UNIFORM SUPERCONVERGENCE ANALYSIS
OF THE DISCONTINUOUS GALERKIN METHOD
FOR A SINGULARLY PERTURBED PROBLEM IN 1-D

ZIQING XIE AND ZHIMIN ZHANG

ABSTRACT. It has been observed from the authors' numerical experiments
(2007) that the Local Discontinuous Galerkin (LDG) method converges uni-
formly under the Shishkin mesh for singularly perturbed two-point boundary
problems of the convection-diffusion type. Especially when using a piecewise
polynomial space of degree k, the LDG solution achieves the optimal conver-
gence rate $k+1$ under the L^2-norm, and a superconvergence rate $2k+1$ for the
one-sided flux uniformly with respect to the singular perturbation parameter ϵ.
In this paper, we investigate the theoretical aspect of this phenomenon under
a simplified ODE model. In particular, we establish uniform convergence rates
$\sqrt{\epsilon}\left(\frac{\ln N}{N}\right)^{k+1}$ for the L^2-norm and $\left(\frac{\ln N}{N}\right)^{2k+1}$ for the one-sided flux inside
the boundary layer region. Here N (even) is the number of elements.

1. INTRODUCTION AND STATEMENT OF THE MAIN RESULT

It is common knowledge in the scientific computing community that discontin-
uous Galerkin (DG) methods, especially the local discontinuous Galerkin (LDG)
method [4], are effective for convection dominated convection-diffusion problems
[2]. Numerical experiments revealed some uniform superconvergence phenomena
of the LDG method for singularly perturbed problems. It has been observed that
(see, e.g., [14]) the LDG method converges uniformly under the Shishkin mesh for
singularly perturbed two-point boundary problems of the convection-diffusion type.
In particular, the rate of convergence for the one-sided flux was found to be of order
$2k + 1$ (comparing with the optimal global rate $k + 1$), which is uniformly valid
with respect to the singular perturbation parameter ϵ. Nevertheless, there is no
theoretical justification of this phenomenon so far. In this paper, we investigate the
uniform convergence/superconvergence properties of the LDG method under the
Shishkin type meshes for a simple one-dimensional model problem on $\Omega = (0, 1)$:

$$(1.1) \qquad\qquad -\epsilon q' + bq = f, \quad q(1) = 1.$$

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To fix the boundary layer at $x = 1$, we assume that $b \geq b_0 > 0$. If $f(x)$ is not the fundamental solution $e^{\int_1^x b/\epsilon}$, the exact solution of (1.1) is

$$(1.2) \qquad q(x) = q(1)e^{\int_1^x b/\epsilon} + \int_x^1 \epsilon^{-1} f(t) e^{\int_t^x b/\epsilon} dt,$$

which can be decomposed into $q = \bar{q} + q_\epsilon$, where

$$(1.3) \qquad |\bar{q}|_{k+1} \leq C, \qquad |q_\epsilon^{(s)}(x)| \leq C \epsilon^{-s} e^{-b_0(1-x)/\epsilon}.$$

Note that when $f \in H^1(0,1)$, the solution can be written as

$$q(x) = [q(1) - \frac{f(1)}{b(1)}]e^{\int_1^x b/\epsilon} + \int_x^1 \frac{f'(t)}{b(t)} e^{\int_t^x b/\epsilon} dt.$$

Our model problem has all the essential properties of the following two-point boundary value problem:

$$(1.4) \qquad -\epsilon u'' + bu' + cu = f, \quad u(0) = 0, \quad u'(1) = 1.$$

Note that when $c = 0$, (1.4) can be decomposed into (1.1) and

$$(1.5) \qquad u' - q = 0, \quad u(0) = 0.$$

When both b and c are constants, we may transfer (1.4) into

$$-\epsilon v'' + \tilde{b}v' = \tilde{f}, \qquad \tilde{b} = b - 2\epsilon t, \quad \tilde{f}(x) = e^{-tx}f(x),$$

by the transformation $u(x) = v(x)e^{tx}$ with t satisfying $-\epsilon t^2 + bt + c = 0$.

The existence of the boundary layer term q_ϵ causes difficulty in the numerical approximation for $\epsilon \ll 1$. It is well known that the standard continuous Galerkin method results in non-physical oscillatory numerical solutions for small ϵ under quasi-uniform grids. On the other hand, DG methods are able to avoid the oscillation. However, in order to capture the boundary layer behavior, mesh refinement inside the boundary layer is needed. Towards this end, we first select a transition number $\tau \in (0, 1/2)$, then we partition the interval $(0, 1)$ into N(even) subintervals with $N/2$ of length $H = 2(1-\tau)/N$ in $(0, 1-\tau)$ and $N/2$ of length $h = 2\tau/N$ in $(1-\tau, 1)$. With the nodal points being $0 = x_0 < x_1 < \cdots < x_N = 1$, we define $\Omega_j = (x_{j-1}, x_j)$, $h_j = x_j - x_{j-1}$, and the DG space as

$$V_N^{\epsilon,k} = \{v \in L^2(0,1) : \ v|_{\Omega_j} \in P_k(\Omega_j), \ j = 1, 2, \ldots, N\},$$

where $\mathcal{P}_k$ is the space of polynomials of degree no more than k. In this paper, we consider two different mesh refinement strategies by two different selections of the transition number:

$$(1.6) \qquad \tau = \tau_\epsilon = \frac{\epsilon}{b_0}(k+1)\ln\frac{1}{\epsilon},$$

or

$$(1.7) \qquad \tau = \tau_N = \frac{\epsilon}{b_0}(2k+1)\ln N.$$

The first choice fixes the transition point for a fixed ϵ. It is well known that the boundary layer width is $O(\epsilon \ln \epsilon^{-1})$. The second choice is the so-called Shishkin mesh; the transition point changes with each different N.

We use the standard notation $Q_j^+, Q_j^-, [Q]_j = Q_j^+ - Q_j^-$ to introduce the DG scheme for (1.1): Find $Q \in V_N^{\epsilon,k}$ with $Q_N^+ = 1$ such that for $j = 1, 2, \ldots, N$,

$$(1.8) \qquad \int_{\Omega_j} Q(v' + b\epsilon^{-1}v) - Q_j^+ v_j^- + Q_{j-1}^+ v_{j-1}^+ = \int_{\Omega_j} \epsilon^{-1} f v, \quad \forall v \in V_N^{\epsilon,k};$$

or equivalently, via integration by parts on the above,

$$(1.9) \qquad \int_{\Omega_j} (-Q' + b\epsilon^{-1}Q)v - [Q]_j v_j^- = \int_{\Omega_j} \epsilon^{-1} f v, \quad \forall v \in V_N^{\epsilon,k}.$$

Summing up the last $N - n + 1$ elements, we obtain for any $v \in V_N^{\epsilon,k}$,

(1.10)

$$a_n(Q, v) = \int_{x_{n-1}}^1 Q(v' + b\epsilon^{-1}v) + Q_{n-1}^+ v_{n-1}^+ - Q_N^+ v_N^- + \sum_{j=n}^{N-1} Q_j^+ [v]_j$$

$$= \int_{x_{n-1}}^1 Q(v' + b\epsilon^{-1}v) + Q_{n-1}^+ v_{n-1}^+ - Q_N^+ v_N^+ + \sum_{j=n}^{N-1} Q_j^+ [v]_j = \int_{x_{n-1}}^1 \epsilon^{-1} f v;$$

(1.11)

$$a_n(Q, v) = \int_{x_{n-1}}^1 (-Q' + b\epsilon^{-1}Q)v - \sum_{j=n}^{N-1} [Q]_j v_j^- = \int_{x_{n-1}}^1 \epsilon^{-1} f v.$$

Here we require $Q_N^- = Q_N^+$ and $v_N^- = v_N^+$ according to the boundary condition at $x_N = 1$. Due to the ODE nature, the solution can be obtained via backward solving.

Remark. The LDG method [4], when applied to the full scale two-point boundary problem (1.4), contains two important steps [2]: 1) transfer the original problem to a first-order ODE system by introducing (1.5); and 2) choose numerical fluxes for Q and u differently (one up-winding and another down-winding) according to the location of the boundary layer.

Although numerical evidence indicates that the LDG method has the same convergence and superconvergence rate for the full scale problem, the theoretical analysis for the ODE model does not carry over to the full scale case when b and c are not constants.

It is worthy to point out that the LDG method, when applied to our simplified ODE model (1.1), is equivalent to the DG time-stepping method introduced in [12, Chapter 14].

Now we are ready to state our main result.

Theorem 1.1. *Let q be the solution of* (1.1). *Assume that $b \geq b_0 > 0$ and f are sufficiently smooth such that the solution q satisfies the regularity* (1.3). *Let $Q \in V_N^{\epsilon,k}$ be the numerical solution of* (1.10) *(or* (1.11)). *Then there is a constant $C(k)$, depending on k, but independent of N and ϵ, such that*
1) If the transition point is given by (1.6), *then*

$$(1.12) \qquad \|q - Q\| \leq C(k)(\sqrt{\epsilon} \left(\frac{\ln \epsilon^{-1}}{N} \right)^k + \frac{1}{N^{k+1}});$$

furthermore, if $q = q_\epsilon$ and $x_{n-1} \in [1 - \tau, 1)$, then

$$\|q - Q\| \leq C(k)\sqrt{\epsilon} \left(\frac{\ln \epsilon^{-1}}{N} \right)^k,$$

(1.13) $$|(q - Q)_{n-1}^+| \leq C(k) \left(\frac{\ln \epsilon^{-1}}{N} \right)^{2k+1}, \qquad n = 1, 2, \ldots, N.$$

2) *If the transition point is given by* (1.7), *then*

(1.14) $$\|q - Q\| \leq C(k)(\sqrt{\epsilon} \left(\frac{\ln N}{N} \right)^k + \frac{1}{N^{k+1}});$$

furthermore, if $q = q_\epsilon$ and $x_{n-1} \in [1 - \tau, 1)$, then

$$\|q - Q\| \leq C(k)\sqrt{\epsilon} \left(\frac{\ln N}{N} \right)^k,$$

(1.15) $$|(q - Q)_{n-1}^+| \leq C(k) \left(\frac{\ln N}{N} \right)^{2k+1}, \qquad n = 1, 2, \ldots, N.$$

Here we use the simplified notation $\| \cdot \|$ to denote the L^2-norm on the whole domain $\Omega = (0, 1)$. Later on, we will use an index to indicate the L^2-norm on a subdomain when necessary.

Prior to a formal proof (which will be postponed to the next two sections), we will make some remarks on the literature.

First we compare the LDG with the standard finite element method (under the Shishkin mesh), which has the following error bound (see [15]) for problem (1.4):

$$\|u - u_N\|_{\epsilon,N} \leq C[\frac{1}{N^k} + (\frac{\ln N}{N})^{k+1}].$$

Here

$$\|v\|_{\epsilon,N}^2 = |v|_{\epsilon,N}^2 + \|v\|^2, \quad |v|_{\epsilon,N}^2 = \epsilon \sum_{i=1}^{N} h_i \sum_{j=1}^{k} w_j v'(x_{ij})^2,$$

where x_{ij} are the Gaussian points in element (x_{j-1}, x_j). We see that the LDG does a better job in approximating $q = u'$. Note the ϵ factor in defining the discrete norm $|v|_{\epsilon,N}$.

To the best of our knowledge, (1.13) or (1.15) is the first such kind of theoretical result for DG methods. We would like to emphasize that the central point here is the ϵ independence. Otherwise, there are many results in the literature which have norms of the solution appearing in the error bounds. Here we list a few.

In the early 1970's, Douglas and Dupont proved that the continuous Galerkin approximation for the two-point boundary value problem

$$-au'' + bu' + cu = f, \qquad u(0) = 0 = u(1),$$

superconverges at all nodal points for $u \in H^r$:

$$|(u - u_h)(x_j)| \leq Ch^{k+s}\|u\|_{s+1}, \qquad s = \min(k, r - 1),$$

compared with the global optimal rate h^{s+1}. This result is sharp [5]. Especially when $k = r - 1$, the convergent rate is improved from h^{k+1} to h^{2k}.

Almost at the same time, Wheeler proposed the following flux approximation for the continuous Galerkin approximation:

$$\Gamma_j = x_j^{-1}[a(u_h', 1)_{(0,x_j)} + (bu_h' + cu_h + f, x)_{(0,x_j)}]$$

to obtain

$$|\Gamma_j - a(x_j)u'(x_j)| \le Ch^{k+s}\|u\|_{s+1}.$$

Especially when $k = r - 1$, the convergent rate is improved from h^k to h^{2k} [13].

With the higher regularity assumption $u \in H^{3k+2}$, Thomée proved that a DG approximation to the ODE system $u' + Au = f$ at nodes converges at rate h^{2k+1}; see [12, Theorem 12.3 (p.189)]. Here, the regularity assumption can be relaxed to $u \in H^{k+1}$ [1, Theorem 7.3.1 (p.248)].

For the DG approximation of a class of transient hyperbolic equations, under locally translation-invariant meshes, Cockburn et al. proved that a simple convolution post-processing improves the rate of convergence from $h^{k+1/2}$ to h^{2k+1} [3].

More recently, Celiker and Cockburn proved that the numerical trace (including flux) of the LDG approximation to problem $-\epsilon u'' + au' = f$ converges at the rate h^{2k+1} for $\epsilon = O(1)$ [2].

Of course, all these results are for regular problems and the term $\|u\|_{s+1}$ appears in the error bounds. By the regularity, we would have $\|u\|_{s+1} \approx \epsilon^{-s-1/2}$, which makes the error bounds meaningless for $\epsilon \ll 1$. Nevertheless, our earlier numerical experiments demonstrated $2k + 1$ order uniform superconvergence for flux at the nodal points under the Shishkin mesh for singularly perturbed problems in 1-D [14], which sparked the motivation for our investigation in this direction.

To end this introduction, we would like to point out that there are vast amounts of literature on numerical approximations for singularly perturbed problems; see, e.g., [6, 7, 8, 9, 10, 11] and the references therein. A central theme is uniform convergence.

2. FUNDAMENTAL INTERPOLATION PROPERTY FOR THE SINGULAR TERM

When comparing singularly perturbed and non-singularly-perturbed problems, the main difficulty for singularly perturbed problems is the uniform approximation for the boundary layer term q_ϵ. In this section, we define projection type interpolants and establish their approximation property for each of the aforementioned choices of τ.

1) For $\tau = \tau_\epsilon$, we define $\pi_\epsilon^\pm \colon C^0 \to V_N^{\epsilon,k}$, such that

(2.1)
$$\int_{\Omega_j} (q_\epsilon - \pi_\epsilon^\pm q_\epsilon)r = 0, \quad \forall r \in \mathcal{P}_{k-1}(\Omega_j);$$
$$(q_\epsilon - \pi_\epsilon^\pm q_\epsilon)_{j-1}^\pm = 0, \qquad j = 1, \ldots, N.$$

We call π_ϵ^- (π_ϵ^+) the right (left) Radau interpolation operator.

2) For $\tau = \tau_N$, we define $\pi_N^\pm \colon C^0 \to V_N^{\epsilon,1}(0, 1 - \tau_N) \cup V_N^{\epsilon,k}(1 - \tau_N, 1)$, such that outside the boundary layer region,

(2.2) $$\int_{\Omega_j} (q_\epsilon - \pi_N^\pm q_\epsilon) = 0, \qquad (q_\epsilon - \pi_N^\pm q_\epsilon)_{j-1}^\pm = 0, \qquad j = 1, \ldots, N/2,$$

and $\pi_N^\pm = \pi_\epsilon^\pm$ inside the boundary layer region.

In order to simplify the notation, we discuss only $\pi_N = \pi_N^+$ and $\pi_\epsilon = \pi_\epsilon^+$ and all results are valid for π^- as well.

First, we define

$$\bar{q}_{\epsilon,j} = \frac{1}{|\Omega_j|} \int_{\Omega_j} q_\epsilon.$$

Then, we can write explicitly for $x \in [x_{j-1}, x_j)$, $j = 1, 2, \ldots, N/2$:

$$(2.3) \qquad \pi_N q_\epsilon(x) = q_\epsilon(x_{j-1}) + \frac{2}{H}(\bar{q}_{\epsilon,j} - q_\epsilon(x_{j-1}))(x - x_{j-1}).$$

It is straightforward to verify that for $j = 1, 2, \ldots, N/2$,

$$(2.4) \qquad \|\pi_N q_\epsilon\|_{L^2(\Omega_j)}^2 = \frac{5H}{6}(q_\epsilon(x_{j-1}) - \bar{q}_{\epsilon,j})^2 + \frac{H}{2}(q_\epsilon(x_{j-1})^2 + \bar{q}_{\epsilon,j}^2),$$

$$(2.5) \qquad \|\pi_N q_\epsilon\|_{L^1(\Omega_j)} = H|q_\epsilon(x_{j-1}) - 2\bar{q}_{\epsilon,j}|.$$

To estimate the approximation for the singular part, we note that

$$e^{-b_0\tau/\epsilon} = \begin{cases} \epsilon^{k+1}, & \tau = \tau_\epsilon, \\ N^{-2k-1}, & \tau = \tau_N. \end{cases}$$

A direct calculation yields, for $r = 1, 2$,

$$(2.6) \qquad \int_0^{1-\tau} e^{-rb_0(1-x)/\epsilon}dx \le \frac{\epsilon}{rb_0} \begin{cases} \epsilon^{r(k+1)}, & \tau = \tau_\epsilon, \\ N^{-r(2k+1)}, & \tau = \tau_N, \end{cases}$$

$$(2.7) \qquad \int_{1-\tau}^1 e^{-rb_0(1-x)/\epsilon}dx \le \frac{\epsilon}{rb_0}.$$

Then by the regularity assumption (1.3), we have

$$(2.8) \qquad |q_\epsilon|_{W_r^{k+1}(0,1-\tau_\epsilon)}^r \le C\epsilon^{-r(k+1)} \int_0^{1-\tau_\epsilon} e^{-rb_0(1-x)/\epsilon}dx \le \frac{C\epsilon}{rb_0},$$

$$(2.9) \qquad \|q_\epsilon\|_{L^r(0,1-\tau_N)}^r \le C \int_0^{1-\tau_N} e^{-rb_0(1-x)/\epsilon}dx \le \frac{C\epsilon}{rb_0}N^{-r(2k+1)},$$

$$(2.10) \qquad |q_\epsilon|_{W_r^{k+1}(1-\tau,1)}^r \le C\epsilon^{-r(k+1)} \int_{1-\tau}^1 e^{-rb_0(1-x)/\epsilon}dx \le \frac{C}{rb_0}\epsilon^{-r(k+1)+1}.$$

Lemma 2.1. *Let q_ϵ satisfy the regularity assumption (1.3) and let π_N be defined by (2.2). Then we have, for $r = 1, 2$,*

$$(2.11) \qquad \|\pi_N q_\epsilon\|_{L^r(0,1-\tau_N)}^r \lesssim \frac{\epsilon}{N^{r(2k+1)}}.$$

Proof. From (2.4), we have

$$(2.12) \qquad \|\pi_N q_\epsilon\|_{L^r(0,1-\tau_N)}^r \approx H \sum_{j=1}^{N/2} (q_\epsilon^r(x_{j-1}) + \bar{q}_{\epsilon,j}^r).$$

By the regularity assumption (1.3),

$$(2.13) \qquad H \sum_{j=1}^{N/2} q_\epsilon^r(x_{j-1}) \;<\; C \int_0^{1-\tau_N} e^{-rb_0(1-x)/\epsilon} dx < \frac{C\epsilon}{rb_0} N^{-r(2k+1)};$$

$$H \sum_{j=1}^{N/2} \bar{q}_{\epsilon,j}^r \;<\; CH \sum_{j=1}^{N/2} \left(\frac{1}{H} \int_{x_{j-1}}^{x_j} e^{-b_0(1-x)/\epsilon} dx \right)^r$$

$$\;<\; C \sum_{j=1}^{N/2} \int_{x_{j-1}}^{x_j} e^{-rb_0(1-x)/\epsilon} dx$$

$$(2.14) \qquad \;=\; C \int_0^{1-\tau_N} e^{-rb_0(1-x)/\epsilon} dx < \frac{C\epsilon}{rb_0} N^{-r(2k+1)}.$$

Substituting (2.13) and (2.14) into (2.12), we obtain (2.11). $\qquad\qquad\square$

Now we are ready to prove the main interpolation result.

Theorem 2.2. *Let q_ϵ satisfy the regularity assumption (1.3), π_ϵ and π_N be defined by (2.1) and (2.2), respectively. Then we have, for $r = 1, 2$,*

$$(2.15) \qquad \|q_\epsilon - \pi_N q_\epsilon\|_{L^r(0,1-\tau_N)} \lesssim \frac{\epsilon^{1/r}}{N^{2k+1}};$$

$$(2.16) \qquad \|q_\epsilon - \pi_N q_\epsilon\|_{L^r(1-\tau_N,1)} \lesssim \epsilon^{1/r} \left(\frac{2k+1}{b_0} \cdot \frac{\ln N}{N} \right)^{k+1};$$

$$(2.17) \qquad \|q_\epsilon - \pi_\epsilon q_\epsilon\|_{L^r(0,1)} \lesssim \epsilon^{1/r} \left(\frac{k+1}{b_0} \cdot \frac{\ln \epsilon^{-1}}{N} \right)^{k+1}.$$

Proof. First, (2.15) follows directly from the triangle inequality, (2.9), (2.10), and Lemma 2.1.

Note that $q_\epsilon - \pi_N q_\epsilon = 0$ if q_ϵ is a polynomial of degree no more than k. By the Bramble-Hilbert Lemma and (2.10),

$$\|q_\epsilon - \pi_N q_\epsilon\|_{L^r(1-\tau_N,1)} \le Ch^{k+1} |q_\epsilon|_{W_r^{k+1}(1-\tau_N,1)} \le C\epsilon^{1/r} \left(\frac{h}{\epsilon} \right)^{k+1}.$$

Then (2.16) follows by using

$$\frac{h}{\epsilon} = \frac{2k+1}{b_0} \cdot \frac{\ln N}{N}.$$

Next, applying (2.8) and (2.10),

$$\|q_\epsilon - \pi_\epsilon q_\epsilon\|_{L^r(0,1)}^r \;=\; \|q_\epsilon - \pi_\epsilon q_\epsilon\|_{L^r(0,1-\tau_\epsilon)}^r + \|q_\epsilon - \pi_\epsilon q_\epsilon\|_{L^r(1-\tau_\epsilon,1)}^r$$

$$\le\; CH^{r(k+1)} |q_\epsilon|_{W_r^{k+1}(0,1-\tau_\epsilon)}^r + Ch^{r(k+1)} |q_\epsilon|_{W_r^{k+1}(1-\tau_\epsilon,1)}^r$$

$$\le\; C\epsilon \left(N^{-r(k+1)} + (\frac{h}{\epsilon})^{r(k+1)} \right)$$

$$(2.18) \qquad \le\; C\epsilon \left(\frac{k+1}{b_0} \cdot \frac{\ln \epsilon^{-1}}{N} \right)^{r(k+1)}.$$

Note that in this case,

$$\frac{h}{\epsilon} = \frac{k+1}{b_0} \cdot \frac{\ln \epsilon^{-1}}{N}.$$

Finally, (2.17) follows by taking the r-roots of (2.18). $\qquad\qquad\square$

3. Proof of the main result

We denote $a(\cdot,\cdot) = a_1(\cdot,\cdot)$. Note that the scheme is consistent, i.e., the exact solution q satisfies (1.10) and (1.11), and we have the orthogonality property

$$(3.1) \qquad a_n(q - Q, v) = 0, \qquad \forall v \in V_N^{\epsilon,k}.$$

To establish the stability of the scheme, we set $v = Q$ in both (1.10) and (1.11) to obtain

$$(3.2) \quad \begin{aligned} a_n(Q,Q) &= \int_{x_{n-1}}^1 QQ' + \epsilon^{-1}\int_{x_{n-1}}^1 bQ^2 + (Q_{n-1}^+)^2 - (Q_N^+)^2 + \sum_{j=n}^{N-1} Q_j^+[Q]_j \\ &= \int_{x_{n-1}}^1 \epsilon^{-1}fQ; \end{aligned}$$

$$(3.3) \quad a_n(Q,Q) = -\int_{x_{n-1}} Q'Q + \epsilon^{-1}\int_{x_{n-1}}^1 bQ^2 - \sum_{j=n}^{N-1}[Q]_jQ_j^- = \int_{x_{n-1}}^1 \epsilon^{-1}fQ.$$

Combining (3.2) and (3.3), we have the following stability result:

$$(3.4) \quad a_n(Q,Q) = \epsilon^{-1}\int_{x_{n-1}}^1 bQ^2 + \frac{1}{2}\sum_{j=n}^{N-1}[Q]_j^2 + \frac{1}{2}[(Q_{n-1}^+)^2 - (Q_N^+)^2] = \int_{x_{n-1}}^1 \epsilon^{-1}fQ.$$

By the orthogonality property, we have

$$a_n(Q - q_I, Q - q_I) = a_n(q - q_I, Q - q_I),$$

for any interpolant $q_I \in V_N^{\epsilon,k}$.

Letting $v = Q - q_I$ satisfy $(q - q_I)_j^+ = 0$, we obtain, by (1.10),

$$(3.5) \quad b_0\|v\|^2 \le \epsilon a(v,v) = \epsilon a(q - q_I, v) = \epsilon\left(\int_0^{1-\tau} + \int_{1-\tau}^1\right)(q - q_I)(v' + b\epsilon^{-1}v).$$

Recall that $b \ge b_0 > 0$.

Based on (1.3), we write

$$q - q_I = \bar{q} - \bar{q}_I + q_\epsilon - q_{\epsilon,I}.$$

The estimate for the regular part is standard:

$$(3.6) \quad \epsilon a(\bar{q} - \bar{q}_I, v) \le C'(\epsilon N^{-k} + N^{-k-1})|\bar{q}|_{k+1}\|v\| \le C(\epsilon N^{-k} + N^{-k-1})\|v\|,$$

$$(3.7) \qquad\qquad\qquad \|\bar{q} - \bar{q}_I\| \le CN^{-k-1}.$$

Proof of (1.12). For $\tau = \tau_\epsilon$, we set $q_{\epsilon,I} = \pi_\epsilon^+ q_\epsilon$. By (3.5) and the inverse inequality,

$$\|v'\|_{(0,1-\tau)} \le CH^{-1}\|v\|_{(0,1-\tau)}, \quad \|v'\|_{(1-\tau,1)} \le Ch^{-1}\|v\|_{(1-\tau,1)},$$

we have

$$(3.8) \quad \begin{aligned} \epsilon a(q_\epsilon - q_{\epsilon,I}, v) &\le C(\epsilon H^k + H^{k+1})|q_\epsilon|_{k+1,(0,1-\tau)}\|v\|_{(0,1-\tau)} \\ &\quad + C(\epsilon h^k + h^{k+1})|q_\epsilon|_{k+1,(1-\tau,1)}\|v\|_{(1-\tau,1)} \\ &\le C\sqrt{\epsilon}\left(\frac{\ln \epsilon^{-1}}{N}\right)^k\|v\|, \end{aligned}$$

by (2.8) and (2.10). Applying (3.6) and (3.8) in (3.5) results in

$$b_0\|v\|^2 = b_0\|Q - q_I\|^2 \leq C(\sqrt{\epsilon}(\frac{\ln \epsilon^{-1}}{N})^k + \frac{1}{N^{k+1}})\|v\|.$$

Canceling $\|v\|$ on both ends, we obtain

$$\|Q - q_I\| \leq C(\sqrt{\epsilon}(\frac{\ln \epsilon^{-1}}{N})^k + \frac{1}{N^{k+1}}).$$

Using (2.17) and (3.7), we obtain

$$(3.9) \qquad \|q - Q\| \leq \|q - q_I\| + \|q_I - Q\| \leq C(\sqrt{\epsilon}(\frac{\ln \epsilon^{-1}}{N})^k + \frac{1}{N^{k+1}}). \qquad \square$$

Proof of (1.14). When $\tau = \tau_N$, we set $q_{\epsilon,I} = \pi_N^+ q_\epsilon$. By Theorem 2.2 and the inverse inequality,

$$(3.10) \qquad \epsilon \int_0^{1-\tau} (q_\epsilon - q_{\epsilon,I})(v' + b\epsilon^{-1}v) \leq \|q_\epsilon - q_{\epsilon,I}\|_{(0,1-\tau_N)}(\epsilon\|v'\| + \|v\|)$$

$$\leq C\sqrt{\epsilon}N^{-2k-1}(N\epsilon + 1)\|v\|.$$

On the other hand, with $2h = \tau/N$, we have, by the inverse estimate $\|v'\| \leq Ch^{-1}\|v\|$,

(3.11)
$$\epsilon \int_{1-\tau}^1 (q_\epsilon - q_{\epsilon,I})(v' + b\epsilon^{-1}v) \leq C(\epsilon h^k + h^{k+1})|q_\epsilon|_{k+1,(1-\tau,1)}\|v\| \leq C\sqrt{\epsilon}(\frac{\ln N}{N})^k\|v\|.$$

Combining (3.10) and (3.11) yields (3.8) for this case. The rest is similar as in the proof of (1.12). $\qquad \square$

Proof of (1.15). Let $u_\epsilon = e^{\int_x^{x_{n-1}} b/\epsilon}$ with $x_{n-1} \in [1 - \tau, 1)$. Then u_ϵ satisfies

$$u_\epsilon' + b\epsilon^{-1}u_\epsilon = 0, \qquad u_\epsilon(x_{n-1}) = 1.$$

Setting $v = u_\epsilon$ in

$$a_n(q - Q, v) = \int_{x_{n-1}}^1 (q - Q)(v' + b\epsilon^{-1}v) + (q - Q)_{n-1}^+ v_{n-1}^+ + \sum_{j=n}^{N-1} (q - Q)_j^+ [v]_j,$$

we have, by the orthogonality property,

$$(q - Q)_{n-1}^+ = a_n(q - Q, u_\epsilon) = a_n(q - Q, u_\epsilon - \pi_N^- u_\epsilon)$$

$$(3.12) \qquad = \int_{x_{n-1}}^1 [-(q - q_I)' + b\epsilon^{-1}(q - Q)](u_\epsilon - \pi_N^- u_\epsilon).$$

Here π_N^- is the projection operator defined in Section 2, and we have used

$$a_n(q - Q, v) = \int_{x_{n-1}}^1 [-(q - Q)' + b\epsilon^{-1}(q - Q)]v - \sum_{j=n}^{N-1} [q - Q]_j v_j^-.$$

We already have, from a previous argument,

$$\Big| \int_{x_{n-1}}^1 [-(q_\epsilon - q_{\epsilon,I})' + b\epsilon^{-1}(q_\epsilon - Q)]v \Big| \leq C(k)\epsilon^{-1/2}(\frac{\ln N}{N})^k\|v\|.$$

By Theorem 2.2, we have

$$\|u_\epsilon - \pi_N^- u_\epsilon\|_{(x_{n-1},1)} \leq C(k)\sqrt{\epsilon}\left(\frac{\ln N}{N}\right)^{k+1}.$$

Therefore,

$$(3.13) \quad |\int_{x_{n-1}}^1 [-(q_\epsilon - q_{\epsilon,I})' + b\epsilon^{-1}(q_\epsilon - Q)](u_\epsilon - \pi_N^- u_\epsilon)| \leq C(k)\left(\frac{\ln N}{N}\right)^{2k+1}.$$

When $q = q_\epsilon$, we apply (3.13) to the right-hand side of (3.12) to obtain

$$|(q - Q)_{n-1}^+| \leq C(k)\left(\frac{\ln N}{N}\right)^{2k+1},$$

which is (1.15). The proof of (1.13) is similar. $\qquad\qquad\square$

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College of Mathematics and Computer Science, Hunan Normal University, People's Republic of China

Department of Mathematics, Wayne State University, Detroit, Michigan 48202