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Uniform Quotient Mappings of the Plane

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1. Introduction

Let X and Y be metric spaces. As is well known, a mapping $f: X \to Y$ is said to be *uniformly continuous* if there is a continuous increasing function $\Omega(r)$, $r \geq 0$ with $\Omega(0) = 0$, so that $d(f(u), f(v)) \leq \Omega(d(u, v))$ for all u and v; or, in other words, $f(\mathrm{B}_r(x)) \subset \mathrm{B}_{\Omega(r)}(f(x))$ for all $x \in X$ and $r > 0$. (We use $\mathrm{B}_r(x)$ to denote the open ball with radius r and center x in the appropriate space.) The mapping f is called *co-uniformly continuous* if there is a continuous increasing function $\omega(r)$, $r > 0$ with $\omega(r) > 0$ for $r > 0$, so that $\mathrm{B}_{\omega(r)}(f(x)) \subset f(\mathrm{B}_r(x))$. The continuity and monotonicity assumptions are made here for convenience and, if not assumed, can be achieved by changing the original functions $\Omega(r)$ and $\omega(r)$. The only necessary requirement is that the limit of $\Omega(r)$ is zero as $r \to 0$.

A surjective mapping f is said to be a *uniform quotient mapping* if it is uniformly continuous and co-uniformly continuous. In other words, f from X onto Y is a uniform quotient mapping if and only if $f \times f: X \times X \to Y \times Y$ maps the uniform neighborhoods of the diagonal in $X \times X$ onto the uniform neighborhoods of the diagonal in $Y \times Y$. Note that if $f: X \to Y$ is uniformly continuous and co-uniformly continuous then f (which of course is open) maps X to a closed set; hence the image of X is both closed and open. Consequently, if Y is connected then f is automatically surjective. Note also that if f is continuous and open and K is a compact subset of X, then for each $r > 0$ there is $\omega(r) > 0$ such that $\mathrm{B}_{\omega(r)}(f(x)) \subset f(\mathrm{B}_r(x))$ is satisfied for x in K. In particular, a continuous open mapping on a compact space is co-uniformly continuous. Finally, if f is uniformly continuous and co-uniformly continuous, then for all $Z \subset Y$ the restriction of f to $f^{-1}(Z)$, when considered as a mapping into Z, is also uniformly continuous and co-uniformly continuous; moreover, the image of every component of $f^{-1}(Z)$ is a component of Z provided that the balls of X are connected and $Z \subset f(X)$ is open. A discussion of the notion of co-uniform continuity and uniform quotient mappings (in the context of general uniform spaces) can be found in [J]. For normed spaces, the moduli always satisfy $\Omega(r) \geq Cr$ and $\omega(r) \leq cr$ for suitable C and c. If $\Omega(r) \leq Cr$ (more precisely, if Ω can be chosen to satisfy $\Omega(r) \leq Cr$ for some $0 < C < \infty$ and all $r > 0$), then we say that f is *Lipschitz*.

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Similarly, if $\omega(r) \geq cr$ then we say that f is *co-Lipschitz*. A surjective mapping that is both Lipschitz and co-Lipschitz is called a *Lipschitz quotient mapping.*

In a recent paper [BJLPS] we dealt with these notions for general Banach spaces X and Y. Here we are interested mainly in the case where $X = Y$ is the plane. (As a matter of notation we shall consider the plane both as $\mathbb{R}^2$ and as the complex plane $\mathbb{C}$. When we consider it as $\mathbb{R}^2$ we use $\|\cdot\|$ to denote the Euclidean norm; when we consider the plane as $\mathbb{C}$ we use $|\cdot|$ for that purpose.)

Nontrivial examples of Lipschitz quotient mappings from the plane to itself are $f_n(re^{i\theta}) = re^{in\theta}$, $n = 1, 2, \dots$. Our main aim is to show that these examples are in a sense typical for general uniform quotient mappings of the plane. We prove, under some conditions on Ω and ω, that any uniform quotient mapping f of the plane is of the form $f = P \circ h$, where h is a homeomorphism of the plane and P a polynomial. In the preceding examples, $f_n = P_n \circ h_n$ where $h_n(re^{i\theta}) = r^{1/n}e^{i\theta}$ and $P_n(z) = z^n$. Conversely, we show that for any given P there is a homeomorphism h of the plane so that $P \circ h$ is even a Lipschitz quotient mapping.

We prove the theorem just mentioned in case Ω and ω satisfy at least one of the following three conditions:

(1) Ω is arbitrary and $\omega \geq cr$—that is, f is uniformly continuous and co-Lipschitz;
(2) ω is arbitrary and $\Omega(r)/\sqrt{r} \to 0$ as $r \to 0$;
(3) there are c, C, p, q with $q < 1 + p$ such that, for $0 < r < 1$, $\omega(r) \geq cr^q$ and $\Omega(r) \leq Cr^p$.

The proofs of parts 1 and 2 of this theorem constitute most of Section 2; the proof of part 3 is contained in Section 4. The main arguments of the proofs presented here involve checking that, under each of our three assumptions, $f^{-1}(y)$ is a discrete set for every y. Once this is done, the representation theorem (Theorem 2.8) is proved using a result of Stoilow [S] that gives a topological characterization of analytic functions.

We actually show that, for every uniform quotient mapping of the plane, there is a number N such that the set $f^{-1}(y)$ has at most N connected components for every y. Assumption (1), (2), or (3) is then used to prove that every such component is a singleton.

In Section 3 we present an example showing that some restrictions on the moduli are required. More precisely, there is a uniform quotient mapping f of the plane onto itself with moduli of power type that maps an interval to zero. Such a mapping cannot, of course, be a superposition of a homeomorphism and a polynomial. As a corollary to this example we also obtain (in Remark 3.3) a relatively simple construction of an example of a continuous open monotone mapping of the plane onto itself which is not a homeomorphism. Such an example was first given by Anderson [A], although his construction is much more complicated.

Theorem 2.8 applies only to mappings defined on the entire plane. However, under assumption (3) we also prove (in Section 4) a local result. For every uniform quotient mapping f from a domain in the plane into the plane satisfying (3), $f^{-1}(y)$ is discrete for every y in the range. Example 4.1 shows that some restriction on the relation between p and q is needed; it fails for $p = 1$, $q = 3$. The same example shows that assumption (2) cannot guarantee that $f^{-1}(y)$ is discrete if f is only assumed to be defined on a domain in the plane.

Section 5 deals with Lipschitz and uniform quotient mappings from $\mathbb{R}^2$ to $\mathbb{R}$. The analysis here is much simpler. We show in particular that, for uniform quotient mapping f from $\mathbb{R}^2$ to $\mathbb{R}$, $\mathbb{R}^2 \setminus f^{-1}(y)$ has a bounded number of components for y ranging over $\mathbb{R}$. If f is a Lipschitz quotient then also $f^{-1}(y)$ has a bounded number of components.

The methods of proof in this paper are particular to the plane. One can ask many natural questions concerning uniform quotient mappings from $\mathbb{R}^n$ to $\mathbb{R}^m$, $n \geq \max(m, 3)$. This area of research is wide open. Some comments on these questions as well as results in the infinite-dimensional situation are presented in [BJLPS].

2. Global Results

We begin with a restatement of Proposition 4.3 of [BJLPS].

PROPOSITION 2.1. *Let $f : \mathbb{R}^2 \to \mathbb{R}^2$ be a continuous and co-Lipschitz mapping. Then, for every $y \in \mathbb{R}^2$, the set $f^{-1}(y)$ is discrete.*

We repeat the proof from [BJLPS]. We first state the following simple lemma concerning the lifting of Lipschitz curves.

LEMMA 2.2. *Suppose that $f : \mathbb{R}^n \to X$ is continuous and co-Lipschitz with constant 1, $f(x) = y$. Suppose also that $\xi : [0, \infty) \to X$ is a curve with Lipschitz constant 1 and that $\xi(0) = y$. Then there is a curve $\phi : [0, \infty) \to \mathbb{R}^n$ with Lipschitz constant 1 such that $\phi(0) = x$ and $f(\phi(t)) = \xi(t)$ for $t \geq 0$.*

Proof. For $m = 1, 2, \dots,$ define $\phi_m(0) = x$. By induction, assuming that $f\big(\phi_m\big(\frac{k}{m}\big)\big) = \xi\big(\frac{k}{m}\big)$, choose $\phi_m\big(\frac{k+1}{m}\big)$ such that $\big\|\phi_m\big(\frac{k+1}{m}\big) - \phi_m\big(\frac{k}{m}\big)\big\| \leq \frac{1}{m}$ and $f\big(\phi_m\big(\frac{k+1}{m}\big)\big) = \xi\big(\frac{k+1}{m}\big)$. Extend $\phi_m(t)$ to a Lipschitz curve $\phi_m : [0, \infty) \to \mathbb{R}^n$ having Lipschitz constant 1. The limit ϕ of any convergent subsequence of ϕ_m has the desired properties. $\square$

Proof of Proposition 2.1. Without loss of generality, assume that $\mathrm{B}_r(f(x)) \subset f(\mathrm{B}_r(x))$ for every x in $\mathbb{R}^2$ and every $r > 0$ and that $y = 0$ and $f(0) = 0$. Let $u_k = e^{k\pi i/3}$ and $S = \{tu_k : t \geq 0,\ k = 0, 2, 4\}$. Let also $0 < \delta < 1$ be such that $\|x\|, \|y\| \leq 2$ and $\|x - y\| < \delta$ imply that $\|f(x) - f(y)\| < 1/2$.

For each $x \in \mathrm{B}_1(0) \cap f^{-1}(0)$ and $k = 1, 3, 5$, use Lemma 2.2 to choose $\phi_{k,x} : [0, \infty) \to \mathbb{R}^2$ having Lipschitz constant 1 such that $\phi_{k,x}(0) = x$ and $f(\phi_{k,x}(t)) = tu_k$ for $t \geq 0$. Let $D_{k,x}$ be the component of $\mathbb{R}^2 \setminus f^{-1}(S)$ containing $\phi_{k,x}(0, \infty)$. Noting that $\mathrm{B}_\delta(\phi_{k,x}(1)) \subset D_{k,x} \cap \mathrm{B}_3(0)$, a comparison of areas shows that the set of all such $D_{k,x}$ has at most $9\delta^{-2}$ elements. Suppose now that $\mathrm{B}_1(0) \cap f^{-1}(0)$ has more than $N = [(9\delta^{-2})^3]$ elements. Then it contains elements $x \neq y$ such that $\{D_{1,x}, D_{3,x}, D_{5,x}\} = \{D_{1,y}, D_{3,y}, D_{5,y}\}$. Hence $D_{k,x} = D_{k,y}$ for $k = 1, 3, 5$, since the (connected) image of $D_k := D_{k,x}$ contains u_k and so can contain no other u_j, and we infer that there are simple curves

$\psi_k : [0,1] \to \mathbb{R}^2$ such that $\psi_k(0) = x$, $\psi_k(1) = y$, and $\psi_k(t) \in D_k$ for $0 < t < 1$. For each pair $k, l = 1, 3, 5$ of different indices, let $G_{k,l}$ be the interior of the Jordan curve $(\psi_k - \psi_l)$ (difference in the sense of oriented curves). If $j \neq k, l$, we note that $G_{k,l} \cap D_j = \emptyset$ since otherwise D_j would be bounded. In particular, $G_{1,3} \cap \partial G_{3,5} = \emptyset$, so either $G_{1,3} \subset G_{3,5}$ or $G_{1,3} \cap G_{3,5} = \emptyset$. In the former case, we would get a contradiction from $\psi_1(0,1) \subset G_{3,5}$ because $\psi_1(0,1) \subset D_1$. In the latter case, $\partial(G_{1,5}) = \partial(\overline{G_{1,3} \cup G_{3,5}})$. This intuitively clear fact follows, for example, from the theorem about θ-curves (see e.g. [K, Ch. 10, Sec. 61, II, Thm. 2]) or from Schoenflies's extension theorem. It follows that $G_{1,5} \supset G_{1,3}$ and so we have a contradiction from $\psi_3(0,1) \subset G_{1,5}$. □

REMARK. If we assume in addition that f is uniformly continuous, then a more careful analysis of the proof shows that the cardinality of $f^{-1}(y)$ is finite and moreover is bounded, independently of y, by a constant depending only on the co-Lipschitz constant of f and its modulus of uniform continuity. We do not expand on this since we shall present a different proof of it, using Lemma 2.7. See the beginning of the proof of Theorem 2.8.

We now come to the main result of this paper, which is a version of Proposition 2.1 in which the co-Lipschitz condition is weakened to mere co-uniformity but the continuity assumption is strengthened to uniform continuity with modulus strictly better than $\sqrt{r}$.

THEOREM 2.3. *Let $f : \mathbb{R}^2 \to \mathbb{R}^2$ satisfy $\mathrm{B}_{\omega(r)}(f(z)) \subset f(\mathrm{B}_r(z)) \subset \mathrm{B}_{\Omega(r)}(f(z))$ for all $r > 0$ and $z \in \mathbb{R}^2$, where $\Omega(r), \omega(r) : [0,\infty) \to [0,\infty)$ are continuous strictly increasing functions such that $\Omega(0) = \omega(0) = 0$. If $\lim_{r\to 0} \Omega(r)/\sqrt{r} = 0$, then the inverse images of points under f are discrete. Moreover, there is a number N depending only on Ω and ω such that the cardinality of $f^{-1}(y)$ is bounded by N for all $y \in \mathbb{R}^2$.*

For the proof we need a sequence of lemmas. In all of these lemmas (2.4–2.7) we assume that $f : \mathbb{R}^2 \to \mathbb{R}^2$ satisfies

$$\mathrm{B}_{\omega(r)}(f(z)) \subset f(\mathrm{B}_r(z)) \subset \mathrm{B}_{\Omega(r)}(f(z)) \tag{2.1}$$

for all $r > 0$ and $z \in \mathbb{R}^2$, where $\Omega(r), \omega(r) : [0,\infty) \to [0,\infty)$ are continuous strictly increasing functions such that $\Omega(0) = \omega(0) = 0$. The additional assumption, $\lim_{r\to 0} \Omega(r)/\sqrt{r} = 0$, is not used in these lemmas.

LEMMA 2.4. *Let $f : \mathbb{R}^2 \to \mathbb{R}^2$ satisfy* (2.1). *For every $r_0 > 0$ there is a constant $R_0 = R_0(r_0) < \infty$ depending only on Ω, ω, and r_0 such that, for every $y \in \mathbb{R}^2$ and every $r \geq r_0$, every component of $f^{-1}(\mathrm{B}_r(y))$ has diameter at most $R_0 r$.*

Proof. If not, then for every $k = 1, 2, \dots$ there exist functions $f_k : \mathbb{R}^2 \to \mathbb{R}^2$ satisfying $f_k(0) = 0$ and $\mathrm{B}_{\omega(s)}(f_k(z)) \subset f_k(\mathrm{B}_s(z)) \subset \mathrm{B}_{\Omega(s)}(f_k(z))$ for all $s > 0$ and $z \in \mathbb{R}^2$ as well as numbers $r_k \geq r_0$ such that the component C_k of $f_k^{-1}(\mathrm{B}_{r_k}(0))$ containing zero has diameter at least $k r_k$. Observe (or see [BJLPS, Rem. 3.3]) that a uniformly continuous g is Lipschitz for large distances in the sense that

$\|g(z_1) - g(z_2)\| \le 2\Omega(1)\|z_1 - z_2\|$ if $\|z_1 - z_2\| \ge 1$. Similarly, for a co-uniformly continuous function g, $g(\mathrm{B}_s(z)) \supset \mathrm{B}_{s\omega(1)/2}(g(z))$ for $s \ge 1$. Hence there is a subsequence of $f_k(kr_k z)/kr_k$ converging to a Lipschitz and co-Lipschitz $g\colon \mathbb{R}^2 \to \mathbb{R}^2$. It follows that $g^{-1}(0)$ contains a connected set of diameter at least $1/2$, in contradiction to Proposition 2.1. $\square$

Lemma 2.5. *Let $f\colon \mathbb{R}^2 \to \mathbb{R}^2$ satisfy* (2.1). *For every $x \in \mathbb{R}^2$ and every unit vector u, there is a closed unbounded set $\Gamma_{x,u}$ such that $x \in \Gamma_{x,u}$, $f(\Gamma_{x,u}) = \{f(x) + tu;\ t \ge 0\}$, and $\{w \in \Gamma_{x,u};\ \|f(w) - f(x)\| \le \tau\}$ is compact and connected for every $\tau \ge 0$.*

Proof. For $m = 1, 2, \dots$, define $\phi_m(0) = x$. By induction, assuming that $f\bigl(\phi_m\bigl(\frac{k}{m}\bigr)\bigr) = f(x) + \frac{k}{m}u$, choose $\phi_m\bigl(\frac{k+1}{m}\bigr)$ such that $\bigl\|\phi_m\bigl(\frac{k+1}{m}\bigr) - \phi_m\bigl(\frac{k}{m}\bigr)\bigr\| \le \omega^{-1}\bigl(\frac{1}{m}\bigr)$ and $f\bigl(\phi_m\bigl(\frac{k+1}{m}\bigr)\bigr) = f(x) + \frac{k+1}{m}u$. Extend $\phi_m(t)$ to a Lipschitz curve $\phi_m\colon [0,\infty) \to \mathbb{R}^2$ having Lipschitz constant at most $m\omega^{-1}\bigl(\frac{1}{m}\bigr)$. Since $\frac{k}{m} = \bigl\|f\bigl(\phi_m\bigl(\frac{k}{m}\bigr)\bigr) - f(x)\bigr\| \le \Omega\bigl(\bigl\|\phi_m\bigl(\frac{k}{m}\bigr) - x\bigr\|\bigr)$, $\|\phi_m(t) - x\| \to \infty$ as $t \to \infty$.

For any $t \ge 0$ of the form $\frac{k}{m}$ choose the largest $s_m(t)$ such that $f(\phi_m(s_m(t))) = f(x) + tu$. By Lemma 2.4, if m is large enough then $\operatorname{diam}(\phi_m[0, s_m(t)]) \le R_0 \cdot (1 + t/2)$, where $R_0 = R_0(1)$. Let m_j be chosen so that, for every rational $t > 0$, $s_{m_j}(t)$ is eventually defined and the sequence $\phi_{m_j}[0, s_{m_j}(t)]$ of continua converges to a continuum C_t. Note that $f(C_t) = [f(x), f(x) + tu]$ and that $t' > t$ implies $C_t \supset C_{t'} \cap f^{-1}[f(x), f(x) + tu]$. In particular, $\Omega(\|w - x\|) \ge t$ for $w \in C_{t'} \setminus C_t$, which shows that $\Gamma_{x,u} = \bigcup_t C_t$ is closed and unbounded. Clearly $x \in \Gamma_{x,u}$ and $f(\Gamma_{x,u}) = \{f(x) + tu;\ t \ge 0\}$. Moreover,

$$\{w \in \Gamma_{x,u};\ f(w) \in [f(x), f(x) + \tau u]\} = \bigcap_{t > \tau} C_t,$$

so it is compact and connected. $\square$

Lemma 2.6. *Let $f\colon \mathbb{R}^2 \to \mathbb{R}^2$ satisfy* (2.1). *Suppose that a, b belong to different components of $f^{-1}(y)$, $r \ge 4(R_0 + \|a - b\|)$, where $R_0 = R_0(1)$ of Lemma 2.4 and u is a unit vector. Then*

$$\begin{aligned}\Omega^{-1}\biggl(\frac{r}{6R_0}\biggr) \le{}& \operatorname{dist}(\{z \in \Gamma_{a,u};\ \|z - a\| \ge r\}, \{z \in \Gamma_{b,u};\ \|z - b\| \ge r\}) \\ &+ \operatorname{dist}(\{z \in \Gamma_{a,-u};\ \|z - a\| \ge r\}, \{z \in \Gamma_{b,-u};\ \|z - b\| \ge r\}).\end{aligned}$$

Proof. Note that, by Lemma 2.5, the sets whose distances we estimate are always nonempty. Suppose that

$$\begin{aligned}&\operatorname{dist}(\{z \in \Gamma_{a,u};\ \|z - a\| \ge r\}, \{z \in \Gamma_{b,u};\ \|z - b\| \ge r\}) \\ &\qquad + \operatorname{dist}(\{z \in \Gamma_{a,-u};\ \|z - a\| \ge r\}, \{z \in \Gamma_{b,-u};\ \|z - b\| \ge r\}) < \Omega^{-1}\biggl(\frac{r}{6R_0}\biggr).\end{aligned}$$

Case I: $\Gamma_{a,u} \cap \Gamma_{b,u} \ne \emptyset$ and $\Gamma_{a,-u} \cap \Gamma_{b,-u} \ne \emptyset$. Then, for sufficiently large τ, $A = \{w \in \Gamma_{a,u} \cup \Gamma_{b,u};\ \|f(w) - y\| \le \tau\}$ and $B = \{w \in \Gamma_{a,-u} \cup \Gamma_{b,-u};\ \|f(w) - y\| \le \tau\}$ are continua. But $\{a, b\} \subset A \cap B \subset f^{-1}(y)$, so $A \cap B$ is not connected since a and b belong to different components of $f^{-1}(y)$. Hence, by [K, Ch. 10, Sec. 61, I, Thm. 5], $\mathbb{R}^2 \setminus (A \cup B)$ has a bounded component G. But then $f(G)$ is bounded, open, and with boundary contained in $f(A \cup B)$, hence in the line $L = \{y + tu;\ t \in \mathbb{R}\}$, which is impossible. Note that the proof actually shows that there is at most one direction u for which $\Gamma_{a,u} \cap \Gamma_{b,u} \neq \emptyset$.

Case II: $\Gamma_{a,u} \cap \Gamma_{b,u} = \emptyset$ *and* $\Gamma_{a,-u} \cap \Gamma_{b,-u} \neq \emptyset$. Choose a segment D such that $D \cap \Gamma_{a,u} = \{p\}$ and $D \cap \Gamma_{b,u} = \{q\}$, where $\|p - a\| \geq r/2$, $\|q - b\| \geq r/2$, and $\operatorname{diam}(D) < \Omega^{-1}(r/6R_0)$. Let $A \subset \Gamma_{a,u} \cup \Gamma_{a,-u} \cup \Gamma_{b,-u} \cup \Gamma_{b,u}$ be a minimal continuum containing p and q. Since $A \cap f^{-1}(\{y + tu;\ t > 0\})$ is disconnected, $A \cap f^{-1}(\{y + tu;\ t \leq 0\}) \neq \emptyset$. By Lemma 2.4, $f(p) = y + su$ where $s \geq r/2R_0$, and we infer that $\operatorname{diam}(f(A)) \geq r/2R_0$.

From [K, Ch. 10, Sec. 62, V, Thm. 6] we infer that $A \cup D$ is the boundary of a bounded component, say G, of its complement. Since f is open, $f(G)$ is open, and since its boundary is contained in $L \cup \mathrm{B}_{\Omega(\|q-p\|)}(f(p))$ (recall that $L = \{y + tu;\ t \in \mathbb{R}\}$), we infer that $f(G) \subset \mathrm{B}_{\Omega(\|q-p\|)}(f(p)) \subset \mathrm{B}_{r/6R_0}(f(p))$. But $\operatorname{diam}(f(G)) \geq \operatorname{diam} f(A) \geq r/2R_0$.

Case III: $\Gamma_{a,u} \cap \Gamma_{b,u} \neq \emptyset$ *and* $\Gamma_{a,-u} \cap \Gamma_{b,-u} = \emptyset$. Symmetric to Case II.

Case IV: $\Gamma_{a,u} \cap \Gamma_{b,u} = \emptyset$ *and* $\Gamma_{a,-u} \cap \Gamma_{b,-u} = \emptyset$. Choose segments D^+, D^- such that $D^+ \cap \Gamma_{a,u} = \{p^+\}$ and $D^+ \cap \Gamma_{b,u} = \{q^+\}$, where $\|p^+ - a\| \geq r/2$, $\|q^+ - b\| \geq r/2$, $D^- \cap \Gamma_{a,-u} = \{p^-\}$ and $D^- \cap \Gamma_{b,-u} = \{q^-\}$, where $\|p^- - a\| \geq r/2$, $\|q^- - b\| \geq r/2$, and $\operatorname{diam}(D^+) + \operatorname{diam}(D^-) < \Omega^{-1}(r/6R_0)$. Note that $\Omega(\|p^+ - p^-\|) \geq r/R_0$ (otherwise, $\|f(p^+) - f(p^-)\| < r/R_0$ so one of $\|f(p^+) - y\|$ or $\|f(p^-) - y\|$ is less than $r/2R_0$, in contradiction to Lemmas 2.4 and 2.5 and the choice of p^+, p^-), so $D^+ \cap D^- = \emptyset$. Let $A \subset \Gamma_{a,u} \cup \Gamma_{a,-u}$ be a minimal continuum containing p^+, p^-, and let $B \subset \Gamma_{b,u} \cup \Gamma_{b,-u}$ be a minimal continuum containing q^+, q^-.

Clearly, $\operatorname{diam}(f(A)) \geq r/R_0$. Noting that A and $D^+ \cup B \cup D^-$ are minimal continua whose intersection is $\{p^+, p^-\}$, we infer from [K, Ch. 10, Sec. 62, V, Thm. 6] that $A \cup D^+ \cup B \cup D^-$ is the boundary of a bounded component, say G, of its complement. Since f is open, $f(G)$ is open, and since its boundary is contained in $L \cup \mathrm{B}_{\Omega(\operatorname{diam}(D^+))}(f(p^+)) \cup \mathrm{B}_{\Omega(\operatorname{diam}(D^-))}(f(p^-))$, we infer that $f(G) \subset \mathrm{B}_{\Omega(\operatorname{diam}(D^+))}(f(p^+)) \cup \mathrm{B}_{\Omega(\operatorname{diam}(D^-))}(f(p^-))$. Since these two discs are disjoint and $f(G)$ is connected, it is contained in one of them. But $\operatorname{diam}(f(G)) \geq \operatorname{diam} f(A) \geq r/R_0$ is bigger than the diameter of either of these discs. This contradiction concludes the proof. $\square$

LEMMA 2.7. *Let* $f : \mathbb{R}^2 \to \mathbb{R}^2$ *satisfy* (2.1). *Then there is an* $N < \infty$, *depending only on* Ω *and* ω, *such that, for each* $y \in \mathbb{R}^2$, *the cardinality of the set of components of* $f^{-1}(y)$ *is at most* N.

Proof. Choose $a \in f^{-1}(y)$. If s is large enough then, applying Lemma 2.6 with $r = 4(R_0 + s)$, we have that the number of components of $f^{-1}(y)$ which meet a disc of radius s around a cannot be greater than the largest number of elements of a set $M \subset \{x;\ \|x - a\| \leq 5(R_0 + s)\} \times \{y;\ \|y - a\| \leq 5(R_0 + s)\}$ that has all ℓ_1 distances larger than or equal to $\Omega^{-1}(4(R_0 + s)/6R_0)$.

We may assume that, for $t \geq 1$, $\Omega(t) \leq 2\Omega(1)t$. Homogeneity now implies that, if s is large enough, then the number of elements of M is at most the number of couples of points in a disc of radius 1 whose mutual ℓ_1 distances are not smaller than some positive number c (depending only on $\Omega(1)$ and R_0). This number is a bound for N. $\square$

Proof of Theorem 2.3. Let n be the largest number for which one can find y with n components (say, $H_1, \dots, H_n$) of $f^{-1}(y)$, of which at least one (say, H_1) is nontrivial. Let y and $H_1, \dots, H_n$ realize this maximum and let $G_i \supset H_i$ be open, with disjoint closures. For z sufficiently close to y, say $\|z - y\| < \delta$, $f^{-1}(z)$ meets each G_i; moreover, for δ sufficiently small, the component of $f^{-1}(z)$ meeting G_i must be contained in G_i, since otherwise we would have a contradiction by taking the limit of such components as $z \to y$. Denoting $H_i(z) = f^{-1}(z) \cap G_i$, we thus have $f^{-1}(z) \supset \bigcup_{i=1}^{n} H_i(z)$, and $H_i(z)$ are components of $f^{-1}(z)$. The component $H_1(z)$ is nontrivial for z close to y (otherwise, since H_1 is nontrivial and f is a uniform quotient mapping, G_1 would contain an arbitrarily large number of components for z close to y, in contradiction to Lemma 2.7). Also, the maximality of n implies that G_1 contains only one component $H_1(z)$. Let u, v be two different points of H_1, and let L be their perpendicular bisector. If $\|z - y\| < \delta_1 = \min\{\delta, \omega(\|v - u\|/4)\}$, then $f^{-1}(z)$ meets L, so $f(L)$ has nonempty interior. This is not possible if $\lim_{r\to 0} \Omega(r)/\sqrt{r} = 0$; an easy way to see this is to compare the cardinality of a maximal ε-separated set of points in a segment of L and in its image. $\square$

It now follows from a deep theorem of Stoilow that uniform quotient mappings satisfying the assumptions of either Proposition 2.1 or Theorem 2.3 are topologically equivalent to polynomials.

THEOREM 2.8. *Let $f\colon \mathbb{R}^2 \to \mathbb{R}^2$ satisfy one of the following three assumptions.*

(i) *f is uniformly continuous and co-Lipschitz*;

(ii) *f is uniformly continuous with modulus of continuity Ω satisfying $\Omega(r)/\sqrt{r} \to 0$ as $r \to 0$, and f is also co-uniformly continuous*; *or*

(iii) *there are c, C, p, q with $q < 1 + p$ such that f is uniformly continuous with modulus of continuity Ω satisfying $\Omega(r) \le Cr^p$ for $0 < r < 1$, and f is also co-uniformly continuous with modulus of co-uniform continuity ω satisfying $\omega(r) \ge cr^q$ for $0 < r < 1$.*

Then $f = P \circ h$, where h is a homeomorphism of $\mathbb{R}^2$ and P is a polynomial.

Proof of 2.8(i) and 2.8(ii). By Stoilow's theorem [S, p. 121], since f is discrete and open it follows that $f = P \circ h$, with h a homeomorphism of $\mathbb{R}^2$ onto a (simply connected) domain G in $\mathbb{R}^2$ and P an analytic function. (In the formulation of Stoilow's theorem in [S], the image is a Riemann surface but the uniformization theorem—see e.g. [FK, p. 195]—implies that it must be a simply connected domain in the plane.) By Proposition 2.1 and Lemma 2.7, the inverse image of each point under f, satisfying assumption (i), is finite (even bounded independently of the point). The same is true also under assumption (ii), by Theorem 2.3. Thus, $P^{-1}(y) \cap G$ also is finite for each point y. We shall show that G is necessarily $\mathbb{R}^2$, so that P is an entire function with the property that the inverse image of each point is finite. It then follows that P is a polynomial.

We now prove that G, the image of h, is the entire plane. First notice that $f(z) \to \infty$ as $z \to \infty$. Indeed, there would otherwise be a sequence z_n such that $z_n \to \infty$ and $f(z_n) \to a$. Since f is co-uniformly continuous, f will take the value a in the disc of radius 1 around z_n for all large enough n. This contradicts the fact that $f^{-1}(a)$ is finite.

If $f = P \circ h$ with P analytic and $h(\mathbb{R}^2) = G \neq \mathbb{R}^2$ then, since G is simply connected, we may assume without loss of generality that G is the unit disc. It follows from the previous paragraph that $P(z)$ tends to infinity as $|z| \to 1$. Since P has only finitely many zeros in the unit disc, by dividing the Blaschke product corresponding to the zeros by P we obtain an analytic function in the disc tending to zero as $|z| \to 1$. By the maximum principle, this is impossible. $\square$

The proof of Theorem 2.8 under assumption (iii) is delayed to the end of Section 4.

REMARK. Note that the homeomorphism h in the representation $f = P \circ h$ is determined up to a transformation of the form $h \to ah + b$ for some complex a and b (and then necessarily P is determined up to a change of variable $z \to az + b$). Indeed, if $P \circ h = Q \circ g$ for polynomials P and Q and homeomorphisms h and g, then P and Q must have the same degree (which is equal to the maximal cardinality of a preimage of a point under f). If $w = gh^{-1}(z)$ and Q is invertible in a neighborhood of w, then gh^{-1} is analytic in a neighborhood of z. It is then necessarily a polynomial of degree 1; this follows easily from the equation $P(z) = Q(gh^{-1}(z))$. Since there are only finitely many exceptional points (the preimages under gh^{-1} of the zeroes of Q'), it follows that gh^{-1}, being a homeomorphism of the plane onto itself and analytic except at finitely points, must be a linear function.

We also have a converse statement to Theorem 2.8.

PROPOSITION 2.9. *Let P be a polynomial in one complex variable with complex coefficients. Then there is a homeomorphism h of the plane such that $f = P \circ h$ is a Lipschitz quotient mapping.*

Sketch of Proof. Assume without loss of generality that $P(z) = z^n + a_{n-1}z^{n-1} + a_{n-2}z^{n-2} + \cdots + a_0$. We first show how to find a homeomorphism h that makes $f = P \circ h$ Lipschitz and co-uniformly continuous. Fix a (large) $R > 0$ and define h by

$$h(z) = \begin{cases} |z|^{1/n} e^{i \arg(z)} & \text{if } 2R \le |z|, \\ \left(\frac{2R-|z|}{R}|z| + \frac{|z|-R}{R}|z|^{1/n}\right) e^{i \arg(z)} & \text{if } R \le |z| < 2R, \\ z & \text{if } |z| \le R. \end{cases}$$

It is easy to see that h is a homeomorphism of $\mathbb{R}^2$ onto itself. Also, h is Lipschitz on the ball of radius $3R$ about zero and is co-uniformly continuous on the same ball in the sense that $\mathrm{B}_{\omega(r)}(h(x)) \subset h(\mathrm{B}_r(x))$ for an appropriate $\omega(r) > 0$ and all x in the ball of radius $3R$ about zero. Since P is Lipschitz on the image of that ball (as it is on any compact domain), $f = P \circ h$ is Lipschitz on the ball of radius $2R$ about zero. Outside that ball,

$$f(z) = |z|e^{in\arg(z)} + a_{n-1}|z|^{(n-1)/n}e^{i(n-1)\arg(z)} + a_{n-2}|z|^{(n-2)/n}e^{i(n-2)\arg(z)} + \cdots + a_0,$$

which is checked to be Lipschitz in this domain.

Since any polynomial is an open mapping, a simple compactness argument (mentioned in the introduction) shows that any polynomial is co-uniformly continuous when restricted to any bounded domain. It follows that f is co-uniformly continuous when restricted to the ball of radius $3R$ about zero. The special form of f outside the ball of radius $2R$ about zero shows that, if R is large enough, f is even co-Lipschitz there.

Assume now that R is such that, in addition to the foregoing implicit requirements on its size, all the zeros of P' are in a ball of radius $R/4$ about zero. We now show how to adjust h on a ball of radius $R/2$ about zero so as to remain Lipschitz and be also co-Lipschitz.

Let $z_1, \dots, z_m$ be the distinct zeros of P'. Let $r > 0$ be such that $\mathrm{B}_{3r}(z_j)$, $j = 1, \dots, m$, are pairwise disjoint and all contained in $\mathrm{B}_{R/2}(0)$. There is no problem with the co-Lipschitzity of h outside the union of these balls. Fix any $j = 1, \dots, m$. Then, by taking an even smaller $r > 0$, we may also assume that in $\mathrm{B}_{3r}(z_j)$ one can write $P(z) = (z - z_j)^k Q(z) + a$, where Q does not vanish on $\mathrm{B}_{3r}(z_j)$. Next, modify the definition of h on $\mathrm{B}_{3r}(z_j)$ as follows. For $z = z_j + se^{i\theta}$,

$$h(z) = \begin{cases} z_j + s^{1/k}e^{i\theta} & \text{if } 0 \le s \le r, \\ z_j + \left(\frac{2r-s}{r}s^{1/k} + \frac{s-r}{r}s\right)e^{i\theta} & \text{if } r < s < 2r, \\ z & \text{if } 2r \le s \le 3r. \end{cases}$$

We do this on each of the balls $\mathrm{B}_{3r}(z_j)$, leaving h as it was outside the union of the balls. □

3. The Example

Here we give an example showing that, without some restrictions on the moduli of uniform continuity or co-uniform continuity of a uniform quotient mapping of the plane to itself, the conclusions of Theorems 2.3 and 2.8 no longer hold.

Lemma 3.1. *Given $d \ge c > 0$ and $a \in \mathbb{R}^2$ with $\|a\| \le d$, there is a mapping $g\colon \mathbb{R}^2 \to \mathbb{R}^2$ such that:*

(i) $g([0, a]) \subset [0, g(a)]$ *and* $\|g(a)\| \le c/4$;
(ii) *the Lipschitz constants of g and of g^{-1} are less than or equal to* $36(d/c)^2$;
(iii) *for all $z \in \mathbb{R}^2$ and $r \ge 2c$, $g(\mathrm{B}_r(z))$ is $6c$-dense in* $\mathrm{B}_{r+d}(z)$;
(iv) *for all $z \in \mathbb{R}^2$ and $r \ge 2c$, $g(\mathrm{B}_r(z))$ is $6c$-dense in* $\mathrm{B}_r(g(z))$;
(v) *for all $z \in R^2$,* $\|g(z) - z\| \le 2d$.

Proof. Rotate the coordinate system so that a is a positive multiple of $(c, -4d)$. Define continuous 1-periodic $\eta\colon R \to R$ by $\eta(0) = \eta(1/2) = 0$, $\eta(1/4) = 1$, $\eta(3/4) = -1$, and η is affine on every component of $R \setminus (1/4)Z$. Let $\varphi(x, y) = (x, y + d\eta(x/c))$, $\psi(x, y) = (x + d\eta(y/c), y)$, and $g = \psi \circ \varphi$.

(i) On the segment $I = \{ (\tau c, -4\tau d) : 0 \le \tau \le 1/4 \}$ we have $\varphi(\tau c, -4\tau d) = (\tau c, 0)$, and on the segment $J = \{ (\tau c, 0) : 0 \le \tau \le 1/4 \}$ we have $\psi(\tau c, 0) = (\tau c, 0)$. Hence I is mapped affinely on J. Since $a \in I$, this shows (i). (ii) is obvious.

(iii, iv) Let $z = (u, v)$ and let k and l be the integer parts of u/c and v/c, respectively. For every $m = 0, 1, \dots,$ the g image of

$$P_m = [(k-m)c, (k+1+m)c] \times [(l-m)c, (l+1+m)c]$$

is contained in

$$Q_m = [(k-m)c - d, (k+1+m)c + d] \times [(l-m)c - d, (l+1+m)c + d]$$

and meets every square $[pc, (p+1)c] \times [qc, (q+1)c]$ that lies inside Q_m, so it is $4c$-dense in Q_m. Choosing the largest m such that $P_m \subset \mathrm{B}_r(z)$ (i.e., m is the integer part of $(r/2c) - 1$), we have that $Q_m \supset \mathrm{B}_{r+d-2c}(z)$, which proves (iii). To prove (iv), we note that $g(z) \in Q_0$, so $Q_m \supset \mathrm{B}_{r-2c}(g(z))$. (v) is obvious. $\square$

To illustrate the complexity of the seemingly simple mapping of Lemma 3.1, Figure 1 sketches the image under such a g of the boundary of the square with vertices $(0, 0)$, $(1, 0)$, $(1, 1)$, $(0, 1)$, where $c = 1$ and $d = 1.5$.

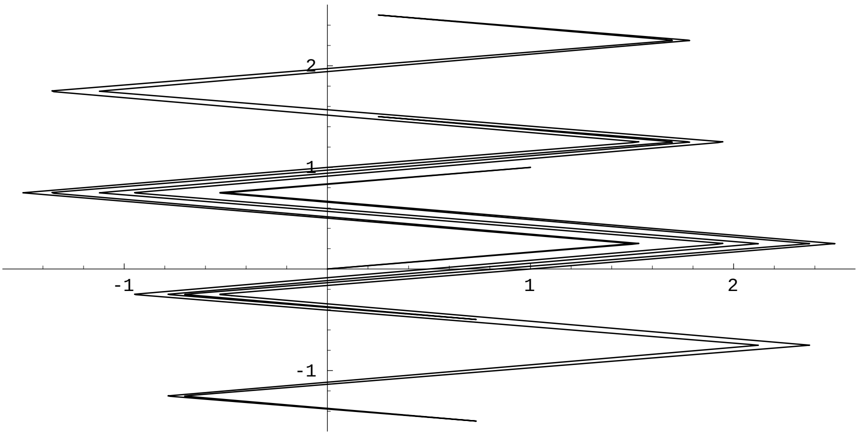

Figure 1

In the following example we use $\mathrm{B}(x, t)$ to denote $\mathrm{B}_t(x)$.

EXAMPLE 3.2. *There is a mapping $f : \mathbb{R}^2 \to \mathbb{R}^2$ such that $f^{-1}(0)$ contains a segment and, for every $z \in \mathbb{R}^2$ and $r > 0$,*

$$\mathrm{B}(f(z), C_1 \min(r^\beta, r)) \subset f(\mathrm{B}(z, r)) \subset \mathrm{B}(f(z), C_2 \max(r^\alpha, r)),$$

where α, β, C_1, C_2 are positive constants.

Proof. Let $c_0 = 1/8$ and $c_{k+1} = 48^{-4k-1} c_k^5$. Let $a_1 = (0, 1)$ and let g_1 be the function obtained by Lemma 3.1 with $c = c_1$, $d = 8c_0$, and $a = a_1$. Recursively put $a_{k+1} = g_k(a_k)$, note that $\|a_{k+1}\| \le c_k$, and let g_{k+1} be the function obtained by Lemma 3.1 with $c = c_{k+1}$, $d = 8c_k$, and $a = a_{k+1}$.

Define $f_1 = g_1$ and $f_{k+1} = g_{k+1} \circ f_k$. Then $\|f_{k+1} - f_k\| \le 16c_k$, so the sequence f_k converges uniformly to a continuous $f : \mathbb{R}^2 \to \mathbb{R}^2$. In particular, $f(z) = 0$ for all $z \in [0, a_1]$, so $f^{-1}(0)$ contains $[0, a_1]$. By (ii) of Lemma 3.1, for each k the Lipschitz constants of f_k and of its inverse do not exceed $48^{2k}/c_k^2$. In particular, $f_k(\mathrm{B}(z, r)) \supset \mathrm{B}(f_k(z), 48^{-2k} c_k^2 r)$.

Let $z \in R^2$ and $0 < r < c_1$. Find the least k such that $r \ge 48^{-k} c_k$, and let $s = 48^{-2k} c_k^2 r$. Then $f_k(\mathrm{B}(z, r)) \supset \mathrm{B}(f_k(z), s)$ and $s \ge 48^{-3k} c_k^3 \ge 2c_{k+1}$.

We prove that, for every $n > k$, $f_n(\mathrm{B}(z, r))$ is $6c_n$-dense in $\mathrm{B}(f_{k+1}(z), s)$. For $n = k + 1$, this follows from (iv) of Lemma 3.1. If it holds for some n and if $y \in \mathrm{B}(f_{k+1}(z), s)$, choose $x \in \mathrm{B}(z, r)$ such that $\|y - f_n(x)\| \le 6c_n$. Let $t = 48^{-2n} c_n^3$ and let $u \in \mathrm{B}(z, r)$ be such that $x \in \mathrm{B}(u, t) \subset \mathrm{B}(z, r)$. Then $\|f_n(u) - f_n(x)\| \le c_n$ and $f_n(\mathrm{B}(u, t)) \supset \mathrm{B}(f_n(u), 48^{-4n} c_n^5) \supset \mathrm{B}(f_n(u), 2c_{n+1})$. Hence $f_{n+1}(\mathrm{B}(z, r)) \supset g_{n+1}(\mathrm{B}(f_n(u), 2c_{n+1}))$ is $6c_{n+1}$-dense in $\mathrm{B}(f_n(u), 8c_n)$. Since $y \in \mathrm{B}(f_n(u), 8c_n)$, the set $f_{n+1}(\mathrm{B}(z, r))$ contains a point $6c_{n+1}$ close to y.

Using that $\|f(z) - f_{k+1}(z)\| \le 16 \sum_{j=k+1}^{\infty} c_j \le s/2$, we conclude that $f(\mathrm{B}(z, r)) \supset \mathrm{B}(f_{k+1}(z), s) \supset \mathrm{B}(f(z), s/2)$. Moreover, $s/2 \ge r^\beta$ if $\beta > 11$ and r is sufficiently small.

Given any x, y and any k, we have

$$\|f(x) - f(y)\| \le 32 \sum_{j=k}^{\infty} c_j + \mathrm{Lip}(f_k)\|x - y\| \le 64c_k + 48^{2k}\|x - y\|/c_k^2.$$

If $c_{k+1}^3 \le \|x - y\| \le c_k^3$, this gives $\|f(x) - f(y)\| \le 48^{2k+2} c_k \le \|x - y\|^\alpha$ if $\alpha < 1/15$ and $\|x - y\|$ is sufficiently small. $\square$

REMARK 3.3. We now show how to modify any nontrivial uniform quotient mapping f of the plane onto itself to obtain a simple construction of an example of a continuous open monotone mapping of the plane onto itself which is not a homeomorphism. ("Monotone" means that the inverse image of each point is a continuum; such a mapping was first constructed by Anderson [A].) Toward this end we first observe that the complement of the inverse image under f of any open disc is connected; indeed, recalling that inverse images of discs are bounded, the opposite would allow us to find first a bounded component C of the complement of $f^{-1}(\mathrm{B}_r(y))$ and then a bounded open set $V \supset C$ whose boundary would lie entirely in $f^{-1}(\mathrm{B}_r(y))$ and thence to conclude that f, being continuous and mapping the boundary of V to $\mathrm{B}_r(y)$, cannot be open. Next we observe that the argument from the beginning of the proof of Theorem 2.3 provides us with an open disc $\mathrm{B}_r(y)$ and a bounded open set G containing a nontrivial component of $f^{-1}(y)$ such that, for every $z \in \mathrm{B}_r(y)$, there is exactly one component H_z of $f^{-1}(z)$ meeting $\bar{G}$ and such that this component is, in fact, contained in G. Then $U = \bigcup_{z \in \mathrm{B}_r(y)} H_z$ is a component of $f^{-1}(\mathrm{B}_r(y))$ and so, by our first observation, it is homeomorphic to the whole plane. Hence it suffices to point out that f is clearly a nontrivial monotone map of U to $\mathrm{B}_r(y)$.

4. Local Results

If one relaxes the assumptions of Theorem 2.3 by changing the domain of f from $\mathbb{R}^2$ to a bounded domain, the conclusion fails in a very strong sense.

EXAMPLE 4.1. *Define* $f\colon \mathbb{R}^2 \to \mathbb{R}^2$ *by* $f(x,y) = y^2(\cos(x/y), \sin(x/y))$ *when* $y \neq 0$ *and* $f(x,0) = 0$. *Then* f *is Lipschitz on bounded sets and for each* M *there exists* $\delta = \delta(M) > 0$ *such that, if* z *is in* $\mathbb{R}^2$ *and* $|z| \leq M$, *then* $f(\mathrm{B}_r(z)) \supset \mathrm{B}_{\delta r^3}(f(z))$ *for all* $r \leq 1$.

Proof. That f is Lipschitz on bounded sets follows by taking partial derivatives. To check the second statement, assume that $f(x_0, y_0) = s_0(\cos\theta_0, \sin\theta_0)$ and, without loss of generality, that $y_0 \geq 0$. Assume first that $r \leq y_0/2 \wedge 1$.

We would like to show that, for an appropriate δ, $f(\mathrm{B}_r(x_0, y_0))$ contains the set $S = \{s(\cos\theta, \sin\theta);\ |s - s_0| < \delta r^3,\ |\theta - \theta_0| < \delta r^2/s\}$ and thus a ball of radius $\delta' r^3$ around $s_0(\cos\theta_0, \sin\theta_0)$. We shall actually show that, for an appropriate δ, $f([x_0 - r, x_0 + r] \times [y_0 - r^2/4M, y_0 + r^2/4M]) \supset S$, which is clearly enough.

Notice that:

(1) for every $0 < t \leq y_0$,

$$[(y_0 - t)^2, (y_0 + t)^2] \supset [s_0 - ty_0, s_0 + ty_0];$$

(2) for $0 < t \leq y_0/2$ and $|y - y_0| \leq t$,

$$\left|\frac{x_0}{y} - \frac{x_0}{y_0}\right| \leq t\frac{|x_0|}{yy_0} \leq t\frac{2|x_0|}{y^2} \leq \frac{2Mt}{y^2};$$

(3) for a fixed y and any positive u,

$$\{\arg(f(x,y));\ |x - x_0| \leq u\} \supset \left[\frac{x_0}{y} - \frac{u}{y}, \frac{x_0}{y} + \frac{u}{y}\right] \pmod{2\pi}.$$

Taking $t = r^2/4M$ in (2) and $u = r$ in (3), for a fixed y and for $s = y^2$ we have

$$\{\arg(f(x,y));\ |x - x_0| \leq r\} \supset \left[\frac{x_0}{y_0} - \frac{r^2}{2s}, \frac{x_0}{y_0} + \frac{r^2}{2s}\right] \pmod{2\pi}.$$

Finally, applying (1) for $t = r^2/4M$, we obtain

$$\left[\left(y_0 - \frac{r^2}{4M}\right)^2, \left(y_0 + \frac{r^2}{4M}\right)^2\right] \supset \left[s_0 - \frac{r^3}{4M}, s_0 + \frac{r^3}{4M}\right].$$

This settles the case of $r \leq y_0/2 \wedge 1$. If $r \leq (20y_0) \wedge 1$ then $f(\mathrm{B}_r(z)) \supset f(\mathrm{B}_{r/40}(z)) \supset \mathrm{B}_{\delta' r^3}(f(z))$, so we are left only with the case $20y_0 < r \leq 1$. In this case, for every y with $|y - y_0| < r/20$, the set $\{x/y;\ |x - x_0| < r\} \pmod{2\pi}$ contains all possible arguments. It follows that $f(\mathrm{B}_r(x_0, y_0))$ contains $\mathrm{B}_{y_0^2 + r^2/400}(0)$ and, in particular, $\mathrm{B}_{r^2/400}(f(x_0, y_0))$. □

REMARK. One can generalize this example. For $\beta > 0$ and $\alpha \ge 1$, let $f(x, y) = |y|^\beta(\cos(x/|y|^\alpha), \sin(x/|y|^\alpha))$. One can check that, when restricting f to a bounded domain, the modulus of uniform continuity of f is bounded by Cr (i.e., f is Lipschitz) if $\beta \ge 1 + \alpha$ and by $Cr^{\beta/(1+\alpha)}$ if $\beta \le 1 + \alpha$. The modulus of co-uniform continuity is bounded from below by $cr^{\beta\vee(1+\beta/\alpha)}$. In particular, minimizing over $\beta = 1+\alpha$ yields a function that is Lipschitz on bounded domains and has modulus of co-uniform continuity bounded from below by $cr^{2.62}$ on bounded domains.

In spite of Example 4.1, one can show that—under some restriction concerning the relation between the modulus of uniform continuity of f and its modulus of co-uniform continuity—a local form of Theorem 2.3 still holds.

PROPOSITION 4.2. *Suppose that $p < 1+q$, $G \subset \mathbb{R}^2$ is open, and $f : G \to \mathbb{R}^2$ is such that $\mathrm{B}_{cr^p}(f(z)) \subset f(\mathrm{B}_r(z)) \subset \mathrm{B}_{Cr^q}(f(z))$ whenever $\mathrm{B}_r(z) \subset G$ and $r \le 1$. Then the inverse images of points under f are discrete.*

Proof. It suffices to assume that $c = C = 1$, $p > q$, $0 \in G$, $f(0) = 0$, and (for some $r_0 > 0$) $\mathrm{B}_{3r_0}(0) = G$ and to show that $\mathrm{B}_{r_0}(0) \cap f^{-1}(0)$ is finite.

LEMMA 4.3. *There exist a positive constant a and a strictly increasing function $h : [0, a] \to [0, \infty]$ such that, whenever $x \in \mathrm{B}_{r_0}(0) \cap f^{-1}(0)$ and $u \in \mathbb{R}^2$ is a unit vector, there is a curve $\phi : [0, a] \to \mathrm{B}_{2r_0}(x)$ with Lipschitz constant 1 such that $\phi(0) = x$, $\|f(\phi(t))\| \ge h(t)$, and $f(\phi(t)) \in \bigcup_{s>0} \mathrm{B}_{s/4}(su)$ for $t \in (0, a]$.*

Proof. Choose $p - 1 < \alpha < q/(p - q)$. Fix a sufficiently large m and choose $x_m \in \mathrm{B}_{m^{-\alpha/p}}(x)$ such that $f(x_m) = m^{-\alpha}u$. For $k < m$, recursively choose $x_k \in \mathrm{B}_{\alpha^{1/p}k^{-(\alpha+1)/p}}(x_{k+1}) \cap \mathrm{B}_{2r_0}(x)$ such that $f(x_k) = k^{-\alpha}u$; the construction stops when either $k = 1$ or no such x_k exists. If x_k is defined, we use that $(\alpha + 1)/p > 1$ to estimate $\sum_{j=k}^{m-1}\|x_{j+1} - x_j\| \le c_1k^{1-(\alpha+1)/p}$. Noting that $k^{-\alpha} - (k + 1)^{-\alpha} \le \alpha k^{-(\alpha+1)}$, we infer that there is an integer k_0 independent of x such that x_{k_0} is defined (as long as m is large enough). Since $\alpha < q/(p - q)$, we can enlarge k_0 if necessary to ensure that $\alpha^{q/p}k^{-(\alpha+1)q/p} \le k^{-\alpha}/4$ for $k \ge k_0$.

Let $\phi_m(t)$, $t \in [0, a_m]$, be the arc-length parameterization of the path obtained by joining $x, x_m, x_{m-1}, \dots, x_{k_0}$ (in this order) by the linear segments $[x, x_m]$ and $[x_{k+1}, x_k]$, $k = 0, \dots, m - 1$. Then $a_m \le m^{-\alpha/p} + c_1k_0^{1-(\alpha+1)/p}$, so we may find a subsequence of a_m converging to $a \le c_1k_0^{1-(\alpha+1)/p}$ such that the corresponding subsequence of ϕ_m converges to a Lipschitz curve $\phi : [0, a] \to \mathbb{R}^2$. Then $\phi(0) = x$ and $f(\phi(a)) = k_0^{-\alpha}u$, so $a \ge a_0 > 0$ where a_0 is independent of x and u. For any $0 < t \le a$ denote $s_m = \min\{a_m, t\}$, and for any sufficiently large m choose $k_m \ge k_0$ such that $\phi_m(s_m) \in [x_{k_m+1}, x_{k_m}]$; note first that $s_m \le c_1k_m^{1-(\alpha+1)/p}$ and hence a suitable subsequence of k_m has a limit $k \le (t/c_1)^{p/(p-1-\alpha)}$. Moreover, $\|f(\phi_m(s_m)) - k_m^{-\alpha}u\| \le \|\phi_m(s_m) - x_{k_m}\|^q \le \alpha^{q/p}k_m^{-(\alpha+1)q/p} \le k_m^{-\alpha}/4$, which, upon taking the limit as $m \to \infty$, gives that $\|f(\phi(t)) - k^{-\alpha}u\| \le k^{-\alpha}/4$. Hence, $f(\phi(t)) \in \bigcup_{s>0} \mathrm{B}(su, s/4)$ and $\|f(\phi(t))\| \ge (t/c_1)^{-p\alpha/(p-1-\alpha)}/2$ for any $t \in (0, a]$. $\square$

Let $r = h(a)^{1/q}/4$ and assume, as we may, that $3r \le 1$. We may also assume that $t^q \ge h(t)$ for $0 < t \le a$; in particular, $r \le a/4$. Denoting $d = (h(r)/5)^{1/q}$, we show that $M = \mathrm{B}_{r_0}(0) \cap f^{-1}(0)$ has at most $N = ((4r_0 + d)/d)^6$ elements. Assume that M has more than N elements.

Let $u_k = e^{k\pi i/3}$. For each $x \in M$ and $k = 1, 3, 5$, choose $\phi_{k,x}\colon [0, a] \to \mathrm{B}_{2r_0}(x)$ with Lipschitz constant 1 such that $\phi_{k,x}(0) = x$, $\|f(\phi_{k,x}(t))\| \ge h(t)$, and $f(\phi_{k,x}(t)) \in \bigcup_{s>0} \mathrm{B}_{s/4}(su_k)$ for $t > 0$. Note that the last statements also show that $f(\phi_{k,x}(t)) \ne 0$ if $t \ne 0$.

The triples $(\phi_{1,x}(r), \phi_{3,x}(r), \phi_{5,x}(r))$, $x \in M$, belong to the product of discs of radius $2r_0$; since $N > ((2r_0 + d/2)/(d/2))^6$, we infer that there are $x, y \in M$ with $x \ne y$ such that $\|\phi_{k,x}(r) - \phi_{k,y}(r)\| < d$ for $k = 1, 3, 5$.

Whenever $z \in [\phi_{k,x}(r), \phi_{k,y}(r)]$ we have $\|f(z) - f(\phi_{k,x}(r))\| \le d^q \le h(r)/5$. Finding $s > 0$ such that $f(\phi_{k,x}(r)) \in \mathrm{B}_{s/4}(su_k)$, we infer from $\|f(\phi_{k,x}(r))\| \ge h(r)$ that $h(r) \le 5s/4$, and we conclude that $f(z) \in \mathrm{B}_{s/4+h(r)/5}(su_k) \subset \mathrm{B}_{s/2}(su_k)$. We also note that this implies $f(z) \ne 0$.

Let L_k be a simple curve joining x and y and lying in the set $[\phi_{k,x}(r), \phi_{k,y}(r)] \cup \phi_{k,x}[0, r] \cup \phi_{k,y}[0, r]$. By the theorem on θ-curves [K, Ch. 10, Sec. 61, II, Thm. 2], one of these curves (say, L_k) lies, with the exception of its end points, entirely in the bounded component C of the complement of the remaining two. By connectedness, $\phi_{k,x}(0, a] \subset C$. Since the ϕ have Lipschitz constant 1 and since $\operatorname{diam}(C) \le 2r + d \le 3r$, we arrive at $(3r)^q \ge \|f(\phi_{k,x}(a)) - f(\phi_{k,x}(0))\| \ge h(a) > (3r)^q$—a contradiction. □

As a simple corollary, we now have the following.

Proof of Theorem 2.8(iii). If f satisfies the assumptions of Theorem 2.8(iii) then applying first Lemma 2.7 and then Proposition 4.2 with $G = \mathbb{R}^2$ we obtain that, for some $N < \infty$ and for all $y \in \mathbb{R}^2$, $f^{-1}(y)$ is a set consisting of at most N elements. The proof of the other two cases of Theorem 2.8 can now be carried over also for this case. □

5. Nonlinear Quotient Mappings from $\mathbb{R}^2$ to $\mathbb{R}$

Notice that there is no uniform quotient mapping from $\mathbb{R}^k$ to $\mathbb{R}^n$ for $k < n$. One way to see this is to observe that such a mapping would be Lipschitz and co-Lipschitz for large distances, which leads to a contradiction when looking at the maximal number of disjoint balls of a certain radius contained in a ball of a larger radius. Thus, the simplest case of Lipschitz and uniform quotient mappings between Euclidean spaces is that of mappings from $\mathbb{R}^2$ to $\mathbb{R}$ (since from $\mathbb{R}$ to $\mathbb{R}$ they are all one-to-one). In this section we briefly discuss this case, which is not entirely trivial, as we shall see. The main result is that, for Lipschitz quotient mappings, the inverse image of a point has finitely many components. Before we start, consider the following two examples (Figure 2) of Lipschitz quotient mappings from $\mathbb{R}^2$ to $\mathbb{R}$. In both cases the mapping f is the distance from the solid lines multiplied, in each component of the complement of the solid lines, by the sign indicated.

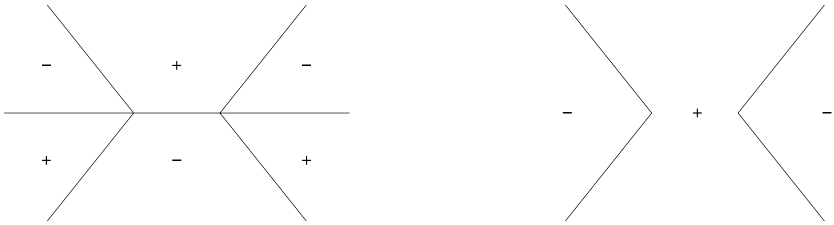

Figure 2

Note that $f^{-1}(0)$ has one component in the first example and two in the second. It is easy to draw examples with an arbitrary finite number of components. Notice also that $\mathbb{R}^2 \setminus f^{-1}(0)$ has six components in the first example and three in the second.

PROPOSITION 5.1. *Let* $f : \mathbb{R}^n \to \mathbb{R}$ *be a uniform quotient mapping satisfying* (2.1). *Then, for each* $t \in \mathbb{R}$, *the number of components of* $\mathbb{R}^n \setminus f^{-1}(t)$ *is finite and bounded by a function of* n, $\omega(\cdot)$, *and* $\Omega(\cdot)$ *only.*

Proof. According to the remarks made in the introduction, each component of $\mathbb{R}^n \setminus f^{-1}(0)$ is mapped by f onto a component of $\mathbb{R} \setminus \{0\}$, that is, onto either $(0, \infty)$ or $(-\infty, 0)$. Recall that f is Lipschitz and co-Lipschitz for large distances and let L and δ, depending only on the moduli of uniform and co-uniform continuity, be such that $\mathrm{B}_{\delta r}(f(z)) \subset f(\mathrm{B}_r(z)) \subset \mathrm{B}_{Lr}(f(z))$ for all $z \in \mathbb{R}^n$ and all $r \geq 1$.

Let $D_1, \ldots, D_k$ be distinct components of $\mathbb{R}^n \setminus f^{-1}(0)$ intersecting $\mathrm{B}_r(0)$ for some $r \geq 1$. Increasing r, we may also assume that there are $x_i \in D_i \cap \mathrm{B}_r(0)$ with $|f(x_i)| > 1$ for $i = 1, \ldots, k$. Note that each D_i intersects $\partial \mathrm{B}_{2r}(0)$. Moreover, there is a $y_i \in D_i \cap \partial \mathrm{B}_{2r}(0)$ such that $|f(y_i)| \geq \delta r$. Indeed, assuming $f(x_i) > 1$, there is an $x_i^1 \in \mathrm{B}_1(x_i)$ such that $f(x_i^1) > 1 + \delta$. Note that, if $L \leq 1$ as we may assume, $\mathrm{B}_1(x_i) \subset D_i$. There is an $x_i^2 \in \mathrm{B}_1(x_i^1)$ such that $f(x_i^2) > 1 + 2\delta$; again, $\mathrm{B}_1(x_i^1) \subset D_i$. Continuing this way at least $[r + 1]$ times (and interpolating between the last two points) yields a $y_i \in D_i \cap \partial \mathrm{B}_{2r}(0)$ with $f(y_i) \geq 1 + \delta[r] \geq \delta r$. It follows that $\mathrm{B}_{\delta r/L}(y_i) \subset D_i$, and we get k disjoint balls of radius $\delta r/L$ included in a ball of radius $3r$. Consequently, $k \leq (3L/\delta)^n$. □

We now aim to prove (in Proposition 5.4) that, for each $t \in \mathbb{R}$, every component of $f^{-1}(t)$ separates the plane. We first need two lemmas.

LEMMA 5.2. *Let* $f : \mathbb{R}^n \to \mathbb{R}$ *be a continuous open mapping. Then, for every* $t \in \mathbb{R}$, *no component of* $f^{-1}(t)$ *is bounded.*

Proof. Assume that A is a compact component of $f^{-1}(0)$. Let U be an open bounded connected set containing A whose boundary does not meet $f^{-1}(0)$. One way to obtain such a set is to let r be such that $A \subset \mathrm{B}_r(0)$ and let B be the union of $\mathbb{R}^n \setminus \mathrm{B}_r(0)$ with all the components of $f^{-1}(0)$ meeting $\mathbb{R}^n \setminus \mathrm{B}_r(0)$ (which is a closed set). Both B and A are components of $B \cup f^{-1}(0)$, so there is an open set $V \subset \mathbb{R}^n \setminus B$ that contains A and whose boundary does not intersect $B \cup f^{-1}(0)$. Now let $U \subset V$ be the component containing A.

Next we would like to make sure that the boundary of U is connected. Toward this end, look at the complement (in $\mathbb{R}^n$) of the unbounded component of $\mathbb{R}^n \setminus U$. Replace U with the component of this set containing A. By [K, Ch. 8, Sec. 57, II, Thm. 6], the boundary of this set is connected. We now have an open bounded connected set containing A whose connected boundary does not meet $f^{-1}(0)$. The boundary of such a set and thus also the set itself is mapped by f into either $(0, \infty)$ or $(-\infty, 0)$—a contradiction. □

LEMMA 5.3. *Let $f : \mathbb{R}^2 \to \mathbb{R}$ be a Lipschitz quotient mapping. Then, for each $t \in \mathbb{R}$ and for each ball* B, *the number of components of $f^{-1}(t)$ intersecting* B *is finite.*

Proof. Assume that $\mathrm{B}_r(f(x)) \subset f(\mathrm{B}_r(x)) \subset \mathrm{B}_{Lr}(f(x))$ for all $x \in \mathbb{R}^2$ and all $r > 0$. Assume also that the number of components of $f^{-1}(0)$ intersecting $\mathrm{B}_r(0)$ is infinite, and fix $0 < \varepsilon < r/(2 + L)$. Then there are infinitely many components A_i such that the distance between any two of $A_i \cap \mathrm{B}_r(0)$ is less than ε. Because all the A_i are unbounded, we can find two of them (say, A_1 and A_2) such that the distance between $A_1 \cap \partial \mathrm{B}_{3r}(0)$ and $A_2 \cap \partial \mathrm{B}_{3r}(0)$ is less than ε.

Let $y \in A_1 \cap \mathrm{B}_r(0)$ and $z \in A_1 \cap \partial \mathrm{B}_{3r}(0)$ be such that $A_2 \cap \mathrm{B}_\varepsilon(y) \neq \emptyset \neq A_2 \cap \mathrm{B}_\varepsilon(z)$. Arguing similarly to Case IV in the proof of Lemma 2.6, we obtain a bounded connected open set G that meets the discs $\mathrm{B}_\varepsilon(y)$ and $\mathrm{B}_\varepsilon(z)$ and whose boundary is contained in $A_1 \cup A_2 \cup \mathrm{B}_\varepsilon(y) \cup \mathrm{B}_\varepsilon(z)$. The latter property of G gives that $|f(x)| < L\varepsilon$ on ∂G, while the former gives that $\{ x \in G : 2r - \varepsilon < |x| < 2r + \varepsilon \}$ is nonempty and open and hence contains a point u with $|u| = 2r$. By Lemma 2.2, we may find a curve $\phi : [0, \infty) \to \mathbb{R}^2$ with Lipschitz constant 1 and $\phi(0) = u$ and such that $f(\phi(t)) = f(u) + t \operatorname{sign}(f(u))$. Since this curve is clearly unbounded, there is a $\tau > 0$ such that $\phi(\tau)$ lies on the boundary of G; then $\phi(\tau) \in \mathrm{B}_\varepsilon(y) \cup \mathrm{B}_\varepsilon(z)$, because $f(\phi(\tau)) \neq 0$ and f is zero on $A_1 \cup A_2$. Hence $L\varepsilon > |f(\phi(\tau))| \geq \tau \geq \|\phi(\tau) - \phi(0)\| \geq r - 2\varepsilon$, which contradicts the choice of ε. □

PROPOSITION 5.4. *Let $f : \mathbb{R}^2 \to \mathbb{R}$ be a Lipschitz quotient mapping. Then, for each $t \in \mathbb{R}$, each component of $f^{-1}(t)$ separates the plane.*

Proof. Let A be a component of $f^{-1}(t)$. By Lemma 5.3, $f^{-1}(t) \setminus A$ is closed. Let G be the component of $\mathbb{R}^2 \setminus (f^{-1}(t) \setminus A)$ containing A; G is an open and connected set. Assuming now that A does not separate the plane, we claim that A also does not separate G. Indeed, $G \setminus A = G \cap (\mathbb{R}^2 \setminus A)$ and both sets in the intersection are connected, so we can apply [K, Ch. 8, Sec. 57, II, Thm. 2] to deduce that $G \setminus A$ is connected. But it is impossible that A separates G: If $G \setminus A$ is connected then f maps it to either $(0, \infty)$ or $(-\infty, 0)$, but near any point of A there exist points whose images are positive and points whose images are negative. This contradiction finishes the proof. □

Propositions 5.1 and 5.4 now imply the following corollary.

COROLLARY 5.5. *Let $f : \mathbb{R}^2 \to \mathbb{R}$ be a Lipschitz quotient mapping. Then, for each $t \in \mathbb{R}$, $f^{-1}(t)$ has a bounded number of components. The upper bound of the number of components depends only on the Lipschitz and co-Lipschitz constants of f.*

There are two unsettled problems related to the material of this section. One is whether one can weaken the assumptions of Lipschitz quotient to uniform quotient in the appropriate places. The other question is to what extent the number of components of $f^{-1}(t)$ or of $\mathbb{R}^2 \setminus f^{-1}(t)$ is independent of t. An examination of the examples given here shows that these numbers may depend on t but leaves the possibility that, after excluding finitely many t, they are constants.

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