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Uniformly distributed orbits of certain flows on homogeneous spaces

Nimish A. Shah

School of Mathematics, Tata Institute of Fundamental Research, Homi Bhabha Road, Bombay 400005, India

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1 Introduction

Let G be a connected Lie group, Γ be a lattice in G and $U = \{u_t\}_{t \in \mathbb{R}}$ be a unipotent one-parameter subgroup of G, viz. Adu is a unipotent linear transformation for all $u \in U$. Consider the flow induced by the action of U (on the left) on G/Γ. Such a flow is referred as a unipotent flow on the homogeneous space G/Γ. The study of orbits of unipotent flows has been the subject of several papers. For a nilpotent group G, a result of Green [13] implies that if U has one dense orbit in G/Γ then every orbit of U is uniformly distributed with respect to a G-invariant measure on G/Γ. In the case when $G = SL(2, \mathbb{R})$, it was proved by Hedlund that every orbit of the unipotent (horocycle) flow is either dense or periodic; periodic orbits exist only when G/Γ is non-compact. For a co-compact lattice, this result was strengthened by Fürstenberg [11] proving that every orbit is uniformly distributed with respect to a G-invariant measure. For non-uniform lattices in $SL(2, \mathbb{R})$, using a classification of invariant measures obtained by Dani in [2], Dani and Smillie [3] proved that every non-periodic orbit is uniformly distributed. There are also various results obtained on orbit closures and invariant measures etc. of larger subgroups consisting of unipotent elements, especially the horospherical subgroups. Recently, there was a spurt in the area initiated by Margulis' proof (cf. [15], see also [7]) of Oppenheim conjecture on values of quadratic forms at integral points using the study of unipotent flows. The reader is referred to the survey articles by Dani [4] and Margulis [14] for various related developments.

We now note some conjectures expected to hold for orbits of a unipotent flow, namely the U-action on G/Γ as above (though we restrict to U being a one-parameter subgroup, the first two conjectures are expected to hold for any subgroup generated by unipotent elements contained in it). A conjecture due to Raghunathan on orbit closures states the following:

Conjecture 1. *For every $x \in G/\Gamma$, there exists a closed subgroup F such that $\overline{Ux} = Fx$.*

For $G = SL(2, \mathbb{R})$ the conjecture follows from the result of Hedlund mentioned above. It was recently verified for "generic" unipotent flows in the case of

$G=SL(3,\mathbb{R})$ [8]. Upto certain easy modifications, these are the only cases of semisimple groups where the conjecture is known to hold.

The following measure theoretic analogue of the above conjecture was recently proved by Ratner [19].

Conjecture 2. *For any finite ergodic U-invariant measure σ on G/Γ, there exists a closed subgroup F containing U and $x \in G/\Gamma$ such that Fx is closed and σ is a F-invariant measure supported on Fx.*

In this paper we shall be concerned with the following conjecture which strengthens Conjectures 1 and 2.

Conjecture 3. *For every $x \in G/\Gamma$, there exists a closed subgroup F such that Fx is closed, Fx admits a F-invariant probability measure σ and the U-orbit through x is uniformly distributed with respect to σ; that is, for all bounded continuous functions f on G/Γ,*

$$\lim_{T\to\infty} \frac{1}{T}\int_0^T f(u_t x)dt = \int_{G/\Gamma} f d\sigma .$$

It may be observed that if the U-orbit of $x \in G/\Gamma$ is uniformly distributed with respect to the F-invariant probability measure on Fx as above then $\overline{Ux}=Fx$. Thus Conjecture 3 includes Conjecture 1. Also since for any ergodic invariant measure there are "generic points" [10], Conjecture 3 is stronger than Conjecture 2.

Our object is to prove the following result on the asymptotic behaviour of the orbits of certain unipotent flows. Specifically, we choose G to be a reductive Lie group and let U be a *regular* unipotent one-parameter subgroup of G, in the sense that U is contained in a unique maximal unipotent subgroup of G (see Theorem 4.3 for other equivalent conditions) and prove the following.

(1.1) **Main Theorem.** *Let G be a connected reductive Lie group, Γ be a lattice in G and $U=\{u_t\}_{t\in\mathbb{R}}$ be a regular unipotent one-parameter subgroup of G. Let $X=G/\Gamma$ and let Y be the subset of X consisting of all points y such that Fy is closed for some proper closed subgroup F of G, containing U. Let $x \in X\setminus Y$. Then, given $\varepsilon>0$, there exists a compact subset $K\subset X\setminus Y$ such that for all $T>0$,*

$$\frac{1}{T}\ell\{t\in[0,T] \mid u_t x \in K\} > 1-\varepsilon ,$$

where ℓ is the Lebesgue measure on $\mathbb{R}$.

Together with Ratner's classification of finite ergodic U-invariant measures, the theorem enables us to describe geometrically the set of points whose U-orbits are uniformly distributed with respect to a G-invariant measure; in particular, we are able to conclude the validity of Conjecture 3 for regular unipotent one-parameter subgroups when either G/Γ is compact or the $\mathbb{R}$-rank of $[G,G]$ is 1. The results may be stated as follows:

(1.2) **Corollary.** *Let G, Γ, and U be as in the Main theorem. Then for every $x \in X\setminus Y$, the U-orbit of x is uniformly distributed with respect to the G-invariant probability measure on G/Γ. In particular, all these orbits are dense in G/Γ.*

(1.3) **Corollary.** *Let G, Γ, and U be as in the Main theorem. Suppose further that G/Γ is compact. Then Conjecture 3 holds (for all $x \in X$).*

(1.4) **Corollary.** *Let G be a connected reductive Lie group such that the $\mathbb{R}$-rank of $[G, G]$ is 1. Let Γ be a lattice in G and U be any unipotent one-parameter subgroup of G. Then Conjecture 3 holds (for all $x \in X$).*

Like the earlier results on unipotent flows [14], the above results on uniformly distributed orbits raise certain interesting possibilities of application to Diophantine approximation. In particular, given a nondegenerate indefinite quadratic form Q on $\mathbb{R}^3$ which is not a multiple of a rational quadratic form and $\varepsilon > 0$ one can get lower estimates, for all larger $r \in \mathbb{R}$, for the number of solutions $x \in \mathbb{Z}^3$, $\|x\| < r$ for the inequality $|Q(x)| < \varepsilon$. On account of the somewhat incomplete nature of the results currently obtained and some new ideas involved in the proof, we shall deal with the applications elsewhere.

The method of proof of the Main theorem is an adaptation of the ideas developed in [3] and the Appendix of [8]. In this method one relates thin neighbourhoods of subsets of Y as above to certain subsets of linear G-spaces and uses polynomial behaviour of orbits of one-parameter groups of unipotent linear transformations in vector spaces, to study properties of U-orbits on G/Γ.

The paper is organized as follows. In Sect. 2 we prove that if $x \in G/\Gamma$ and F is the smallest closed subgroup of G containing U such that Fx is closed then Fx admits a finite F-invariant measure and the U-action on Fx is ergodic. This result, which would also be of general interest, is used in Sect. 3 to show that any F as above comes from a special class of subgroups. The conclusion is used in Sect. 5 to give a geometric description of the set Y defined as in the Main theorem. Using this description we complete the proof of the Main theorem in Sect. 6. The Sect. 4 is an independent section devoted to a discussion on regular unipotent elements.

2 Finite volume, ergodicity and Zariski density

Let G be a connected Lie group, Γ be a lattice in G and L be a subgroup such that the unipotent one-parameter subgroups of G contained in L generate L. Let $X = G/\Gamma$. In this section we note some properties related to closed orbits of the form Fx, where $x \in X$ and F is a closed subgroup containing L such that $\overline{Lx} = Fx$.

(2.1) *Notation.* For $x \in X$ and a subgroup $F \subset G$ define

$$F_x = \{g \in F \mid gx = x\}.$$

We first note the following.

(2.2) **Lemma.** *Let F and H be Lie subgroups of G, Z_1 and Z_2 be closed orbits of F and H, respectively in X and let $Z = Z_1 \cap Z_2$. Then every orbit of $F \cap H$ in Z is both open and closed in Z. In particular, for any $x \in X$ there exists a unique smallest Lie subgroup F such that $L \subset F$ and Fx is closed.*

Proof. Let $z \in Z$. Then $Fz = Z_1$ and $Hz = Z_2$ are closed. Therefore, $F/F_z \simeq Fz$ and $H/H_z \simeq Hz$. Also G_z, F_z, and H_z are discrete. Therefore, there exists a neighbourhood Ω of the identity e in G such that $\Omega\Omega^{-1} \cap G_z = \{e\}$, $(Fz \cap \Omega z) = (F \cap \Omega)z$ and $(Hz \cap \Omega z) = (H \cap \Omega)z$. This implies that $(Fz \cap Hz \cap \Omega z) = (F \cap H \cap \Omega)z$. Hence $(F \cap H)z$ is open in $Fz \cap Hz = Z$ for every $z \in Z$. Now $(F \cap H)z$ is closed, because its complement in Z is the union of open $F \cap H$ orbits in Z and Z is closed. □

One of our aims in this section is to prove the following:

(2.3) **Theorem.** *For $x \in X$ let F be the smallest subgroup of G such that $L \subset F$ and Fx is closed. Then*

(a) *F_x is a lattice in F and*

(b) *L acts ergodically on Fx with respect to the F-invariant probability measure. In particular, Fx contains a dense L-orbit.*

We recall some preliminaries and a result due to Margulis before going to the proof of the theorem.

(2.4) **Definition.** A subgroup H of G is said to have *property* (D) if for every locally finite H-invariant measure σ on X, there exist measurable H-invariant subsets X_i, $i \in \mathbb{N}$ such that $\sigma(X_i) < \infty$ for all $i \in \mathbb{N}$ and $X = \bigcup_{i \in \mathbb{N}} X_i$.

In particular, if H has property (D) then every locally finite ergodic H-invariant measure on X is finite.

(2.5) **Proposition** [5, Theorem 4.3]. *Any unipotent subgroup $U \subset G$ has property* (D). □

(2.6) **Definition.** Let F be a topological group, $H \subset F$ and $L \subset F$. We say that the triple (F, H, L) has the *Mautner property* if the following condition is satisfied: for any continuous unitary representation of F on a Hilbert space $\mathscr{H}$, if a vector $\xi \in \mathscr{H}$ is fixed by L then it is also fixed by H.

The following Proposition is a slight modification of Theorem 1.1 in [16].

(2.7) **Proposition.** *Let F be a Lie group and L be a subgroup such that the unipotent one-parameter subgroups contained in L generate L. Then there exists a closed normal subgroup H of F such that* (i) *$L \subset H$ and* (ii) *the triple (F, H, L) has the Mautner property.*

Proof. Let U be a unipotent one-parameter subgroup contained in L. By Theorem 1.1 of [16], there exists a normal subgroup $H_U \subset F$ such that (a) (F, H_U, U) has the Mautner property and (b) the image of $\mathrm{Ad}(U)$ in the automorphism group of the Lie algebra of F/H_U is relatively compact. For each $u \in U$, $\mathrm{Ad}\, u$ is a unipotent transformation of the Lie algebra of F, therefore the image of U in F/H_U is in the center. Hence the group UH_U is normal in F and (F, UH_U, U) has the Mautner property.

Suppose unipotent one-parameter subgroups $U_1, \ldots, U_n$ generate L. Let $H_1, \ldots, H_n$ be normal subgroups of F such that $U_i \subset H_i$ and the triples (F, H_i, U_i) have the Mautner property for all $1 \leqq i \leqq n$. Then $H = \overline{H_1 \ldots H_n}$ satisfies the conditions (i) and (ii). □

The proof of Theorem 2.3 depends on the following observation by Margulis.

(2.8) **Lemma** [14, Remarks 3.12]. *Suppose $H \subset G$ admits a Levi decomposition $H = S \cdot N$, where S is a semisimple group without compact factors and N is the unipotent radical of H. Then H has property* (D).

Proof. Let σ be a locally finite H-invariant measure on X. H admits a left regular unitary representation on $\mathscr{L}^2(X, \sigma)$.

Let W be a maximal unipotent subgroup of S. Then $W \cdot N$ is a unipotent subgroup of G. By Proposition 2.5 there exists a measurable $W \cdot N$ invariant partition $\{X_i\}_{i \in \mathbb{N}}$ of X such that $\sigma(X_i) < \infty$ for all $i \in \mathbb{N}$. If χ_i denotes the characteristic function of X_i then χ_i is a $W \cdot N$ invariant function in $\mathscr{L}^2(X, \sigma)$. By

Proposition 2.7 there exists a normal subgroup Q of G containing $W \cdot N$ such that χ_i is Q invariant for all $i \in \mathbb{N}$. Since S is a semisimple group without compact factors, $S \subset Q$. Hence X_i is H invariant for all $i \in \mathbb{N}$. This completes the proof. □

Now we describe the group theoretic structure of a closed subgroup generated by unipotent one-parameter subgroups.

(2.9) **Lemma.** *Let $H \subset G$ be a closed subgroup which is generated by unipotent one-parameter subgroups contained in it. Then H admits a Levi decomposition $H = S \cdot N$, where S is a semisimple group with no compact factors and N is the unipotent radical of H.*

Proof. It is enough to prove the lemma for the adjoint group of G. Therefore we may assume that $G \subset GL(n, \mathbb{R})$ and its unipotent elements are unipotent linear transformations. By Levi decomposition $H = S \cdot R$, where S is a connected semisimple group and R is the radical of H (cf. [18, Sect. P.1.3]). Suppose H_1 is a normal subgroup of H containing R such that H/H_1 is a compact semisimple group. Note that under a surjective morphism a unipotent element projects to a unipotent element. Since compact semisimple groups contain no nontrivial unipotent elements, by hypothesis $H = H_1$. This shows that S has no compact factors.

To prove the other part we argue as follows; we refer the reader to [18, Preliminaries 2] for the results used in the argument.

Let $\mathbf{H}$ be the smallest algebraic $\mathbb{R}$-subgroup of $GL(n, \mathbb{C})$ containing H. Let $\mathbf{N}$ be the unipotent radical of $\mathbf{H}$. By Levi decomposition there exists a connected semisimple $\mathbb{R}$-subgroup $\mathbf{S} \subset \mathbf{H}$ such that $\mathbf{S} \cdot \mathbf{N}$ is a normal subgroup of $\mathbf{H}$ and $\mathbf{T} = \mathbf{H}/(\mathbf{S} \cdot \mathbf{N})$ is an algebraic $\mathbb{R}$-torus. Now the projection of any unipotent element of $\mathbf{H}$ in $\mathbf{T}$ is unipotent. But any algebraic torus contains only semisimple elements. Hence by hypothesis $H \subset \mathbf{S} \cdot \mathbf{N}$. By minimality of $\mathbf{H}$, $\mathbf{H} = \mathbf{S} \cdot \mathbf{N}$.

Since H normalizes the Lie subalgebra $\mathfrak{r}$ corresponding to its radical R, by definition $\mathbf{H}$ normalizes $\mathfrak{r} \otimes \mathbb{C}$. Hence R is contained in the radical of $\mathbf{H}$. Since the radical of $\mathbf{H}$ is unipotent, R consists of unipotent linear transformations. This completes the proof. □

We also need the following lemma.

(2.10) **Lemma.** *Let F be a Lie group, Λ be a discrete subgroup of F and H be a normal subgroup of F such that $\overline{H\Lambda} = F$. Then H acts ergodically on $(F/\Lambda, \sigma)$, where σ is a locally finite F-semi-invariant measure on F/Λ with the modular function of F as its character (cf.* [18, Sect. 1.4]*).*

Proof. The proof of Lemma 8.2 in [2] goes through as it is, if we replace $\mathcal{L}^2(F/\Lambda, \sigma)$ by the spce of locally integrable functions on $(F/\Lambda, \sigma)$. □

Proof of Theorem 2.3. By Proposition 2.7 there exists a smallest closed normal subgroup H of F containing L such that the triple (F, H, L) has the Mautner property.

Since H is normal in F, HF_x is a subgroup of F. If $H_1 = \overline{HF_x}$ then $H_1 \supset H$ and $H_1 x$ is closed in Fx. By minimality of F, $H_1 = F$. Hence $\overline{HF_x} = F$.

Let H' be the closure of the group generated by all unipotent one-parameter subgroups of G contained in H. Then $L \subset H'$ and H' is normal in F. Therefore, by the hypothesis on H, $H' = H$.

Let σ be a locally finite F-semi-invariant measure on F/F_x with a character Δ_F, where Δ_F is a modular function of F. If $\mathfrak{f}$ is the Lie subalgebra corresponding to F then $\Delta_F(f) = |\det(\mathrm{Ad} f|_{\mathfrak{f}})|$ for all $f \in F$.

Since H is the closure of a subgroup generated by unipotent one-parameter subgroups, $\Delta_F(H)=1$. Therefore, σ is H-invariant. By Lemma 2.10, H acts ergodically on $(F/F_x, \sigma)$. Since Fx is closed, the natural inclusion $F/F_x \hookrightarrow X$ is proper. Therefore, we may treat σ as a locally finite ergodic invariant measure of H on X. By Lemmas 2.8 and 2.9, H has property (D). Hence σ is finite. But a finite F-semi-invariant measure σ must be F-invariant. Now by the Mautner property of the triple (F, H, L), L also acts ergodically on (Fx, σ). □

Our next aim is to show that F as in Theorem 2.3 is contained in the Zariski closure of F_x (cf. Corollary 2.13). We first prove the following result related to Borel's density theorem.

(2.11) **Proposition.** *Let F be a connected Lie group, $\Delta \neq \{e\}$ be a discrete subgroup of F and $U=\{u_t\}_{t\in\mathbb{R}}$ be a one-parameter subgroup of F such that $\overline{U\Delta}=F$. Let $\varrho: F \to GL(E)$ be a finite dimensional representation of F such that $\varrho(U)$ consists of unipotent linear transformations of E. Then every Δ-stable subspace of E is also F-stable.*

Proof (cf. [9, Proposition 9]). Let W be Δ-stable subspace of E. Passing to a suitable exterior power of ϱ, we may assume that $\dim(W)=1$. Let $\bar{\varrho}: F \to PGL(E)$ be the projective linear representation of F on the projective space $\mathbb{P}^1(E)$. Let $w \in W\setminus\{0\}$ and $\varphi: \mathbb{R} \to E$ be the map given by $\varphi(t)=\varrho(u_t)w$ for all $t \in \mathbb{R}$. Fix an orthonormal basis $\{e_1, \ldots, e_n\}$ of E with respect to some inner product. Since $\varrho(U)$ consists of unipotent linear transformations, there exist polynomials $\varphi_1, \ldots, \varphi_n$ on $\mathbb{R}$ such that $\varphi(t)=\sum_{i=1}^{n} \varphi_i(t)e_n$ for all $t \in \mathbb{R}$. Now $\varphi_i^2(t) \Big/ \sum_{j=1}^{n} \varphi_j^2(t)$ converges as $t \to \infty$ for $1 \leqq i \leqq n$. Hence $\lim_{t\to\infty} \varphi(t)/\|\varphi(t)\|=p$ for some $p \in E\setminus\{0\}$.

If $\dim F=1$ then $U=F$ and there exists $t_0 \in \mathbb{R}\setminus\{0\}$ such that $u_{kt_0} \in \Delta$ for all $k \in \mathbb{N}$. Therefore, $\varrho(\Delta)w=w$ and $\varphi(kt_0)=w$ for all $k \in \mathbb{N}$. Since φ is a polynomial function, it must be constant. Thus $\varrho(F)w=w$ in this case.

Suppose $\dim F>1$. Since $\overline{U\Delta}=F$, for every $f \in F$, there exist sequences $\{t_k\}_{k\in\mathbb{N}} \subset \mathbb{R}$, and $\{\delta_k\}_{k\in\mathbb{N}} \subset \Delta$ such that $t_k \to \infty$ and $u_{t_k}\delta_k \to f$ as $k \to \infty$. For $x \in E\setminus\{0\}$, let $\bar{x}$ denote its image in $\mathbb{P}^1(E)$. Since $\bar{\varrho}(\Delta)\bar{w}=\bar{w}$,

$$\bar{\varrho}(f)\bar{w}=\lim_{k\to\infty} \bar{\varrho}(u_{t_k}\delta_k)\bar{w}=\lim_{k\to\infty} \bar{\varrho}(u_{t_k})\bar{w}=\lim_{k\to\infty} \overline{\varphi(t_k)}=\bar{p}\,.$$

Putting $f=$identity we get $\bar{p}=\bar{w}$. Hence $\bar{\varrho}(F)\bar{w}=\bar{w}$. □

(2.12) **Corollary.** *Let F be a connected Lie subgroup of G and L be a subgroup of F generated by unipotent one-parameter subgroups $U_1, \ldots, U_k$ of G. Suppose $\overline{Lx}=Fx$. Let $\varrho: F \to GL(E)$ be a finite dimensional representation such that for each $1 \leqq i \leqq k$, $\varrho(U_i)$ consists of unipotent linear transformations of E. Then every F_x-stable subspace of E is F-stable.*

Proof. Fix i. Let F_i be the smallest closed subgroup of F such that $F_i \supset U_i$ and $F_i x$ is closed. By Theorem 2.3 there exists $g_i \in F_i$ such that $\overline{U_i g_i x}=F_i g_i x=F_i x$. Let $\Delta=F_x$ and $\Delta_i=\Delta\cap F_i$. Then $\overline{U_i g_i \Delta_i g_i^{-1}}=F_i$.

Now let W be a Δ-stable subspace of E. Let $W_i=\varrho(g_i)W$. Then W_i is stabilized by $(g_i\Delta_i g_i^{-1})$. By Proposition 2.11, W_i is stabilized by F_i. Since $g_i^{-1} \in F_i$, we have $W_i=\varrho(g_i^{-1})W_i=W$. Therefore, F_i stabilizes W. This happens for each $i=1, \ldots, k$. Therefore, $\varrho(L)W=W$ and hence $\varrho(L\Delta)W=W$. Now by continuity of ϱ we have $\varrho(F)W=W$. □

(2.13) **Corollary.** *Let the notation be as above. Suppose further that G is the component of identity in $\mathbf{G}_{\mathbb{R}}=\mathbf{G}\cap GL(n,\mathbb{R})$, where $\mathbf{G}\subset GL(n,\mathbb{C})$ is an algebraic* *subgroup defined over* $\mathbb{R}$. *Let F be a connected Lie subgroup of G and L be a subgroup generated by all algebraic unipotent one-parameter subgroups contained in F. Suppose* $x \in X$ *is such that* $\overline{Lx} = Fx$. *Then F is contained in the Zariski closure of* F_x *in* $GL(n, \mathbb{R})$.

Proof. Let P_d be the space of real polynomials of degree $\leqq d$ defined on $M(n, \mathbb{R})$, the space of $n \times n$ matrices with real entries. Consider the representation ϱ of $GL(n, \mathbb{R})$ on P_d defined as follows: for $g \in G$, $p \in P_d$ and $x \in M(n, \mathbb{R})$, $[\varrho(g)p](x) = p(g^{-1}x)$. Clearly, $\varrho(g)p \in P_d$. Since $\varrho: GL(n, \mathbb{R}) \to GL(P_d)$ is an algebraic morphism, ϱ preserves algebraic unipotent subgroups. Thus ϱ restricted to F satisfies the conditions of Corollary 2.12. Define $I_d = \{p \in P_d \mid p(\delta) = 0$ for all $\delta \in F_x\}$. Since F_x is a group, F_x stabilizes I_d. Therefore, by Corollary 2.12, for all $f \in F$ and $p \in I_d$ we have $\varrho(f^{-1})p \in I_d$ and hence $p(f) = [\varrho(f^{-1})p](e) = 0$. Thus $p(f) = 0$ for all $f \in F$, $p \in I_d$ and $d \geqq 0$. This shows that F is in the Zariski closure of F_x in $GL(n, \mathbb{R})$.

3 On subgroups with closed orbits

(3.1) **Proposition.** *Let G be a connected semisimple Lie group without compact factors, with trivial center and of* $\mathbb{R}$*-rank* $= 1$. *Let* Γ *be a lattice in G and* $L \neq \{e\}$ *be a subgroup generated by unipotent elements of G contained in L. Let* $x \in G/\Gamma$ *and suppose that* $\overline{Lx} = Fx$ *for a connected Lie subgroup* $F \subset G$. *Then either*

(a) *F is a reductive group with compact center, or*
(b) *F is a unipotent subgroup of G.*

Proof. $G = \mathbf{G}^0_{\mathbb{R}}$ for a semisimple algebraic $\mathbb{R}$-group $\mathbf{G}$ (cf. [21, Sect. 3.1.6]). Let $\mathbf{F}$ be the smallest algebraic $\mathbb{R}$-subgroup of $\mathbf{G}$ containing F.

Suppose that the unipotent radical $R_u(\mathbf{F})$ is trivial. Then $\mathbf{F}^0_{\mathbb{R}}$ is a reductive group (cf. [18, Prelim. 2.5]). Since the $\mathbb{R}$-rank of G is 1 and the commutator subgroup of $\mathbf{F}^0_{\mathbb{R}}$ is noncompact, this also implies that the center of $\mathbf{F}^0_{\mathbb{R}}$ is compact. Since F is Zariski dense in $\mathbf{F}$, the radical of F is contained in the radical of $\mathbf{F}^0_{\mathbb{R}}$. Hence F is a reductive group with compact center.

Suppose the unipotent radical of $\mathbf{F}^0_{\mathbb{R}}$ is nontrivial. Let N be a maximal unipotent subgroup of G containing $R_u(\mathbf{F}^0_{\mathbb{R}})$. Then $F \subset P = N_G(N)$, the normalizer of N in G (cf. [18, Sect. 12.6]). Since N contains all unipotent elements in P, $L \subset N$. If $F = \overline{L}$ then we are through. Otherwise, since $\overline{Lx} = Fx$, there exist sequences $\{l_i\} \subset L$ and $\{f_i\} \subset F$ such that $l_i \to \infty$, $f_i \to e$ and $l_i x = f_i x$ for all $i \in \mathbb{N}$. Now for all large i, $\gamma_i = f_i^{-1} l_i \in F_x \setminus \{e\}$. Since P is a minimal parabolic subgroup of G, P admits a decomposition $P = Z_G(A) \cdot N$, where A is a Cartan subgroup of G and $Z_G(A)$ is the centralizer of A in G. There exists Y in the Lie algebra of A such that if $q = \exp(-Y)$ then for all $l \in N$, $q^n l q^{-n} \to e$ as $n \to \infty$. Therefore there exists an increasing sequence $\{n_i\}_{i \in \mathbb{N}}$ such that $q^{n_i} \gamma_i q^{-n_i} \to e$ as $i \to \infty$. For each $j \in \mathbb{N}$, put $\gamma'_j = e$ if $j < n_1$, and put $\gamma'_j = \gamma_{n_i}$ if $n_i \leqq j < n_{i+1}$. Then $q^n \gamma'_n q^{-n} \to e$ as $n \to \infty$. Therefore by Lemmas 3.5 and 3.6 in [12], $N \cap G_x \neq \emptyset$. Now by Lemmas 3.16 and 3.17 in [12], $N/N \cap G_x$ is compact, equivalently Nx is compact. This shows that $\overline{Lx} \subset Nx$. Hence $F \subset N$. This completes the proof. □

In view of arithmeticity of irreducible lattices in semisimple groups of rank greater than 1, we formulate the next result in the setup of algebraic groups.

(3.2) **Proposition.** *Let* $G = \mathbf{G}^0_{\mathbb{R}}$ *for an algebraic* $\mathbb{Q}$*-group* $\mathbf{G} \subset SL(n, \mathbb{C})$, $\Gamma = G \cap SL(n, \mathbb{Z})$ *and L be a subgroup generated by algebraic unipotent one-parameter subgroups of G contained in L. Suppose* $\overline{L\Gamma} = F\Gamma$ *for a connected Lie subgroup F of*

G. Let $\mathbf{F}$ *be the smallest algebraic* $\mathbb{Q}$*-group containing L. Then the radical of* $\mathbf{F}$ *is a unipotent algebraic* $\mathbb{Q}$*-group and* $F = \mathbf{F}^0_{\mathbb{R}}$.

Proof. By the arguments as in Lemma 2.9, it follows that the radical of $\mathbf{F}$ is a unipotent algebraic $\mathbb{Q}$-group. Therefore $\mathbf{F}$ admits no nontrivial character defined over $\mathbb{Q}$. Let $\Lambda = SL(n, \mathbb{Z})$ and $\Delta = \mathbf{F}^0_{\mathbb{R}} \cap \Lambda$. Then the natural inclusion map $\mathbf{i}: \mathbf{F}^0_{\mathbb{R}}/\Delta \to SL(n, \mathbb{R})/\Lambda$ is proper (cf. [18, Sect. 10.15]). Let $\mathbf{j}: G/\Gamma \to SL(n, \mathbb{R})/\Lambda$ be the natural inclusion map. Now $\mathbf{F}^0_{\mathbb{R}}\Gamma = \mathbf{j}^{-1}(\mathbf{F}^0_{\mathbb{R}}\Lambda)$ is closed in G. Therefore $F \subset \mathbf{F}^0_{\mathbb{R}}$. Hence $\overline{L\Delta} = F\Delta$ in $\mathbf{F}^0_{\mathbb{R}}/\Delta$. Since $\mathbf{i}$ is proper, $\overline{L\Lambda} = F\Lambda$.

Note that Λ is a lattice in $SL(n, \mathbb{R})$ (cf. [18, Sect. 10.5]). Let $\mathbf{F}'$ be the Zariski closure of Δ in $\mathbf{G}$. Therefore $\mathbf{F}'$ is defined over $\mathbb{Q}$ (cf. [21, Sect. 3.1.8]). By Corollary 2.13, $F \subset \mathbf{F}'$. Since $L \subset F$, by minimality of $\mathbf{F}$ we have $\mathbf{F}' = \mathbf{F}$. Hence F is Zariski dense in $\mathbf{F}$.

By Levi decomposition $F = S \cdot R$, where S is a semisimple group and R is the radical of F. Let $\mathbf{S}$ and $\mathbf{R}$ be the smallest algebraic $\mathbb{R}$-groups containing S and R, respectively. Now $\mathbf{S}$ normalizes $\mathbf{R}$ and $\mathbf{R}$ is a solvable group. Therefore $\mathbf{F} = \mathbf{S} \cdot \mathbf{R}$. Since the radical of $\mathbf{F}$ is unipotent, $\mathbf{S}$ is a semisimple group and $\mathbf{R}$ is a unipotent group. Therefore $R = \mathbf{R}^0_{\mathbb{R}}$ (cf. [18, Sect. P.2.2]). Now S is a connected normal Zariski dense subgroup of the semisimple group $\mathbf{S}^0_{\mathbb{R}}$. Hence $S = \mathbf{S}^0_{\mathbb{R}}$. Thus $F = \mathbf{S}^0_{\mathbb{R}} \cdot \mathbf{R}^0_{\mathbb{R}} = \mathbf{F}^0_{\mathbb{R}}$. □

(3.3) *Notation.* For a semisimple Lie group G and a parabolic subgroup P, define ${}^0P = \{g \in P \mid \det(\mathrm{Ad}\, g|_{\mathfrak{w}}) = 1\}$, where $\mathfrak{w}$ is the Lie subalgebra corresponding to the unipotent radical of P.

(3.4) **Lemma.** *Let* $G = \mathbf{G}^0_{\mathbb{R}}$ *for a semisimple algebraic* $\mathbb{Q}$*-group* $\mathbf{G}$, $\Gamma = G \cap SL(n, \mathbb{Z})$ *and* $U = \mathbf{U}^0_{\mathbb{R}}$ *for a unipotent algebraic* $\mathbb{Q}$*-subgroup* $\mathbf{U} \subset \mathbf{G}$. *Then there exists a parabolic* $\mathbb{Q}$*-subgroup* $\mathbf{P} \subset \mathbf{G}$ *such that if* $P = \mathbf{P}^0_{\mathbb{R}}$ *and* $W = \mathrm{R}_u(P)$ *then* (a) $U \subset W$ *and* $\mathrm{N}_G(U) \subset P$, (b) $W/W \cap \Gamma$ *is compact and* (c) ${}^0P\Gamma$ *is closed.*

Proof. Let $\mathbf{P}$ be the maximal algebraic $\mathbb{Q}$-subgroup of $\mathbf{G}$ such that $U \subset \mathrm{R}_u(\mathbf{P})$ and $\mathrm{N}_G(U) \subset \mathbf{P}$. Let $\mathbf{W} = \mathrm{R}_u(\mathbf{P})$. Then $\mathbf{W}$ is a $\mathbb{Q}$-group, $\mathbf{P}_1 = \mathrm{N}_{\mathbf{G}}(\mathbf{W})$ is a $\mathbb{Q}$-group, $\mathbf{P} \subset \mathbf{P}_1$ and $\mathbf{W} \subset \mathrm{R}_u(\mathbf{P}_1)$. By the maximality $\mathbf{P}_1 = \mathbf{P}$. Therefore $\mathbf{P}$ is a parabolic subgroup of $\mathbf{G}$ (cf. [18, Sect. 12.8]). This proves (a).

Let $W = \mathbf{W}^0_{\mathbb{R}}$. Since $\mathbf{W}$ is defined over $\mathbb{Q}$, $W/W \cap \Gamma$ is compact (cf. [18, Sect. 2.13]).

Let $\mathfrak{g}$ and $\mathfrak{w}$ be Lie algebras corresponding to G and W, respectively. Let $r = \dim(\mathfrak{w})$ and $\varrho: G \to GL(\wedge^r \mathfrak{g})$ be the r-th exterior of the Adjoint representation $\mathrm{Ad}: G \to GL(\mathfrak{g})$. Let $\{e_1, \ldots, e_r\}$ be a basis of $\mathfrak{w}$ and put $p = e_1 \wedge \ldots \wedge e_r$. Then for all $a \in P$, $ap = (\mathrm{Ad}\, a)e_1 \wedge \ldots \wedge (\mathrm{Ad}\, a)e_r = (\det \mathrm{Ad}\, a|_{\mathfrak{w}})p$. Therefore, ${}^0P = \{g \in G \mid \varrho(g)p = p\}$.

Since $\mathbf{G}$ and $\mathbf{W}$ are defined over $\mathbb{Q}$, $\mathfrak{g}$ and $\mathfrak{w}$ admit compatible rational structures. Now $\wedge^r \mathfrak{g}$, $\wedge^r \mathfrak{w}$ and ϱ are defined over $\mathbb{Q}$. Fix a rational basis of $\wedge^r \mathfrak{g}$. For a nonzero rational vector $p \in \wedge^r \mathfrak{w}$, let $\alpha: G \to \wedge^r \mathfrak{g}$ be the map defined by $\alpha(g) = \varrho(g)p$ for all $g \in G$. Then α is defined over $\mathbb{Q}$. Therefore $\alpha(\Gamma)$ consists of rational points with bounded denominators in each co-ordinate. Hence $\alpha(\Gamma)$ is discrete and $\Gamma {}^0P = \alpha^{-1}(\alpha(\Gamma))$ is closed. Therefore ${}^0P\Gamma$ is closed. □

The next lemma is useful when we want to change a lattice by a commensurable one.

(3.5) **Lemma.** *Let G be a Lie group and* Γ *and* Γ' *be discrete subgroups of G. Suppose* $\Gamma' \subset \Gamma$ *and* $[\Gamma : \Gamma'] < \infty$. *Let* $\varphi: X' = G/\Gamma' \to X = G/\Gamma$ *be the natural quotient map. Let* $x' \in X'$, $x = \varphi(x') \in X$ *and F and H be closed connected subgroups G. Then the* *following statements hold.*

1. $\overline{Fx'} = Hx' \Rightarrow \overline{Fx} = Hx$.
2. $\overline{Fx} = Hx \Rightarrow Hx'$ *is closed and* $\overline{Fx'}$ *contains an open subset of* Hx'.
3. *If* Γ *is a lattice in* G *and* F *is generated by unipotent one-parameter subgroups of* G *then* $\overline{Fx} = Hx \Rightarrow \overline{Fx'} = Hx'$.

Proof. Since $[\Gamma : \Gamma'] = n < \infty$, the map φ is proper. If $\overline{Fx'} = Hx'$ then by properness of φ, $\overline{Fx} = \overline{\varphi(Fx')} = \varphi(\overline{Fx'}) = \varphi(Hx') = Hs$. This proves 1).

Let $\Gamma'' = \bigcap_{\gamma \in \Gamma} \gamma \Gamma' \gamma^{-1}$. Then Γ'' is a normal subgroup of finite index in Γ. Therefore in view of 1), we may assume that Γ' is normal in Γ for proving 2) and 3). Therefore Γ acts on X' from the right and $X = X'/\Gamma$.

Suppose $\overline{Fx} = Hx$. Let $Z' = \varphi^{-1}(Hx)$ and $E' = \overline{Fx'} \subset Z'$. Now $\varphi(E') = \overline{Fx} = Hx = \varphi(Hx')$. Hence there exits $\{\gamma_1, \ldots, \gamma_n\} \subset \Gamma$ such that $\bigcup_{i=1}^{n} Hx'\gamma_i = Z' = \bigcup_{i=1}^{n} E'\gamma_i$. This shows that Hx' and E' contain open subsets of Z'. Therefore $Hx'\gamma_i$ is open in Z' for each i and hence Hx' is closed in Z'. This proves 2).

Assume the hypothesis in 3). Then Γ' is also a lattice in G. Now since $\overline{Fx'}$ contains an open subset of Hx', H is the smallest Lie subgroup of G such that $F \subset H$ and Hx' is closed. By Theorem 2.3, Hx' contains a dense orbit of F, which must intersect $\overline{Fx'}$. Therefore $\overline{Fx'} = Hx'$. This proves 3). □

(3.6) **Definition.** Let G be a Lie group. We call discrete subgroups Γ and Γ' of G *commensurable* if $[\Gamma : \Gamma \cap \Gamma'] < \infty$ and $[\Gamma' : \Gamma \cap \Gamma'] < \infty$.

(3.7) *Remark.* The Proposition 3.2 and Lemma 3.4 hold when Γ is commensurable with $G \cap SL(n, \mathbb{Z})$. To show this use Lemma 3.5 and Theorem 2.3.

The next lemma is useful when we want to factor agroup a group by a compact normal subgroup.

(3.8) **Lemma.** *Let* $\varrho : G \to G_1$ *be a surjective homomorphism of Lie groups* G *and* G_1. *Let* Γ *be a discrete subgroup of* G. *Suppose* $\Gamma_1 = \varrho(\Gamma)$ *is a lattice in* G_1 *and the canonical quotient map* $\bar{\varrho} : X = G/\Gamma \to X_1 = G_1/\Gamma_1$ *is proper. Let* L *be a subgroup of* G *generated by unipotent one-parameter subgroups contained in* L. *Suppose* ker(ϱ) *normalizes* L. *Let* $L_1 = \varrho(L)$, $x \in X$ *and* $x_1 = \bar{\varrho}(x) \in X_1$. *Then* $\overline{Lx} = Fx$ *for a connected Lie subgroup* F *of* G *if and only if* $\overline{L_1 x_1} = F_1 x_1$ *for a connected Lie subgroup* F_1 *of* G_1. *In this case* $F_1 = \varrho(F)$.

Proof. Since $\bar{\varrho}$ is proper, Γ must be a lattice in G.

If $\overline{Lx} = Fx$ then by properness of ϱ, $\overline{L_1 x_1} = \bar{\varrho}(Lx) = \bar{\varrho}(\overline{Lx}) = \bar{\varrho}(Fx) = \varrho(F)x_1$.

Suppose $\overline{L_1 x_1} = F_1 x_1$. If $H = \varrho^{-1}(F_1)$ then $Hx = \bar{\varrho}^{-1}(F_1 x_1)$ is closed. Let F be the smallest Lie subgroup of H such that $F \supset L$ and Fx is closed. Now $F_1 x_1 = \bar{\varrho}(Hx) \supset \bar{\varrho}(Fx) \supset \bar{\varrho}(\overline{Lx}) = \overline{L_1 x_1} = F_1 x_1$. Therefore since F and F_1 are connected, we have $F_1 = \varrho(F)$. By Theorem 2.3 there exists $y \in Fx$ such that $\overline{Lx} = Fx$. Since $\bar{\varrho}(\overline{Lx}) = \bar{\varrho}(Fx)$ there exists $h \in \ker \varrho$ such that $hy \in \overline{Lx}$. Since ker(ϱ) normalizes L, $\overline{Lx} \supset \overline{Lhy} = h\overline{Ly} = hFx = (hFh^{-1})hy$. Therefore $hFh^{-1} \subset F$ and by dimension consideration $hFh^{-1} = F$. Hence $\overline{Lx} = Fx$. □

(3.9) **Lemma.** *Let* G *be a connected semisimple Lie group without compact factors, with trivial center and of* $\mathbb{R}$*-rank* > 1. *Let* Γ *be an irreducible lattice in* G *and* L *be a subgroup generated by unipotent one-parameter subgroups of* G *contained in* L. *Suppose* $\overline{L\Gamma} = F\Gamma$ *for a connected Lie group* $F \subset G$. *Then the radical of* F *is a unipotent subgroup of* G. *Moreover, if the radical of* F *is nontrivial then there exists a proper parabolic subgroup* P *of* G *such that* (a) $F \subset {}^0P$, (b) ${}^0P\Gamma$ *is closed in* G/Γ, *and* (c) $W/W \cap \Gamma$ *is compact, where* $W = \mathrm{R_u}(P)$.

Proof. By the arithmeticity theorem of Margulis (cf. [21, Theorem 6.1.2]), Γ is an arithmetic lattice in G. That is, there exists a semisimple algebraic $\mathbb{Q}$-group $\mathbf{H}$ and a surjective homomorphism $\varrho: H \to G$ such that $\ker(\varrho)$ is compact and $\varrho(\Lambda)$ is commensurable with Γ, where $H = \mathbf{H}_{\mathbb{R}}^0$ and $\Lambda = H \cap SL(n, \mathbb{Z})$. By Lemma 3.5, there no loss of generality in assuming that $\varrho(\Lambda) = \Gamma$.

There is a normal semisimple subgroup $H_1 \subset H$ such that $\varrho(H_1) = G$ and $\ker(\varrho) \cap H_1$ is discrete. Note that H_1 contains all unipotent one-parameter subgroups of H and $\ker(\varrho)$ commutes with H_1. There exists a subgroup $L' \subset H_1$ such that L' is generated by algebraic unipotent one-parameter subgroups contained in it and $\varrho(L') = L$. If $\bar{\varrho}: H/\Lambda \to G/\Gamma$ is the natural quotient map then $\bar{\varrho}$ is proper. By Lemma 3.8, there exists a subgroup $F' \subset H$ such that $\varrho(F') = F$ and $\overline{L'\Lambda} = F'\Lambda$.

By Proposition 3.2, the radical of F' is $U' = \mathbf{U}_{\mathbb{R}}'^0$ for a unipotent $\mathbb{Q}$-subgroup $\mathbf{U}' \subset \mathbf{H}$. Since ϱ is surjective and $\ker(\varrho)$ is compact, the radical of F is also a unipotent subgroup of G.

If U' is not trivial then by Lemm 3.4 there exists a proper parabolic subgroup P' of H such that a) $L' \subset N_H(U') \subset P'$, b) ${}^0P'\Lambda$ is closed in H/Λ, and c) $W'/W' \cap \Lambda$ is compact, where $W' = \mathrm{R_u}(P')$. Since L' is generated by unipotent one-parameter subgroups, $L' \subset {}^0P'$ and hence $F' \subset {}^0P'$. If $P = \varrho(P')$ then P is a proper parabolic subgroup of G with the required properties. □

(3.10) **Proposition.** *Let G be a connected semisimple Lie group without compact factors and with trivial center. Let Γ be a lattice in G and L be a subgroup generated by unipotent one-parameter subgroups contained in L. If $x \in G/\Gamma$ and $\overline{Lx} = Fx$ for a connected Lie subgroup $F \subset G$ then one of the following possibilities holds.*

1. *F is a reductive group with compact center.*
2. *There is a proper parabolic subgroup P of G such that* (a) $F \subset {}^0P$, (b) ${}^0Px$ *is closed and* (c) $\mathrm{R_u}(P)x$ *is compact.*

Proof. Let $G_x = \Lambda$. For G there exists a direct product decomposition $G = G_1 \ldots G_n$ such that $\Lambda_k = G_k \cap \Lambda$ is an irreducible lattice in G_k for $1 \leqq k \leqq n$ and $[\Lambda : \Lambda_1 \ldots \Lambda_n] < \infty$ (cf. [18, Sect. 5.22]). By Lemma 3.5, without loss of generality we may assume that $\Lambda = \Lambda_1 \ldots \Lambda_n$. Then $G/\Lambda \simeq G_1/\Lambda_1 \times \ldots \times G_n/\Lambda_n$. Let $\varphi_k: G \to G_k$ be the projection homomorphism of G onto G_k and $\bar{\varphi}_k: G/\Lambda \to G_k/\Lambda_k$ be the natural projection. Let $x_k = \bar{\varphi}_k(x)$, $L_k = \varphi_k(L)$ and $F_k = \varphi_k(F)$. By Theorem 2.3, Fx supports a finite F invariant measure σ. Now the projection of σ on $F_k x_k$ is a finite F_k invariant measure. Therefore $F_k \cap \Lambda_k$ is a lattice in F_k. Hence $F_k x_k$ is closed in G_k/Λ_k (cf. [18, Sect. 1.13]). Now $\overline{L_k x_k} = F_k x_k$.

Let R be the radical of F. Then $R_k = \varphi_k(R)$ is the radical of F_k. If F_k is reductive for all k then by Proposition 3.1 or Lemma 3.9, R_k is compact and abelian for all k. Hence R is compact and abelian. In this case 1) holds.

Now suppose F_k is not reductive for some k. Then by Proposition 3.1 or Lemma 3.9, there exists a parabolic subgroup $P_k \subsetneqq G_k$ such that (a) $F_k \subset {}^0P$, (b) ${}^0Px_k$ is closed and (c) $\mathrm{R_u}(P_k)x_k$ is compact. If $P = \left(\prod_{j \neq k} G_j\right) P_k$ then P is a parabolic subgroup of G, ${}^0P = \left(\prod_{j \neq k} G_j\right) {}^0P$, and $\mathrm{R_u}(P) = \mathrm{R_u}(P_k)$. Now (a) $F \subset {}^0P$, (b) ${}^0Px = \bar{\varphi}_k^{-1}({}^0P_k x_k)$ is closed and (c) in view of the natural inclusion $G_k/\Lambda_k \hookrightarrow G/\Lambda$, $\mathrm{R_u}(P)x = \mathrm{R_u}(P_k)x_k$ is compact. □

4 Regular unipotent elements

(4.1) **Definition.** Let G be a connected reductive Lie group. A unipotent element of G is called *regular* if it is contained in a unique maximal unipotent subgroup of G. A unipotent one-parameter subgroup of G is called *regular* if it contains a regular unipotent element.

(4.2) **Lemma.** *Let $\varrho: G \to G'$ be a homomorphism of connected reductive Lie groups. Let $u \in G$ be a unipotent element.*

1. *If ϱ is surjective and u is regular in G then $\varrho(u)$ is regular unipotent in G'.*
2. *If $\ker \varrho \subset Z(G)$ and $\varrho(u)$ is regular unipotent in G' the u is regular in G.*

Proof. 1) follows immediately from the definition.

Suppose $\ker \varrho \subset Z(G)$. Since G is connected and reductive, $G = Z(G)S$, where S is a connected semisimple group. Now any maximal unipotent subgroup of G is of the form $Z(G)N$, where N is a maximal unipotent subgroup of S. Since S is semisimple and connected, $\varrho(N)$ is a unipotent subgroup of G' (cf. [21, Sect. 3.4.2]). Now 2) easily follows from the definition. □

(4.3) **Theorem** (cf. [20, Theorem 3.7]). *Let G be a connected semisimple Lie group with trivial center, A be a Cartan subgroup of G and $P_0 = MAN$ be a minimal parabolic subgroup of G, where M is a maximal compact connected subgroup of $Z_G(A)$ and N is a maximal unipotent subgroup of G. Let u be a unipotent element of G. Then the following statements are equivalent.*

(a) *u is regular.*

(b) *u belongs to a unique conjugate of a given parabolic subgroup of G.*

(c) *If $u \in N$ and $u = \exp\left(\sum_{\alpha \in R^+} X_\alpha\right)$ then for every simple root α, $X_\alpha + X_{2\alpha} \neq 0$.*

Here R^+ is the set of positive roots associated to the parabolic pair (P_0, A) and X_α is an element of the α-root space of A.

Proof. We refer to [18, Chap. 12] for the results used in this proof. First note the following. $G = \mathbf{G}_{\mathbb{R}}^0$ for an algebraic $\mathbb{R}$-group $\mathbf{G}$. If P is a parabolic subgroup of G then $P = \mathbf{P} \cap G$. Let $W = R_u(P)$. There exists a semisimple $\mathbb{R}$-subgroup $\mathbf{S} \subset \mathbf{P}$ such that if $S = \mathbf{S}_{\mathbb{R}}^0$ then any unipotent element of P is contained in $S \cdot W$ (cf. [18, Prelim. 2.5]). Therefore any maximal unipotent subgroup of P is of the form $V \cdot W$, where V is a maximal unipotent subgroup of S. Since S is a semisimple group, any two maximal unipotent subgroups of S are conjugates. Therefore, any two maximal unipotent subgroups of P are conjugate by an element of P. Moreover, any maximal unipotent subgroup of G is conjugate to a subgroup of P.

Now assume (a). Let N' be a maximal unipotent subgroup of G containing u. Suppose there exist $g_1, g_2 \in G$ such that $g_1 u g_1^{-1}, g_2 u g_2^{-1} \in P$. By regularity of $g_i u g_i^{-1}$ and the observations made above, $g_i N' g_i^{-1} \subset P$ for $i = 1, 2$. Now there exists $p \in P$ such that $g_1 N' g_1^{-1} = p g_2 N' g_2^{-1} p^{-1}$. Since the normalizer of a maximal unipotent subgroup of P is contained in P, we have $p g_2 g_1^{-1} \in P$. Hence $g_1^{-1} P g_1 = g_2^{-1} P g_2$. Thus (a) $\Rightarrow$ (b).

Assume the contrary to (c). Suppose $u \in P_0$, $u = \exp\left(\sum_{\alpha \in R^+} X_\alpha\right)$ and for some simple root β, $X_\beta + X_{2\beta} = 0$. Let $X_{-\beta} \neq 0$ be an element of the $(-\beta)$-root space. If $\alpha \in R^+$ and $\alpha \neq \beta$ or 2β then for every $k \in \mathbb{N}$, either $\alpha - k\beta \in R^+$ or $\alpha - k\beta$ is not a root. Since $\operatorname{ad}^k X_{-\beta}(X_\alpha)$ belongs to the $(\alpha - k\beta)$-root space, we have $\operatorname{Ad} g\left(\sum_{\alpha \in R^+} X_\alpha\right) = \sum_{\alpha \in R^+} Y_\alpha$, where $g = \exp(X_{-\beta})$ and Y_α belongs to the α-root space. Therefore, $g u g^{-1} \in P_0$. Hence $u \in P_0 \cap g P_0 g^{-1}$. But $g \notin P_0$. This contradicts (b). Thus (b) $\Rightarrow$ (c).

Let $u \in N$ be such that $u = \exp\left(\sum_{\alpha \in R^+} X_\alpha\right)$ and $X_\beta + X_{2\beta} \neq 0$ for every simple root β. Suppose for some $g \in G$, $u \in g N g^{-1}$. By Bruhat decomposition of G, $g = p_1 w^* p_2$

where $p_1, p_2 \in P_0$ and $w^* \in \mathrm{N}_G(A)$ represents a Weyl group element $w \in \mathrm{W}(G, A)$. Therefore $gug^{-1} = (p_2^{-1} w^{*-1} p_1^{-1}) u (p_1 w^* p_2) \in N$. Hence $w^{*-1}(p_1^{-1} u p_1) w^* \in P_0$. Now $u' = p_1 u p_1^{-1} = \exp\left(\sum_{\alpha \in R^+} X'_\alpha\right) \in P_0$. Suppose $p_1 = amn$, where $a \in A$, $m \in M$ and $n \in N$. Then for every simple root β, $X'_\beta = \mathrm{Ad}(am) X_\beta$ and when $X_\beta = 0$, $X'_{2\beta} = \mathrm{Ad}(am) X_{2\beta}$. Now $w^{*-1} u' w^* = \exp\left(\sum_{\alpha \in R^+} Y'_{w(\alpha)}\right) \in P_0$, where $Y'_{w(\alpha)} = (\mathrm{Ad}\, w^*) X'_\alpha$ belongs to the $w(\alpha)$-root space of A. Hence $w(\alpha) \in R^+$ unless $X'_\alpha = 0$. But $X'_\beta + X'_{2\beta} \neq 0$ for every simple β; therefore w stabilizes the positive Weyl chamber. Since the Weyl group acts simply transitively on the Weyl chambers of A, w is an identity element of $\mathrm{W}(G, A)$. Therefore $w^* \in \mathrm{Z}_G(A) \subset P_0$. Hence $gNg^{-1} = N$. Since any two maximal unipotent subgroups of G are conjugate, (c) $\Rightarrow$ (a). □

(4.4) *Remark.* Suppose G is a semisimple group of $\mathbb{R}$-rank 1. Then in the notation of the Theorem 4.3, R^+ contains only one simple root. Therefore, any non-trivial unipotent element of G is regular.

(4.5) **Lemma.** *Let G be a semisimple group with trivial center and let $H \subset G$ be a reductive algebraic subgroup of G. If H contains a regular unipotent element of G then $\mathrm{Z}_G(H)$ is compact.*

Proof. Let the notation be as in Theorem 4.3. Let $u \in H$ be a regular unipotent element of G. We may assume that $u \in N$. Then $\mathrm{Z}_G(u) \subset P_0$. By Theorem 4.3(c), it is easy to see that $\mathrm{Z}_G(u) \subset MN$. Therefore $\mathrm{Z}_G(H) \subset MN$. But $\mathrm{Z}_G(H)$ is reductive group (cf. [18, Sect. P.2.6]). Hence it must be compact. □

5 Union of lower dimensional homogeneous closures

Let G be a semisimple Lie group with trivial center and no compact factors, Γ be a lattice in G and $U = \{u_t\}_{t \in \mathbb{R}}$ be a regular unipotent one-parameter subgroup of G.

Let $X = G/\Gamma$ and $Y = \{y \in X \mid Fy$ is closed for a connected Lie subgroup F of G such that $\dim F < \dim G$ and $U \subset F\}$.

Take $y \in Y$ and let F be the smallest Lie subgroup of G such that $U \subset F$ and Fy is closed. Then $\dim F < \dim G$. By Theorem 2.3 there exists $x \in Fy$ such that $\overline{Ux} = Fx = Fy$. By Proposition 3.10, either 1) F is a connected reductive group with compact center or 2) there exists a proper parabolic subgroup P of G such that (a) $U \subset {}^0P$, (b) ${}^0Px$ is closed and (c) $\mathrm{R}_u(P)x$ is compact.

Consider the first possibility. Let F^{nc} denote the maximal connected normal semisimple subgroup of F with no compact factors. Let $H = \mathrm{N}_G(F^{\mathrm{nc}})$. Then H is an open subgroup of $\mathbb{R}$-points of a real algebraic group. Hence H has finitely many components (cf. [18, Prelim. 2.4]). Since F^{nc} is connected and semisimple, $H^0 \subset \mathrm{Z}_G(F^{\mathrm{nc}}) F^{\mathrm{nc}}$. By Lemma 4.5, $\mathrm{Z}_G(F^{\mathrm{nc}})$ is compact. Therefore Hx is closed and $H^{\mathrm{nc}} = F^{\mathrm{nc}}$.

Let $\mathscr{R}$ be the set of all pairs of the form (H, x), where $x \in Y$ and H is a reductive subgroup of G such that (a) $U \subset H$, (b) Hx is closed, (c) $\dim H < \dim G$ and (d) there exists a connected reductive subgroup F with compact center such that $H = \mathrm{N}_G(F^{\mathrm{nc}})$ and $\overline{Ux} = Fx$.

Let $\mathscr{P}$ be the set of all pairs of the form $({}^0P, x)$, where $x \in Y$ and P is a parabolic subgroup of G such that (a) $U \subset H$, (b) Hx is closed, (c) $\dim H < \dim G$, and (d) there

(5.1) *Remark.* Let $\mathscr{F} = \mathscr{P} \cup \mathscr{R}$. Then from the above discussion,

$$Y = \bigcup_{(F, x) \in \mathscr{F}} Fx.$$

(5.2) **Lemma.** *There is a countable subset* $\mathscr{F}^*$ *of* $\mathscr{F}$ *such that if* $(F',x')\in\mathscr{F}$ *then there exist* $g\in G$ *and* $(F,x)\in\mathscr{F}^*$ *such that* $F'=gFg^{-1}$ *and* $x'=gx$.

Proof. Let $\mathscr{D}$ be the set of connected Lie subgroups of G such that if $A\in\mathscr{D}$ then (i) either A is a reductive group with compact center or A is unipotent subgroup of G and (ii) $A\cap\Gamma$ is a Zariski dense lattice in A. In both cases A is the identity component of a real algebraic group. Moreover $A\cap\Gamma$ is finitely generated (cf. [18, Sects. 2.10, 13.20, 13.25]). Thus A is completely determined by finitely many elements of Γ. Hence $\mathscr{D}$ is countable.

Let $(H_1,x_1),(H_2,x_2)\in\mathscr{R}$. Let F_1 and F_2 be closed connected subgroups such that $\overline{Ux_1}=F_1x_1$ and $\overline{Ux_2}=F_2x_2$. Let $g_1,g_2\in G$ be such that $x_1=g_1\Gamma$ and $x_2=g_2\Gamma$. Then by the definition of $\mathscr{R}$ and Corollary 2.13, $g_1F_1g_1^{-1}, g_2F_2g_2^{-1}\in\mathscr{D}$. Suppose if $g_1F_1g_1^{-1}=g_2F_2g_2^{-1}$ then $g_1\mathrm{N}_G(F_1^{\mathrm{nc}})g_1^{-1}=g_2\mathrm{N}_G(F_2^{\mathrm{nc}})g_2^{-1}$. By definition $H_1=\mathrm{N}_G(F_1^{\mathrm{nc}})$ and $H_2=\mathrm{N}_G(F_2^{\mathrm{nc}})$. If we put $g=g_1g_2^{-1}$ then $H_1=gH_2g^{-1}$ and $x_1=gx_2$. Thus by countability of $\mathscr{D}$ we can choose a countable subset $\mathscr{R}^*$ of $\mathscr{R}$ with the property that for every $(F',x')\in\mathscr{R}$, there exits $g\in G$ such that $(g^{-1}F'g, g^{-1}x')\in\mathscr{R}^*$.

Take $({}^0P,x)\in\mathscr{P}$. Then $\mathrm{R_u}(P)x$ is compact. Take $g\in G$ such that $x=g\Gamma$. Then by Theorem 2.1 in [18], $g\mathrm{R_u}(P)g^{-1}\in\mathscr{D}$. Since P is a parabolic subgroup, $P=\mathrm{N}_G(\mathrm{R_u}(P))$. Now argue as in the above case. □

(5.3) *Notation.* For a Lie subgroup F of G containing U define

$$L(F)=\{g\in G\,|\,gFg^{-1}\supset U\}.$$

Note that $\mathrm{N}_G(U)L(F)\mathrm{N}_G(F)=L(F)$.

(5.4) *Remark.* Let $({}^0P,x)\in\mathscr{P}$. If $g\in L({}^0P)$ then $U\subset P\cap gPg^{-1}$. Since U contains a regular unipotent element of G, by Theorem 4.3, $L({}^0P)=P$.

(5.5) *Notations.* Define $\mathscr{F}_0=\mathscr{P}\cap\mathscr{F}^*$ and for all $n\in\mathbb{N}$ define

$$\mathscr{F}_n=\{(H,x)\in\mathscr{R}\cap\mathscr{F}^*\,|\,\dim(H^{\mathrm{nc}})=n\}.$$

Put $Y_{-1}=\emptyset$ and for all $n\in\mathbb{N}\cup\{0\}$ define

$$Y_n=Y_{n-1}\cup\Big(\bigcup_{(F,x)\in\mathscr{F}_n}L(F)x\Big).$$

The next corollary is a direct consequence of the above discussion.

(5.6) **Corollary.** $Y_n=Y$ *for all* $n\geqq\dim G-1$. □

6 Proof of the main theorem

We follow the notations of Sect. 5. To prove the main theorem (for G as above) it is enough to prove the following.

(6.1) **Theorem.** *Let* $n\in\mathbb{N}\cup\{0,-1\}$. *Given* $\varepsilon>0$ *and a point* $y\in X\setminus Y_n$ *there exists a compact subset* $K\subset X\setminus Y_n$ *such that for all* $T>0$,

$$(1/T)\ell\{t\in[0,T]\,|\,u_ty\in K\}>1-\varepsilon.$$

For $n=-1$, that is when $Y_n=\emptyset$, the theorem is essentially proved in [5, 6] and the above form is deduced in Proposition 1.8 of [8]. For $n=0$ it is proved in the Appendix of [8]. In this paper we exploit the techniques of [8] to study the case of $n\in\mathbb{N}$. We shall assume the theorem for $n=-1$ and give a proof by induction on $n\in\mathbb{N}\cup\{0\}$.

(6.2) **Lemma.** *For each $(F, x) \in \mathscr{F}$ there is a finite dimensional real vector space E equipped with a linear G action and a point $p \in E$ such that*

(a) *F is the isotopy subgroup of p,*

(b) *if $(F, x) \in \mathscr{R}$ then Gp is closed, and*

(c) *if $(F, x) = ({}^0P, x) \in \mathscr{P}$ then $ap = \det(\operatorname{Ad} a|_{\mathbf{w}})p$ for all $a \in P$, where $\mathbf{w}$ is the Lie subalgebra corresponding to $R_u(P)$.*

Proof. First note that $G = \mathbf{G}^0_{\mathbb{R}}$ for an algebraic $\mathbb{R}$-group $\mathbf{G}$. Suppose $(F, x) \in \mathscr{R}$. By the definition $F = G \cap \mathbf{H}$ for a reductive algebraic $\mathbb{R}$-subgroup $\mathbf{H} \subset \mathbf{G}$. Now Proposition 7.7 in [1] provides such a representation.

For $(F, x) = ({}^0P, x) \in \mathscr{P}$ such a representation is constructed explicitly in the proof of Lemma 3.4. □

(6.3) **Lemma.** *Let $({}^0P, x) \in \mathscr{P}$ and (E, p) be a linear G-space with a distinguished vector as in Lemma 6.2. Let $W = R_u(P)$ and $\mathbf{w}$ be the Lie subalgebra corresponding to W. Then the following holds.*

1. *If $a \in P$ then $\operatorname{Vol}(W/W_{ax}) = \det(\operatorname{Ad} a|_{\mathbf{w}})|\operatorname{Vol}(W/W_x)$, the volumes of the quotients being understood to be relative to a fixed Haar measure on W.*
2. *If $ax = a'x$ for some $a, a' \in P$ then $ap = \pm a'p$.*
3. *For a sequence $\{a_i\}_{i \in \mathbb{N}} \subset P$, if $a_i p \to 0$ as $i \to \infty$ then no subsequence of $\{a_i x\}_{i \in \mathbb{N}}$ converges in X.*

Proof. Let σ be a Haar measure on W. Let $\mathscr{S}$ be a fundamental domain for the lattice W_x in W. Then $\operatorname{Vol}(W/W_x) = \sigma(\mathscr{S})$. Since $W_{ax} = aW_x a^{-1}$, the set $a\mathscr{S}a^{-1}$ is a fundamental domain for W_{ax}. Therefore $\operatorname{Vol}(W/W_{ax}) = \sigma(a\mathscr{S}a^{-1}) = |\det(\operatorname{Ad} a|_{\mathbf{w}})|\sigma(\mathscr{S})$. Hence 1) holds.

Now 2) is an immediate consequence of 1) and the condition (c) of Lemma 6.2.

Now suppose that $a_i p \to 0$. Then by 1), $\operatorname{Vol}(W/W_{a_i x}) \to 0$. Since $W_{a_i x} = a_i W_x a_i^{-1}$, there exists a sequence $\{\gamma_i\}_{i \in \mathbb{N}} \subset W_x \subset G_x$, such that $\gamma_i \neq e$ and $a_i \gamma_i a_i^{-1} \to e$. Hence $\{a_i x\}_{i \in \mathbb{N}}$ is divergent in X (cf. [18, Sect. 1.12]). This proves 3). □

(6.4) *Remark.* In the notation of Lemma 6.3, $0 \in \overline{Pp}$. Therefore if X is compact then $\mathscr{P} = \emptyset$.

The following lemma will enable us to apply induction in the proof of Theorem 6.1.

(6.5) **Lemma.** *Let $(H, x) \in \mathscr{R}$. Define*

$$Z = \{z \in Y \mid z \in l_1 Hx \cap l_2 Hx \text{ and } l_1 H \neq l_2 H \text{ for some } l_1,\ l_2 \in L(H)\}.$$

Then $Z \subset Y_{n-1}$, where $n = \dim(H^{\mathrm{nc}})$.

Proof. Let $z \in l_1 Hx \cap l_2 Hx$ be such that $l_1 H \neq l_2 H$, where $l_1, l_2 \in L(H)$. Now $l_i Hx = (l_i H l_i^{-1})(l_i x)$ for $i = 1, 2$. Since Hx is closed, by Lemma 2.2 if $F = (l_1 H l_1^{-1}) \cap (l_2 H l_2^{-1})$ then Fz is closed. By the definition of $L(H)$, $U \subset F$. Let F_0 be the smallest Lie subgroup of G such that $U \subset F_0$ and $F_0 z$ is closed. If F_0 is not reductive then by definition $z \in Y_0$.

Suppose F_0 is reductive. Since $F_0 \subset F$, we have $F_0^{\mathrm{nc}} \subset (l_1 H^{\mathrm{nc}} l_1^{-1})$. If $\dim(F_0^{\mathrm{nc}}) < \dim(H^{\mathrm{nc}})$ then $z \in Y_{n-1}$.

Now suppose $\dim(F_0^{\mathrm{nc}}) = \dim(H^{\mathrm{nc}})$. Since F_0^{nc} and H^{nc} are connected, $F_0^{\mathrm{nc}} = l_1 H^{\mathrm{nc}} l_1^{-1} = l_2 H^{\mathrm{nc}} l_2^{-1}$. Therefore, $l_1^{-1} l_2 \in N_G(H^{\mathrm{nc}}) = H$, a contradiction. This completes the proof. □

The following notations are used in Sect. 6.8 to Sect. 6.13.

(6.6) *Notations.* Let $(F, x) \in \mathscr{F}$ and (E, p) be a linear G-space with a distinguished vector, satisfying the properties mentioned in Lemma 6.2.

Let

$$L=\{v\in E \mid Uv=v\}$$

be the fixed point space of U in E. For a subset $S\subset E$ define

$$S(x)=\{gx\in X \mid g\in G \text{ and } gp\in S\}.$$

(6.7) *Remark.* If $g\in G$ and $gp\in L$ then $U(gp)=gp$. Therefore $g^{-1}Ug\subset F$ and hence $g\in L(F)$. Thus $L(x)=L(F)x$.

(6.8) **Lemma.** *Let $\Delta=\{(gx, gF)\in X\times G/F \mid g\in G\}$. Let $\psi:\Delta\to X\times E$ be the map defined by $\psi(gx, gF)=(gx, gp)$ for all $g\in G$. Then ψ is injective and proper.*

Proof. First note that ψ is well defined and injective. Also ψ is G-equivariant with respect to the obvious G-actions.

Suppose $(F,x)\in\mathscr{R}$. In this case Gp is closed. Therefore the map $\alpha: G/F\to E$ defined by $\alpha(gF)=gp$ for all $g\in G$ is proper. Hence ψ is proper.

Suppose $(F,x)=({}^0P,x)\in\mathscr{P}$. By Iwasawa decomposition there is a maximal compact subgroup K_0 of G such that $G=K_0P$. Let $\Delta_P=\{(ax, a{}^0P)\in\Delta \mid a\in P\}$. Then $\Delta=K_0\Delta_P$. Let $\psi_r:\Delta_P\to X\times E$ be the restriction of ψ to Δ_P. Since K_0 is compact and ψ is G-equivariant, it is enough to show that ψ_r is proper. Since $\overline{Pp}=Pp\cup\{0\}$, the properness of ψ_r follows immediately from 3) of Lemma 6.3. □

(6.9) **Corollary.** *For $y\in X$, the set $S=\{gp\in E \mid g\in G$ and $y=gx\}$ is closed.*

Proof. Since Fx is closed, the set G_xF is closed in G. If $y=gx$ for some $g\in G$ then the set $\Delta_y=(y, gG_xF)\subset\Delta$ is closed. For the map ψ as in Lemma 6.8, $\psi(\Delta_y)=(y,S)$. Since ψ is proper, S is closed. □

(6.10) **Lemma.** *Let $\varphi: G/F_x\to X\times E$ be the map defined by $\varphi(gF_x)=(gx, gp)$ for all $g\in G$. Then φ is proper.*

Proof. Let $\pi: G/F_x\to X\times G/F$ be the map defined by $\pi(gF_x)=(gx, gF)$ for all $g\in G$. Then $\varphi=\psi\circ\pi$. In view of Lemma 6.8, it is enough to prove that π is proper.

Let $\{g_i\}\subset G$ be a sequence such that $\{g_ix\}$ and $\{g_iF\}$ are convergent. Then there are sequences $\{\delta_i\}\subset G_x$ and $\{f_i\}\subset F$ such that $\{g_i\delta_i\}$ and $\{g_if_i\}$ converge in G. Therefore $\{f_i^{-1}\delta_i\}=\{(g_if_i)^{-1}(g_i\delta_i)\}$ converges in G. Hence $f_i^{-1}x$ converges in X. Since Fx is closed, the map $\mathbf{i}: F/F_x\to X$ given by $\mathbf{i}(fF_x)=fx$ for all $f\in F$, is a homeomorphism on to its image. Therefore $\{f_i^{-1}F_x\}$ converges in F/F_x and hence $\{g_iF_x\}=\{(g_if_i)f_i^{-1}F_x\}$ converges in G/F_x. This shows that π is proper. □

(6.11) **Proposition.** *Let $n\in\mathbb{N}\cup\{0\}$ and $(F,x)\in\mathscr{F}_n$. Let K be a compact subset of $X\setminus Y_{n-1}$ and C be a compact subset of L. Then there exists a neighbourhood Ω of C in E such that if $g, g'\in G$, $gx=g'x\in K$ and $gp, g'p\in\Omega$ then $gp=\pm g'p$.*

Proof. Suppose this is not true. Then there exist sequences $\{g_i\}$ and $\{g_i'\}$ contained in G and points $y\in K$ and $c, c'\in C$ such that the following holds. (i) For each $i\in\mathbb{N}$, $y_i=g_ix=g_i'x\in K$ and $g_ip\pm g_i'p$. (ii) As $i\to\infty$, $y_i\to y$, $g_ip\to c$ and $g_i'p\to c'$.

Since the map φ as in Lemma 6.10 is proper, the sequences $\{g_iF_x\}$ and $\{g_i'F_x\}$ have convergent subsequences in G/F_x. Passing to subsequences and replacing g_i and g_i' by appropriate elements of g_iF_x, and $g_i'F_x$ we may assume that $g_i\to g$ and $g_i'\to g'$ for some $g, g'\in G$. Now $g_i^{-1}g_i'\in G_x$ for all $i\in\mathbb{N}$. Since $g_i^{-1}g_i'\to g^{-1}g'$ and G_x is discrete, there exists $\delta\in G_x$ such that $g_i'=g_i\delta$ for all large i. Since $g_ip\neq\pm g_i'p=\pm g_i\delta p$, we have $\delta p\neq\pm p$. Hence $gp\neq\pm g'p$. Since $gp=c$, $g'p=c'\in L$, by Remark 6.7, $g, g'\in L(F)$. Also $y=gx=g'x$.

Suppose $(F,x)=({}^0P,x)\in\mathscr{P}$. By Remark 5.4, $g, g'\in L({}^0P)=P$. Now $gx=g'x$ but $gp\neq\pm g'p$. This contradicts 2) of Lemma 6.3.

Suppose $(F,x)=(H,x)\in\mathscr{R}$. Since $g, g'\in L(H)$, $y\in gHx\cap g'Hx$ and $gH\neq g'H$, by Lemma 6.5, $y\in Y_{n-1}$. This contradicts the fact that $y\in K$ and $K\cap Y_{n-1}=\emptyset$. This completes the proof. □

The following Lemma about polynomial growth is very useful in studying the dynamical behaviour of orbits of a unipotent flow in a vector space.

(6.12) **Lemma** [8, Lemma A.8]. *Let $n\in\mathbb{N}$ and $\varepsilon>0$ be given. Then for any $\alpha>0$ there exists a $\beta\in(0,\alpha)$ and for any $\beta>0$ there exists an $\alpha>\beta$ such that the following condition is satisfied: if φ is a polynomial on $\mathbb{R}$ of degree atmost n such that $|\varphi(0)|\geqq\alpha$ then there exists $t\in(1, 1+\varepsilon)$ such that $|\varphi(t)|\geqq\beta$.* □

(6.13) **Proposition.** *Let $n\in\mathbb{N}\cup\{0\}$ and $(F,x)\in\mathscr{F}_n$. Let $y\in X\setminus Y_n$, K be a compact subset of $X\setminus Y_{n-1}$ and C be a compact subset of L. Then, given $\varepsilon>0$, there exists a neighbourhood Ψ of C in E such that for all $T>0$,*

$$\ell\{t\in[0,T]\mid u_t y\in\Psi(x)\cap K\}<\varepsilon T.$$

Proof. Let $L^\perp$ be a complementary subspace of L in E. Let $c=\dim L$, $d=\dim L^\perp$ and $m=c+d=\dim E$. Let $\{e_1,\ldots,e_c\}$ be a basis of L and $\{f_1,\ldots,f_d\}$ be a basis of $L^\perp$. For $r>0$ define

$$I(r)=\left\{\sum_{i=1}^{c}\xi_i e_i\,\middle|\,|\xi_i|<r \text{ for all } 1\leqq i\leqq c\right\}\subset L$$

and

$$J(r)=\left\{\sum_{j=1}^{d}\xi_j f_j\,\middle|\,|\xi_j|<r \text{ for all } 1\leqq j\leqq d\right\}\subset L^\perp.$$

Let $\beta>0$ be such that $I(\beta)\supset C$. By Lemma 6.12 there exists $\alpha>\beta$ such that if φ is a polynomial on $\mathbb{R}$ of degree atmost m and if $|\varphi(0)|\geqq\alpha$ then there exists $t\in(1,1+\varepsilon/(2m^2))$ such that $|\varphi(t)|\geqq\beta$.

Let $C'=\overline{I(\alpha)}$. By Proposition 6.11 there exists a neighbourhood Ω' of C' in E such that if $g, g'\in G$, $gx=g'x\in K$ and $gp, g'p\in\Omega'$, then $gp=\pm g'p$. Since $y\notin Y_n\supset L(x)$, by Corollary 6.9 there exists $a>0$ such that if $\Omega=I(\alpha)\times J(a)$ then $y\notin\Omega(x)$ and $\bar{\Omega}\subset\Omega'$.

By Lemma 6.12 there exists $0<b<a$ such that if φ is a polynomial on $\mathbb{R}$ of degree atmost m and if $|\varphi(0)|\geqq a$ then there exists $t\in(1,1+\varepsilon/(2m^2))$ such that $|\varphi(t)|\geqq b$. Let $\Psi=I(\beta)\times J(b)$. Then $\Psi\subset\Omega$ and Ψ is a neighbourhood of C in E.

For $g\in G$ and $T>0$ define,

$$A(g)=\{t\in(0,T)\mid u_t(gp)\in\Omega\}.$$

Note that $A(g)$ is a union of open intervals.

Step 1. If $gp\notin\Omega$, $I=(r,s)$ is a connected component of $A(g)$ and $s'\in(r,s]$ then

$$\ell\{t\in[r,s']\mid u_t(gx)\in\Psi(x)\cap K\}<\varepsilon(s'-r).$$

Proof. Since $gp\notin\Omega$, $u_r(gp)\in\bar{\Omega}\setminus\Omega$. Therefore replacing g by $u_r g$, we may assume that $r=0$. Now if $t\in[0,s]$ and $u_t(gx)\in\Psi(x)\cap K$ then $u_t(gp)\in\Omega$ and there exists $g'\in G$ such that $g'x=(u_t g)x$ and $g'p\in\Psi$. By the choice of Ω, $(u_t g)p=\pm g'p$. Since Ψ is symmetric about the origin, $u_t(gp)\in\Psi$. Therefore

$$\ell\{t\in[0,s']\mid u_t(gx)\in\Psi(x)\cap K\}<\ell\{t\in[0,s']\mid u_t(gp)\in\Psi\}. \tag{1}$$

Note that U acts by unipotent linear transformations on E (cf. [21, Sect. 3.4.2]). Therefore there exist polynomials $\varphi_1,\ldots,\varphi_c$ and $\psi_1,\ldots,\psi_d$ on $\mathbb{R}$ of degree atmost

$m(=\dim E)$ such that for all $t\in\mathbb{R}$,

$$u_t(gp)=\sum_{i=1}^{c}\varphi_i(t)e_i+\sum_{j=1}^{d}\psi_j(t)f_j.$$

Let $A_1=\{t\in(0,s')\mid u_t(gp)\in\Psi\}$. Then A_1 is open. Let (a_0,b_0) be a connected component of A_1. Since $gp\notin\Psi$, $u_{a_0}(gp)\in\bar{\Psi}\setminus\Psi$. Therefore there exist $i\in\{1,\dots,c\}$ such that $|\varphi_i(a_0)|=\beta$ or $j\in\{1,\dots,d\}$ such that $|\psi_j(a_0)|=b$. Hence A_1 consists of atmost $2m^2$ connected components. Now $A_1=(a_1,b_1)\cup\dots\cup(a_l,b_l)$, where $l\leqq 2m^2$ and $b_k\leqq a_{k+1}$ for $1\leqq k<l$.

Take $k\in\{1,\dots,l\}$. Since $u_r(gp)\notin\Omega$, there exist $i\in\{1,\dots,c\}$ such that $|\varphi_i(0)|\geqq\alpha$ or $j\in\{1,\dots,d\}$ such that $|\psi_j(0)|\geqq a$. Now by the choice of α and b, there exists $t\in(1,1+\varepsilon/(2m^2))$ such that $\varphi_i(ta_k)\geqq\beta$ or $\psi_j(ta_k)\geqq b$. Therefore, $0<b_k<ta_k$. Hence $(b_k-a_k)<(\varepsilon/(2m^2))a_k<(\varepsilon/(2m^2))s'$. Thus

$$\ell\{t\in[0,s']\mid u_t(gp)\in\Psi\}=\ell(A_1)=\sum_{k=1}^{l}(b_k-a_k)<\varepsilon s'. \tag{2}$$

Since $r=0$, Step 1 follows from (1) and (2).

Let

$$G_{x,y}=\{g\in G\mid gx=y\}.$$

Fix $T>0$. Let $\mathscr{C}$ be the collection of pairs of the form (I,g), where $g\in G_{x,y}$ and I is a connected component of $A(g)$. For an interval $I=(r,s)\subset\mathbb{R}$ define $J(I)=(r,s']$, where

$$s'=\sup\{t\in[r,s]\mid u_ty\in K\}.$$

Step 2. We can choose a subcollection $\mathscr{C}_0\subset\mathscr{C}$ satisfying the following conditions.

1. $\{t\in[0,T]\mid u_ty\in\Psi(x)\cap K\}\subset\bigcup_{(I,g)\in\mathscr{C}_0}J(I)$.
2. *For $(I_1,g_2),(I_2,g_2)\in\mathscr{C}_0$ if $(I_1,g_1)\neq(I_2,g_2)$ then $J(I_1)\cap J(I_2)=\emptyset$.*

Proof. For $t>0$ if $u_ty\in K\cap\Psi(x)$ then there exists $(I,g)\in\mathscr{C}$ such that $t\in J(I)$. Suppose $J(I_1)\cap J(I_2)\neq\emptyset$ for some $(I_1,g_1),(I_2,g_2)\in\mathscr{C}$. Then there exists $t_0\in\bar{I_1}\cap\bar{I_2}$ such that $u_{t_0}y\in K$ and $u_{t_0}g_1p, u_{t_0}g_2p\in\bar{\Omega}\subset\Omega'$. By the choice of Ω', we have $u_{t_0}(g_1p)=\pm u_{t_0}(g_2p)$. Thus $u_t(g_1)p=\pm u_t(g_2)p$ for all $t\in\mathbb{R}$. Since Ω is symmetric about the origin, $A(g_1)=A(g_2)$ and hence $I_1=I_2$. Now it is clear how to choose $\mathscr{C}_0\subset\mathscr{C}$ so that 1) and 2) are satisfied.

Thus by Step 1 and Step 2,

$$\ell\{t\in[0,T]\mid u_ty\in\Psi(x)\cap K\}\leqq\varepsilon\Big(\sum_{(I,g)\in\mathscr{C}_0}\ell(J(I))\Big)\leqq\varepsilon T.\quad\square$$

Proof of Theorem 6.1. For $n=-1$ the theorem is same as Proposition 1.8 in [8].

Take $n\in\mathbb{N}\cup\{0\}$. Then by Notations 5.5 and Remark 6.7,

$$Y_n=Y_{n-1}\cup\Big(\bigcup_{(F,x)\in\mathscr{F}_n}L(x)\Big).$$

Now $\mathscr{F}_n$ is countable and for each $(F,x)\in\mathscr{F}_n$ the set L is a countable union of compact subsets. Therefore, for each $i\in\mathbb{N}$ we can choose a 3-tuple (F_i,x_i,C_i) such that (a) $(F_i,x_i)\in\mathscr{F}_n$, (b) C_i is a compact subset of L_i, and (c) $\bigcup_{(F,x)\in\mathscr{F}_n}L(x)=\bigcup_{i\in\mathbb{N}}C_i(x_i)$.

By induction there exists a compact subset $K'\subset X\setminus Y_{n-1}$ such that for all $T>0$,

$$\ell\{t\in[0,T]\mid u_ty\in K'\}>(1-\varepsilon/2)T. \tag{3}$$

By Proposition 6.13, for each $i \in \mathbb{N}$ there exists a neighbourhood Ψ_i of C_i in E_i such that for all $T>0$,

$$\ell\{t\in[0,T] \mid u_t y\in \Psi_i(x)\cap K'\}<(\varepsilon/2^{i+1})T. \tag{4}$$

Let $K=K'\setminus\left(\bigcup_{i\in\mathbb{N}}\Psi_i(x)\right)$. Then $K\subset X\setminus Y_n$. By (3) and (4),

$$(1/T)\ell\{t\in[0,T] \mid u_t y\in K\}>1-\varepsilon/2-\sum_{i\in\mathbb{N}}\varepsilon/2^{i+1}=1-\varepsilon. \quad \square$$

(6.14) *Reduction to semisimple case.* Let G be a connected reductive Lie group; that is, the adjoint action of G on its lie algebra is completely reducible. Let G^* be the adjoint group of G. Then G^* admits a direct product decomposition $G^*=C\cdot G'$, where C and G' are normal (and hence semisimple) subgroups of G^*, C is compact and G' is a semisimple group with trivial center and no compact factors. Let $\varrho: G\to G'$ be the projection homomorphism of G onto G'.

Let Γ be a lattice in G. Let $\bar{\varrho}: X=G/\Gamma\to X'=G'/\varrho(\Gamma)$ be the map defined by $\bar{\varrho}(\Gamma)=\varrho(g)\varrho(\Gamma)$ for all $g\in G$.

(6.15) **Lemma** [2, Lemma 9.1]. *$\varrho(\Gamma)$ is a lattice in G' and $\bar{\varrho}$ is proper.*

Let U be a regular unipotent one parameter subgroup of G. Then $U'=\varrho(U)$ is a regular unipotent one parameter subgroup of G'. Let $Y=\{y\in X \mid Fy$ is closed for a connected Lie subgroup F of G such that $\dim F<\dim G$ and $U\subset F\}$. Let $Y'\subset X'$ be similarly defined with respect to $U'\subset G'$.

(6.16) **Lemma.** (i) $\bar{\varrho}^{-1}(Y')\subset Y$. (ii) *If $Y\neq X$ then $\bar{\varrho}(Y)=Y'$.*

Proof. (i) follows from properness of $\bar{\varrho}$.

Now let $y\in Y$. Let F be a connected Lie subgroup of G such that $U\subset F$, $\dim F<\dim G$ and Fy is closed. Since $\bar{\varrho}$ is a proper map, if $F'=\varrho(F)$ and $y'=\bar{\varrho}(y)$ then $F'y'=\bar{\varrho}(Fy)$ is closed. Also $U'\subset F'$. To show that $y'\in Y'$ we need to show that $\dim F'<\dim G'$.

Suppose $\dim F'=\dim G'$. Since G' is connected, $F'=G'$. Therefore $G=F\cdot\ker\varrho$. Since $\ker\varrho$ commutes with U, $U\subset gFg^{-1}$ for all $g\in G$. Also $(gFg^{-1})(gy)=gFy$ is closed for all $g\in G$. Therefore $gy\in Y$ for all $g\in G$. Hence $\bar{\varrho}(Y)\subset Y'$ if $Y\neq X$. $\square$

Proof of the main theorem 1.1. Let ϱ, G', $U'=\{u'_t\}_{t\in\mathbb{R}}$, X', Y' etc. be as defined earlier. Let $x'=\bar{\varrho}(x)$. Then $x'\in X'\setminus Y'$ by Lemma 6.16. By Corollary 5.6 and Theorem 6.1, there exists a compact subset $K'\subset X'\setminus Y'$ such that for all $T>0$,

$$(1/T)\ell\{t\in[0,T] \mid u'_t x'\in K'\}>1-\varepsilon.$$

Let $K=\bar{\varrho}^{-1}(K')$. Then K is compact and by Lemma 6.16, $K\subset X\setminus Y$. Now $u_t x\in K$ if and only if $u'_t x'\in K'$. Hence for all $T>0$,

$$(1/T)\ell\{t\in[0,T] \mid u_t y\in K\}>1-\varepsilon. \quad \square$$

7 Deductions

(7.1) **Corollary.** *Let G be a connected reductive Lie group, Γ be a lattice in G and $U=\{u_t\}_{t\in\mathbb{R}}$ be a regular unipotent one-parameter subgroup of G. Let $x\in X=G/\Gamma$ and F be the smallest Lie subgroup of G containing U such that Fx is closed. If F is a reductive group then the U-orbit through x is uniformly distributed with respect to the F-invariant propability measure supported on Fx.*

Proof. By Theorem 2.3, F_x is a lattice in F. By Lemma 4.2, U is a regular unipotent one-parameter subgroup of F. Suppose if $F\neq G$ then $\dim F<\dim G$ and since

$F/F_x \simeq Fx$, the corollary will follow by induction. Now in the notation of the main theorem, we may assume that $x \in X \setminus Y$.

Let $\bar{X} = X \cup \{\infty\}$ be the one point compactification of X. For $T > 0$ let ν_T be the measure on $\bar{X}$ such that for all continuous functions f on $\bar{X}$,

$$\int_{\bar{X}} f d\nu_T = \frac{1}{T} \int_0^T f(u_t x) dt .$$

Note that the space of Borel probability measures on a compact second countable space is compact with respect to the weak* topology. Now to prove uniform distribution of the $\{u_t\}$-orbit through x, it is enough to prove that whenever for a sequence $T_i \to \infty$ the sequence of measures ν_{T_i} converges (in the weak* topology), the limit measure ν is supported on X and is G-invariant.

First we claim that $\nu(Y \cup \{\infty\}) = 0$. Let $\varepsilon > 0$ be given. By the main theorem, there exists a compact set $K \subset X \setminus Y$ such that for all $T > 0$,

$$(1/T)\ell\{t \in [0, T] \mid u_t y \in K\} > 1 - \varepsilon .$$

Now $\Omega = \bar{X} \setminus K$ is a neighbourhood of $Y \cup \{\infty\}$ and $\nu_T(\Omega) < \varepsilon$ for all $T > 0$. Hence $\nu(\Omega) < \varepsilon$. This proves the claim.

It is easy to see that ν is U-invariant. Therefore there exist a partition of X into U-invariant subsets X_C, $C \in \xi$, probability measures π_C on X_C and a probability measure π on ξ such that (a) for almost all $C \in \xi$, π_C is U-ergodic invariant, and (b) for any measurable $A \subset X$, $A \cap X_C$ is measurable for almost all $C \in \xi$ and $\nu(A) = \int_{\xi} \pi_C(A \cap C) d\pi(C)$ (cf. [2, Sect. 1.4]). Since $\nu(Y) = 0$, we have $\pi_C(Y) = 0$ for almost all $C \in \xi$. By Ratner's Theorem [19], the preceeding observation implies that for almost all C, π_C is G-invariant. Hence ν is invariant under the action of G. This completes the proof. □

(7.2) *Remark.* Corollary 1.2 is a particular case of Corollary 7.1.

Proof of Corollary 1.3. Let the notation be as in Sect. 6.14. Let $x \in X$ and $x' = \bar{\varrho}(x) \in X'$. Let F be the smallest Lie subgroup of G such that $U \subset F$ and Fx is closed. If F' is the smallest Lie subgroup of G' such that $U' \subset F'$ and $F'x'$ is closed then $F_1 = \varrho(F)$. Now X' is compact. Therefore by Remark 6.4, F' is reductive. Hence by the definition of ϱ, F is reductive. Now apply Corollary 7.1 to complete the proof. □

Proof of Corollary 1.4. Let the notations be as in Sect. 6.14. For $x \in X$ let x', F and F' be defined as in the proof of Corollary 1.3. If F' is reductive then F is reductive and we can apply Corollary 7.1 to complete the proof.

Otherwise by Proposition 3.1, F' is a unipotent subgroup of G' and $F'x'$ is compact. Since $F' = \varrho(F)$, it follows that $F = C \cdot W$, where C is a connected compact normal semisimple subgroup of F and W is the nilpotent radical of F. Therefore $U \subset W$ and $C \subset Z_G(W)$. Since W is normal in F, $R = \overline{WF_x}$ is a subgroup of F and $\overline{Wx} = R^0 x$. By a theorem of Auslander (applied to F/F_x), R^0 is a solvable group (cf. [18, Sect. 8.24]). From the structure of F it is clear that R^0 is actually a nilpotent group and by the definition of F, $R^0 = F$. Now the uniform distribution of the U-orbit through x follows from a result of Green (cf. [13], see also [17, Theorem 5]) about flows on compact nilmanifolds. This completes the proof. □

(7.3) *Remark.* Let $G = SL(3, \mathbb{R})$, Γ be a lattice in G and U be any unipotent one parameter subgroup of G. Then using the method of this paper, it is not very difficult to verify the validity Conjecture 3 in this case.

(7.4) *Remark.* Let $G=SO(n,1)$, Γ be a lattice in G and L be a closed subgroup generated by unipotent one-parameter subgroups contained in L. In this case using the methods of [15, 7, 8] and the method of the proof of the main theorem the author is able to verify Conjecture 1 (for L in place of U), without using the classification of invariant measures involved in the proof of the above corollaries.

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Note added in proof. Subsequent to writing this paper the author received a preprint from M. Ratner, where Conjecture I has been settled.