}$
C_n, $(n \ge 2)$	$\begin{cases} \gamma_i = 2\alpha_i + \cdots + 2\alpha_{n-1} + \alpha_n \text{ for } 1 \le i \le N-1 \\ \gamma_N = \alpha_n \end{cases}$
D_n, $(n \ge 4)$	$\begin{cases} \gamma_1 = \alpha_1 + \cdots + \alpha_{n-2} + \alpha_n \\ \gamma_2 = \alpha_1 + \cdots + \alpha_{n-1} \\ \gamma_{2i-1} = \alpha_i + \cdots + \alpha_{n-i} + 2\alpha_{n-i+1} + \cdots + 2\alpha_{n-2} + \alpha_{n-1} + \alpha_n \\ \gamma_{2i} = \alpha_i + \cdots + \alpha_{n-i} \text{ for } 2 \le i \le [\frac{n}{2}] \end{cases}$
E_6	$\begin{cases} \gamma_1 = \alpha_1 + \alpha_2 + \alpha_3 + 2\alpha_4 + 2\alpha_5 + \alpha_6 \\ \gamma_2 = \alpha_1 + \alpha_2 + 2\alpha_3 + 2\alpha_4 + \alpha_5 + \alpha_6 \\ \gamma_3 = \alpha_2 + \alpha_3 + 2\alpha_4 + \alpha_5 \\ \gamma_4 = \alpha_2 \end{cases}$
E_7	$\begin{cases} \gamma_1 = \alpha_1 + \alpha_2 + \alpha_3 + 2\alpha_4 + 2\alpha_5 + \alpha_6 + \alpha_7 \\ \gamma_2 = \alpha_1 + \alpha_2 + 2\alpha_3 + 2\alpha_4 + \alpha_5 + \alpha_6 + \alpha_7 \\ \gamma_3 = \alpha_1 + \alpha_2 + 2\alpha_3 + 2\alpha_4 + 2\alpha_5 + \alpha_6 \\ \gamma_4 = \alpha_1 + \alpha_2 + \alpha_3 + 2\alpha_4 + \alpha_5 + \alpha_6 \\ \gamma_5 = \alpha_2 + \alpha_3 + 2\alpha_4 + \alpha_5 \\ \gamma_6 = \alpha_2 + \alpha_3 + 2\alpha_4 + 2\alpha_5 + 2\alpha_6 + \alpha_7 \\ \gamma_7 = \alpha_2 \end{cases}$
E_8	$\begin{cases} \gamma_1 = \alpha_1 + 2\alpha_2 + 3\alpha_3 + 5\alpha_4 + 4\alpha_5 + 3\alpha_6 + 2\alpha_7 + \alpha_8 \\ \gamma_2 = \alpha_1 + 2\alpha_2 + 2\alpha_3 + 3\alpha_4 + 3\alpha_5 + 2\alpha_6 + \alpha_7 + \alpha_8 \\ \gamma_3 = \alpha_1 + 2\alpha_2 + 2\alpha_3 + 3\alpha_4 + 2\alpha_5 + 2\alpha_6 + \alpha_7 \\ \gamma_4 = \alpha_1 + \alpha_2 + \alpha_3 + 2\alpha_4 + 2\alpha_5 + 2\alpha_6 + 2\alpha_7 + \alpha_8 \\ \gamma_5 = \alpha_1 + \alpha_2 + 2\alpha_3 + 2\alpha_4 + \alpha_5 + \alpha_6 + \alpha_7 + \alpha_8 \\ \gamma_6 = \alpha_1 + \alpha_2 + 2\alpha_3 + 2\alpha_4 + 2\alpha_5 + \alpha_6 + \alpha_7 \\ \gamma_7 = \alpha_1 + \alpha_2 + \alpha_3 + 2\alpha_4 + \alpha_5 \\ \gamma_8 = \alpha_1 + \alpha_3 + \alpha_4 + \alpha_5 + \alpha_6 \end{cases}$
F_4	$\begin{cases} \gamma_1 = \alpha_1 + 2\alpha_2 + 4\alpha_3 + 2\alpha_4 \\ \gamma_2 = \alpha_1 + 2\alpha_2 + 2\alpha_3 + 2\alpha_4 \\ \gamma_3 = \alpha_1 + 2\alpha_2 + 2\alpha_3 \\ \gamma_4 = \alpha_1 \end{cases}$
G_2	$\begin{cases} \gamma_1 = 3\alpha_1 + 2\alpha_2 \\ \gamma_2 = \alpha_1 \end{cases}$

We set $\eta(\Phi)$ to be the half sum of the roots in a maximal strongly orthogonal system of Φ. Recall that $\eta(\Phi)$ does not depend on the choice of a maximal strongly orthogonal system.

Φ	$\eta(\Phi)$
A_n	$\begin{cases} \sum_{i=1}^{(n-1)/2} \frac{i}{2}\alpha_i + \sum_{i=(n+1)/2}^{n} \frac{(n-i+1)}{2}\alpha_i & \text{for } n \text{ odd} \\ \sum_{i=1}^{n/2} \frac{i}{2}\alpha_i + \frac{n}{4}\alpha_{n/2+1} + \sum_{i=n/2+2}^{n} \frac{(n-i+1)}{2}\alpha_i & \text{for } n \text{ even} \end{cases}$
B_n, $(n \geq 2)$	$\displaystyle\sum_{i=1}^{\lfloor n/2 \rfloor} i\alpha_i + \sum_{i=\lfloor n/2+1 \rfloor}^{n} \frac{n}{2}\alpha_i$
C_n, $(n \geq 2)$	$\displaystyle\sum_{i=1}^{n-1} i\alpha_i + \frac{n}{2}\alpha_n$
D_n, $(n \geq 4)$	$\displaystyle\sum_{i=1}^{\lfloor n/2 \rfloor} i\alpha_i + \sum_{i=\lfloor n/2 \rfloor+1}^{n-2} \lfloor \frac{n}{2} \rfloor \alpha_i + \frac{1}{2}\lfloor \frac{n}{2} \rfloor(\alpha_{n-1} + \alpha_n)$
E_6	$\alpha_1 + 2\alpha_2 + 2\alpha_3 + 3\alpha_4 + 2\alpha_5 + \alpha_6$
E_7	$2\alpha_1 + \frac{7}{2}\alpha_2 + 4\alpha_3 + 6\alpha_4 + \frac{9}{2}\alpha_5 + 3\alpha_6 + \frac{3}{2}\alpha_7$
E_8	$4\alpha_1 + 5\alpha_2 + 7\alpha_3 + 10\alpha_4 + 8\alpha_5 + 6\alpha_6 + 4\alpha_7 + 2\alpha_8$
F_4	$2\alpha_1 + 3\alpha_2 + 4\alpha_3 + 2\alpha_4$
G_2	$2\alpha_1 + \alpha_2$

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