Notre Dame Journal of Formal Logic
Volume 35, Number 3, Summer 1994

Uniformization Problems and the Cofinality of the Infinite Symmetric Group

JAMES D. SHARP and SIMON THOMAS

Abstract Assuming Martin's Axiom, we compute the value of the cofinality of the symmetric group on the natural numbers. We also show that Martin's Axiom does not decide the value of the covering number of a related Mycielski ideal.

1 Introduction Suppose that G is a group that is not finitely generated. Then G can be expressed as the union of a chain of proper subgroups. The cofinality of G, written $c(G)$, is defined to be the least cardinal λ such that G can be expressed as the union of a chain of λ proper subgroups. If κ is an infinite cardinal, then $Sym(\kappa)$ denotes the group of all permutations of the set $\kappa = \{\alpha \mid \alpha < \kappa\}$. The following result was proved in Macpherson and Neumann [5].

Theorem 1.1 *If κ is an infinite cardinal, then $c(Sym(\kappa)) > \kappa$.*

This raises the question of computing the exact value of $c(Sym(\kappa))$. In this paper, we shall study the possibilities for the value of $c(Sym(\omega))$. Of course, this question is only interesting if the Continuum Hypothesis is false.

Upon learning of Theorem 1.1, Mekler and Thomas independently pointed out the following easy observation.

Theorem 1.2 *Suppose that $M \vDash \kappa^\omega = \kappa > \omega_1$. Let $\mathbb{P} = Fn(\kappa, 2)$ be the partial order of finite functions from κ to 2. Then $M^{\mathbb{P}} \vDash c(Sym(\omega)) = \omega_1 < 2^\omega = \kappa$.*

Proof: Working inside M, express $\kappa = \bigcup_{\alpha<\omega_1} X_\alpha$ as the union of an increasing chain such that $|X_{\alpha+1} \setminus X_\alpha| = \kappa$ for each $\alpha < \omega_1$. From now on, we shall work inside $M[G] = M^{\mathbb{P}}$. Let $G_\alpha = G \cap Fn(X_\alpha, 2)$ and let $S_\alpha = \{\pi \in Sym(\omega) \mid \pi \in M[G_\alpha]\}$. Then each S_α is a proper subgroup and $Sym(\omega) = \bigcup_{\alpha<\omega_1} S_\alpha$. The result follows easily.

Clearly the above idea admits many variations. A second source of models of set theory with $c(Sym(\omega)) < 2^\omega$ is provided by short scales.

July 6, 1993; revised October 7, 1994

Definition 1.3 If $f : \omega \to \omega$ and $g : \omega \to \omega$, then we define $f <^* g$ iff there exists $n_0 \in \omega$ such that $f(n) < g(n)$ for all $n \geq n_0$.

Definition 1.4 If λ is a regular cardinal, then $S = \{ f_\alpha : \omega \to \omega \mid \alpha < \lambda \}$ is a λ-scale if:

(i) f_α is a strictly increasing function for each $\alpha < \lambda$;
(ii) $f_\alpha <^* f_\beta$ whenever $\alpha < \beta < \lambda$;
(iii) for each $g : \omega \to \omega$, there exists $\alpha < \lambda$ such that $g <^* f_\alpha$.

Theorem 1.5 *If a λ-scale exists, then $c(Sym(\omega)) \leq \lambda$.*

Proof: Let $\{ f_\alpha \mid \alpha < \lambda \}$ be a λ-scale. By passing to a suitable subsequence if necessary, we can suppose that $f_\alpha \circ f_\alpha <^* f_{\alpha+1}$ for all $\alpha < \lambda$. Let L be the set of all limit ordinals δ such that $\delta < \lambda$. If $\delta \in L$, define

$$S_\delta = \{\pi \in Sym(\omega) \mid \text{There exists } \alpha < \delta \text{ such that } \pi, \pi^{-1} <^* f_\alpha\}.$$

Then each S_δ is a proper subgroup and $Sym(\omega) = \bigcup_{\delta \in L} S_\delta$.

One of the main theorems in this paper is that it is also consistent that $c(Sym(\omega)) = 2^\omega > \omega_1$. In order to explain our approach to this result, it will be helpful to sketch the proof of Theorem 1.1. First we need to introduce some notation. If $G \leq Sym(\Omega)$ and $\Gamma \subseteq \Omega$, then $G_{\{\Gamma\}}$ and $G_{(\Gamma)}$ denote respectively the setwise and pointwise stabilizers of Γ in G. If λ is a (possibly finite) cardinal, then $[\Omega]^\lambda = \{\Gamma \subseteq \Omega \mid |\Gamma| = \lambda\}$. The proof of Theorem 1.1 relies on the following result.

Lemma 1.6 *[5] Let $G \leq Sym(\kappa)$. Suppose that there exists $X \in [\kappa]^\kappa$ such that $G_{\{X\}}$ induces $Sym(X)$ on X. Then there exists $\pi \in Sym(\kappa)$ such that $\langle G, \pi \rangle = Sym(\kappa)$.*

Proof Proof of Theorem 1.1: Suppose that $Sym(\kappa) = \bigcup_{\alpha<\lambda} G_\alpha$ for some $\lambda \leq \kappa$. Express $\kappa = \bigcup_{\alpha<\lambda} X_\alpha$ as the disjoint union of λ sets such that $X_\alpha \in [\kappa]^\kappa$ for each $\alpha < \lambda$. Lemma 1.6 implies that for each $\alpha < \lambda$, there exists $\pi_\alpha \in Sym(X_\alpha)$ such that $g \restriction X_\alpha \neq \pi_\alpha$ for all $g \in G_\alpha$. Define $\pi \in Sym(\kappa)$ by $\pi \restriction X_\alpha = \pi_\alpha$ for each $\alpha < \lambda$. Then $\pi \notin G_\alpha$ for all $\alpha < \lambda$, which is a contradiction.

We would like to adapt this argument so as to reach the stronger conclusion that $c(Sym(\omega)) = 2^\omega$. Suppose then that $Sym(\omega) = \bigcup_{\alpha<\lambda} G_\alpha$ for some $\lambda < 2^\omega$. By Lemma 1.6, there exists a function π with domain $[\omega]^\omega \times \lambda$ such that $\pi(X, \alpha) \in Sym(X)$ and $g \restriction X \neq \pi(X, \alpha)$ for all $g \in G_\alpha$. To reach a contradiction, it is enough to find an element $\pi \in Sym(\omega)$ such that the set

$$\{\alpha < \lambda \mid \text{ There exists } X \in [\omega]^\omega \text{ such that } \pi \restriction X = \pi(X, \alpha)\}$$

is cofinal in λ. In order that such an element π might exist, it is necessary to exercise some care in the choice of the elements $\pi(X, \alpha)$. For example, if each $\pi(X, \alpha)$ consists of an infinite cycle which acts transitively on X, then it is obvious that no such π exists. However, it is possible to choose each $\pi(X, \alpha)$ so that it contains no infinite cycles. If we do this, there seems no obvious reason why such an element π should not exist. Of course, we cannot hope to prove the existence of such an element in ZFC; and the main challenge is to discover which extra set-theoretic hypotheses will suffice to produce such an element.

In order to gain insight into this problem, we shall also consider a purely combinatorial version.

Definition 1.7 The Mycielski ideal $\mathcal{B}_2$ on $\mathcal{P}(\omega)$ consists of the subsets $X \subseteq \mathcal{P}(\omega)$ such that for every $A \in [\omega]^\omega$, $\mathcal{P}(A) \neq \{B \cap A \mid B \in X\}$.

Arguing as in the proof of Theorem 1.1, it is easily seen that $\mathcal{B}_2$ is a σ-ideal. The covering number of $\mathcal{B}_2$, written $cov(\mathcal{B}_2)$, is defined to be the least cardinal λ such that $\mathcal{P}(\omega) = \bigcup_{\alpha<\lambda} X_\alpha$, where each $X_\alpha \in \mathcal{B}_2$. The value of $cov(\mathcal{B}_2)$ is intimately connected with the following set-theoretic hypothesis.

The Uniformization Principle (U_λ) Suppose that $\pi : [\omega]^\omega \times \lambda \to \mathcal{P}(\omega)$ satisfies $\pi(A, \alpha) \in \mathcal{P}(A)$ for all $(A, \alpha) \in dom\, \pi$. Then there exists $S \in \mathcal{P}(\omega)$ such that the set

$$\{\alpha < \lambda \mid \text{ There exists } A \in [\omega]^\omega \text{ such that } S \cap A = \pi(A, \alpha)\}$$

is cofinal in λ.

Proposition 1.8 $cov(\mathcal{B}_2) = \min\{\lambda \mid U_\lambda \textit{ is false }\}$.

Proof: Suppose that $cov(\mathcal{B}_2) = \lambda$. Express $\mathcal{P}(\omega) = \bigcup_{\alpha<\lambda} X_\alpha$ as an increasing union, where each $X_\alpha \in \mathcal{B}_2$. Let $\pi : [\omega]^\omega \times \lambda \to \mathcal{P}(\omega)$ be a function such that $\pi(A, \alpha) \in \mathcal{P}(A)$ and $S \cap A \neq \pi(A, \alpha)$ for all $S \in X_\alpha$. Then π is a counterexample to U_λ. Conversely, suppose that π is a counterexample to U_λ. Define $R : [\omega]^\omega \to \lambda$ by

$$R(S) = \sup\{\alpha < \lambda \mid \text{ There exists } A \in [\omega]^\omega \text{ such that } S \cap A = \pi(A, \alpha)\}.$$

Let $X_\alpha = \{S \mid R(S) \leq \alpha\}$. Then each $X_\alpha \in \mathcal{B}_2$ for all $\alpha < \lambda$ and $\mathcal{P}(\omega) = \bigcup_{\alpha<\lambda} X_\alpha$.

In the statement of the following result, PFA is the Proper Forcing Axiom. (An extremely clear account of this axiom can be found in Baumgartner [2].)

Theorem 1.9 *[3]* (PFA) $cov(\mathcal{B}_2) = 2^\omega$.

Proof: Suppose that the result is false. By Velickovic [10], PFA implies that $2^\omega = \omega_2$. Hence we can express $\mathcal{P}(\omega) = \bigcup_{\alpha<\omega_1} X_\alpha$, where each $X_\alpha \in \mathcal{B}_2$. Let $\mathbb{P}$ denote the Prikry-Silver notion of forcing. Thus each condition $p \in \mathbb{P}$ is a function with values 0 and 1, defined on a co-infinite subset of ω. It is well known that $\mathbb{P}$ is proper. If $S \in \mathcal{P}(\omega)$, then χ_S denotes the characteristic function of S. For each $\alpha < \omega_1$, let D_α consist of the conditions $p \in \mathbb{P}$ which satisfy:

(1) there exists $A \in [\text{ dom } p]^\omega$ such that $p \upharpoonright A \neq \chi_S \upharpoonright A$ for all $S \in X_\alpha$.

Clearly each D_α is dense in $\mathbb{P}$. By PFA, there exists a filter $G \subseteq \mathbb{P}$ such that $G \cap D_\alpha \neq \varnothing$ for all $\alpha < \omega_1$. Let $g = \cup G$, and let $S \in \mathcal{P}(\omega)$ satisfy $\chi_S = g$. Then $S \notin X_\alpha$ for all $\alpha < \omega_1$, which is a contradiction.

With a fairly substantial amount of effort, it is possible to modify the proof of Theorem 1.9 so as to obtain the conclusion that PFA also implies that $c(Sym(\omega)) = 2^\omega$. This raises the question of whether these results can be proved from the strictly weaker hypothesis of $MA + \neg CH$ (Martin's Axiom plus the negation of the Continuum Hypothesis.) The answer is somewhat surprising.

Theorem 1.10 $((MA))$ $c(Sym(\omega)) = 2^\omega$.

Theorem 1.11 *[3] $MA + \neg CH + cov(\mathcal{B}_2) = \omega_1$ is consistent with ZFC.*

Theorem 1.10 will be proved in Section 2. A slight strengthening of Theorem 1.11 will be proved in Section 4. Given the obvious similiarity between the two problems, it is initially puzzling why $MA + \neg CH$ should settle one of them while leaving the other undecided. We shall explain why this happens, by pointing out an important difference between the two problems.

Definition 1.12 Let $A \in [\omega]^\omega$ and let $\{a_n \mid n < \omega\}$ be the increasing enumeration of A.

(i) If $\pi \in Sym(\omega)$, then $\pi^A \in Sym(A)$ is defined by $\pi^A(a_n) = a_{\pi(n)}$ for each $n < \omega$.
(ii) If $f : \omega \to 2$ then $f^A : A \to 2$ is defined by $f^A(a_n) = f(n)$ for each $n < \omega$.

Suppose once again that $Sym(\omega) = \bigcup_{\alpha<\lambda} G_\alpha$ for some $\lambda < 2^\omega$. Let $\Omega \in [\omega]^\omega$ be co-infinite, and for each $\alpha < \lambda$ choose $\pi_\alpha \in Sym(\omega)$ such that $g \restriction \Omega \neq \pi_\alpha^\Omega$ for all $g \in G_\alpha$. If $A \in [\omega]^\omega$ is also co-infinite, then there exists $\varphi \in Sym(\omega)$ such that $\varphi[A] = \Omega$ and $\varphi \restriction A$ is order-preserving. Thus there exists $\beta < \lambda$ such that for all $\beta \le \alpha < \lambda$, $g \restriction A \neq \pi_\alpha^A$ for all $g \in G_\alpha$. We conclude that the following holds:

Corollary 1.13 *There exists a set $\{\pi_\alpha \mid \alpha < \lambda\} \subseteq Sym(\omega)$ and a function π with domain $[\omega]^\omega \times \lambda$ such that:*

(i) $\pi(A, \alpha) \in Sym(A)$ and $g \restriction A \neq \pi(A, \alpha)$ for all $g \in G_\alpha$;
(ii) for each co-infinite $A \in [\omega]^\omega$, there exists $\beta(A) < \lambda$ such that $\pi(A, \alpha) = \pi_\alpha^A$ for all $\beta(A) \le \alpha < \lambda$.

A major difference in our problems is the uniformity given by Corollary 1.13 (ii). We can confirm that this is an essential point as follows. Consider the following set-theoretic hypothesis.

The Weak Uniformization Principle (wU_λ) Suppose that $\pi : [\omega]^\omega \times \lambda \to \mathcal{P}(\omega)$ satisfies $\pi(A, \alpha) \in \mathcal{P}(A)$ for all $(A, \alpha) \in \operatorname{dom} \pi$. Suppose further that there exists $\{f_\alpha : \omega \to 2 \mid \alpha < \lambda\}$ such that for each co-infinite $A \in [\omega]^\omega$, there exists $\beta(A) < \lambda$ such that $\chi_{\pi(A,\alpha)} \restriction A = f_\alpha^A$ for all $\beta(A) \le \alpha < \lambda$. Then there exists $S \in \mathcal{P}(\omega)$ such that the set

$$\{\alpha < \lambda \mid \text{There exists } A \in [\omega]^\omega \text{ such that } S \cap A = \pi(A, \alpha)\}$$

is cofinal in λ.

Theorem 1.14 $((MA))$ *For all $\lambda < 2^\omega$, wU_λ holds.*

This theorem will be proved in Section 2. It is perhaps interesting to mention that we make use of permutation groups in the proof of Theorem 1.14. In fact, the same idea forms the heart of the proofs of both Theorem 1.10 and Theorem 1.14.

Some readers may feel that the uniformity condition in wU_λ is not the most natural one.

The Symmetric Uniformization Principle (sU_λ) Suppose that $\pi : [\omega]^\omega \times \lambda \to \mathcal{P}(\omega)$ satisfies $\pi(A, \alpha) \in \mathcal{P}(A)$ for all $(A, \alpha) \in \operatorname{dom} \pi$. Suppose further that for each

$A, B \in [\omega]^\omega$, there exists a bijection $f : A \to B$ such that $f[\pi(A, \alpha)] = \pi(B, \alpha)$ for all $\alpha < \lambda$. Then there exists $S \in \mathcal{P}(\omega)$ such that the set

$$\{\alpha < \lambda \mid \text{ There exists } A \in [\omega]^\omega \text{ such that } S \cap A = \pi(A, \alpha)\}$$

is cofinal in λ.

Theorem 1.15 $MA + \neg CH + \neg sU_{\omega_1}$ *is consistent with* ZFC.

This theorem will be proved in Section 4. The proof of Theorem 1.15 is quite similar to that of Theorem 2.3 of Cichoń, Roskanowski, Steprans and Węglorz [3]. (Our proof was found independently. We learned of [3] after completing an earlier version of this paper.) However, there are some differences in our approach. The point of the proof is to gradually adjoin a difficult uniformization problem. This corresponds to a universal statement, and hence there is the possibility that later stages of our forcing construction might destroy our earlier work. To avoid this difficulty, in Section 3 we find an existential reformulation of the problem by considering difficult uniformization problems in models of PFA. Finally the proof in Section 4 has been written in the form of an ω_2-Baire forcing over a model of PFA. This allows us to avoid some of the notational complexities of an iterated forcing construction, and to present the combinatorial heart of the proof in an uncluttered fashion. It also allows us to point out an easy but striking "$\diamond$-like" argument which is useful in this kind of problem.

Our notation mainly follows that of Kunen [4]. Thus if $\mathbb{P}$ is a notion of forcing and $p, q \in \mathbb{P}$, then $q \leq p$ means that q is a strengthening of p. If M is the ground model, then we often denote the generic extension by $M^{\mathbb{P}}$ if we do not wish to specify a particular generic filter $G \subseteq \mathbb{P}$. If we want to emphasize that the term t is to be interpreted in the model N, then we write t^N; for example, ω_2^N or $Sym^N(\omega)$. We shall refer to the following internal forcing axioms.

$FA_\kappa(\mathbb{P})$	If $\mathcal{D}$ is a family of dense subsets of $\mathbb{P}$ with $\lvert\mathcal{D}\rvert \leq \kappa$, then there exists a filter $G \subseteq \mathbb{P}$ such that $G \cap D \neq \varnothing$ for every $D \in \mathcal{D}$.
$FA(\mathbb{P})$	$FA_\kappa(\mathbb{P})$ for all $\kappa < 2^\omega$.
MA_κ	$FA_\kappa(\mathbb{P})$ for all c.c.c. $\mathbb{P}$.
MA	MA_κ for all $\kappa < 2^\omega$.
PFA	$FA_{\omega_1}(\mathbb{P})$ for all proper $\mathbb{P}$.

If $A = \{a_i \mid i < \lambda\}$ is a set with the given enumeration, then $[A]^n$ will sometimes be identified with the set of n-tuples $\langle a_{i_1}, \ldots, a_{i_n}\rangle$ with $i_1 < \cdots < i_n$. If A and B are sets, then ${}^A B = \{f \mid f : A \to B\}$.

2 The Cofinality of the Infinite Symmetric Group

In this section, we will prove the following result.

Theorem 2.1 (MA_κ) $c(Sym(\omega)) > \kappa$.

Corollary 2.2 *Let* $M \models GCH$, *and suppose that* $\lambda \leq \theta$ *are regular uncountable cardinals in* M. *Then there exists a c.c.c. poset* $\mathbb{P}$ *such that* $M^{\mathbb{P}} \models c(Sym(\omega)) = \lambda \leq \theta = 2^\omega$.

Proof: Let M_α, $\alpha \le \lambda$, be an iterated finite support c.c.c. construction such that $M_0 = M$ and $M_\alpha \vDash MA + 2^\omega = \theta$ for all $0 < \alpha < \lambda$. Then $M_\lambda \vDash MA_\kappa$ for all $\kappa < \lambda$. Hence $M_\lambda \vDash c(Sym(\omega)) \ge \lambda$. Since $Sym^{M_\lambda}(\omega) = \bigcup_{\alpha<\lambda} Sym^{M_\alpha}(\omega)$, it follows that $M_\lambda \vDash c(Sym(\omega)) = \lambda$.

Definition 2.3 Let $g \in {}^\omega\omega$ be a strictly increasing function.

(a) $P_g = \prod_{n<\omega} Sym(F_n)$, where $F_0 = g(0)$ and $F_n = g(n)\backslash g(n-1)$ for all $n \ge 1$.
(b) Define $g_0, g_1 \in {}^\omega\omega$ by $g_0(n) = g(2n)$ and $g_1(n) = g(2n+1)$. Then $Q_g = \langle P_{g_0}, P_{g_1}\rangle$.

Lemma 2.4 (ZFC) *Suppose that* $g \in {}^\omega\omega$ *is strictly increasing and that* $\pi \in Sym(\omega)$ *satisfies:*

for all $n < \omega$, *if* $\ell \in g(n)$ *then* $\pi(\ell), \pi^{-1}(\ell) \in g(n+1)$.

Then $\pi \in Q_g$.

Proof: Let $I_0 = g_0(0) = g(0)$

$$I_n = g_0(n)\backslash g_0(n-1) = g(2n)\backslash g(2n-2) \quad n \ge 1.$$
$$J_0 = g_1(0) = g(1)$$
$$J_n = g_1(n)\backslash g_1(n-1) = g(2n+1)\backslash g(2n-1) \quad n \ge 1.$$

We shall construct by induction on $n < \omega$ a sequence of finite permutations $\varphi_0 \subseteq \varphi_1 \subseteq \cdots \subseteq \varphi_n \subseteq \cdots$ satisfying the following conditions:

(a) $\varphi_0 = \varnothing$;
(b) $\varphi_{n+1} \in \prod_{i\le n} Sym(J_i)$ for $n \ge 0$;
(c) $\pi \circ \varphi_{n+1} \restriction g_0(n) \in \prod_{i \le n} Sym(I_i)$ for $n \ge 0$.

Suppose inductively that we have constructed φ_n for some $n \ge 0$. If $n = 0$, let $J_{n-1} = \varnothing$.

Claim 2.5 *For all* $\ell \in I_n \cap J_{n-1}$, $\pi \circ \varphi_n(\ell) \in I_n$.

Proof: We can suppose that $n > 0$. Let $\ell \in I_n \cap J_{n-1}$. Then $\varphi_n(\ell) \in J_{n-1} = g(2n-1)\backslash g(2n-3)$. By (2.4), $\pi \circ \varphi_n(\ell) \in g(2n) = g_0(n)$. Since $\pi \circ \varphi_n \restriction g_0(n-1) \in \prod_{i \le n-1} Sym(I_i)$ and $\ell \notin g_0(n-1)$, we must have that $\pi \circ \varphi_n(\ell) \in g_0(n)\backslash g_0(n-1) = I_n$.

Thus we need only ensure that $\varphi_{n+1} \restriction J_n \in Sym(J_n)$ satisfies:

for all $\ell \in I_n \cap J_n$, $\pi \circ \varphi_{n+1}(\ell) \in I_n$.

Define $\Phi_n \in Sym(\omega)$ by

$$\begin{aligned} \Phi_n(\ell) &= \varphi_n(\ell), \qquad \ell \in \operatorname{dom} \varphi_n \\ &= \ell, \qquad\qquad \ell \notin \operatorname{dom} \varphi_n. \end{aligned}$$

Let $\psi = \pi \circ \Phi_n$. Then $\psi[\bigcup_{i<n} I_i] = \bigcup_{i<n} I_i$. Define $A = \{\ell \in I_n \mid \psi(\ell) \notin I_n\}$ and $B = \{\ell \in \omega\backslash I_n \mid \psi(\ell) \in I_n\}$. Clearly $|A| = |B|$, say $A = \{a_i \mid 1 \le i \le t\}$ and $B =$

$\{b_i \mid 1 \leq i \leq t\}$. Also, by Claim 2.5, $\psi(\ell) = \pi(\ell)$ for all $\ell \in A \cup B$. Let $b \in B$. Then $\pi(b) \in I_n$ and (2.4) implies that $b \in J_n$. Thus $B \subseteq J_n \backslash I_n$. Define

$$\varphi_{n+1} \restriction J_n = \prod_{1 \leq i \leq t} (a_i\, b_i) \in Sym(J_n).$$

Then it is easily checked that φ_{n+1} satisfies our requirements.

Finally let $\varphi = \bigcup_{n<\omega} \varphi_n$. Then $\varphi \in \prod_{n<\omega} Sym(J_n)$ and $\pi \circ \varphi \in \prod_{n<\omega} Sym(I_n)$.

The proof of the following result is essentially just the well-known argument that MA_κ implies that the dominating number $\underset{\sim}{d}$ is greater than κ.

Lemma 2.6 $((MA_\kappa))$ *Suppose that $\mathcal{F} \subseteq {}^\omega\omega$ and that $|\mathcal{F}| \leq \kappa$. Then there exists a strictly increasing $g \in {}^\omega\omega$ satisfying:*

for all $f \in \mathcal{F}$ there exists $m < \omega$ such that for all $m \leq n < \omega$,
if $\ell \in g(n)$ then $f(\ell) \in g(n+1)$.

Proof: Let $\mathbb{P}_\mathcal{F}$ be the set of all pairs $\langle p, F\rangle$ such that:

(a) there exists $n < \omega$ such that $p : n \to \omega$ is strictly increasing;
(b) $F \in [\mathcal{F}]^{<\omega}$.

We order $\mathbb{P}_\mathcal{F}$ by setting $\langle p_1, F_1\rangle \leq \langle p_0\, F_0\rangle$ if and only if:

(c) $p_1 \supseteq p_0$ and $F_1 \supseteq F_0$;
(d) for all $n \in dom\, p_1 \backslash dom\, p_0$ and $f \in F_0$, if $\ell \in p_1(n-1)$ then $f(\ell) \in p_1(n)$.

Clearly $\mathbb{P}_\mathcal{F}$ is c.c.c. For each $f \in \mathcal{F}$, the set $D_f = \{\langle p, F\rangle \mid f \in F\}$ is dense in $\mathbb{P}_\mathcal{F}$. It is also easy to see that for each $m < \omega$, the set $E_m = \{\langle p, F\rangle \mid m \subseteq dom\, p\}$ is dense in $\mathbb{P}_\mathcal{F}$. So the result follows by a simple application of MA_κ.

From now on, we suppose that $c(Sym(\omega)) = \lambda < \kappa^+$ and that $Sym(\omega) = \bigcup_{i<\lambda} G_i$.

Definition 2.7 Let $H \leq Sym(\omega)$ and $\Omega \in [\omega]^\omega$.

(a) $H(\Omega)$ is the subgroup of $Sym(\Omega)$ induced on Ω by $H_{\{\Omega\}}$.
(b) $H^\Omega = \{h^\Omega \mid h \in H\}$. (See Definition 1.12.)

Lemma 2.8 $((MA_\kappa))$ *There exists a strictly increasing $g \in {}^\omega\omega$ such that $P_g^\Omega \not\leq G_i(\Omega)$ for every co-infinite $\Omega \in [\omega]^\omega$ and every $i < \lambda$.*

Proof: Suppose not. Then for every strictly increasing $g \in {}^\omega\omega$ and every co-infinite $\Omega \in [\omega]^\omega$, there would exist an $i < \lambda$ such that $P_g^\Omega \leq G_i(\Omega)$.

To see this let $g \in {}^\omega\omega$ be strictly increasing and let $\Omega \in [\omega]^\omega$ be co-infinite. By assumption, there would exist a co-infinite $\Omega' \in [\omega]^\omega$ and an $i < \lambda$ such that $P_g^{\Omega'} \leq G_i(\Omega')$. Then there would exist a $\sigma \in Sym(\omega)$ such that $\sigma \restriction \Omega$ was an order-preserving bijection between Ω and Ω'. We could suppose that $\sigma \in G_i$ and this would imply that $P_g^\Omega \leq G_i(\Omega)$.

Now fix a co-infinite $\Omega \in [\omega]^\omega$. For each $i < \lambda$, choose $\pi_i \in Sym(\omega)$ such that $\pi_i^\Omega \notin G_i(\Omega)$. Let $\Gamma = \langle \pi_i \mid i < \lambda\rangle$. Applying Lemma 2.6 to Γ, we obtain a strictly increasing $g \in {}^\omega\omega$ such that:

for all $\pi \in \Gamma$ there exists $m < \omega$ such that for all $m \leq n < \omega$, if $\ell \in g(n)$ then $\pi(\ell),\ \pi^{-1}(\ell) \in g(n+1)$.

For each $t < \omega$, define $g_t \in {}^{\omega}\omega$ by $g_t(n) = g(n+t)$. By Lemma 2.4, for each $\pi \in \Gamma$ there exists $t < \omega$ such that $\pi \in Q_{g_t}$. The first paragraph implies that for each $t < \omega$ there exists $i < \lambda$ such that $Q_{g_t}^{\Omega} \leq G_i(\Omega)$. By Theorem 1.1, $cf(\lambda) > \omega$. Hence there exists $i < \lambda$ such that $Q_{g_t}^{\Omega} \leq G_i(\Omega)$ for all $t < \omega$. But then $\pi_i^{\Omega} \in G_i(\Omega)$, which is a contradiction.

The above argument actually yields the following slightly stronger result.

Lemma 2.9 $((MA_\kappa))$ *There exists a strictly increasing $g \in {}^{\omega}\omega$ and a subgroup $\Gamma \leq P_g$ such that*

(a) $|\Gamma| = \lambda$;

(b) $\Gamma^{\Omega} \not\leq G_i(\Omega)$ *for every co-infinite $\Omega \in [\omega]^{\omega}$ and every $i < \lambda$.*

Fix such a function $g \in {}^{\omega}\omega$ and such a subgroup $\Gamma = \langle \sigma_i \mid i < \lambda \rangle \leq P_g$. Let $T \subseteq [\omega]^{<\omega}$ be a complete binary tree, with ordering $\prec$, which satisfies the following conditions:

(i) The elements of T are pairwise disjoint and $\cup T = \omega$.

(ii) If $a,\ b \in T$ and $a \prec b$, then $\max(a) < \min(b)$.

(iii) If $a \in Lev_n(T)$, the n^{th} level of T, then a is a set of $g(n) - g(n-1)$ consecutive natural numbers.

(If n=0, then we set g(n-1) = 0.)

If η is a branch of T, then $\eta(n) = \eta \cap Lev_n(T)$.

Definition 2.10 A permutation $\Phi \in Sym(\omega)$ is *level preserving* iff for each $n < \omega$ and $a \in Lev_n(T)$, $\Phi[a] \in Lev_n(T)$ and $\Phi \restriction a$ is order-preserving.

Definition 2.11 Let $\mathcal{B} = \{\eta_i \mid i < \lambda\}$ be a set of branches of T. Then $LP(\mathcal{B})$ is the group of all level-preserving permutations Φ such that there exists a permutation $\varphi \in Sym(\lambda)$ such that:

for all $i < \lambda$, there exists $m < \omega$ such that
$\Phi[\eta_i(n)] = \eta_{\varphi(i)}(n)$ for all $m \leq n < \omega$.

$\psi_{\mathcal{B}} : LP(\mathcal{B}) \to Sym(\lambda)$ denotes the associated homorphism such that $\psi_{\mathcal{B}}(\Phi) = \varphi$.

Lemma 2.12 $((MA_\kappa))$ *There exists a set $\mathcal{B} = \{\eta_i \mid i < \lambda\}$ of branches of T and a permutation $\Pi \in Sym(\omega)$ such that the following conditions hold:*

(i) The homomorphism $\psi_{\mathcal{B}} : LP(\mathcal{B}) \to Sym(\lambda)$ is surjective.

(ii) For each $i < \lambda$, let $\cup \eta_i = B_i = \{b_\ell^i \mid \ell < \omega\}$. Then for each $i < \lambda$, there exists $t < \omega$ such that $\Pi(b_\ell^i) = \sigma_i^{B_i}(b_\ell^i)$ for all $t \leq \ell < \omega$.

Before proving Lemma 2.12, we shall first show how to complete the proof of Theorem 2.1.

Proof Proof of Theorem 2.1: By Theorem 1.1, there exists $i < \lambda$ such that

$$\psi_{\mathcal{B}} \restriction G_i \cap LP(\mathcal{B}) \to Sym(\lambda)$$

is surjective. We may also suppose that $\Pi \in G_i$ and that G_i contains the countable subgroup Fin (ω) of all finite permutations. Let $j < \lambda$ be arbitrary. Then there exists $\Phi_j \in G_i \cap LP(\mathcal{B})$ and $m < \omega$ such that $\Phi_j[\eta_0(n)] = \eta_j(n)$ for all $m \leq n < \omega$. Let $\Pi_j = \Phi_j^{-1} \circ \Pi \circ \Phi_j \in G_i$. Then there exists $t < \omega$ such that $\Pi_j(b^0_\ell) = \sigma_j^{B_0}(b^0_\ell)$ for all $t \leq \ell < \omega$. Adjusting Π_j by an element of Fin(ω) if necessary, it follows that $\sigma_j^{B_0} \in G_i(B_0)$. But then $\Gamma^{B_0} \leq G_i(B_0)$, which is a contradiction.

We return to the proof of Lemma 2.12. The following result was proved in Section 3 of Shelah and Thomas [7]. (For the rest of this section, $[\mathcal{B}]^n$ will be identified with the set of n-tuples $\langle \eta_{i_1}, \dots, \eta_{i_n} \rangle$ such that $i_1 < \cdots < i_n$.)

Lemma 2.13 $((MA_\kappa))$ *Let $\mathcal{B} = \{\eta_i \mid i < \lambda\}$ be a set of branches of T. Suppose that there exists a set of functions $\{d_n \mid d_n : [\mathcal{B}]^n \to \omega\}$ which satisfies the following condition: If $\langle \tau_1, \dots, \tau_n \rangle$, $\langle \theta_1, \dots, \theta_n \rangle \in [\mathcal{B}]^n$ and $d_n(\tau_1, \dots, \tau_n) = d_n(\theta_1, \dots, \theta_n)$, then there exists $k < \omega$ such that:*

(a) $\tau_i(k-1) = \theta_i(k-1)$for $1 \leq i \leq n$;
(b) $\tau_i(k-1) \neq \tau_j(k-1)$ for $1 \leq i < j \leq n$;
(c) if $\tau_i \neq \theta_i$, then $\tau_i(k) \neq \theta_i(k)$ for $1 \leq i \leq n$.

Then $\psi_{\mathcal{B}} : LP(\mathcal{B}) \to Sym(\lambda)$ is surjective.

Now we let $P_g = \prod_{n<\omega} Sym(F_n)$.

Lemma 2.14 $((MA_\kappa))$ *There exists a set $\mathcal{B} = \{\eta_i \mid i < \lambda\}$ of branches of T, equipped with a set of functions $\{d_n \mid d_n : [\mathcal{B}]^n \to \omega\}$ satisfying the conditions of Lemma 2.13 and a function $H : \mathcal{B} \to \omega$ satisfying*

if $i, j < \lambda$ and $n \geq \max\{H(\eta_i), H(\eta_j)\}$, then $\eta_i(n) = \eta_j(n)$ implies that $\sigma_i \restriction F_n = \sigma_j \restriction F_n$.

Proof: Let $\mathbb{P}$ be the partial ordering consisting of elements of the form $p = \langle f, h, \langle d_n \mid n < \omega \rangle \rangle$ satisfying the following conditions:

(1) There exists $X \in [\lambda]^{<\omega}$ and $m \in \omega$ such that

 (a) $dom(f) = X \times m$;
 (b) $h : X \to \omega$;
 (c) $d_n : [X]^n \to \omega$.

 (In particular, $d_n = \varnothing$ if $|X| < n$.)

(2) For $(\alpha, n) \in dom\, f$, $f_\alpha(n) = f(\alpha, n) \in Lev_n(T)$ and f_α is a branch of $\bigcup_{n<m} Lev_n(T)$.
(3) If $\alpha, \beta \in X$ are distinct, then $f_\alpha \neq f_\beta$.
(4) If $\alpha, \beta \in X$ and $\max\{h(\alpha), h(\beta)\} \leq n < m$, then $f_\alpha(n) = f_\beta(n)$ implies that $\sigma_\alpha \restriction F_n = \sigma_\beta \restriction F_n$.
(5) If $\langle \alpha_1, \dots, \alpha_n \rangle$, $\langle \beta_1, \dots, \beta_n \rangle \in [X]^n$ and $d_n(\alpha_1, \dots, \alpha_n) = d_n(\beta_1, \dots, \beta_n)$, then there exists $k \in m$ such that:

 (a) $f_{\alpha_i}(k-1) = f_{\beta_i}(k-1)$ for $1 \leq i \leq n$;
 (b) $f_{\alpha_i}(k-1) \neq f_{\alpha_j}(k-1)$ for $1 \leq i < j \leq n$;
 (c) If $\alpha_i \neq \beta_i$, then $f_{\alpha_i}(k) \neq f_{\beta_i}(k)$ for $1 \leq i \leq n$.

The ordering on $\mathbb{P}$ is the obvious one. A straightforward $\triangle$-system argument shows that $\mathbb{P}$ is c.c.c., and the result follows by an easy application of MA_κ.

Proof Proof of Lemma 2.12: If $\mathcal{B} = \{\eta_i \mid i < \lambda\}$ is the set of branches given by Lemma 2.14, then the homorphism $\psi_{\mathcal{B}} : LP(\mathcal{B}) \to Sym(\lambda)$ is surjective. Hence it is enough to find an element $\Pi \in Sym(\omega)$ which satisfies the second condition of Lemma 2.12.

Let $i < \lambda$ and $n < \omega$; and let $F_n = \{a_r \mid 1 \leq r \leq |F_n|\}$ and $\eta_i(n) = \{b_r | 1 \leq r \leq |F_n|\}$ be the increasing enumerations. Define $\tilde{\sigma}_{i,n} \in Sym(\eta_i(n))$ by $\tilde{\sigma}_{i,n}(b_r) = b_s$ iff $\sigma_i(a_r) = a_s$. Now define $\Pi \in Sym(\omega)$ as follows. Let $\ell \in \omega$. If there exists $i < \lambda$ and $n < \omega$ such that $H(\eta_i) \leq n$ and $\ell \in \eta_i(n)$, then $\Pi(\ell) = \tilde{\sigma}_{i,n}(\ell)$. Otherwise, $\Pi(\ell) = \ell$. It is easily seen that Π satisfies our requirements.

Theorem 2.15 ((MA_κ)) *For all $\lambda \leq \kappa$, wU_λ holds.*

Proof: As the proof is very similar to that of Theorem 2.1, we shall just sketch the main points of the argument. Clearly we can restrict our attention to the case when λ is regular and uncountable. Suppose that $\pi : [\omega]^\omega \times \lambda \to \mathcal{P}(\omega)$ satisfies the hypotheses of wU_λ with respect to $\{f_i \mid i < \lambda\} \subseteq {}^\omega 2$. Let $T = \langle \omega, \prec \rangle$ be a complete binary tree with the property that $a \prec b$ implies $a < b$. Then $\Phi \in Sym(\omega)$ is level-preserving iff $\Phi \in \prod_{n<\omega} Sym(Lev_n(T))$. If B is a branch of T, then $B(n)$ denotes the unique element of $B \cap Lev_n(T)$.

Arguing as above, there exists a set $\mathcal{B} = \{B_i \mid i < \lambda\}$ of branches of T satisfying the following conditions:

(a) the homomorphism $\psi_{\mathcal{B}} : LP(\mathcal{B}) \to Sym(\lambda)$ is surjective;

(b) there exists a subset $S \in [\omega]^\omega$ such that whenever $i < \lambda$ there exists $m < \omega$ such that $\chi_S(B_i(n)) = f_i^{B_i}(B_i(n))$ for all $m \leq n < \omega$.

There exists $\varphi \in Sym(\lambda)$ and $X \in [\lambda]^\lambda$ such that $\chi_{\pi(B_i,\varphi(i))} \restriction B_i = f_{\varphi(i)}^{B_i}$ for all $i \in X$. Let $\Phi \in LP(\mathcal{B})$ satisfy $\psi_{\mathcal{B}}(\Phi) = \varphi$, and let $T = \Phi^{-1}[S]$. Then for each $i \in X$, there exists $m_i < \omega$ such that $\chi_T(B_i(n)) = f_{\varphi(i)}^{B_i}(B_i(n))$ for all $m_i \leq n < \omega$. There exists $Y \in [X]^\lambda$, $m < \omega$ and $\{B(\ell) \mid \ell \leq m\}$ such that:

(1) $m_i = m$ for all $i \in Y$;

(2) $B_i(\ell) = B(\ell)$ for all $i \in Y$ and $\ell \leq m$;

(3) $f_{\varphi(i)}^{B_i}(B(\ell)) = f_{\varphi(j)}^{B_j}(B(\ell))$ for all $i, j \in Y$ and $\ell \leq m$.

Adjusting the finite set $T \cap \{B(\ell) \mid \ell \leq m\}$ if necessary, we can suppose that $T \cap B_i = \pi(B_i, \varphi(i))$ for all $i \in Y$. This completes the proof of wU_λ.

3 Difficult Uniformization Problems

Definition 3.1 $\{\langle A_\alpha, B_\alpha \rangle \mid \alpha < \theta\}$ is a *uniformization problem* if $A_\alpha \subseteq B_\alpha \subseteq \omega$ for each $\alpha < \theta$. If $S \in \mathcal{P}(\omega)$, then $I(S) = \{\alpha < \theta \mid S \cap B_\alpha = A_\alpha\}$ is called the *solution set* of S.

The following result will be useful in Section 4.

Theorem 3.2 ($(MA + SOCA)$) *If $\theta < 2^\omega$ and $\{\langle A_\alpha, B_\alpha \rangle \mid \alpha < \theta\}$ is a uniformization problem, then the following are equivalent.*

(1) For every $S \in \mathcal{P}(\omega)$, $I(S)$ is countable.

(2) There exists a function $d : \theta \to \omega$ such that if $\alpha < \beta < \theta$ satisfy $d(\alpha) = d(\beta)$, then $A_\alpha \cap B_\beta \neq A_\beta \cap B_\alpha$.

Here $SOCA$ is the following set-theoretic hypothesis, which was introduced in Abraham, Rubin and Shelah [1].

The Semiopen Coloring Axiom If X is an uncountable set of reals and $[X]^2 = K_0 \sqcup K_1$ is a partition with K_0 open in the product topology, then X contains an uncountable subset Y such that $[Y]^2 \subseteq K_i$ for some $i \in \{0, 1\}$. (Y is said to be i-homogeneous.)

In [1], it was shown that $MA + SOCA + 2^\omega = \kappa$ is consistent with ZFC for any regular $\kappa > \omega_1$. And in [8], Todorcevic proved that $SOCA$ is a consequence of PFA.

We now begin the proof of Theorem 3.2. Let $\theta < 2^\omega$ and let $U = \{\langle A_\alpha, B_\alpha\rangle \mid \alpha < \theta\}$ be a uniformization problem.

Lemma 3.3 $((ZFC))$ *Clause (2) of Theorem 3.2 implies Clause (1).*

Proof: Let $S \in \mathcal{P}(\omega)$. If $\alpha, \beta \in I(S)$ are distinct, then

$$A_\alpha \cap B_\beta = S \cap B_\alpha \cap B_\beta = A_\beta \cap B_\alpha \text{ and so } d(\alpha) \neq d(\beta).$$

Thus $|I(S)| \leq \omega$.

Lemma 3.4 $((MA + SOCA))$ *Clause (1) of Theorem 3.2 implies Clause (2).*

Proof: We shall require the following claim:

Claim 3.5 *Suppose that $X \subseteq U$ is an uncountable subset. Define a partition $[X]^2 = K_0 \sqcup K_1$ by $\{\langle A_\alpha, B_\alpha\rangle, \langle A_\beta, B_\beta\rangle\} \in K_0$ iff $A_\alpha \cap B_\beta \neq A_\beta \cap B_\alpha$. Then there exists an uncountable* 0*-homogeneous subset $Y \subseteq X$.*

Proof: Clearly K_0 is open in $[X]^2$. Thus if the claim fails, then there exists an uncountable 1-homogeneous subset $Z \subseteq X$. Let $I = \{\alpha < \theta \mid \langle A_\alpha, B_\alpha\rangle \in Z\}$ and let $S = \bigcup_{\alpha \in I} A_\alpha$. We claim that $S \cap B_\alpha = A_\alpha$ for all $\alpha \in I$, which is a contradiction. Clearly $A_\alpha \subseteq S \cap B_\alpha$. For the converse, suppose that $n \in S \cap B_\alpha$. Then there exists $\beta \in I$ such that $n \in A_\beta \cap B_\alpha = A_\alpha \cap B_\beta$, and so $n \in A_\alpha$.

Let $\mathbb{D}$ consist of all finite function $d : \theta \to \omega$ such that if α, β are distinct elements of *dom* d with $d(\alpha) = d(\beta)$, then $A_\alpha \cap B_\beta \neq A_\beta \cap B_\alpha$. By MA, it is enough to show that $\mathbb{D}$ is c.c.c. Suppose that $\{d_i \mid i < \omega_1\}$ is an antichain. By a $\triangle$-system argument and a counting argument, we can assume that the following hold:

(a) There exists an integer ℓ such that *dom* $d_i = \{\alpha_1^i, \dots, \alpha_\ell^i\}$ and a function $f : \{1, \dots, \ell\} \to \omega$ such that $d_i(\alpha_k^i) = f(k)$ for all $i < \omega_1$ and $1 \leq k \leq \ell$.

(b) The sets *dom* d_i form a $\triangle$-system with root R, and the elements of R lie in corresponding positions in each increasing enumeration $\alpha_1^i < \cdots < \alpha_\ell^i$.

(c) There exists an integer n such that if $1 \leq j < k \leq \ell$ and $f(j) = f(k)$, then

$$A_{\alpha_j^i} \cap B_{\alpha_k^i} \cap n \neq A_{\alpha_k^i} \cap B_{\alpha_j^i} \cap n$$

for all $i < \omega_1$. Furthermore, for each such pair, there are fixed subsets $s, t \subseteq n$ such that $A_{\alpha_j^i} \cap B_{\alpha_k^i} \cap n = s$ and $A_{\alpha_k^i} \cap B_{\alpha_j^i} \cap n = t$ for all $i < \omega_1$.

Applying Claim 3.5 to $X_0 = \{\langle A_{\alpha_k^i},\ B_{\alpha_k^i}\rangle \mid i < \omega_1\}$ where k is the least integer such that $\alpha_k^i \in dom\ d_i \setminus R$, there exists an uncountable $I_1 \subseteq \omega_1$ such that if i, j are distinct elements of I_1, then $A_{\alpha_k^i} \cap B_{\alpha_k^j} \neq A_{\alpha_k^j} \cap B_{\alpha_k^i}$. Continuing in this manner, we obtain an uncountable $I \subseteq \omega_1$ such that if i, j are distinct elements of I and $\alpha_k^i \neq \alpha_k^j$, then $A_{\alpha_k^i} \cap B_{\alpha_k^j} \neq A_{\alpha_k^j} \cap B_{\alpha_k^i}$. But then $d_i \cup d_j \in \mathbb{D}$, which is a contradiction.

We end this section by showing that Theorem 3.2 cannot be proved from MA alone. We shall make use of the following result, which was pointed out to us by the referee.

Theorem 3.6 (($MA + \neg CH$)) *Suppose that there exists an uncountable set X of reals and a partition $[X]^2 = K_0 \sqcup K_1$ such that:*

(i) K_0 and K_1 are both open; and

(ii) X contains no uncountable homogeneous subsets.

Then there exists a uniformization problem $\{\langle A_\alpha, B_\alpha\rangle \mid \alpha < \omega_1\}$ such that:

(a) for every $S \in \mathcal{P}(\omega)$, $I(S)$ is countable; and

(b) for every function $d : \omega_1 \to \omega$, there exist $\alpha < \beta < \omega_1$ such that $d(\alpha) = d(\beta)$ and $A_\alpha \cap B_\beta = A_\beta \cap B_\alpha$.

Proof: We can suppose that X is a subset of ${}^{\omega}2$ of cardinality ω_1; say $X = \{r_\alpha \mid \alpha < \omega_1\}$. For each $t \in {}^{<\omega}2$, let $U_t = \{r \in {}^{\omega}2 \mid t \subset r\}$ be the corresponding basic open subset of ${}^{\omega}2$. By passing to a suitable subset if necessary, we can suppose that the following condition holds.

(7) If $\alpha < \omega_1, t \in {}^{<\omega}2$ and $r_\alpha \in U_t$,
then there exist uncountably many β such that $r_\beta \in U_t$.

For each $\alpha < \omega_1$, let $B_\alpha = \{t \in {}^{<\omega}2 \mid t \subset r_\alpha\}$. Then $\{B_\alpha \mid \alpha < \omega_1\}$ is an almost disjoint family of subsets of ${}^{<\omega}2$. Let $\mathbb{P} = \prod_{\alpha<\omega_1} F_n(B_\alpha, 2)$ be the finite support product. For each $p \in \mathbb{P}$, $supt(p)$ will denote the support of p. If $p \in \mathbb{P}$, then we shall write $p = \langle p_\alpha\rangle_{\alpha<\omega_1}$, where $p_\alpha \in F_n(B_\alpha, 2)$. Let $\mathbb{Q}$ consist of all conditions $p \in \mathbb{P}$ which satisfy the following property.

If $\alpha, \beta \in supt(p)$ are distinct, then $B_\alpha \cap B_\beta \subseteq \text{dom } p_\alpha \cap \text{dom } p_\beta$;
and $p_\alpha \restriction B_\alpha \cap B_\beta = p_\beta \restriction B_\alpha \cap B_\beta$ iff $\{r_\alpha, r_\beta\} \in K_0$.

Then it is easily checked that $\mathbb{Q}$ is c.c.c. Using (7), we see that for each $\gamma < \omega_1$, the set $D_\gamma = \{p \in \mathbb{Q} \mid \text{There exists } \gamma < \delta < \omega_1 \text{ such that } \delta \in supt(p)\}$ is dense in $\mathbb{Q}$. Also notice that if $p \in \mathbb{Q}$, $\alpha \in supt(p)$ and $t \in B_\alpha$, then there exists $q \leq p$ such that $t \in \text{dom } q_\alpha$. Applying MA, let G be a suitably generic filter on $\mathbb{Q}$. Let

$$I = \{\alpha < \omega_1 \mid \text{ There exists } p \in G \text{ such that } \alpha \in supt(p)\}$$

and for each $\alpha \in I$, let

$$A_\alpha = \{t \in b_\alpha \mid \text{ There exists } p \in G \text{ such that } p_\alpha(t) = 1\}.$$

Then $|I| = \omega_1$; and if $\alpha, \beta \in I$ are distinct, then $A_\alpha \cap B_\beta = A_\beta \cap B_\alpha$ iff $\{r_\alpha, r_\beta\} \in K_0$. It follows easily that the uniformization problem $\{\langle A_\alpha, B_\alpha\rangle \mid \alpha \in I\}$ satisfies our requirements.

Thus to prove that Theorem 3.2 cannot be proved from MA, it is enough to show that $MA + \neg CH + \neg wSOCA$ is consistent, where $wSOCA$ is the following weak version of $SOCA$.

wSOCA If X is an uncountable set of reals and $[X]^2 = K_0 \sqcup K_1$ is a partition with *both* K_0 and K_1 open in the product topology, then X contains an uncountable homogeneous subset.

It is easily seen that $wSOCA$ implies that if $f \subseteq \mathbb{R} \times \mathbb{R}$ is an uncountable function, then there exists an uncountable monotonic function $g \subseteq f$ (for example, see Theorem 1.2 of [1]). Abraham and Shelah have proved that it is consistent with $MA + \neg CH$ that there exists an uncountable $f \subseteq \mathbb{R} \times \mathbb{R}$ which does not include any uncountable monotonic function (see Section 8 of [1]). Hence Theorem 3.2 is not a consequence of MA.

4 The Symmetric Uniformization Principle In this section, we shall prove Theorem 1.15. Following the example of Velickovic in [9] and [11], we shall accomplish this by forcing with an ω_2-Baire poset $\mathbb{Q}$ over a model of PFA. Recall that $\mathbb{Q}$ is said to be ω_2-Baire if the intersection of each family of ω_1 dense open subsets of $\mathbb{Q}$ is dense. In this case, $\mathbb{Q}$ does not adjoin any new ω_1-sequences. Hence if $V \vDash PFA$, then $V^{\mathbb{Q}} \vDash MA + 2^\omega = \omega_2$.

Theorem 4.1 *Suppose that $V \vDash PFA$. Then there exists an ω_2-Baire poset $\mathbb{Q}$ such that $V^{\mathbb{Q}} \vDash \neg sU_{\omega_1}$.*

For the rest of this section, we shall assume PFA. Let $H(\omega_1)$ be the set of all sets which are hereditarily of cardinality less than ω_1. Let $H(\omega_1) = \bigcup_{\alpha<\omega_2} M_\alpha$ be a smooth strictly increasing chain of transitive elementary submodels such that $|M_\alpha| = \omega_1$ for all $\alpha < \omega_2$. (This chain is only introduced for the very crudest bookkeeping purposes.) Let $E \in \mathcal{P}(\omega)$ be the set of all even numbers, and let $\{E_\alpha \mid \alpha < \omega_1\}$ be an independent family of subsets of E.

Definition 4.2 The poset $\mathbb{R}$ consists of all functions p satisfying the following conditions.

(a) There exists $\alpha < \omega_2$ such that $dom\ p = ([\omega]^\omega \cap M_\alpha) \times \omega_1$, and $p(X, \gamma) \in \mathcal{P}(X)$ for all $(X, \gamma) \in dom\ p$.
(b) For each $S \in \mathcal{P}(\omega)$, there exist only countably many $(X, \gamma) \in dom\ p$ such that $S \cap X = p(X, \gamma)$.
(c) For each $X \in [\omega]^\omega \cap M_\alpha$, there exists a bijection $g : E \to X$ such that $g[E_\gamma] = p(X, \gamma)$ for all $\gamma < \omega_1$.

$\mathbb{R}$ almost serves our requirements. However, in the course of our argument, we shall need to show that a certain function q is an element of $\mathbb{R}$. Conditions (a) and (c) will be clear. By Theorem 3.2, Condition (b) is equivalent to the existence of a function $d : dom\ q \to \omega$ such that if $(X, \gamma),\ (Y, \delta) \in dom\ q$ are distinct and $d(X, \gamma) = d(Y, \delta)$, then $X \cap q(Y, \delta) \neq Y \cap q(X, \gamma)$. We shall show that d exists by proving that the partial order $\mathbb{D}$ of finite approximations to such a function is c.c.c. The next two definitions are designed to prevent the growth of uncountable antichains in $\mathbb{D}$.

Definition 4.3 Let $p \in \mathbb{R}$. A *dangerous sequence* for p of width $\ell \in \omega$ consists of a set $\{\langle A_n^t,\ B_n^t\rangle_{1 \le t \le \ell} \mid n < \omega\}$ satisfying the following conditions:

(a) If $\langle A_n^t,\ B_n^t\rangle = \langle A_m^s,\ B_m^s\rangle$, then $(t, n) = (s, m)$.
(b) For each $1 \le t \le \ell$ and $n < \omega$, there exists $(X, \gamma) \in dom\, p$ such that $B_n^t = X$ and $A_n^t = p(X, \gamma)$.
(c) For each $n < m < \omega$, there exists $1 \le t \le \ell$ such that $A_n^t \cap B_m^t = A_m^t \cap B_n^t$.

Definition 4.4 The poset $\mathbb{Q}$ consists of all functions $p \in \mathbb{R}$ such that the following further condition is satisfied:

Suppose that $dom\, p = ([\omega]^\omega \cap M_\alpha) \times \omega_1$, $\delta < \alpha$ and that $\Phi = \{\langle A_n^t,\ B_n^t\rangle_{1\le t\le \ell} \mid n < \omega\} \in M_\delta$ is a dangerous sequence for p. Suppose further that $\Phi \cup \{\langle C^t,\ D^t\rangle_{1\le t\le\ell}\}$ is a dangerous sequence for p such that $\langle D^t\rangle_{1\le t\le\ell} \notin M_\delta$. Then there exists a nonempty $P \subseteq \{1, \ldots, \ell\}$ and a sequence of finite sets $\langle F^t\rangle_{t\in P}$ such that for all $n < \omega$, either:

(1) there exists $t \in \{1, \ldots, \ell\}\backslash P$ such that $A_n^t \cap D^t = B_n^t \cap C^t$; or
(2) there exists $t \in P$ such that $B_n^t \cap D^t \subseteq F^t$.

Clearly $\mathbb{Q}$ is σ-closed and hence proper. Our intention is that we will eventually begin to deal with each dangerous sequence. But suppose that $p \in \mathbb{Q}$ with $dom\, p = ([\omega]^\omega \cap M_\alpha) \times \omega_1$ and that Φ is a dangerous sequence for p. Then for each $\beta < \omega_2$, there exists a subsequence $\Phi_\beta \subseteq \Phi$ such that $\Phi_\beta \notin M_\beta$.

In particular, this is true when $|\beta\backslash\alpha| = \omega_1$, and so there still appears to be plenty of scope for the slow growth of an uncountable dangerous sequence. An amusing argument, which perhaps deserves to be called *The Empty Box*, will eliminate this unpleasant possibility.

Theorem 4.1 is an immediate consequence of the following two results.

Lemma 4.5 *For each $\alpha < \omega_2$, the set*

$$D_\alpha = \{p \in \mathbb{Q} \mid dom\, p = ([\omega]^\omega \cap M_\beta) \times \omega_1 \textit{ for some } \alpha \le \beta < \omega_2\}$$

is dense in $\mathbb{Q}$.

Lemma 4.6 *$\mathbb{Q}$ is ω_2-Baire.*

Proof Proof of Lemma 4.5: Suppose that $p \in \mathbb{Q}$ and that $dom\, p = ([\omega]^\omega \cap M_\gamma) \times \omega_1$ for some $\gamma < \alpha$. Let $\mathcal{F} = ([\omega]^\omega \cap M_\alpha)\backslash([\omega]^\omega \cap M_\gamma)$. For each $B \in \mathcal{F}$, let $\mathbb{P}^B$ be the set of all finite injective functions $p : E \to B$, ordered by reverse inclusion. Let $\mathbb{P} = \prod_{B\in\mathcal{F}} \mathbb{P}^B$ be the finite support product. For each $q = \langle q^B\rangle_{B\in\mathcal{F}} \in \mathbb{P}$, let $S(q) = \{B \mid q^B \neq \varnothing\}$. Let $\mathbb{C}$ consist of all conditions $\langle q,\ d\rangle$ which satisfy the following conditions:

(1) $q = \langle q^B\rangle_{B\in\mathcal{F}} \in \mathbb{P}$.
(2) There exists a finite subset $X \subset \omega_1$ such that $d : S(q) \times X \to \omega$.
(3) Whenever $(A, i), (B, j) \in dom\, d$ are distinct elements with $d(A, i) = d(B, j)$, then there exists $n \in$ ran $q^A \cap$ ran q^B such that

$$(q^A)^{-1}(n) \in E_i \text{ iff } (q^B)^{-1}(n) \notin E_j.$$

It is routine to check that $\mathbb{C}$ is c.c.c. For each $C \in \mathcal{P}(\omega) \cap M_\alpha$ and each $(B, i) \in \mathcal{F} \times \omega_1$, let $D^C_{(B,i)}$ consist of those $\langle q, d\rangle \in \mathbb{C}$ such that there exists $n \in$ ran q^B with $(q^B)^{-1}(n) \in E_i$ iff $n \notin C$. Clearly each $D^C_{(B,i)}$ is dense in $\mathbb{C}$.

Now let $\Phi = \{\langle A_n^t, B_n^t\rangle_{1\le t\le \ell} \mid n < \omega\} \in M_\alpha$ be a dangerous sequence for p. Suppose that:

(i) $P \subseteq \{1, \dots, \ell\}$ is a nonempty subset;
(ii) for each $t \in \{1, \dots, \ell\}\setminus P$, $\langle C^t,\ D^t\rangle = \langle p(X, \gamma),\ X\rangle$ for some $(X, \gamma) \in dom\, p$;
(iii) $\langle D^t,\ i_t\rangle \in \mathcal{F} \times \omega_1$ for $t \in P$ are distinct;
(iv) there *does not* exist a sequence $\langle F^t\rangle_{t\in P}$ of finite sets such that for all $n < \omega$, either:

  (a) there exists $t \in \{1, \dots, \ell\}\setminus P$ such that $A_n^t \cap D^t = B_n^t \cap C^t$; or
  (b) there exists $t \in P$ such that $B_n^t \cap D^t \subseteq F^t$.

Let $\mathcal{D}(\Phi, P, \dots)$ consist of those $\langle q, d\rangle \in \mathbb{C}$ such that there exists $n < \omega$ satisfying:

$$(\alpha)\ A_n^t \cap D^t \neq B_n^t \cap C^t \text{ for all } t \in \{1, \dots, \ell\}\setminus P;$$

$$(\beta) \text{ for all } t \in P,\ \text{ there exists } m_t \in\ \operatorname{ran} q^{D^t} \cap B_n^t$$

such that $(q^{D^t})^{-1}(m_t) \in E_{i_t}$ iff $m_t \notin A_n^t$.

It is easily checked that each $\mathcal{D}(\Phi, P, \dots)$ is dense in $\mathbb{C}$.

By PFA, there exists a filter $G \subseteq \mathbb{C}$ which intersects each of the ω_1 dense sets mentioned above. Define a function

$$p \subseteq p^+ : ([\omega]^\omega \cap M_\alpha) \times \omega_1 \to \mathcal{P}(\omega)$$

by specifying for each $(B, i) \in \mathcal{F} \times \omega_1$ that $n \in p^+(B, i)$ iff there exists $\langle q, d\rangle \in G$ and $m \in E_i$ such that $q^B(m) = n$. Also define a function $D : \mathcal{F} \times \omega_1 \to \omega$ by $D(B, i) = s$ iff there exists $\langle q, d\rangle \in G$ such that $d(B, i) = s$. Using Lemma 3.3, we see that $p^+ \in \mathbb{R}$. Suppose that $\delta < \alpha$ and that $\Phi = \{\langle A_n^t, B_n^t\rangle_{1\le t\le\ell} \mid n < \omega\} \in M_\delta$ is a dangerous sequence for p^+. If $(B, i) \in \mathcal{F} \times \omega_1$, then the dense sets of the form $D^C_{(B,i)}$ ensure that $p^+(B, i) \notin M_\alpha$. Hence Φ is already a dangerous sequence for p. Suppose that $\Phi \cup \{\langle C^t, D^t\rangle_{1\le t\le\ell}\}$ is a dangerous sequence for p^+ such that $\langle D^t\rangle_{1\le t\le\ell} \notin M_\delta$. If $\langle D^t\rangle_{1\le t\le\ell} \in M_\gamma$, then (4.4) must hold since $p \in \mathbb{Q}$. On the other hand, if $\langle D^t\rangle_{1\le t\le\ell} \notin M_\gamma$ then the dense sets of the form $\mathcal{D}(\Phi, P, \dots)$ ensure that (4.4) holds. Thus $p^+ \in \mathbb{Q}$.

Proof Proof of Lemma 4.6: Suppose that D_ξ, $\xi < \omega_1$, are dense open subsets of $\mathbb{Q}$ and that $p \in \mathbb{Q}$. We must find $q \le p$ such that $q \in \bigcap_{\xi<\omega_1} D_\xi$. Until further notice, we shall work within $V^{\mathbb{Q}}$. Let G be a generic filter of $\mathbb{Q}$ such that $V^{\mathbb{Q}} = V[G]$, and let $g = \cup G$. Then $g : [\omega]^\omega \times \omega_1 \to \mathcal{P}(\omega)$; and for every $\alpha < \omega_2^V$, $g_\alpha = g \restriction ([\omega]^\omega \cap M_\alpha) \times \omega_1 \in \mathbb{Q}$. Let $\mathbb{D}$ consist of all finite functions $d : [\omega]^\omega \times \omega_1 \to \omega$ such that if $(S, i), (T, j) \in dom\, d$ are distinct and $d(S, i) = d(T, j)$, then $g(S, i) \cap T \neq g(T, j) \cap S$.

Claim 4.7 *$\mathbb{D}$ is c.c.c. in $V^{\mathbb{Q}}$.*

Proof: Suppose not. Then, arguing as in the proof of Lemma 3.4, we see that there exists a sequence $\{\langle A_\alpha^t, B_\alpha^t\rangle_{1\le t\le\ell} \mid \alpha < \omega_1\}$ for some $\ell < \omega$ satisfying the following conditions.

(a) If $\langle A_\alpha^t, B_\alpha^t\rangle = \langle A_\beta^s, B_\beta^s\rangle$, then $(t, \alpha) = (s, \beta)$.

(b) For each $1 \le t \le \ell$ and $\alpha < \omega_1$, there exists $(X, \gamma) \in [\omega]^\omega \times \omega_1$ such that $B^t_\alpha = X$ and $A^t_\alpha = g(X, \gamma)$.

(c) For each $\alpha < \beta < \omega_1$, there exists $1 \le t \le \ell$ such that $A^t_\alpha \cap B^t_\beta = A^t_\beta \cap B^t_\alpha$.

We suppose that the sequence has been chosen so that ℓ is minimal.

Suppose that there exists $\delta < \omega_2^V$ and an uncountable $I \subseteq \omega_1$ such that $B^t_\alpha \in M_\delta$ for all $1 \le t \le \ell$ and $\alpha \in I$. By Theorem 3.2 applied to $g_\delta \in \mathbb{Q}$, there exists a function $\Delta \in V$ such that:

(1) $\Delta : ([\omega]^\omega \cap M_\delta) \times \omega_1 \to \omega$;

(2) if $(X, i), (Y, j) \in dom\, \Delta$ are distinct and $\Delta(X, i) = \Delta(Y, j)$, then

$$g_\delta(X, i) \cap Y \neq g_\delta(Y, j) \cap X.$$

Define $\Delta^* : I \to {}^\ell\omega$ by $\Delta^*(\alpha) = \langle \Delta(B^t_\alpha, \gamma^t_\alpha)\rangle_{1 \le t \le \ell}$ where $A^t_\alpha = g_\delta(B^t_\alpha, \gamma^t_\alpha)$. Then there exists an uncountable $J \subseteq I$ such that $\Delta^*(\alpha) = \Delta^*(\beta)$ for all $\alpha, \beta \in J$. But then if $\alpha, \beta \in J$ are distinct, we have that $A^t_\alpha \cap B^t_\beta \neq A^t_\beta \cap B^t_\alpha$ for all $1 \le t \le \ell$, which is a contradiction.

Thus if $\delta < \omega_2^V$, then there exist only countably many $\alpha < \omega_1$ such that $B^t_\alpha \in M_\delta$ for all $1 \le t \le \ell$. Since $\mathbb{Q}$ is proper, $cf(\omega_2^V) > \omega$. Hence we can suppose that for all $\gamma < \omega_1$, there exists $\delta < \omega_2^V$ such that $\{\langle A^t_\alpha, B^t_\alpha\rangle_{1 \le t \le \ell} \mid \alpha < \gamma\} \in M_\delta$ and $\langle B^t_\gamma\rangle_{1 \le t \le \ell} \notin M_\delta$. Applying Definition 4.4, for each $\omega \le \gamma < \omega_1$ there exists a nonempty subset $P_\gamma \subseteq \{1, \dots, \ell\}$ and a sequence of finite sets $\langle F^t_\gamma\rangle_{t \in P_\gamma}$ such that for all $\beta < \gamma$ either

(1) there exists $t \in \{1, \dots, \ell\} \setminus P_\gamma$ such that $A^t_\beta \cap B^t_\gamma = B^t_\beta \cap A^t_\gamma$; or

(2) there exists $t \in P_\gamma$ such that $B^t_\beta \cap B^t_\gamma \subseteq F^t_\gamma$.

We can suppose that there is a fixed set P and a fixed sequence $\langle F^t\rangle_{t \in P}$ such that $P_\gamma = P$ and $\langle F^t_\gamma\rangle_{t \in P_\gamma} = \langle F^t\rangle_{t \in P}$ for all $\omega \le \gamma < \omega_1$. We can also suppose that there exist integers $n_t \in \omega \setminus F^t$ for $t \in P$ such that $n_t \in B^t_\gamma$ for all $\omega \le \gamma < \omega_1$. But this means that whenever $\omega \le \beta < \gamma < \omega_1$, then there exists $t \in \{1, \dots, \ell\} \setminus P$ such that $A^t_\beta \cap B^t_\gamma = B^t_\beta \cap A^t_\gamma$. This contradicts the minimality of ℓ.

In particular, $\mathbb{Q} * \mathbb{D}$ is proper. For the remainder of the proof, we shall work inside V. For each $\alpha < \omega_2$, let $h_\alpha : \omega_1 \to ([\omega]^\omega \cap M_\alpha) \times \omega_1$ be a bijection. For each $\xi, \upsilon < \omega_1$ let $\mathcal{D}_{\xi\upsilon}$ consist of those conditions $\langle q, d\rangle \in \mathbb{Q} * \mathbb{D}$ which satisfy:

(1) $q \in D_\xi$;

(2) if α_ξ is the least ordinal such that $q \restriction ([\omega]^\omega \cap M_{\alpha_\xi}) \times \omega_1 \in D_\xi$, then $h_{\alpha_\xi}(\upsilon) \in dom\, d$.

Clearly $\mathbb{D}_{\xi\upsilon}$ is dense in $\mathbb{Q} * \mathbb{D}$. By PFA, there exists a filter $\langle p, \varnothing\rangle \in H \subseteq \mathbb{Q} * \mathbb{D}$ such that $H \cap \mathbb{D}_{\xi\upsilon} \neq \varnothing$ for all $\xi, \upsilon < \omega_1$. For each $\xi < \omega_1$, there exists $\langle q_\xi, \varnothing\rangle \in H$ such that $q_\xi \in D_\xi$ and $q' \notin D_\xi$ for all $q_\xi < q' \in \mathbb{Q}$. Let $q = \cup_{\xi < \omega_1} q_\xi$. We can suppose that $p \subseteq q$. Then q satisfies all of our requirements, except possibly Condition 4.2 (b). Define $\Delta : dom\, q \to \omega$ by

$$\Delta(X, \gamma) = n \text{ iff there exists } \langle q_\xi, d\rangle \in H \text{ such that } d(X, \gamma) = n.$$

Using Lemma 3.3, Δ witnesses the fact that q satisfies Condition 4.2(b). Thus $q \in \mathbb{Q}$.

5 Concluding Remarks In the previous sections, we have shown that cov $(\mathcal{B}_2) = 2^\omega$ is independent of $MA + \neg CH$. It is straightforward to eliminate the use of PFA from both directions of this result.

Theorem 5.1 *Let $M \models GCH$, and suppose that $\kappa > \omega_1$ is a regular cardinal in M.*

(a) There exists a c.c.c. poset $\mathbb{Q}$ such that $M^{\mathbb{Q}} \models MA + 2^\omega = \kappa +$ cov $(\mathcal{B}_2) = 2^\omega$.
(b) There exists a c.c.c. poset $\mathbb{R}$ such that $M^{\mathbb{R}} \models MA + 2^\omega = \kappa +$ cov $(\mathcal{B}_2) = \omega_1$.

Proof Sketch Proof:

(a) Let $S, \mathbb{P}$ denote Sacks forcing and Prikry-Silver forcing respectively. In [9], Velickovic constructed a c.c.c. poset $\mathbb{Q}^*$ such that $M^{\mathbb{Q}^*} \models MA + 2^\omega = \kappa + FA(S)$. Velickovic pointed out in the Introduction of [9] that it is routine to modify his construction so as to obtain a c.c.c. poset $\mathbb{Q}$ such that $M^{\mathbb{Q}} \models MA + 2^\omega = \kappa + FA(\mathbb{P})$. Arguing as in the proof of Theorem 1.9, we see that $\mathbb{Q}$ satisfies our requirements.

(b) We perform an iterated finite support c.c.c. construction $M_\alpha, \alpha \le \kappa$, with $M_0 = M$. The odd stages of the construction are devoted to ensuring that $M_\kappa \models MA + 2^\omega = \kappa$. At even stages $2\alpha = 2\beta + 2$, we use the c.c.c. poset $\mathbb{C}$ from the proof of Lemma 4.5 to adjoin $\Pi_{2\beta+2} : ([\omega]^\omega \cap (M_{2\beta+2} \backslash M_{2\beta})) \times \omega_1 \to \mathcal{P}(\omega)$.

At limit stages α of uncountable cofinality, we use the poset $\mathbb{D}$ from the proof of Lemma 4.6 to adjoin $d : \bigcup_{\beta<\alpha} dom\, \Pi_{2\beta+2} \to \omega$. An easy modification of the proof of Claim 4.7 shows that $\mathbb{D}$ is c.c.c. Finally $\Pi = \bigcup_{\beta<\kappa} \Pi_{2\beta+2}$ is a counterexample to sU_{ω_1} and hence $M_\kappa \models$ cov $(\mathcal{B}_2) = \omega_1$.

Putting together Theorem 1.10 and Theorem 1.11, we see that

$$\text{cov } (\mathcal{B}_2) < c(Sym(\omega)) \text{ is consistent with } ZFC.$$

Theorem 5.2 *$c(Sym(\omega)) <$ cov $(\mathcal{B}_2)$ is consistent with ZFC.*

Proof: Let $M \models CH$. Then there exists an ω_1-scale $S = \{ f_\alpha \mid \alpha < \omega_1 \} \in M$. Let $\mathbb{P}$ be the countable support iteration of length ω_2 which adjoins a sequence of ω_2 Prikry-Silver reals iteratively. Then it is easily seen that $M^{\mathbb{P}} \models$ cov $(\mathcal{B}_2) = \omega_2$. By Shelah [6] V 4.3, S remains an ω_1-scale in $M^{\mathbb{P}}$. Thus Theorem 1.5 implies that $c(Sym(\omega)) = \omega_1$.

REFERENCES

[1] Abraham, U., M. Rubin and S. Shelah, "On the consistency of some partition theorems for continuous colorings, and the structure of $\aleph$-dense real order types," *Annals of Pure and Applied Logic*, vol. 29 (1985), pp. 123–206. Zbl 0585.03019 MR 87d:03132 3, 3

[2] Baumgartner, J. E., "Applications of the Proper Forcing Axiom," pp. 913–959 in *Handbook of set-theoretic topology*, edited by K.Kunen and J. Vaughan, North-Holland, Amsterdam, 1984. Zbl 0556.03040 MR 86g:03084 1

[3] Cichoń, J., A. Roskanowski, J. Steprans and B. Węglorz, "Combinatorial properties of the ideal $\mathcal{B}_2$," *Journal of Symbolic Logic*, vol. 58, (1993), pp. 42–54. Zbl 0782.03024 MR 94d:03098 1.9, 1.11, 1, 1

[4] Kunen K., *Set Theory* North-Holland, Amsterdam, 1980. Zbl 0443.03021 MR 82f:03001 1

[5] Macpherson, H. D. and Peter M. Neumann, "Subgroups of infinite symmetric groups," *Journal of the London Mathematical Society (2)* vol. 42 (1990), pp. 64–84. Zbl 0668.20005 MR 92d:20006 1, 1.6

[6] Shelah, S., "Proper Forcing," *Lecture Notes in Mathematics 940*, Springer-Verlag, Berlin, 1982. Zbl 0495.03035 MR 84h:03002

[7] Shelah, S. and S. Thomas, "Implausible subgroups of infinite symmetric groups," *Bulletin of the London Mathematical Society*, vol. 20 (1988), pp. 313–318. Zbl 0644.20009 MR 90a:20009 2

[8] Todorcevic, S., "Partition Problems in Topology," *Contemporary Mathematics*, vol. 84, American Mathematical Society, Providence,1989. Zbl 0659.54001 MR 90d:04001 3

[9] Velickovic, B., "CCC posets of perfect trees," *Compositio Mathematica*, vol. 79 (1991), pp. 279–294. Zbl 0735.03023 MR 92k:03023 4

[10] Velickovic, B., "Forcing Axioms and Stationary Sets," *Advances in Mathematics*, vol. 94 (1992), pp. 256–284. Zbl 0785.03031 MR 93k:03045 1

[11] Velickovic, B., "OCA and automorphisms of $\mathcal{P}(\omega)/fin$," *Topology and its Applications*, vol. 49 (1993), pp. 1–13. Zbl 0785.03033 MR 94a:03080 4

Department of Mathematics
Rutgers University
New Brunswick, N.J. 08903