UNIFORM STABILIZATION OF A NONLINEAR DISPERSIVE SYSTEM

By

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Abstract. The purpose of this work is to study the internal stabilization of a coupled system of two generalized Korteweg-de Vries equations under the effect of a localized damping term. To obtain the decay we use multiplier techniques combined with compactness arguments and reduce the problem to prove a unique continuation property for weak solutions. A locally exponential decay result is derived.

1. Introduction. Let us consider the following Cauchy problem for the dispersive model

$$\begin{cases} u_t + u_{xxx} + a_3 v_{xxx} + u u_x + a_1 v v_x + a_2 (uv)_x = 0, \quad x \in \mathbb{R}, t > 0 \\ b_1 v_t + r v_x + v_{xxx} + b_2 a_3 u_{xxx} + v v_x + b_2 a_2 u u_x + b_2 a_1 (uv)_x = 0, \\ u(x,0) = u_0(x) \quad \text{and} \quad v(x,0) = v_0(x), \end{cases} \quad (1.1)$$

where $r, a_1, a_2, a_3, b_1, b_2$ are real constants with $b_1, b_2 > 0$ and r can be assumed to be very small (see, for instance, [5, 27]). The unknowns u and v are real valued functions of the variables x and t.

This system was derived by Gear and Grimshaw in [9] as a model to describe strong interactions of two long internal gravity waves in a stratified fluid, where the two waves are assumed to correspond to different modes of the linearized equations of motion. It has the structure of a pair of KdV equations with both linear and nonlinear coupling terms and has been the object of intensive research in recent years (see, for instance, [1–3, 5, 7, 8, 11, 14, 18, 19, 21, 27, 30]). Particularly, we refer to [5, 27] for an extensive

discussion on the physical relevance of the system and on the existence and uniqueness of the solution.

The analysis developed in the works mentioned above shows that solutions of (1.1) satisfy the following conservation law

$$\int_{\mathbb{R}} (b_2 u^2 + b_1 v^2) dx = \int_{\mathbb{R}} (b_2 (u_0)^2 + b_1 (v_0)^2) dx,$$

which allows us to conclude that the total energy associated to the model is conserved along every trajectory. In many real situations, however, one cannot neglect energy dissipation mechanisms, especially for the long-time behavior. In this context, several energy dissipation mechanisms were derived and, depending on the physical situation, they must be taken into account, at least, as a perturbation. It is our purpose here to establish this as a fact, at least in the context of a damped dispersive system. More precisely, we study the exponential decay of solutions of system (1.1) under the presence of a localized damping term represented by a function $a = a(x)$, when $a_1 = a_2$:

$$\begin{cases} u_t + u_{xxx} + a_3 v_{xxx} + u u_x + a_1 v v_x + a_1 (uv)_x + a(x)u = 0, \quad x \in \mathbb{R}, t > 0 \\ b_1 v_t + r v_x + v_{xxx} + b_2 a_3 u_{xxx} + v v_x + b_2 a_1 u u_x + b_2 a_1 (uv)_x + a(x)v = 0, \\ u(x,0) = u_0(x) \text{ and } v(x,0) = v_0(x). \end{cases} \quad (1.2)$$

We also assume that

$$r, a_1, a_3, b_1 \text{ and } b_2 \text{ are real constants with } 0 < a_3^2 b_2 < 1 \text{ and } b_1, b_2 > 0.$$

According to the works mentioned above, the assumption $0 < a_3^2 b_2 < 1$ combined with some conservation law satisfied by the solutions allow us to obtain a priori estimates leading to the global well-posedness results. The constant r is a non-dimensional constant parameter that can be assumed to be very small (see, for instance, [5, 27]). Condition $a_1 = a_2$ is technical and will be used to simplify some computations.

The total energy associated to (1.2) is given by

$$E(t) = \frac{1}{2} \int_{\mathbb{R}} (b_2 u^2 + b_1 v^2) dx, \quad (1.3)$$

and we can verify that

$$\frac{d}{dt} \frac{1}{2} \int_{\mathbb{R}} (b_2 u^2 + b_1 v^2) dx = - \int_{\mathbb{R}} a(x)(b_2 u^2 + v^2) dx \leq 0. \quad (1.4)$$

Thus, if $a(x) \geq a_0 > 0$ almost everywhere in $\mathbb{R}$, then it is easy to prove that the energy $E(t)$ decays exponentially as $t \to \infty$. However, the problem of stabilization when the damping is effective only on a subset of $\mathbb{R}$ is much more subtle. In this paper we are concerned with this problem; therefore we assume that $a = a(x)$ is a real-valued function that satisfies the conditions

$$\begin{cases} a \in L^\infty(\mathbb{R}) \text{ is a positive function and } a(x) \geq a_0 > 0 \text{ a. e. in } \omega, \\ \text{where } \omega \text{ contains a set of the form } (-\infty, \alpha) \cup (\beta, +\infty), \text{ for some } \alpha, \beta \in \mathbb{R}. \end{cases} \quad (1.5)$$

Under the above conditions, we obtain the main result of this work:

THEOREM 1.1. Let $a = a(x)$ be any damping function satisfying (1.5). Then, if $(u_0, v_0) \in [L^2(\mathbb{R})]^2$, system (1.2) is locally uniformly exponential stable; that is, for any $R > 0$, there exist positive constants $C = C(R)$ and $\alpha = \alpha(R)$ satisfying

$$E(t) \leq C(R)E(0)e^{-\alpha(R)t}, \ \forall \ t > 0,$$

provided $E(0) \leq R$.

Since the energy associated to the model is decreasing, the problem of proving the exponential decay of solutions can be stated in the following equivalent form: Find $T > 0$ and $C > 0$ such that

$$E(0) \leq C \int_0^T \int_{\mathbb{R}} a(x)(b_2 u^2 + v^2)dxdt \qquad (1.6)$$

holds for every finite energy solution of (1.2). If the above inequality holds, from (1.4) we have that $E(T) \leq \gamma E(0)$ with $0 < \gamma < 1$, which combined with the semigroup property allow us to derive the exponential decay of $E(t)$. Our analysis extends Theorem 1.1 in [6] where the same issue was addressed for the corresponding scalar models. It is also worth mentioning some results obtained for the scalar KdV equation in connection with the analysis developed here [10, 12, 13, 16, 22, 24–26]. In fact, in system (1.2), the problem was addressed only in a bounded domain [3, 15, 21], and estimate (1.6) was proved following the ideas introduced for the analysis of the corresponding scalar case.

The proof of our main result combines multipliers and the so-called compactness-uniqueness argument that leads one to apply a unique continuation result. It is precisely at that point when we use the assumption (1.5) on the support of the damping function. Indeed, using multipliers, estimate (1.6) will not hold directly since lower-order additional terms will appear. So, to absorb them we shall use the compactness-uniqueness argument which reduces the problem to show that the solution that satisfies $a(x)u = a(x)v = 0$ a.e. for all time t has to be the trivial one. This problem can be viewed as a unique continuation problem since $a(x)u = a(x)v = 0$ implies that $(u, v) \equiv (0, 0)$ in $\{a(x) > 0\} \times (0, T)$. However, the existing unique continuation result (see [21]) does not apply directly since we are dealing with weak solutions (with initial data in $[L^2(\mathbb{R})]^2$). To overcome this problem we use the fact that ω satisfies (1.5) to get a compactly supported (in space) solution of the Cauchy problem. We then proceed as in [21], Corollary 3, showing that the solution is smooth. This allows applying the unique continuation results in [21] to conclude that $u = v \equiv 0$. The main difficulty in this context comes from the structure of the nonlinear terms and the loss of compactness in the whole line. Both problems require more delicate analysis and lead us to estimate the solutions in terms of the energy estimates concentrated on bounded sets of the form $\{|x| \leq r\} \times (0, T)$.

Similar conclusions remain valid when we consider model (1.2) posed on a quarter plane with homogeneous boundary conditions, i.e.,

$$u(0, t) = v(0, t) = 0, \qquad t > 0.$$

The initial boundary value problems for KdV type models arise naturally in modeling small-amplitude long waves in a channel with a wavemaker mounted at one end (see, for instance, [4]). Such mathematical formulations have received considerable attention in

the past, and a satisfactory theory of global well-posedness is available in the literature. Here, making the same assumptions on the coefficients a_i, b_i and r, we can also prove the exponential decay of the energy if we consider $\omega = (0, \alpha) \cup (\beta, +\infty)$ for some $\alpha, \beta \in \mathbb{R}^+$. Our analysis extends the work [12] where the same issue was addressed for the corresponding scalar model.

In both cases, where either $x \in \mathbb{R}$ or $\mathbb{R}^+$, the problem is open when $\omega = (x_0, +\infty)$, for some $x_0 > 0$, for example. This is probably a purely technical problem that could be overcome by proving unique continuation results for weak solutions. But, as far as we know, this remains to be done.

The paper is organized as follows: in Section 2 we present some preliminaries, the global well-posedness result and some a priori estimates. Section 3 is devoted to the proof of Theorem 1.1, and in Section 4 we present this theorem in the case of the quarter plane.

2. Preliminary results. The first result of this section is concerned with the global well-posedness of (1.2). We follow closely the arguments developed in [27] for the study of the corresponding conservative system, i.e., when $a \equiv 0$.

We introduce the Hilbert space

$$X = [L^2(\mathbb{R})]^2$$

endowed with the inner product

$$((u, v), (\varphi, \psi))_X = \frac{b_2}{b_1} \int_{\mathbb{R}} u\varphi \, dx + \int_{\mathbb{R}} v\psi \, dx.$$

For $s, b \in \mathbb{R}$, we also introduce the Bourgain spaces $X_{s,b}$ related to the system (1.2) as follows:

$$X_{s,b}^1 = \{f : \|(1 + |\tau - \xi^3|)^b (1 + |\xi|)^s \hat{f}(\tau, \xi)\|_X < \infty\}$$

$$X_{s,b}^2 = \{f : \|(1 + |\tau - \xi^3 + r\xi|)^b (1 + |\xi|)^s \hat{f}(\tau, \xi)\|_X < \infty\}.$$

Here, $\hat{f}$ denotes the Fourier transform of f in both x and t variables:

$$\hat{f}(\xi, t) = \frac{1}{2\pi} \int_{\mathbb{R}^2} e^{-i(x\xi + t\tau)} f(x, t) \, dx \, d\tau.$$

We also observe that, for $b > \frac{1}{2}$,

$$X_{s,b}^1, \ X_{s,b}^2 \subset \mathcal{C}(\mathbb{R}; H^s(\mathbb{R})).$$

The following basic lemma will be needed:

LEMMA 2.1. For $\varepsilon > 0$ sufficiently small, there exists $\varepsilon' = \dfrac{\varepsilon}{1 - \varepsilon} > 0$ such that

$$\|f\|_{H_t^{-\frac{1}{2} + \varepsilon}} \leq \|f\|_{L_t^{1 + \varepsilon'}}$$

for all $f \in L_t^{1 + \varepsilon'}$.

Proof. If $\dfrac{1}{p} = \dfrac{1}{2} - s$, where $p > 2$ and $s > 0$, there exists a positive constant $c > 0$ such that

$$||f||_{L_t^p} \le c\,||f||_{H_t^s}.$$

Consequently, for the corresponding dual spaces we have

$$||f||_{H_t^{-s}} \le c'\,||f||_{L_t^{p'}}$$

for some $c' > 0$, where $\dfrac{1}{p} + \dfrac{1}{p'} = 1$. Thus, letting $s = \dfrac{1}{2} - \varepsilon$, with $0 < \varepsilon < \frac{1}{2}$, we get $p = \dfrac{1}{\varepsilon}$ and $p' = \dfrac{1}{1-\varepsilon} = 1 + \varepsilon'$ for some $\varepsilon' > 0$, since ε is sufficiently small. $\square$

Then, the following holds.

THEOREM 2.2. *Let $T > 0$. For any $(u_0, v_0) \in X$, problem (1.2) admits a unique mild solution $u \in \mathcal{C}([0, T]; X)$.*

Proof. The proof is obtained following the arguments developed in [27], where the same problem was addressed when $a \equiv 0$. Therefore, we omit the details.

First, observe that system (1.2) can be written as

$$\begin{cases} b_1 U_t + A U_{xxx} + R U_x + C(U)U_x + B(x)U = 0 \\ U(0) = U_0, \end{cases} \tag{2.1}$$

where $U = (u, v)$,

$$A = \begin{pmatrix} b_1 & b_1 a_3 \\ b_2 a_3 & 1 \end{pmatrix}, \quad R = \begin{pmatrix} 0 & 0 \\ r & 0 \end{pmatrix},$$

$$C(U) = \begin{pmatrix} b_1(u + a_1 v) & b_1(a_1 u + a_1 v) \\ b_2(a_1 u + a_1 v) & b_2 a_1 u + v \end{pmatrix} \text{ and } B(x) = \begin{pmatrix} b_1 a(x) & 0 \\ 0 & a(x) \end{pmatrix}.$$

To find the solution we should solve the integral equation corresponding to (2.1),

$$W(t) = S(t)U_0 - \int_0^t S(t-s)[C(U)U_x(s) + B(x)U(s)]ds, \tag{2.2}$$

where $S(t) = e^{(-t\partial^3)}$ is the unitary group associated to the linear problem. This will be done by applying a fixed point argument in the Hilbert space $X^1_{0,\frac{1}{2}+} \times X^2_{0,\frac{1}{2}+}$ introduced above. Therefore, we consider the following cut-off version of (2.2),

$$W(t) = \psi_T(t)S(t)U_0 - \psi_T(t)\int_0^t S(t-s)[C(U)U_x(s) + B(x)U(s)]ds, \tag{2.3}$$

with the function $\psi_T(t) = \psi(\frac{t}{T})$ and the function $\psi \in \mathcal{C}_0^\infty(\mathbb{R})$ given by

$$\psi_T(t) = \begin{cases} 1, & |t| < 1, \\ 0, & |t| \ge 2. \end{cases}$$

Hence, taking into account the bilinear estimates obtained in [27], Proposition 4.1, the proof will be complete if we prove that

$$\|S(-t)B(x)U\|_{X^1_{0,\frac{1}{2}+} \times X^2_{0,\frac{1}{2}+}} \le c\|U\|_{X^1_{0,\frac{1}{2}+} \times X^2_{0,\frac{1}{2}+}}, \tag{2.4}$$

for some positive constant c. Indeed, in [27] the fixed point principle was applied when $B \equiv 0$.

Let $0 \le t \le T$. Then, (2.4) is obtained following the arguments developed in [20], estimate (3.48):

$$\|S(-t)a(x)u\|_{H^{-\frac{1}{2}+}(0,T;X)} = \|\|a(x)u\|_X\|_{H^{-\frac{1}{2}+}} = \|a(x)u\|_{X^1_{0,-\frac{1}{2}+}}$$
$$\le c\|a(x)u\|_{L^{1+}(0,T;L^2(\mathbb{R}))} \le c\|a\|_{L^\infty(\mathbb{R})}\|u\|_{L^{1+}(0,T;L^2(\mathbb{R}))}$$
$$\le T^\gamma\|a\|_{L^\infty(\mathbb{R})}\|u\|_{L^\infty(0,T;L^2(\mathbb{R}))} \le cT^\gamma\|a\|_{L^\infty(\mathbb{R})}\|u\|_{X^1_{0,-\frac{1}{2}+}},$$

for some $c > 0$, where $\gamma = \frac{1}{1+}$. The other inequality is obtained in the same way.

The above discussion allows us to conclude the local well-posedness of (1.2) in X. To prove that the system is globally well-posed we use the energy dissipation law proved in the next proposition. $\qquad\square$

Now we establish a series of estimates that will be useful in the proof of our main result.

PROPOSITION 2.3. Let u be the solution of problem (1.2) given by Theorem 2.2. Then, for any $T > 0$ and $x_0 \ge 0$, there exists a positive constant $C = C(T, \|(u_0, v_0)\|_X)$ such that

$$\int_0^T \int_{x_0}^{x_0+1} (u_x^2 + v_x^2) dx dt \le C,$$

$$\|(u(.,T), v(.,T))\|_X^2 - \|(u_0, v_0)\|_X^2 = -\frac{2}{b_1} \int_0^T \int_{\mathbb{R}} a(x)\left(b_2 u^2 + v^2\right) dx dt,$$

$$\|(u_0, v_0)\|_X^2 \le \frac{1}{T}\left(\int_0^T \int_{\mathbb{R}} (\frac{b_2}{b_1}u^2 + v^2) dx dt\right) + \frac{2}{b_1} \int_0^T \int_{\mathbb{R}} a(x)(b_2 u^2 + v^2) dx dt.$$

Proof. The second identity, i.e., the energy dissipation law, is obtained by multiplying the first equation of (1.2) by u, the second one by v and integrating over $\mathbb{R} \times (0,T)$. We observe that

$$\int_0^T \int_{\mathbb{R}} uu_t dx dt = \frac{1}{2}\|u(.,T)\|_{L^2(\mathbb{R})}^2 - \frac{1}{2}\|u_0\|_{L^2(\mathbb{R})}^2$$
$$\int_0^T \int_{\mathbb{R}} u^2 u_x dx dt = \frac{1}{3}\int_0^T \int_{\mathbb{R}} \frac{\partial}{\partial x}u^3 dx dt = 0$$
$$\int_0^T \int_{\mathbb{R}} uu_{xxx} dx dt = -\frac{1}{2}\int_0^T \int_{\mathbb{R}} \frac{\partial}{\partial x}u_x^2 dx dt = 0$$
$$\int_0^T \int_{\mathbb{R}} uv_{xxx} dx dt = -\int_0^T \int_{\mathbb{R}} u_x v_{xx} dx dt$$
$$\int_0^T \int_{\mathbb{R}} uvv_x dx dt = \frac{1}{2}\int_0^T \int_{\mathbb{R}} u\frac{\partial}{\partial x}v^2 dx dt = -\frac{1}{2}\int_0^T \int_{\mathbb{R}} u_x v^2 dx dt$$
$$\int_0^T \int_{\mathbb{R}} u(uv)_x dx dt = -\int_0^T \int_{\mathbb{R}} u_x uv dx dt = \frac{1}{2}\int_0^T \int_{\mathbb{R}} u^2 v_x dx dt.$$

Similar computations can be done when we multiply the second equation by v. Since $\int_0^T \int_{\mathbb{R}} u_x v_{xx} dx dt = -\int_0^T \int_{\mathbb{R}} v_x u_{xx} dx dt$, we can multiply the first equation in (1.2) by b_2 and add the resulting identities to obtain

$$\frac{b_2}{2} \|u(.,T)\|^2_{L^2(\mathbb{R})} + \frac{b_1}{2} \|v(.,T)\|^2_{L^2(\mathbb{R})}$$

$$+ b_2 \int_0^T \int_{\mathbb{R}} a(x) u^2 dx dt + \int_0^T \int_{\mathbb{R}} a(x) v^2 dx dt = \frac{b_2}{2} \|u_0\|^2_{L^2(\mathbb{R})} + \frac{b_1}{2} \|v_0\|^2_{L^2(\mathbb{R})},$$

and the result follows. A proof of the third identity is similar. We first multiply the first equation of (1.2) by $(T-t)b_2 u$ and add it to the second one multiplied by $(T-t)v$. Performing integration by parts we get

$$\frac{b_2}{2} \int_0^T \int_{\mathbb{R}} u^2 dx dt + \frac{b_1}{2} \int_0^T \int_{\mathbb{R}} v^2 dx dt - \frac{Tb_2}{2} \int_{\mathbb{R}} (u_0)^2 dx - \frac{Tb_1}{2} \int_{\mathbb{R}} (v_0)^2 dx$$

$$= -b_2 \int_0^T \int_{\mathbb{R}} (T-t)a(x) u^2 dx dt - \int_0^T \int_{\mathbb{R}} (T-t)a(x) v^2 dx dt,$$

that is,

$$\frac{b_2}{2} \int_{\mathbb{R}} (u_0)^2 dx + \frac{b_1}{2} \int_{\mathbb{R}} (v_0)^2 dx \leq \frac{1}{T} \left(\frac{b_2}{2} \int_0^T \int_{\mathbb{R}} u^2 dx dt + \frac{b_1}{2} \int_0^T \int_{\mathbb{R}} v^2 dx dt \right)$$

$$+ \int_0^T \int_{\mathbb{R}} a(x)(b_2 u^2 + v^2) dx dt.$$

The above identity allows us to obtain the bound for (u_0, v_0) in X.

To prove the first inequality, we introduce a convenient cut-off function. Let $\psi_0 \in \mathcal{C}^\infty(\mathbb{R})$ be a nondecreasing function such that $\psi_0(x) = 0$ for $x \leq \frac{1}{2}$ and $\psi_0(x) = 1$ for $x \geq 1$. For $\alpha \geq 0$ we set $\psi_\alpha(x) = x^\alpha \psi_0(x)$ and note that $\psi_\alpha \in \mathcal{C}^\infty(\mathbb{R})$ and $\psi'_\alpha(x) \geq 0$ for any $x \in \mathbb{R}$.

We multiply the first equation of (1.2) by $b_2 u \psi_\alpha(x - x_0)$ and the second one by $v \psi_\alpha(x - x_0)$ to obtain the identity

$$\int_{\mathbb{R}} (b_2 u u_t + b_1 v v_t) \psi_\alpha(x - x_0) dx + \int_{\mathbb{R}} (b_2 u u_{xxx} + v v_{xxx}) \psi_\alpha(x - x_0) dx$$

$$+ \int_{\mathbb{R}} (a_3 b_2 u v_{xxx} + a_3 b_2 v u_{xxx}) \psi_\alpha(x - x_0) dx + \int_{\mathbb{R}} (b_2 u^2 u_x + v^2 v_x) \psi_\alpha(x - x_0) dx$$

$$+ \int_{\mathbb{R}} (a_1 b_2 u v v_x + a_1 b_2 u v u_x) \psi_\alpha(x - x_0) dx + \int_{\mathbb{R}} a_1 b_2 u (uv)_x \psi_\alpha(x - x_0) dx \qquad (2.5)$$

$$+ \int_{\mathbb{R}} a_1 b_2 v (uv)_x \psi_\alpha(x - x_0) dx + \int_{\mathbb{R}} r v v_x \psi_\alpha(x - x_0) dx$$

$$+ \int_{\mathbb{R}} a(x)(b_2 u^2 + v^2) \psi_\alpha(x - x_0) dx = 0.$$

The next steps are devoted to splitting (2.5). First observe that

$$\int_{\mathbb{R}} uu_{xxx}\psi_\alpha(x-x_0)dx = -\int_{\mathbb{R}} u_{xx}(u_x\psi_\alpha + u\psi'_\alpha)dx$$

$$= -\frac{1}{2}\int_{\mathbb{R}} (u_x^2)_x\psi_\alpha(x-x_0)dx + \int_{\mathbb{R}} u_x(u_x\psi'_\alpha + u\psi''_\alpha)dx$$

$$= \frac{3}{2}\int_{\mathbb{R}} u_x^2\psi'_\alpha(x-x_0)dx - \frac{1}{2}\int_{\mathbb{R}} u^2\psi'''_\alpha(x-x_0)dx.$$

Then, we have

$$\int_{\mathbb{R}} (b_2 uu_{xxx} + vv_{xxx})\psi_\alpha(x-x_0)dx$$

$$= \frac{3}{2}\int_{\mathbb{R}} (b_2 u_x^2 + v_x^2)\psi'_\alpha(x-x_0)dx - \frac{1}{2}\int_{\mathbb{R}} (b_2 u^2 + v^2)\psi'''_\alpha(x-x_0)dx. \tag{2.6}$$

A similar computation gives us that

$$\int_{\mathbb{R}} (a_3 b_2 vu_{xxx} + a_3 b_2 uv_{xxx})\psi_\alpha(x-x_0)dx$$

$$= -a_3 b_2\int_{\mathbb{R}} u_{xx}(v_x\psi_\alpha + v\psi'_\alpha)dx - a_3 b_2\int_{\mathbb{R}} v_{xx}(u_x\psi_\alpha + u\psi'_\alpha)dx \tag{2.7}$$

$$= 3a_3 b_2\int_{\mathbb{R}} u_x v_x\psi'_\alpha(x-x_0)dx - a_3 b_2\int_{\mathbb{R}} uv\psi'''_\alpha(x-x_0)dx.$$

Finally, we have

$$\int_{\mathbb{R}} (b_2 u^2 u_x + v^2 v_x)\psi_\alpha(x-x_0)dx = -\frac{1}{3}\int_{\mathbb{R}} (b_2 u^3 + v^3)\psi'_\alpha(x-x_0)dx \tag{2.8}$$

and

$$\int_{\mathbb{R}} a_1 b_2 u(uv)_x\psi_\alpha(x-x_0)dx + \int_{\mathbb{R}} a_1 b_2 v(uv)_x\psi_\alpha(x-x_0)dx$$

$$= -\int_{\mathbb{R}} a_1 b_2 uvv_x\psi_\alpha(x-x_0)dx - \int_{\mathbb{R}} a_1 b_2 v^2 u\psi'_\alpha(x-x_0)dx \tag{2.9}$$

$$-\int_{\mathbb{R}} a_1 b_2 uvu_x\psi_\alpha(x-x_0)dx - \int_{\mathbb{R}} a_1 b_2 u^2 v\psi'_\alpha(x-x_0)dx.$$

Combining (2.5) and the above estimates (2.6)-(2.10), we obtain

$$\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}} (b_2 u^2 + b_1 v^2)\psi_\alpha(x-x_0)dx + \frac{3}{2}\int_{\mathbb{R}} (b_2 u_x^2 + v_x^2)\psi'_\alpha(x-x_0)dx$$

$$= \frac{1}{2}\int_{\mathbb{R}} (b_2 u^2 + v^2)\psi'''_\alpha(x-x_0)dx - 3a_3 b_2\int_{\mathbb{R}} u_x v_x\psi'_\alpha(x-x_0)dx$$

$$+a_3 b_2\int_{\mathbb{R}} uv\psi'''_\alpha(x-x_0)dx + \frac{1}{3}\int_{\mathbb{R}} (b_2 u^3 + v^3)\psi'_\alpha(x-x_0)dx \tag{2.10}$$

$$+a_1 b_2\int_{\mathbb{R}} u^2 v\psi'_\alpha(x-x_0)dx + a_1 b_2\int_{\mathbb{R}} uv^2\psi'_\alpha(x-x_0)dx$$

$$+\frac{r}{2}\int_{\mathbb{R}} v^2\psi'_\alpha(x-x_0)dx - \int_{\mathbb{R}} a(x)(b_2 u^2 + v^2)\psi_\alpha(x-x_0)dx.$$

Now we estimate the terms on the right-hand side of (2.10):

$$\frac{1}{3} \int_{\mathbb{R}} \left(b_2 u^3 + v^3\right) \psi'_\alpha(x - x_0) dx \leq \frac{1}{3} \sup_{x \in \mathbb{R}} \left| u \sqrt{\psi'_\alpha(x - x_0)} \right| \int_{\mathbb{R}} u^2 \sqrt{\psi'_\alpha(x - x_0)} dx$$

$$+ \frac{1}{3} \sup_{x \in \mathbb{R}} \left| v \sqrt{\psi'_\alpha(x - x_0)} \right| \int_{\mathbb{R}} v^2 \sqrt{\psi'_\alpha(x - x_0)} dx. \tag{2.11}$$

To estimate $\sup_{x \in \mathbb{R}} |u(x,t) \sqrt{\psi'_\alpha(x - x_0)}|$ we use the following inequality:

$$\sup_{x \in \mathbb{R}} w^2(x) \leq \frac{1}{2} \int_{\mathbb{R}} |w(x)||w'(x)| dx, \ \forall\, w \in H^1(\mathbb{R}).$$

Then, letting $w = u(x,t) \sqrt{\psi'_\alpha(x - x_0)}$ it follows that

$$\sup_{x \in \mathbb{R}} |u \sqrt{\psi'_\alpha}| \leq \frac{1}{\sqrt{2}} \left(\int_{\mathbb{R}} |u \sqrt{\psi'_\alpha}| \, |u_x \sqrt{\psi'_\alpha} + \frac{u \psi''_\alpha}{2 \sqrt{\psi'_\alpha}} | dx \right)^{\frac{1}{2}}$$

$$\leq \frac{1}{\sqrt{2}} \left(\int_{\mathbb{R}} u_x^2 \psi'_\alpha dx \right)^{\frac{1}{4}} \left(\int_{\mathbb{R}} u^2 \psi'_\alpha dx \right)^{\frac{1}{4}} + \frac{1}{2} \left(\int_{\mathbb{R}} u^2 \psi''_\alpha dx \right)^{\frac{1}{2}}.$$

For $0 \leq \alpha \leq 1$ and $k \geq 1$, the function ψ_α introduced above satisfies

$$|\psi_\alpha^{(k)}(x)| \leq c_k,$$

where c_k is a positive constant that depends on k. Then, combining Hölder inequality and the energy dissipation law we get

$$\frac{1}{3} \int_{\mathbb{R}} \left(b_2 u^3 + v^3\right) \psi'_\alpha(x - x_0) dx$$

$$\leq c \left[1 + \left(\int_{\mathbb{R}} u_x^2 \psi'_\alpha(x - x_0) dx \right)^{\frac{1}{4}} + \left(\int_{\mathbb{R}} v_x^2 \psi'_\alpha(x - x_0) dx \right)^{\frac{1}{4}} \right] \tag{2.12}$$

$$\leq c \left[1 + \delta \int_{\mathbb{R}} (u_x^2 + v_x^2) \psi'_\alpha(x - x_0) dx \right]$$

for any $\delta > 0$ and $c > 0$ that depends on $E(0)$.

Proceeding as in the previous computations we have

$$\int_{\mathbb{R}} u^2 v \psi'_\alpha(x - x_0) dx \leq \sup_{x \in \mathbb{R}} |v(x,t) \sqrt{\psi'_\alpha(x - x_0)}| \int_{\mathbb{R}} u^2 \sqrt{\psi'_\alpha(x - x_0)} dx$$

$$\leq \left[\frac{1}{\sqrt{2}} \left(\int_{\mathbb{R}} v_x^2 \psi'_\alpha dx \right)^{\frac{1}{4}} \left(\int_{\mathbb{R}} v^2 \psi'_\alpha dx \right)^{\frac{1}{4}} + \frac{1}{2} \left(\int_{\mathbb{R}} v^2 \psi''_\alpha dx \right)^{\frac{1}{2}} \right] \int_{\mathbb{R}} u^2 \sqrt{\psi'_\alpha} dx$$

$$\leq c \left[1 + \delta \int_{\mathbb{R}} v_x^2 \psi'_\alpha(x - x_0) dx \right]$$

for any $\delta > 0$ and $c = c(E(0)) > 0$. Thus,

$$-a_1 b_2 \int_{\mathbb{R}} u^2 v \psi'_\alpha(x - x_0) dx - a_1 b_2 \int_{\mathbb{R}} u v^2 \psi'_\alpha(x - x_0) dx$$

$$\leq c \left[1 + \delta \int_{\mathbb{R}} (u_x^2 + v_x^2) \psi'_\alpha(x - x_0) dx \right] \tag{2.13}$$

for any $\delta > 0$ and $c = c(E(0)) > 0$. Moreover,

$$3a_3b_2 \int_{\mathbb{R}} u_x v_x \psi_\alpha'(x - x_0)dx \leq \frac{3|a_3|\sqrt{b_2}}{2} \int_{\mathbb{R}} (b_2 u_x^2 + v_x^2)\psi_\alpha'(x - x_0)dx. \tag{2.14}$$

Then, combining (2.10) and the estimates (2.11)-(2.14), we deduce that

$$\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}}(b_2u^2+b_1v^2)\psi_\alpha(x-x_0)dx + (\frac{3}{2}(1 - |a_3|\sqrt{b_2}) - c\delta)\int_{\mathbb{R}}(b_2u_x^2 + v_x^2)\psi_\alpha'(x - x_0)dx$$

$$\leq c_1 + c_2 \int_{\mathbb{R}} (u^2 + v^2)\{\psi_\alpha'(x - x_0) + |\psi_\alpha'''(x - x_0)|\}dx$$

$$\tag{2.15}$$

where c_1 and c_2 are positive constants. Now, choosing $\delta > 0$ such that $\frac{3}{2}(1 - a_3^2 b_2) - c\delta > 0$ and using the energy dissipation law, we obtain the first estimate by integrating the identity above with respect to t and considering the form of the function ψ_α when $\alpha = 0$. $\square$

As a consequence of Theorem 2.2, we have a global existence result in $[H^1(\mathbb{R})]^2$.

COROLLARY 2.4. If $a \in W^{1,\infty}(\mathbb{R})$ and $(u_0, v_0) \in [H^1(\mathbb{R})]^2$, then system (1.2) has a unique solution $(u, v) \in L^\infty(0, T; [H^1(\mathbb{R})]^2)$, for all $T > 0$.

Proof. The ideas involved in the proof follow closely the previous arguments and those presented in [5], Proposition 5.3; therefore, it will be omitted. $\square$

3. Proof of the main result. In this section we prove the uniform exponential decay of the total energy $E(t)$.

Proof of Theorem 1.1. As we pointed out in Section 1, to obtain the exponential decay of $E(t)$ we claim that, for any $R > 0$, the following inequality holds:

$$E(0) \leq C \int_0^T \Big[\int_{\mathbb{R}} a(x)(b_2 u^2 + v^2)dx \Big] dt, \tag{3.1}$$

provided that $E(0) \leq R$, where $C = C(R, T)$ is a positive constant. This fact, together with the energy dissipation law and the semigroup property, suffices to obtain the uniform exponential decay. Let us prove (3.1).

From Proposition 2.3, it follows that

$$\|(u_0, v_0)\|_X^2 \leq \frac{1}{T}\left(\int_0^T \int_{\mathbb{R}} (\frac{b_2}{b_1}u^2 + v^2)dxdt\right) + \frac{2}{b_1}\int_0^T \int_{\mathbb{R}} a(x)(b_2 u^2 + v^2)dxdt. \tag{3.2}$$

Thus, in order to show (3.1) it suffices to prove that for any $T > 0$ and $R > 0$, there exists a positive constant $C = C(R, T)$ satisfying

$$\int_0^T \int_{\mathbb{R}} (\frac{b_2}{b_1}u^2 + v^2)dxdt \leq C \int_0^T \int_{\mathbb{R}} a(x)(b_2 u^2 + v^2)dxdt. \tag{3.3}$$

Let us argue by contradiction following the so-called "compactness-uniqueness" argument (see for instance [31]). Suppose that (3.3) is not valid. Then, we can find a sequence of functions $\{U_n\}_{n\in\mathbb{N}} = \{(u_n, v_n)\}_{n\in\mathbb{N}} \in \mathcal{C}([0, T]; X)$ solving (1.2), such that

$$\|(u_n(\cdot, 0), v_n(\cdot, 0))\|_X \leq R \tag{3.4}$$

and

$$\lim_{n\to\infty} \frac{\|(u_n, v_n)\|^2_{L^2(0,T;X)}}{\int_0^T \int_{\mathbb{R}} a(x)(b_2 u_n^2 + v_n^2)dxdt} = +\infty.$$

Let $\lambda_n = \|U_n\|_{L^2(0,T;X)}$ and define $W_n(x,t) = U_n(x,t)/\lambda_n = (y_n, z_n)$. Then, for each $n \in \mathbb{N}$, the function W_n solves

$$\begin{cases} y_{n,t} + y_{n,xxx} + a_3 z_{n,xxx} + \lambda_n(y_n y_{n,x} + a_1 z_n z_{n,x} + a_1(y_n z_n)_x) + a(x)y_n = 0, \\ b_1 z_{n,t} + r z_{n,x} + z_{n,xxx} + b_2 a_3 y_{n,xxx} \\ \qquad\qquad\qquad + \lambda_n(z_n z_{n,x} + b_2 a_1 y_n y_{n,x} + b_2 a_1(y_n z_n)_x) + a(x)z_n = 0, \\ y_n(x,0) = y_{0,n}(x), \quad z_n(x,0) = z_{0,n}(x), \end{cases}$$

$$(3.5)$$

where $x \in \mathbb{R}, t > 0$. Moreover,

$$\|(y_n, z_n)\|_{L^2(0,T;X)} = 1, \quad \forall n \in \mathbb{N}, \tag{3.6}$$

and

$$\int_0^T \int_{\mathbb{R}} a(x)(b_2 y_n^2 + z_n^2)dxdt \longrightarrow 0, \tag{3.7}$$

as $n \to \infty$. Observe that the energy dissipation law and (3.4) guarantee that $\{\lambda_n\}_{n\in\mathbb{N}}$ is bounded. Then, by extracting a subsequence still denoted by the same index n, we can assume that

$$\lambda_n \to \lambda \geq 0. \tag{3.8}$$

Now, from (3.2), (3.6) and (3.7), it follows that

$$W_n(x,0) = U_n(x,0)/\lambda_n = (y_n(x,0), z_n(x,0)) = W_{0,n}(x)$$

is bounded in X. Then, by Proposition 2.3, we deduce that

$$\|(y_n, z_n)\|_{L^2(0,T;[H^1_{loc}(\mathbb{R})]^2)} \leq C, \tag{3.9}$$

for all $n \in \mathbb{N}$, where $C > 0$. On the other hand, since

$$\|y_n z_{n,x}\|_{L^2(0,T;L^1_{loc}(\mathbb{R}))} \leq \|(y_n, z_n)\|_{L^\infty(0,T;[L^2_{loc}(\mathbb{R})]^2)} \|(y_n, z_n)\|_{L^2(0,T;[H^1_{loc}(\mathbb{R})]^2)},$$

due to the above estimates, we obtain a positive constant C such that

$$\|y_n z_{n,x}\|_{L^2(0,T;L^1_{loc}(\mathbb{R}))} \leq C, \quad \forall n \in \mathbb{N}.$$

The remaining nonlinear terms can be estimated in a similar way. Consequently,

$$\{(y_n, z_n)\}_{n\in\mathbb{N}} \text{ is bounded in } L^2(0,T;[H^{-3}_{loc}(\mathbb{R})]^2). \tag{3.10}$$

Indeed, according to (3.5),

$$y_{n,t} = -y_{n,xxx} - a_3 z_{n,xxx} - \lambda_n(y_n y_{n,x} - a_1 z_n z_{n,x} - a_1(y_n z_n)_x) - a(x)y_n,$$

$$b_1 z_{n,t} = -r z_{n,x} - z_{n,xxx} - b_2 a_3 y_{n,xxx}$$
$$\qquad\qquad\qquad - \lambda_n(z_n z_{n,x} - b_2 a_1 y_n y_{n,x} - b_2 a_1(y_n z_n)_x) - a(x)z_n$$

in $\mathcal{D}'(0,T;[H^{-2}(\mathbb{R})]^2)$, and (3.9)-(3.10) guarantee the boundedness of the terms appearing on the right-hand side of both equations in $L^2(0,T;[H^{-2}_{loc}(\mathbb{R})]^2)$. Then, using the

estimates above and classical compactness results [[28], Corollary 4], we obtain a subsequence of $\{(y_n, z_n)\}_{n \in \mathbb{N}}$, still denoted by the same index n, and a function $W = (y, z)$ such that

$$
\begin{cases}
(y_n, z_n) \rightharpoonup (y, z) \text{ weakly } * \text{ in } L^\infty(0, T; [L^2_{loc}(\mathbb{R})]^2) \\
(y_n, z_n) \rightharpoonup (y, z) \text{ weakly in } L^2(0, T; [H^1_{loc}(\mathbb{R})]^2) \\
(y_n, z_n) \to (y, z) \text{ strongly in } L^2(0, T; [L^2_{loc}(\mathbb{R})]^2) \\
(y_n, z_n) \to (y, z) \text{ strongly in } \mathcal{C}([0, T]; [H^{-1}_{loc}(\mathbb{R})]^2).
\end{cases}
\tag{3.11}
$$

In particular,

$$
W_n(x, 0) = W_{0,n}(x) \to W(x, 0) = (y(x, 0), z(x, 0)) := W_0(x).
\tag{3.12}
$$

Since $W \in L^\infty(0, T; [L^2_{loc}(\mathbb{R})]^2) \cap \mathcal{C}([0, T]; [H^{-1}_{loc}(\mathbb{R})]^2)$, from [[29], Chapter III, Lemma 4.1], we have

$$
W \in \mathcal{C}_w([0, T]; [L^2_{loc}(\mathbb{R})]^2),
$$

where $\mathcal{C}_w([0, T]; [L^2_{loc}(\mathbb{R})]^2)$ denotes the space of sequentially weakly continuous functions from $[0, T]$ into $[L^2_{loc}(\mathbb{R})]^2$. Moreover, convergences (3.7) and (3.11) allow us to conclude that

$$
0 = \liminf_{n \to \infty} \left\{ \int_0^T \int_\mathbb{R} a(x)(b_2 y_n^2 + z_n^2) dx dt \right\} \geq \int_0^T \int_K a(x)(b_2 y^2 + z^2) dx dt
\tag{3.13}
$$

for all $K \subset \mathbb{R}$ compact. Also,

$$
\int_0^T \int_{\overline{\omega}^c} (y_n^2 + z_n^2) dx dt \longrightarrow 0.
\tag{3.14}
$$

Consequently, from the structure of ω, (3.6), (3.11) and (3.14), we obtain

$$
\|W\|_{L^2(0,T;X)}^2 = \int_0^T \int_{\overline{\omega}} |W|^2 dx dt + \int_0^T \int_{\overline{\omega}^c} |W|^2 dx dt = 1.
\tag{3.15}
$$

Combining the results above, we can conclude that the weak limit $W = (y, z)$ solves

$$
\begin{cases}
b_1 W_t + A W_{xxx} + R W_x + \lambda C(W) W_x = 0 \\
W \equiv 0 \quad \text{on} \quad \omega \times (0, T) \\
W(x, 0) := W_0(x) \in X,
\end{cases}
\tag{3.16}
$$

where A, R, C and B were introduced in the proof of Theorem 2.2. Then, according to the unique continuation property (UCP) proved in [[21], Corollary 3] for the subset ω, we have $W \equiv 0$ in $\mathbb{R} \times (0, T)$. This contradicts (3.15) and the proof is complete. We note that in order to apply the result proved in [21] we need to show that $W \in L^\infty_{loc}(0, T; [H^1_{loc}(\mathbb{R})]^2)$. But since W is a compactly supported (in space) solution, we can proceed as in [21], Corollary 3, to guarantee that the solution is smooth enough. In this context, Corollary 2.4 is crucial.   □

REMARK 3.1. Following the approach used in [3], Theorem 4.1, we could also obtain similar results on the exponential decay as $t \to \infty$.

4. The quarter plane. Similar conclusions remain valid when we consider model (1.2) posed on a quarter plane with homogeneous boundary conditions:

$$\begin{cases} u_t + u_{xxx} + a_3 v_{xxx} + u u_x + a_1 v v_x + a_1(uv)_x + a(x)u = 0, \\ b_1 v_t + r v_x + v_{xxx} + b_2 a_3 u_{xxx} + v v_x + b_2 a_1 u u_x + b_2 a_1(uv)_x + a(x)v = 0, \\ u(0,t) = v(0,t) = 0, \\ u(x,0) = u_0(x) \ \text{ and } \ v(x,0) = v_0(x), \end{cases} \qquad (4.1)$$

where $x, t \in \mathbb{R}^+$. In this case we assume the same conditions on the coefficients r, a_1, a_3, b_1, b_2 and the function $a = a(x)$ with ω being

$$\omega = (0, \alpha) \cup (\beta, +\infty), \quad \text{for some } \alpha, \beta \in \mathbb{R}^+.$$

Under the above conditions, the total energy associated to (4.1) satisfies

$$\begin{aligned} \frac{d}{dt} \frac{1}{2} \int_{\mathbb{R}^+} (b_2 u^2 + b_1 v^2) dx &= -\big[\frac{b_2}{2} u_x^2(0,t) + \frac{1}{2} v_x^2(0,t) + a_3 b_2 u_x(0,t) v_x(0,t) \big] \\ &\quad - \int_{\mathbb{R}^+} a(x)(b_2 u^2 + v^2) dx = -\frac{1}{2} \left(\sqrt{b_2} u_x(0,t) + \sqrt{a_3^2 b_2} v_x(0,t) \right)^2 \\ &\quad - \frac{1}{2} \left(1 - a_3^2 b_2 \right) v_x^2(0,t) - \int_{\mathbb{R}^+} a(x)(b_2 u^2 + v^2) dx \leq 0, \end{aligned} \qquad (4.2)$$

i.e., the energy is decreasing. So, in the light of the computations performed in the previous section, we can also prove that it decays to zero exponentially. More precisely, we have the following result:

THEOREM 4.1. *If $(u_0, v_0) \in X$, then system (4.1) is locally uniformly exponential stable.*

Indeed, due to (4.2), to obtain the exponential decay it is sufficient to find $C > 0$ and $T > 0$, such that

$$\begin{aligned} E(0) \leq C \int_0^T \Big[\int_{\mathbb{R}} a(x)(b_2 u^2 + v^2) dx + \frac{1}{2} \left(\sqrt{b_2} u_x(0,t) + \sqrt{a_3^2 b_2} v_x(0,t) \right)^2 \\ + \frac{1}{2} \left(1 - a_3^2 b_2 \right) v_x^2(0,t) \Big] dt \end{aligned} \qquad (4.3)$$

holds for every finite energy solution of (4.2). To prove (4.3) we first multiply the first equation of (1.2) by $(T - t)b_2 u$ and add to the second one multiplied by $(T - t)v$. Performing integration by parts we get

$$\begin{aligned} \frac{b_2}{2} \int_0^T \int_{\mathbb{R}^+} u^2 dx dt &+ \frac{b_1}{2} \int_0^T \int_{\mathbb{R}^+} v^2 dx dt + \frac{b_2}{2} \int_0^T (T - t) u_x^2(0,t) dt \\ &+ \frac{1}{2} \int_0^T (T - t) v_x^2(0,t) dt + b_2 a_3 \int_0^T (T - t) u_x(0,t) v_x(0,t) dt \\ &+ b_2 \int_0^T \int_{\mathbb{R}^+} (T - t) a(x) u^2 dx dt + \int_0^T \int_{\mathbb{R}^+} (T - t) a(x) v^2 dx dt \\ &= \frac{T b_2}{2} \int_{\mathbb{R}^+} (u_0)^2 dx + \frac{T b_1}{2} \int_{\mathbb{R}^+} (v_0)^2 dx; \end{aligned}$$

that is,

$$
\begin{aligned}
&\frac{b_2}{2} \int_{\mathbb{R}^+} (u_0)^2 dx + \frac{b_1}{2} \int_{\mathbb{R}^+} (v_0)^2 dx \\
&\leq \frac{1}{T} \left(\frac{b_2}{2} \int_0^T \int_{\mathbb{R}^+} u^2 dx dt + \frac{b_1}{2} \int_0^T \int_{\mathbb{R}^+} v^2 dx dt \right) \\
&+ \frac{1}{2} \int_0^T \left[(\sqrt{b_2} u_x(0,t) + \sqrt{a_3^2 b_2} v_x(0,t))^2 + (1 - a_3^2 b_2) v_x^2(0,t) \right] dt \\
&+ \int_0^T \int_{\mathbb{R}^+} a(x)(b_2 u^2 + v^2) dx dt.
\end{aligned}
\tag{4.4}
$$

Then, to obtain (4.3) we have to prove that for any $T > 0$ and $R > 0$, there exists a constant $C(R,T) > 0$ satisfying

$$
\begin{aligned}
&\frac{b_2}{b_1} \int_0^T \int_{\mathbb{R}^+} u^2 dx dt + \int_0^T \int_{\mathbb{R}^+} v^2 dx dt \\
&\leq C(R,T) \Big(\int_0^T \left[(\sqrt{b_2} u_x(0,t) + \sqrt{a_3^2 b_2} v_x(0,t))^2 + (1 - a_3^2 b_2) v_x^2(0,t) \right] dt \\
&+ \int_0^T \int_{\mathbb{R}^+} 2a(x)(b_2 u^2 + v^2) dx dt \Big),
\end{aligned}
\tag{4.5}
$$

whenever $||(u_0, v_0)||_X \leq R$. We argue by contradiction and suppose that (4.5) is not true. Then, arguing as in (3.4)-(3.15), the problem is reduced to showing the unique continuation property for a function $W = (u,v)$, solution of

$$
\begin{cases}
b_1 W_t + A W_{xxx} + R W_x + \lambda C(W) W_x = 0 \\
W(0,t) = W_x(0,t) = 0 \\
W(x,0) := W_0(x) \in X,
\end{cases}
\tag{4.6}
$$

which satisfies

$$
W \equiv 0 \quad \text{on} \quad \omega \times (0,T).
\tag{4.7}
$$

Here, A, R, C and B were introduced in the proof of Theorem 2.2. Then, applying the unique continuation property proved in [[21], Corollary 3] for the subset ω, we deduce that $W \equiv 0$ in $\mathbb{R}^+ \times (0,T)$. As in the previous case, in order to apply the unique continuation result, we first need to guarantee that $W \in L^\infty(0,T; [H^1_{loc}(\mathbb{R})]^2)$. However, in the absence of Corollary 2.4, we proceed as in [3], Theorem 4.1, to prove that the solution is smooth enough. Again, this is possible because we are dealing with a compactly supported (in space) solution.

It is also important to note that all the estimates needed for the proof are obtained as in Proposition 2.3. In particular, if we suppose that the function $\psi_0 \in \mathcal{C}^\infty(\mathbb{R}^+)$, introduced in the proofs of Proposition 2.3, satisfies $\psi_0(x) = 0$ for $0 \leq x \leq \frac{1}{2}$, then we deduce that

$$
||(u,v)||_{L^2(0,T; [H^1_{loc}(\mathbb{R}^+)]^2)} \leq C
\tag{4.8}
$$

for some $C > 0$. The same argument was used in [12].

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