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UNIFORMLY FAT SETS

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ABSTRACT. In this paper we study closed sets E which are "locally uniformly fat" with respect to a certain nonlinear Riesz capacity. We show that E is actually "locally uniformly fat" with respect to a weaker Riesz capacity. Two applications of this result are given. The first application is concerned with proving Sobolev-type inequalities in domains whose complements are uniformly fat. The second application is concerned with the Fekete points of E.

Introduction. Let $x = (x_1, x_2, \ldots, x_n)$ be a point in Euclidean n space, $\mathbf{R}^n$, with $|x|$ the norm of x. For $\alpha > 0$, let $I_\alpha(x) = |x|^{(\alpha-n)}$ denote the Riesz kernel of order α and put

$$I_\alpha * f(x) = \int_{\mathbf{R}^n} |x-y|^{(\alpha-n)} f(y)\,dy, \qquad x \in \mathbf{R}^n,$$
$$I_\alpha * \mu(x) = \int_{\mathbf{R}^n} |x-y|^{(\alpha-n)}\,d\mu(y), \qquad x \in \mathbf{R}^n,$$

when f is a Lebesgue integrable function and μ a Borel measure on $\mathbf{R}^n$. Here dy denotes Lebesgue measure on $\mathbf{R}^n$. If $E \subseteq \mathbf{R}^n$, $0 < \alpha p < n$ and $p > 1$, define the (α, p) outer Riesz capacity of E by

$$R_{\alpha,p}(E) = \inf\{\|f\|_p^p \colon I_\alpha * f \geq 1 \text{ on } E\},$$

where $\|f\|_p$ is the Lebesgue p norm of f. We note that $\alpha = 1$, $p = 2$, is the classical Newtonian capacity. If $\alpha p = n$, we let

$$R_{\alpha,p}(E) = \inf\{\|f\|_p^p \colon J_\alpha * f \geq 1 \text{ on } E\},$$

where J_α is the truncated Riesz kernel defined by

$$J_\alpha(x) = |x|^{(\alpha-n)} - (100)^{(\alpha-n)}, \qquad |x| \leq 100,$$
$$= 0, \qquad |x| > 100.$$

We shall say that a property holds (α, p) quasi everywhere (abbreviated (α, p) q.e.) on a set E if it holds on E except perhaps for a set of $R_{\alpha,p}$ capacity zero. Next we list some properties of Riesz capacities. To simplify the writing we state these properties only for $0 < \alpha p < n$. However (A)–(C) also hold when $\alpha p = n$ provided I_α is replaced by J_α. If E is a compact set, then

(A) $R_{\alpha,p}(E) = \sup\{\nu(E)^p \colon \operatorname{supp}\nu \subseteq E \text{ and } \|I_\alpha * \nu\|_{p'} \leq 1\}$, for $p' = p/(p-1)$,

(B) $R_{\alpha,p}(E) = \sup\{\nu(E) \colon I_\alpha * (I_\alpha * \nu)^{1/(p-1)} \leq 1 \text{ on } \operatorname{supp}\nu \subseteq E\}$,

(C) there exists a measure μ for which $P = I_\alpha * (I_\alpha * \mu)^{1/(p-1)} \geq 1$, (α, p) q.e., on E and $P = 1$, (α, p) q.e., on $\operatorname{supp}\mu \subseteq E$. Moreover,

$$\mu(E) = \|I_\alpha * \mu\|_{p'}^{p'} = R_{\alpha,p}(E).$$

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(D) If $B(x,r)=\{y\colon |y-x|<r\}$, then $R_{\alpha,p}[B(x,r)]\sim r^{(n-\alpha p)}$, where $\sim$ means the ratio of the two functions is bounded above and below by a positive constant,

(E) If H^m denotes m-dimensional Hausdorff measure on $\mathbf{R}^n$, then

(i) $H^{(n-\alpha p)}(E)<\infty \to R_{\alpha,p}(E)=0$,

(ii) $H^{(n-\alpha p+\varepsilon)}(E)>0 \to R_{\alpha,p}(E)>0$, $\varepsilon>0$,

(F) If $\beta>0$, $q>1$, and either $\beta q<\alpha p$ or $\beta q=\alpha p$ and $\beta<\alpha$, then

$$R_{\beta,q}(E)^{(n-\alpha p)} \le A R_{\alpha,p}(E)^{(n-\beta q)},$$

where A is a positive constant independent of E.

Note from (F) that $R_{\alpha,p}$ is a stronger capacity than $R_{\beta,q}$ in the sense that $R_{\alpha,p}(E)=0 \to R_{\beta,q}(E)=0$. For the proof of (A), (C), (D) and (E) see [**14**]. (B) is given in [**2**] and (F) can be found in [**3**]. Finally we mention that a general survey of nonlinear potential theory as well as an outline of the results in this paper can be found in [**13**].

In classical potential theory a set can be defined to be fat at a point in several equivalent ways. The most natural definition for nonlinear potential theory turns out to be: A set E is said to be (α,p) fat at x for $0<\alpha p<n$ if

$$\int_0^1 [r^{(\alpha p-n)} R_{\alpha,p}(E\cap B(x,r))]^{1/(p-1)} \frac{dr}{r} = +\infty.$$

Otherwise E is said to be (α,p) thin at x.

A similar definition holds for $\alpha p=n$. We shall say that a set E is (α,p) locally uniformly fat for $0<\alpha p\le n$, $p>1$, provided there exist positive constants r_0 and λ such that

$$R_{\alpha,p}(r^{-1}[B(x,r)\cap E]) \ge \lambda \tag{1.1}$$

for every x in E and $0<r<r_0$. Here

$$r^{-1}[B(x,r)\cap E] = \{x+r^{-1}(y-x)\colon y\in B(x,r)\cap E\}.$$

If $0<\alpha p<n$ we note that (1.1) is equivalent to

$$r^{(\alpha p-n)} R_{\alpha,p}[B(x,r)\cap E] \ge \lambda \tag{1.2}$$

for every x in E and $0<r<r_0$, as follows from the dilation properties of Riesz capacities.

If (1.1) holds for $0<r<\infty$ and every x in E we say that E is uniformly fat. In the classical case $\alpha=1$, $p=2$, domains whose complements satisfy (1.1) have been considered in several recent papers (see [**6**, **10**, **18**]).

We note from (F) that a (β,q) locally uniformly fat set is also (α,p) uniformly fat when either $\beta q<\alpha p<n$ or $\alpha p=\beta q<n$ and $\alpha>\beta$. In this paper we show that if a closed set E is locally uniformly fat with respect to a given capacity, then E is also locally uniformly fat with respect to a weaker capacity and corresponding Hausdorff measure. More specifically, we prove

THEOREM 1. *Given $0<\alpha p\le n$, $p>1$, suppose E is closed and (α,p) locally uniformly fat. Then there exists ε, $\lambda_1>0$ depending only on α,p,n,λ, such that whenever $x\in E$, $0<r<r_0$, and $\alpha p-\varepsilon<\beta q<\alpha p$, we have*

$$R_{\beta,q}[E\cap B(x,r)] \ge \lambda_1 r^{(n-\beta q)}, \tag{1.3}$$

$$H^{(n-\alpha p+\varepsilon)}[E\cap B(x,r)] \ge \lambda_1 r^{(n-\alpha p+\varepsilon)}. \tag{1.4}$$

We give two applications of Theorem 1. In order to state the first application we need some notation. Let D be an open set, $C_0^\infty(D)$ the space of infinitely differentiable functions with compact support in D, and $d(x)$ the distance from x to $\mathbf{R}^n - D$. In §5 we prove

THEOREM 2. *Let D be an open set and $1 < p \le n$. If $\mathbf{R}^n - D$ is $(1,p)$ uniformly fat, then*

$$\int_D \left[\frac{u(x)}{d(x)}\right]^p dx \le A \int_D |\nabla u|^p \, dx, \tag{1.5}$$

for each $u \in C_0^\infty(D)$. Here $A > 0$ depends only on p, λ, and n.

In Theorem 2, ∇u denotes the gradient of u. We note that Ancona [**6**] proved Theorem 2 when $p = 2$ and showed that it is best possible for $n = 2$, $p = 2$. In §5 we also point out that (1.5) holds for $p > n$ whenever $D \ne \mathbf{R}^n$. In §6 we examine the extent to which (1.5) implies $\mathbf{R}^n - D$ is uniformly fat when $1 < p \le n$. In case $D = B(0,1) - \{0\}$ and $1 < p < n$, it can be shown using Hardy's inequality on rays that (1.5) holds for some fixed $A > 0$. Clearly $\mathbf{R}^n - D$ is not uniformly fat. Thus (1.5) is not sufficient for $\mathbf{R}^n - D$ to be uniformly fat when $1 < p < n$. However we can prove

THEOREM 3. *If $p = n$ and (1.5) holds for some fixed $A > 0$, then $\mathbf{R}^n - D$ is uniformly fat. Moreover, there exists a compact set $F \subseteq B(0,1)$ with $H^\varepsilon(F) = 0$ for each $\varepsilon > 0$ with the following property: If $D = B(0,1) - F$ and A, p, $1 < p < n$, are given positive numbers, then (1.5) fails to hold for some u in $C_0^\infty(D)$.*

Next let $E \subseteq B(0,1)$ be a compact set, m a given positive integer, and $0 < \alpha < n/2$. Recall that a sequence $(x_i)_1^m$ of m points in E which minimizes

$$\sum_{i \ne j} |y_i - y_j|^{(2\alpha - n)}$$

over all sequences $(y_i)_1^m$ of m points in E, is called a sequence of m Fekete points of E corresponding to the Riesz kernel $I_{2\alpha}$. In §7 we point out that Theorem 1 implies

THEOREM 4. *Let E and $(x_i)_1^m$ be as above for fixed α, $0 < \alpha < n/2$, and suppose that E is $(\alpha, 2)$ locally uniformly fat. Then there exists $\varepsilon, a > 0$ such that*

$$\min_{i \ne j} |x_i - x_j| \ge a(1/m)^{(n - 2\alpha + \varepsilon)}$$

where a and ε depend only on α, n, λ, and r_0.

We note that related theorems have been proved in [**8**, **16**, and **17**]. In §7 we also indicate that Theorem 4 has an analogue for $\alpha = n/2$.

Finally in this section we would like to thank Professor D. R. Adams for many useful conversations regarding Riesz capacities.

2. Proof of Theorem 1 for $\alpha \ge 2$. In the sequel c denotes a positive constant which may only depend on α, p, and n, not necessarily the same at each occurrence. Also $\overline{A}$ and ∂A denote the closure and boundary of the set A. We now begin the proof of Theorem 1. Fix R, $0 < R \le r_0$, and suppose E is (α, p) locally uniformly fat. Given $x_0 \in E$, we claim there exists a compact set F contained in $\overline{B}(x_0, R) \cap E$ with

$$(2.1) \qquad R_{\alpha,p}(r^{-1}[F \cap B(x,r)]) \geq c\lambda,$$

for $0 < r \leq R$ and every $x \in F$. The set in parentheses is defined as in §1. Thus F is (α, p) locally uniformly fat. To construct F let $E_1 = E \cap B(x_0, R/2)$ and inductively let

$$E_m = \left[\bigcup_{x \in E_{m-1}} B(x, 2^{-m}R) \right] \cap E, \qquad m = 2, 3, \ldots.$$

Then if suffices to let F be the closure of $\bigcup_{m=1}^{\infty} E_m$, as is easily shown using (1.1).

The proof of Theorem 1 will be divided into three parts. In §3 we prove Theorem 1 for $0 < \alpha < 2$, $\alpha p < n$. In §4 the proof of Theorem 1 for $\alpha p = n$ is given. In this section we prove Theorem 1 for $\alpha \geq 2$ and $\alpha p < n$. To this end let $P = I_\alpha * (I_\alpha * \mu)^{1/(p-1)}$ be the equilibrium potential for F (see (C) of §1). We first establish certain Wiener-type estimates on P which will in fact imply that P is Hölder continuous on $\mathbf{R}^n$. Using the relationship between P and μ it is then relatively easy to prove (1.3) and (1.4). To establish these estimates we shall need some notation. Let $u = 1 - P$ and for fixed $x_1 \in \operatorname{supp} \mu$ with $P(x_1) = 1$ let

$$M(t) = M(t, u, x_1) = \sup_{x \in B(x_1, t)} u(x), \qquad t > 0.$$

If $\alpha \geq 2$, note that since $\alpha p < n$, and $p > 1$, we must have $n \geq 3$. Also in this case I_α and consequently P are superharmonic in $\mathbf{R}^n$. Thus $u = 1 - P$ is subharmonic and we can write u in $B(x_1, s)$, $s > 0$, in the form $u = h - q$, where h is the least harmonic majorant of u in $B(x_1, s)$ and q is a Green's potential in $B(x_1, s)$. h can be written explicitly in terms of a Poisson integral as

$$(2.2) \qquad h(x) = cs^{-1} \int_{\partial B(x_1, s)} \frac{(s^2 - |x_1 - x|^2)}{|y - x|^n} u(y)\, dH^{n-1}y, \qquad x \in B(x_1, s).$$

For fixed $s \leq R$ and $0 < s_1 \leq \frac{1}{2}s$ we first show that if

$$(2.3) \qquad |h(x) - h(x_1)| \leq \tfrac{1}{4} M(s, h), \qquad x \in B(x_1, s_1),$$

then there exists β, $\frac{3}{4} \leq \beta < 1$, depending only on α, p, n, and λ such that

$$(2.4) \qquad M(\tfrac{1}{8}s_1, u) \leq \beta M(s, h).$$

Here, $M(s, h) = \sup_{x \in B(x_1, s)} h(x)$.

To prove (2.4) under assumption (2.3) note that either $h(x_1) \geq \frac{1}{2}M(s, h)$ or (2.4) holds with $\beta = \frac{3}{4}$ since $u \leq h$ in $B(x_1, s)$. If $h(x_1) \geq M(s, h)/2$, then from (2.3), (C) of §1, and the definition of u, q, h, we deduce

$$(2.5) \qquad q(x) \geq h(x) \geq M(s, h)/4, \qquad (\alpha, p) \text{ q.e. on } F \cap B(x_1, s_1).$$

Next we note for $\alpha > 2$ that

$$\Delta u = -\Delta P = c I_{(\alpha - 2)} * (I_\alpha * \mu)^{1/(p-1)}$$

for H^n almost every x in $\mathbf{R}^n$, while $\Delta u = c(I_2 * \mu)^{1/(p-1)}$ if $\alpha = 2$. If $\alpha > 2$ and

$$g(x, y) = |x - y|^{(2-n)} - \left| (x - x_1)|y - x_1|^2 - (y - x_1)s^2 \right|^{(2-n)} (s|y - x_1|)^{(n-2)},$$

$x, y \in B(x_1, s)$, denotes the Green's function for $B(x_1, s)$ with pole at $y \in B(x_1, s)$, it follows that

$$\begin{aligned} q(x) &= c \int_{B(x_1,s)} g(x,y)[I_{(\alpha-2)} * (I_\alpha * \mu)^{1/(p-1)}](y)\,dy \\ &= c\int_{B(x_1,s_1/2)} \cdots + c\int_{B(x_1,s)-B(x_1,s_1/2)} \cdots \\ &= K_1(x) + K_2(x). \end{aligned}$$

If $\alpha = 2$ replace the first term in brackets by the expression for Δu. Observe that K_2 is a positive harmonic function in $B(x_1, s_1/2)$. Hence by Harnack's inequality either $K_2 < \frac{1}{8}M(s,h)$ at each point of $B(x_1, s_1/4)$ or $K_2 \geq cM(s,h)$ at each point of $B(x_1, \frac{1}{4}s_1)$. If the second possibility occurs it follows easily from $u \leq h$ and $q \geq K_2$ that (2.4) is valid. If the first possibility occurs, then from (2.5) we deduce that $K_1 \geq \frac{1}{8}M(s,h)$, (α, p) q.e. on $F \cap B(x_1, \frac{1}{4}s_1)$.

We continue under the above assumption. Observe for $x \in B(x_1, s_1/4)$, $y \in B(x_1, s_1/2)$, that there exists a positive constant c with

$$|x-y|^{(2-n)} \leq c[|x-y|^{(2-n)} - (s/2)^{(2-n)}] \leq cg(x,y), \tag{2.6}$$

where the last inequality follows easily from the maximum principle for harmonic functions. Also

$$g(x,y) \leq |x-y|^{(2-n)}, \qquad x, y \in B(x_1, s). \tag{2.7}$$

Let $\mu_1 = \mu/B(x_1, s_1/4)$ and $\mu_2 = \mu - \mu_1$. Using (2.7) we get for $x \in B(x_1, s_1/4)$ and $\alpha > 2$

$$\begin{aligned} K_1(x) &= c\int_{B(x_1,s_1/2)} g(x,y)[I_{(\alpha-2)} * (I_\alpha * \mu)^{1/(p-1)}](y)\,dy \\ &\leq c\int_{B(x_1,s_1/2)} |x-y|^{(2-n)}[I_{\alpha-2} * (I_\alpha * \mu_1)^{1/(p-1)}](y)\,dy \\ &\quad + c\int_{B(x_1,s_1/2)} |x-y|^{(2-n)}[I_{\alpha-2} * (I_\alpha * \mu_2)^{1/(p-1)}](y)\,dy \\ &= L_1(x) + L_2(x). \end{aligned}$$

If $\alpha = 2$ replace the last two terms in brackets by the expression for Δu.

We claim there exists $c > 0$ such that

$$L_2(z) \leq cL_2(w), \qquad z, w \in B(x_1, s_1/8). \tag{2.8}$$

To prove this claim use the fact that

$$I_\alpha * \mu_2(z) \leq cI_\alpha * \mu_2(w), \qquad z, w \in B(x_1, 3s_1/16),$$

(since $\operatorname{supp}\mu_2 \subseteq \mathbf{R}^n - B(x_1, s_1/4)$) to deduce that

$$I_{(\alpha-2)} * (I_\alpha * \mu_2)^{1/(p-1)}(z) \leq c[I_{(\alpha-2)} * (I_\alpha * \mu_2)^{1/(p-1)}](w)$$

for z, $w \in B(x_1, 5s_1/32)$ and $\alpha > 2$. Next write L_2 as a sum of integrals over $B(x_1, 5s_1/32)$ and $B(x_1, s_1/2) - B(x_1, 5s_1/32)$, and use the above inequality. There are obvious modifications if $\alpha = 2$. We omit the details. Using (2.8) we find that either $L_2 \leq \frac{1}{16}M(s,h)$ at every point of $B(x_1, s_1/8)$ or $L_2 \geq cM(s,h)$ at each point of $B(x_1, s_1/8)$. If the second alternative occurs, then from (2.6) we deduce that

$$q \geq K_1 \geq cL_2 \geq cM(s,h)$$

on $B(x_1, s_1/8)$, from which (2.4) follows as previously. If the first alternative occurs, then

$$L_1(x) \geq (K_1 - L_2)(x) \geq M(s,h)/16, \qquad (\alpha, p) \text{ q.e., on } F \cap B(x_1, s_1/8). \tag{2.9}$$

In this case since $I_a * I_b$ is a constant multiple of I_{a+b} when $a, b > 0$ and $0 < a + b < n$, we have

$$L_1 \leq cI_\alpha * (I_\alpha * \mu_1)^{1/(p-1)}.$$

It follows from this inequality, (2.9) and the definition of $R_{\alpha,p}$ capacity that

$$\begin{aligned} M(s,h)^p R_{\alpha,p}\left[B\left(x_1, \frac{s_1}{8}\right) \cap F\right] &\leq c \int_{\mathbf{R}^n} (I_\alpha * \mu_1)^{p/(p-1)}(x)\, dx \\ &= c \int_{\mathbf{R}^n} (I_\alpha * \mu_1)^{1/(p-1)}(x)(I_\alpha * \mu_1)(x)\, dx \\ &= c \int_{\mathbf{R}^n} I_\alpha * (I_\alpha * \mu_1)^{1/(p-1)}\, d\mu_1. \end{aligned} \tag{2.10}$$

To proceed further we need the inequality

$$[s_1^{(\alpha p - n)} \mu_1(\mathbf{R}^n)]^{1/(p-1)} \leq c[I_\alpha * (I_\alpha * \mu_1)^{1/(p-1)}] \leq cK_1, \tag{2.11}$$

for each $x \in B(x_1, s_1/4)$. The left inequality follows easily from the fact that $\operatorname{supp} \mu_1 \subseteq B(x_1, s_1/4)$. To prove the right inequality let χ be the characteristic function of $B(x_1, s_1/2)$ and note from (2.6) that for $\alpha > 2$

$$\begin{aligned} K_1(x) &\geq c \int_{\mathbf{R}^n} \chi(y)|x-y|^{(2-n)}[I_{(\alpha-2)} * (I_\alpha * \mu_1)]^{1/(p-1)}(y)\, dy \\ &= c \int_{\mathbf{R}^n} \left[\int_{\mathbf{R}^n} \chi(y)|x-y|^{(2-n)}|y-z|^{(\alpha-2-n)}\, dy\right] (I_\alpha * \mu_1)^{1/(p-1)}(z)\, dz \\ &= \int_{\mathbf{R}^n} T(x,z)(I_\alpha * \mu_1)^{1/(p-1)}(z)\, dz, \end{aligned} \tag{2.12}$$

when $x \in B(x_1, s_1/4)$, where we have used the Tonelli Theorem to interchange the order of integration in the second integral. If $z \in B(x_1, s_1)$, $x \in B(x_1, s_1/4)$, and $|x - z| \geq \frac{1}{8}s_1$, then clearly

$$T(x,z) \geq cs_1^{(\alpha-n)} \geq c|x-z|^{(\alpha-n)}.$$

Otherwise if $A = \{y \in B(x_1, s_1/2) \colon 2|x-z| < |y-x|\}$, then from the geometry of the situation we see for $x \in B(x_1, s_1/4)$

$$T(x,z) \geq c \int_A |x-y|^{(\alpha-n)}\, dy \geq c|x-z|^{(\alpha-n)}.$$

From (2.12) it follows in either case that

$$K_1(x) \geq c \int_{B(x_1, s_1/2)} |x-z|^{(\alpha-n)}(I_\alpha * \mu_1)^{1/(p-1)}(z)\, dz = S(x), \tag{2.13}$$

for each x in $B(x_1, s_1/4)$. Also $I_\alpha * \mu_1(z) \le c\mu_1(\mathbf{R}^n)|x_1 - z|^{(\alpha-n)}$, for z in $\mathbf{R}^n - B(x_1, s_1/2)$. Using this inequality and (2.13) it follows for $x \in B(x_1, s_1/4)$ and $\alpha > 2$ that

$$\begin{aligned} I_\alpha * (I_\alpha * \mu_1)^{1/(p-1)}(x) &\le cS(x) + c[s_1^{(\alpha p-n)}\mu_1(\mathbf{R}^n)]^{1/(p-1)} \\ &\le cS(x) \le cK_1(x). \end{aligned}$$

If $\alpha = 2$, the above inequality follows directly from (2.6). Thus (2.11) is true. Using the right-hand inequality of (2.11) in (2.10) and the fact (see (C) of §1) that

$$K_1(x) \le q(x) = h(x) \le M(s,h),$$

(α, p) q.e. on $\operatorname{supp}\mu_1$, we obtain

$$M(s,h)^p R_{\alpha,p}[B(x_1, s_1/8) \cap F] \le cM(s,h)\mu_1(\mathbf{R}^n).$$

Using the left inequality in (2.11) it follows that

$$M(s,h)\{s_1^{(\alpha p-n)} R_{\alpha,p}[B(x_1, s_1/8) \cap F]\}^{1/(p-1)} \le cK_1 \le cq$$

for each x in $B(x_1, s_1/4)$. The left-hand side of this inequality is bounded below by $c\lambda^{1/(p-1)}M(s,h)$ since F is locally uniformly fat. Hence, if $x \in B(x_1, s_1/8)$, then

$$u(x) \le h(x) - q(x) \le (1 - c\lambda^{1/(p-1)})M(s,h),$$

so (2.4) is valid when $\alpha \ge 2$ and $0 < \alpha p < n$.

Next observe from (2.2) that for $x \in B(x_1, s/2)$, and $u^+ = \max(u, 0)$,

$$\begin{aligned} |\nabla h|(x) &\le cs^{-n} \int_{\partial B(x_1,s)} |u(y)| dH^{n-1}y \le cs^{-n} \int_{\partial B(x_1,s)} u^+(y)\, dH^{n-1}y \\ &\le cs^{-1}M(s,u) = cs^{-1}M(s,h), \end{aligned}$$

where we have used $u(x_1) = 0$ and the sub mean value property of subharmonic functions. Hence there exist $\delta > 0$ such that if $s_1 = 8\delta s$, then (2.3) and consequently (2.4) hold.

To continue the proof of Theorem 1, we now iterate (2.4) starting with $r_0 = R$ and continuing with $r_k = \delta^k R$, $k = 1, 2 \ldots$. Applying (2.4) with $s = r_i$, $1 \le i \le k$, we get since $u \le 1$ and $M(s,h) = M(s,u)$ that $M(r_k, u) \le \beta^k$. Since M is nondecreasing it follows for $r_{k+1} \le r \le r_k$ that

$$M(r,u) \le M(r_k, u) \le c(r/R)^\sigma$$

where $\beta = \delta^\sigma$. From (2.11) with $s = 8r$, $s_1 = \frac{1}{2}s$, and the fact that

$$K_1(x_1) \le q(x_1) = h(x_1) \le M(8r, u)$$

we deduce

$$[r^{(\alpha p-n)}\mu(B(x_1, r))]^{1/(p-1)} \le cM(8r, u), \qquad 0 < r < \infty.$$

Hence,

$$[r^{(\alpha p-n)}\mu(B(x_1, r))]^{1/(p-1)} \le c(r/R)^\sigma, \qquad 0 < r < R. \tag{2.14}$$

We note that $x_1 \in F$ is arbitrary in (2.14) subject to the requirement that $P(x_1) = 1$, which from (C) of §1 is true (α, p) q.e. on $\operatorname{supp}\mu$. We claim that (2.14) holds whenever $x_1 \in \mathbf{R}^n$ for some positive constant c. To see this we note first from a boundedness principle (see [**2**, **9**]) that P is bounded above by a positive constant in $\mathbf{R}^n$ depending only on α, p, and n. From the definition of $\operatorname{supp}\mu$ and (B) of §1 it follows that for $r > 0$

$$R_{\alpha,p}[B(x,r) \cap \operatorname{supp}\mu] \neq 0, \qquad x \in \operatorname{supp}\mu.$$

From this inequality we easily deduce that (2.14) holds for all $x_1 \in \operatorname{supp}\mu$. An easy argument using the compactness of $\operatorname{supp}\mu$ now shows that (2.14) holds for all $x_1 \in \mathbf{R}^n$. We also note from uniform fatness of F and (C) of §1 that

$$\mu[F] \geq c\lambda R^{(n-\alpha p)}. \tag{2.15}$$

Let $0 < \varepsilon < \sigma(p-1)$ and let $\beta > 0$, $q > 1$, be fixed numbers with $\alpha p - \varepsilon \leq \beta q < \alpha p$. Put

$$\nu(G) = R^{(\alpha p - \beta q)}\mu(G),$$

when G is a Borel set. Then (2.14) implies (see [**1**, Theorem 2]) that $I_\beta * (I_\beta * \nu)^{1/(q-1)} \leq c\gamma$, where γ depends only on α, p, n, and λ. This inequality, (B) of §1, (2.15), and the fact that $F \subseteq E \cap \overline{B}(x_0, R)$ imply (1.3) of Theorem 1. To prove (1.4) let $B(x_j, r_j)$, $j = 1, \ldots,$ be a covering of F by balls with $r_j \leq R$, $j = 1, 2 \ldots$. Then from (2.14) and (2.15) we get

$$\begin{aligned} \sum r_j^{(n-\alpha p+\sigma(p-1))} &\geq cR^{\sigma(p-1)} \sum \mu(B(x_j, r_j)) \\ &\geq cR^{\sigma(p-1)}\mu(F) \geq cR^{(n-\alpha p)+\sigma(p-1)}. \end{aligned}$$

Taking the infimum of the left-hand sum over all such coverings we get (1.4) of Theorem 1 with $\varepsilon = \sigma(p-1)$. This completes the proof of Theorem 1 for $\alpha \geq 2$.

3. Proof of Theorem 1 for $0 < \alpha < 2$. We now suppose that $0 < \alpha p < n$, $0 < \alpha < 2$, and shall use the same notation as in §2. The proof of Theorem 1 in this case is harder since now the equilibrium potential P for F need not be superharmonic in the classical sense. However P is still α superharmonic in the sense of Landkof [**12**], and $u = 1 - P$ may be written in the form $u = h - q$ in $B(x_1, s)$ where q is an α Green's potential and $h \geq u$ is the least α harmonic majorant of u in $B(x_1, s)$. Moreover

$$h(x) = \int_{\mathbf{R}^n - B(x_1,s)} Q_s(x,y)u(y)dy \tag{3.1}$$

(see [**12**, Chapter 1, §6, p. 123]) where

$$Q_s(x,y) = \frac{c(s^2 - |x_1 - x|^2)^{\alpha/2}}{(|y - x_1|^2 - s^2)^{\alpha/2}}|x-y|^{-n}, \qquad x \in B(x_1, s),\ y \in \mathbf{R}^n - B(x_1, s),$$

and c is chosen so that the integral in (3.1) is one when $u \equiv 1$. For fixed $s \leq R$ and $0 < s_1 < \frac{1}{2}s$, we first show, as in §2, that if

$$|h(x) - h(x_1)| \leq \tfrac{1}{4}M(s,h), \qquad x \in B(x_1, s_1), \tag{3.2}$$

then there exists β, $\frac{3}{4} \leq \beta < 1$, depending only on α, p, n, and λ such that

$$M(\tfrac{1}{8}s_1, u) \leq \beta M(s,h). \tag{3.3}$$

Recall that $M(s,h) = \sup_{x\in B(x_1,s)} h(x)$. To prove (3.3) we follow closely the proof of (2.4). Observe from (3.2) that if $h(x_1) \le \frac{1}{2}M(s,h)$, then (3.3) holds with $\beta = \frac{3}{4}$ since $u \le h$ in $B(x_1,s)$. Otherwise, from (C) of §1 and (3.2) we deduce

$$q(x) \ge h(x) \ge \tfrac{1}{4}M(s,h), \quad (\alpha,p) \text{ q.e.,} \tag{3.4}$$

on $F \cap B(x_1,s_1)$. Now

$$\begin{aligned} q(x) &= c\int_{B(x_1,s)} g(x,t)(I_\alpha * \mu)^{1/(p-1)}(t)\,dt \\ &= c\int_{B(x_1,s_1/2)} \cdots + c\int_{B(x_1,s)-B(x_1,s_1/2)} \cdots \\ &= K_1 + K_2. \end{aligned}$$

Here $g(x,\cdot)$ is the α Green's function for $B(x_1,s)$. g can be written explicitly as (see [**12**, Chapter 4, §5])

$$g(x,t) = |x-t|^{(\alpha-n)} - \int_{\mathbf{R}^n - B(x_1,s)} Q_s(x,y)|t-y|^{(\alpha-n)}\,dy$$

when $t \in B(x_1,s)$ and $x \in \mathbf{R}^n$. We note that $g(\cdot,t) \equiv 0$ in $\mathbf{R}^n - \overline{B}(x_1,s)$, $g \ge 0$, and g is α harmonic as a function of x in $B(x_1,s_1/2)$ when $t \in B(x_1,s) - B(x_1,s_1/2)$. Equivalently, (3.1) holds in $B(x_1,s_1/2)$ with u, h replaced by $g(\cdot,t)$ and s by $\frac{1}{2}s_1$. Hence K_2 is a positive α harmonic function in $B(x_1,\frac{1}{2}s_1)$ and so satisfies a Harnack-type inequality in $B(x_1,\frac{1}{4}s_1)$, as we see from (3.1). Thus either $K_2 \le \frac{1}{8}M(s,h)$ or $K_2 \ge cM(s,h)$ in $B(x_1,\frac{1}{4}s_1)$. In the second case we see that (3.3) holds. In the first case we find from (3.4) that

$$K_1 \ge \tfrac{1}{8}M(s,h), \quad (\alpha,p) \text{ q.e.} \tag{3.5}$$

on $B(x_1,\frac{1}{4}s_1) \cap F$.

We claim for $x \in B(x_1,s_1/4)$ and $t \in B(x_1,s_1/2)$ that

$$c|x-t|^{(\alpha-n)} \le |x-t|^{(\alpha-n)} - (s/2)^{(\alpha-n)} \le g(x,t). \tag{3.6}$$

The left inequality is easily verified while the right inequality is true because the difference of the above functions is α harmonic in $B(x_1,s/2)$ and ≤ 0 on $\mathbf{R}^n - B(x_1,s/2)$. Moreover, clearly

$$g(x,t) \le |x-t|^{(\alpha-n)}, \qquad x,t \in B(x_1,s). \tag{3.7}$$

Using (3.6), (3.7) in place of (2.6), (2.7), we repeat the argument leading to the proof of (2.4). For completeness we give some details. Let $\mu_1 = \mu/B(x_1,\frac{1}{4}s_1)$ and $\mu_2 = \mu - \mu_1$. Put

$$L_i(x) = 2^{1/(p-1)}\int_{B(x_1,s_1/2)} |x-t|^{(\alpha-n)}(I_\alpha * \mu_i)^{1/(p-1)}(t)\,dt, \qquad i = 1,2.$$

Then from (3.6)–(3.7) we see that $K_1 \le L_1 + L_2$ in $B(x_1,s_1/2)$ and $K_1 \ge cL_2$ in $B(x_1,s_1/4)$. As in §2, L_2 satisfies a Harnack inequality in $B(x_1,s_1/8)$ so as above either (3.3) holds or $L_2 \le \frac{1}{16}M(s_1,h)$ in $B(x_1,s_1/8)$. If the last inequality holds, then from (3.5) we deduce

$$\tfrac{1}{16}M(s,h) \le L_1 \le cI_\alpha * (I_\alpha * \mu_1)^{1/(p-1)},$$

(α, p) q.e., on $F \cap B(x_1, \frac{1}{8}s_1)$. It follows that (compare with (2.10))

$$R_{\alpha,p}[F \cap B(x_1, \tfrac{1}{8}s_1)]M(s,h)^p \le c \int_{\mathbf{R}^n} I_\alpha * (I_\alpha * \mu_1)^{1/(p-1)} d\mu_1. \tag{3.8}$$

Now

$$[\mu_1(\mathbf{R}^n) s_1^{(\alpha p - n)}]^{1/(p-1)} \le c I_\alpha * (I_\alpha * \mu_1)^{1/(p-1)} \le c K_1 \tag{3.9}$$

in $B(x_1, s_1/4)$ as follows easily from (3.6). Also

$$K_1 \le q = h, \tag{3.10}$$

(α, p) q.e. in $\operatorname{supp} \mu_1$.

Using (3.9), (3.10) in (3.8) it follows that

$$\{s_1^{(\alpha p - n)} R_{\alpha,p}[F \cap B(x_1, s_1/8)]\}^{1/(p-1)} M(s,h) \le cK_1 \le cq$$

in $B(x_1, s_1/4)$. Using uniform fatness of F we get (3.3), as in §2.

(3.3) implies as in §2 that there exists δ, $0 < \delta < \frac{1}{32}$, such that if $r_j = \delta^j R$, $j = 0, 1, \ldots$, then

$$M(r_j, u) \le \beta^j. \tag{3.11}$$

The proof is more difficult in this case though since h cannot be estimated above in $B(x_1, r)$ by $M(r, u)$. Observe that (3.11) is trivially true when $j = 0$. To prove (3.11) we first allow δ to vary and shall later fix it at a number satisfying several conditions. To see the requirements on δ suppose for fixed $k \ge -1$ that we have chosen $r_0, \ldots, r_{k+1}$, such that (3.11) holds. Let $s = r_{k+1}$, $0 < \delta^\alpha < \frac{1}{2}$, and put $u^+ = \max(u, 0)$ as previously. Define h relative to u as in (3.1). Then from (3.1), the fact that $0 = u(x_1) \le h(x_1)$, and (3.11), we obtain for $k \ge 1$ and $x \in B(x_1, s/2)$

$$\begin{aligned}
|\nabla h(x)| &\le \frac{c}{s} \int_{\mathbf{R}^n - B(x_1,s)} Q_s(x_1, y)|u(y)|\, dy \\
&\le \frac{2c}{s} \int_{\mathbf{R}^n - B(x_1,s)} Q_s(x_1, y) u^+(y)\, dy \\
&\le \frac{c}{s}\left[\beta^k + \sum_{j=0}^{k-1} \beta^j \int_{\mathbf{R}^n - B(x_1, r_{j+1})} Q_s(x_1, y)\, dy\right] \\
&\le \frac{c}{s}\left[\beta^k + s^\alpha \sum_{j=0}^{k-1} \beta^j \int_{\mathbf{R}^n - B(x_1, r_{j+1})} |y - x_1|^{-(n+\alpha)}\, dy\right] \\
&\le \frac{c}{s}\left[\beta^k + \sum_{j=0}^{k-1} \beta^j \left(\frac{s}{r_{j+1}}\right)^\alpha\right] \le \frac{c}{s}\left[\beta^k + \sum_{j=0}^{k-1} \beta^j (\delta^\alpha)^{(k-j)}\right] \\
&\le \frac{c}{s}[\beta^k + \beta^{k-1}\delta^\alpha] \le \frac{c_1 \beta^k}{s},
\end{aligned} \tag{3.12}$$

where we have used the fact that $\delta^\alpha \le \frac{1}{2} < \frac{3}{4} \le \beta$. If $k = -1, 0$, the above inequality also holds for c_1 large enough as is easily seen. We now fix δ to be the smaller of the numbers: $\frac{1}{32}$, $(\frac{1}{2})^{1/\alpha}$, and $1/250c_1$, where c_1 is the constant in (3.12).

Using (3.12), the fact that $\frac{3}{4} \le \beta < 1$, and the mean value theorem we find for $s_1 = 8\delta s$ that

$$|h(x) - h(x_1)| \le \tfrac{1}{4}\beta^{k+2}, \qquad x \in B(x_1, s_1). \tag{3.13}$$

If $M(s,h) \le \beta^{k+2}$, then (3.11) holds for $j = k+2$ since $s = r_{k+1} > r_{k+2}$ and $u \le h$ in $B(x_1, s)$. Otherwise from (3.13) we see that (3.2) holds. Using (3.3) and the induction hypothesis we again see that (3.11) holds for $j = k+2$. We conclude by induction that (3.11) is valid.

From (3.11) we find as in §2 that there exists $\sigma_1 > 0$ with

$$M(r,u) \le c(r/R)^{\sigma_1}, \qquad 0 < r < R.$$

If $\sigma = \min(\sigma_1, \alpha)$, it follows from the above inequality and (3.1) as in the proof of (3.12) that $h(x_1) \le c(s/R)^{\sigma}$. From this last inequality, (3.9), and (3.10) we see that (2.14) holds also when $0 < \alpha < 2$. Theorem 1 follows from (2.14) just as in §2. We omit the details.

4. Proof of Theorem 1 for $\alpha p = n$. Let J_α be the truncated Riesz kernel defined in §1 and for $\alpha p = n$ let $P = J_\alpha * (J_\alpha * \mu)^{1/(p-1)}$ be the equilibrium potential for

$$F_1 = \{x_0 + R^{-1}(y - x_0) \colon y \in F\}.$$

Let

$$P_1(x) = \int_{B(x_0,50)} J_\alpha(x-y)(J_\alpha * \mu)^{1/(p-1)}(y)\,dy, \qquad x \in \mathbf{R}^n.$$

From the definition of P and P_1 we see that

$$|\nabla(P - P_1)|(x) \le c, \qquad x \in B(x_0, 2).$$

Let $u = 1 - P$ and for fixed $x_1 \in \operatorname{supp}\mu \subseteq \overline{B}(x_0, 1)$ with $P(x_1) = 1$ put $u_1 = P_1(x_1) - P_1$. From the above inequality we see that

$$|u_1 - u|(x) \le |(P_1 - P)(x) - (P_1 - P)(x_1)| \le c_2|x - x_1|, \tag{4.1}$$

when $x \in B(x_0, 2)$.

We note that u_1 is subharmonic in $B(x_0, 2)$ when $\alpha \ge 2$ and u_1 is α subharmonic in $B(x_0, 2)$ when $0 < \alpha < 2$. For fixed s, $0 < s \le 1$, we write $u_1 = h_1 - q_1$, where h_1 is the least harmonic majorant of u_1 in $B(x_1, s)$ when $\alpha \ge 2$ and h_1 is the least α harmonic majorant of u_1 in $B(x_1, s)$, defined as in (3.1), when $0 < \alpha < 2$. Also, q_1 is a Green's potential or α Green's potential in $B(x_1, s)$ depending on whether $\alpha \ge 2$ or $0 < \alpha < 2$. If $0 < s_1 \le s/2$,

$$M(s, h_1) \ge 8c_2 s, \tag{4.2}$$

where $c_2 > 1$ is the constant in (4.1), and

$$|h_1(x) - h_1(x_1)| \le \tfrac{1}{4}M(s,h) \tag{4.3}$$

in $B(x_1, s_1)$.

We show as in §§2–3 that there exists β, $\frac{3}{4} \le \beta < 1$, such that

$$M(s_1/8, u_1) \le \beta M(s, h_1). \tag{4.4}$$

As before we note from (4.3) that either $h_1(x_1) \leq \frac{1}{2}M(s, h_1)$ in which case (4.4) holds with $\beta = \frac{3}{4}$ or $h_1 \geq \frac{1}{4}M(s, h_1)$ in $B(x_1, s_1)$. In this case from (4.2), (4.1) and (C) of §1 we find for $x \in F_1 \cap B(x_1, s_1)$

$$0 \geq u(x) \geq u_1(x) - c_2 s \geq \tfrac{1}{8}M(s, h_1) - q_1(x).$$

Hence

$$q_1 \geq \tfrac{1}{8}M(s, h), \quad (\alpha, p) \text{ q.e., on } F_1 \cap B(x_1, s_1). \tag{4.5}$$

We define K_i, L_i, $i = 1, 2$, as in §§2 and 3 with I_α replaced by J_α and u by u_1. Proceeding as in §§2 and 3 we find that either (4.4) holds or

$$L_1 \geq \tfrac{1}{32}M(s, h_1), \quad (\alpha, p) \text{ q.e on } F_1 \cap B(x_1, \tfrac{1}{8}s_1). \tag{4.6}$$

Let $H(x) = L_1[x_1 + s_1(x - x_1)]$, $x \in B(x_1, 1)$, and put

$$\nu(T) = \mu_1(s_1 T), \quad T \text{ a Borel set},$$

where $s_1 T = \{x_1 + s_1(y - x_1) \colon y \in T\}$ and $\mu_1 = \mu | B(x_1, s_1/4)$.

Changing variables in the integrals defining L_1 in §§2 and 3 we see that

$$H(x) \leq c[J_\alpha * (J_\alpha * \nu)^{1/(p-1)}](x), \qquad x \in B(x_1, 1). \tag{4.7}$$

Using (4.6), (4.7), and (1.1) we deduce as in §§2 and 3 (see (3.8) and (2.10)) that

$$\lambda M(s, h_1)^p \leq c \int_{\mathbf{R}^n} J_\alpha * (J_\alpha * \nu)^{1/(p-1)}\, d\nu.$$

Again as in §§2 and 3 we find

$$J_\alpha * (J_\alpha * \nu)^{1/(p-1)}(x) \leq c L_1[x_1 + s_1(x - x_1)], \qquad x \in B(x_1, \tfrac{1}{4}).$$

Also from (4.1) and (C) of §1, we have

$$L_1 \leq c K_1 \leq c q_1 \leq c h_1 + c c_2 s,$$

(α, p) q.e. on $\operatorname{supp} \mu_1$. Using these inequalities and (4.2) we deduce as previously first that

$$\lambda^{1/(p-1)} M(s, h_1) \leq c q_1, \quad \text{in } B(x_1, \tfrac{1}{4})$$

and thereupon that (4.4) holds.

(4.4) implies there exists δ, $0 < \delta < \frac{1}{32}$, such that if $r_j = \delta^j$, $j = 0, 1, \ldots$, and $c_2 > 1$ is the constant in (4.1), then

$$M(r_j, u_1) \leq 2c_2 \beta^j, \qquad j = 0, 1, \ldots. \tag{4.8}$$

In fact if (4.8) holds for $r_0, r_1, \ldots, r_{k+1}$ $(k \geq -1)$ and $s = r_{k+2}$, then either

$$M(s, h_1) \leq 2c_2 \beta^{k+2}$$

(in which case (4.8) holds with $j = k + 2$ since $u_1 \leq h_1$) or (4.2) is satisfied since $4\delta < \beta$. If (4.2) is true, we argue as in §§2 and 3 with u replaced by u_1 to get (4.8) for $j = k + 2$. Then by induction we obtain (4.8). The rest of the proof of Theorem 1 for $\alpha p = n$ is the same as in the previous sections. The proof of Theorem 1 is now complete.

5. Proof of Theorem 2. Let D be $(1,p)$ uniformly fat, $1 < p \le n$, and let $\{Q_i\}_1^\infty$ be a sequence of Whitney cubes for D with $Q_i \subseteq D$, $i = 1, 2, \dots$, and

$$(4n)^{-1} d(Q_i, \partial D) \le r_i \le d(Q_i, \partial D), \tag{5.1}$$

where r_i, $i = 1, 2, \dots$, denotes the sidelength of Q_i and $d(E, F)$ denotes the distance from the set E to the set F. Let $\tilde{Q}_i$ denote the cube with the same center as Q_i and with sidelength, $100nr_i$. Given $u \in C_0^\infty(D)$ put $v = |u|$. Let h_Q denote the average of the function h over the cube Q. We note that (see [**15**, p. 195])

$$|v(x) - v_Q| \le c \int_Q |\nabla v|(y) |x - y|^{(1-n)}\, dy, \qquad x \in Q. \tag{5.2}$$

From (5.2) it can be shown that

$$\int_Q |v(x) - v_Q|^p \, dx \le c r^p \int_Q |\nabla v|^p \, dx, \tag{5.3}$$

where r is the sidelength of Q.

Observe from (5.1) that $d(x, \partial D) \ge c r_i$, when $x \in Q_i$. Using this observation and (5.3) with $Q = Q_i$, $i = 1, 2, \dots$, and $d(x) = d(\{x\}, \partial D)$, we obtain

$$\begin{aligned} \int_D \left|\frac{u}{d}\right|^p dx &\le c \sum \left(r_i^{-p} \int_{Q_i} v^p \, dx \right) \\ &\le c \left[\sum (r_i^{(n-p)} v_{Q_i}^p) + \int_D |\nabla v|^p \, dx \right]. \end{aligned} \tag{5.4}$$

To estimate v_{Q_i} choose $x_i \in \partial D$ such that $d(\{x_i\}, Q_i) = d(Q_i, \partial D)$. Note from (5.1) and the definition of $\tilde{Q}_i$ that

$$\overline{B}(x_i, r_i) \subseteq \tilde{Q}_i, \qquad i = 1, 2, \dots.$$

Let $q = p - \varepsilon/2$, where ε is as in Theorem 1 for $\alpha = 1$, and put $q' = q/(q-1)$. Let μ be a $R_{1,q}$ capacitary measure for $\overline{B}(x_i, r_i) \cap (\mathbf{R}^n - D)$. Then from (5.2) with $Q = \tilde{Q}_i$, the fact that $v = 0$ on $\mathbf{R}^n - D$, and (C) of §1, we obtain

$$\begin{aligned} & v_{\tilde{Q}_i} \mu[\overline{B}(x_i, r_i) \cap (\mathbf{R}^n - D)] \\ &\quad \le c \int_{\tilde{Q}_i} |\nabla v| (I_1 * \mu)\, dy \le c \| |\nabla v| \chi_i \|_q \| I_1 * \mu \|_{q'} \\ &\quad \le c \| |\nabla v| \chi_i \|_q \{\mu[\overline{B}(x_i, r_i) \cap (\mathbf{R}^n - D)]\}^{1/q'}, \end{aligned} \tag{5.5}$$

where χ_i denotes the characteristic function of $\tilde{Q}_i$, $i = 1, 2, \dots$. Observe from Theorem 1 that

$$\mu[\overline{B}(x_i, r_i) \cap (\mathbf{R}^n - D)] \ge c r_i^{n-q}.$$

Using this inequality in (5.5) and doing some arithmetic we find that

$$v_{\tilde{Q}_i}^p r_i^{(n-p)} \le c \| |\nabla v| \chi_i r_i^{-n/q} \|_q^p r_i^n \le c (f_{\tilde{Q}_i})^{p/q} r_i^n, \tag{5.6}$$

where $f = |\nabla v|^q$. Using (5.6) and $v_{Q_i} \le c v_{\tilde{Q}_i}$ in (5.4) we conclude that

$$\int_D \left|\frac{u}{d}\right|^p dx \le c \left[\sum (f_{\tilde{Q}_i})^{p/q} r_i^n + \int_D |\nabla v|^p \, dx \right]. \tag{5.7}$$

We observe that $\{\tilde{Q}_i\}$ is a sequence of "Carleson" cubes. That is, if Q is a cube

$$\sum_{\tilde{Q}_i \subseteq Q} H^n(\tilde{Q}_i) \le c \sum_{Q_i \subseteq Q} r_i^n \le cH^n(Q).$$

Also if $\tau = p/q > 1$, then $f \in L^\tau(\mathbf{R}^n)$. Applying a well-known lemma originally proved by Carleson when $p = 2$, $n = 1$ (see [**7**, **11**, and **19**, Chapter 7, ex. 4.4]), we obtain

$$\sum (f_{\tilde{Q}_i})^{p/q} r_i^n \le c\|f\|_\tau^\tau = c \int_D |\nabla v|^p \, dx.$$

Using this inequality in (5.7) and the fact that $|\nabla u| = |\nabla v|$ almost everywhere, we obtain Theorem 2.

As mentioned in §1, (1.5) holds for any open set $D \neq \mathbf{R}^n$ when $p > n$. In fact if $u \in C_0^\infty(D)$ and $q = n + \frac{1}{2}(p - n)$, then from (5.2) with $u = v$ and Hölder's inequality we obtain

$$|u(x) - u(y)| \le c\||\nabla u|\chi\|_q r^{(1-n/q)}, \qquad x, y \in Q,$$

where again Q is a cube with sidelength r and χ denotes the characteristic function of Q.

Defining $(\tilde{Q}_i)$ as before and applying the above inequality with $Q = \tilde{Q}_i$ and $u(y) = 0$ we see that (5.6) is still true. Repeating the argument in Theorem 3 from (5.6) on we conclude that (1.5) holds whenever $D \neq \mathbf{R}^n$.

6. Proof of Theorem 3. We first show that Theorem 2 has a converse when $p = n$. The proof is by contradiction. Suppose that D is an open set and $\mathbf{R}^n - D$ is not $(1, n)$ uniformly fat. Then for each $\varepsilon > 0$ there exists $x_0 = x_0(\varepsilon)$ in $\mathbf{R}^n - D$ and $r = r(\varepsilon) > 0$ such that if

$$\Lambda = \{x_0 + r^{-1}(y - x_0) \colon y \in \overline{B}(x_0, r) \cap (\mathbf{R}^n - D)\},$$

then

$$R_{1,n}(\Lambda) \le \varepsilon^n. \tag{6.1}$$

Let

$$\Gamma(\Lambda) = \inf_\phi \left(\int |\nabla \phi|^n \, dx \right)$$

where the infimum is taken over all $\phi \in C_0^\infty(B(x_0, 4))$ with $\phi \equiv 1$ in a neighborhood of Λ. It is well known (see [**4**]) that

$$c^{-1}\Gamma(\Lambda) \le R_{1,n}(\Lambda) \le c\Gamma(\Lambda). \tag{6.2}$$

Using (6.1), (6.2), we see there exists $\psi \in C_0^\infty(B(x_0, 4))$ with $\psi \equiv 1$ in an open set containing Λ and

$$\int_{\mathbf{R}^n} |\nabla \psi|^n \, dx \le c\varepsilon^n. \tag{6.3}$$

Let $\theta \in C_0^\infty[B(x_0, 1)]$ with $\theta \equiv 1$ on $B(x_0, \frac{1}{2})$ and $|\nabla \theta| \le 1000$. Put

$$h(x) = (1 - \psi(x))\theta(x), \qquad x \in \mathbf{R}^n,$$

and observe that $u(x) = h(x_0 + (x - x_0)/r)$, $x \in \mathbf{R}^n$, is in $C_0^\infty(D)$. Also,

$$(6.4)\qquad \begin{aligned}\int_{\mathbf{R}^n} |\nabla u|^n\, dx &= \int_{\mathbf{R}^n} |\nabla h|^n\, dx \\ &\le c\left[\int_{B(x_0,1)} (1-\psi)^n\, dx + \int_{B(x_0,1)} |\nabla\psi|^n\, dx\right] \le c,\end{aligned}$$

where we have used Poincaré's inequality, and (6.3). Again from these inequalities we deduce

$$(6.5)\qquad \begin{aligned}H^n\left[\left\{x\colon \psi(x) \ge \frac{1}{2}\right\}\right] &\le 2^n \int \psi^n\, dx \\ &\le c\int_{B(x_0,4)} |\nabla\psi|^n\, dx \le c\varepsilon^n.\end{aligned}$$

Let $G \subseteq \mathbf{R}$ be the set of all $t \in (0, \frac{1}{2})$ for which

$$H^{n-1}[\{x\colon \psi(x) < \tfrac{1}{2}\} \cap \partial B(x_0, t)] \ge \tfrac{1}{2} H^{n-1}[\partial B(x_0, t)].$$

From Fubini's Theorem and (6.5) we find first that there exists $c_3 > 0$ with

$$\int_{(0,1/2)-G} t^{n-1} dt \le (c_3\varepsilon)^n,$$

and thereupon since $t \to t^{-n}$ is decreasing that

$$(6.6)\qquad \int_G t^{-1}\, dt = \int_G t^{-n} t^{n-1}\, dt \ge \int_{c_3\varepsilon}^{1/2} t^{-1}\, dt = -\ln(2c_3\varepsilon).$$

Now if $d(x) = d(\{x\}, \mathbf{R}^n - D)$, then from (6.6) and the fact that $h \ge \frac{1}{2}$ on G we obtain

$$(6.7)\qquad \begin{aligned}\int_D \left(\frac{u}{d}\right)^n dx &\ge \int_{B(x_0,r)} \frac{u(x)^n}{|x-x_0|^n}\, dx = \int_{B(x_0,1)} \frac{h(x)^n}{|x-x_0|^n}\, dx \\ &\ge c\int_G t^{-1}\, dt \ge -c\ln(2c_3\varepsilon).\end{aligned}$$

Given $A > 0$ we see from (6.4) and (6.7) that for $\varepsilon > 0$, sufficiently small, there exists $u \in C_0^\infty(D)$ for which (1.5) is false. This proves Theorem 3 for $p = n$.

To prove Theorem 3 for $1 < p < n$ requires an explicit construction. To this end let $(m_j)_1^\infty$ be an increasing sequence of positive integers with $m_j \ge j+1$, $j = 1, \ldots,$ to be chosen later. Put $s_0 = 1/2n$ and for $j \ge 1$, let

$$s_j = s_{j-1}(m_j)^{-j}, \quad t_j = (s_{j-1} - s_j)/(m_j - 1).$$

Since $m_j \ge j + 1$ it is easily seen that

$$(6.8)\qquad t_j - 10s_j \ge (2m_j)^{-1} s_{j-1}, \qquad j = 10, 11, \ldots.$$

If $J = [a, b]$ is an interval with $b - a = s_{j-1}$, let $\Phi(J, m_j)$ be the m_j equally spaced closed intervals of length s_j contained in J defined by

$$\Phi(J, m_j) = \{[a + (i-1)t_j, a + (i-1)t_j + s_j]\colon 1 \le i \le m_j\}.$$

Define families K_j of closed intervals inductively as follows:

$$K_0 = [-1/4n, 1/4n], \quad K_j = \{\Phi(J, m_j)\colon J \in K_{j-1}\}, \qquad j \ge 1.$$

Let L_j, $j = 0, 1, \ldots$, be the family of cubes defined by

$$L_j = \{Q = J_1 \times \cdots \times J_n \colon J_i \in K_j,\ 1 \le i \le n\}.$$

Note that L_j, $j = 1, \ldots$, consists of $(m_1 \cdots m_j)^n$ closed cubes of sidelength s_j. Also if $Q \in L_{j-1}$, then

$$\sum_{\substack{T \in L_j \\ T \subseteq Q}} [H^n(T)]^{1/j} = H^n(Q)^{1/j}, \qquad j = 1, 2, \ldots. \tag{6.9}$$

Let

$$F_j = \bigcup_{Q \in L_j} Q, \quad j = 0, 1, \ldots, \qquad F = \bigcap_0^\infty F_j.$$

Let $D_j = B(0,1) - F_j$, $j = 0, 1, \ldots$, and $D = B(0,1) - F$. If $x \in D_j$ let

$$\tilde{u}_j(x) = \begin{cases} 0 \quad \text{when } d(\{x\}, \partial D_j) < s_j, \\ s_j^{-1}[d(\{x\}, \partial D_j) - s_j], \quad \text{when } s_j \le d(\{x\}, \partial D_j) \le 2s_j, \\ 1 \quad \text{when } d(\{x\}, \partial D_j) > 2s_j. \end{cases}$$

Put $u_j = [\tilde{u}_j * \theta_j]\theta$, $j = 0, 1, \ldots$, where

$$\theta_j(x) = (2/s_j)^n \theta[2x/s_j](\|\theta\|_1)^{-1}, \qquad x \in \mathbf{R}^n,$$

and as previously, $\theta \in C_0^\infty(B(0,1))$ with $\theta \equiv 1$ on $B(0, \frac{1}{2})$. From the definition of $\tilde{u}_j$ we see that $u_j \in C_0^\infty(D_j)$,

$$\nabla u_j(x) = 0 \quad \text{if } d(\{x\}, \partial D_j) \ge 3s_j \text{ and } |x| < \tfrac{1}{2},$$

while $\nabla u_j = \nabla \tilde{u}_j * \theta_j$, $j = 0, 1, \ldots$. Using these facts we find that

$$\int_{D_j} |\nabla u_j|^p \, dx \le c(m_1 \cdots m_j)^n (s_j)^{(n-p)}. \tag{6.10}$$

Let $l = l(p, n)$ be the first positive integer such that $1/l < 1 - p/n$. Then from (6.9) observe that

$$\begin{aligned} (m_1 \cdots m_j)^n (s_j)^{(n-p)} &= (m_1 \cdots m_{j-1})^n (s_{j-1})^{n/j} (s_j)^{n(1-p/n-1/j)} \\ &\le (m_1 \cdots m_{j-1})^n (s_{j-1})^{n-p} \\ &\le \cdots \le (m_1, \ldots, m_l)^n (s_l)^{n-p}, \qquad j \ge l. \end{aligned}$$

Using (6.10) it follows that

$$\int_{D_i} |\nabla u_j|^p \, dx \le c(m_1 \cdots m_l)^n (s_l)^{(n-p)}, \qquad j \ge l. \tag{6.11}$$

Now (6.8) implies that the distance between successive cubes in K_j is at least $(2m_j)^{-1} s_{j-1}$, when $j \ge 10$. Using this fact we deduce for $Q \in L_{j-1}$, $d = d(\cdot, \partial D)$, and $\tau = (2m_j)^{-1} s_{j-1}$

$$\begin{aligned} \int_Q \left(\frac{u_j}{d}\right)^p dx &\ge c(m_j)^n \int_{10s_j}^{\tau} \rho^{n-p-1} \, d\rho \ge c(m_j)^n \tau^{n-p} \\ &\ge c(m_j)^p (s_{j-1})^{n-p}. \end{aligned} \tag{6.12}$$

We now choose

$$m_j = [(j+1)(s_{j-1})^{1-n/p}]$$

where $[x]$ denotes the greatest integer $\leq x$. Since $s_j = (s_{j-1})(m_j)^{-j}$, the sequence $(m_j)_1^\infty$ is well defined by induction. Also from (6.11), (6.12), we conclude

$$\int_D (u_j/d)^p \, dx \to +\infty \quad \text{as } j \to \infty$$

while $\int |\nabla u_j|^p \, dx$ remains bounded independently of j. Finally it is easily shown that F has Hausdorff dimension zero. Simply take the cubes in L_j, $j = 1, 2, \ldots$, as coverings and use (6.9). We omit the details. The proof of Theorem 3 is now complete.

7. Proof of Theorem 4. Let E be a compact subset of $B(0,1)$ and α fixed, $0 < \alpha < n/2$. Suppose that E is $(\alpha, 2)$ locally uniformly fat for some λ, $r_0 > 0$. Then clearly,

$$r^{(2\alpha-n)} R_{\alpha,p}[E \cap B(x,r)] \geq c\lambda r_0^{(n-2\alpha)} = \tilde{\lambda}, \tag{7.1}$$

for $0 < r < 2$ and every $x \in E$. Let $(x_i)_1^m$ be a sequence of m Fekete points for E. We assume, as we may, that

$$\min_{i \neq j} |x_i - x_j| = |x_1 - x_2|.$$

Let

$$P(x) = \frac{1}{(m-1)} \sum_{i=2}^{m} |x - x_i|^{(2\alpha - n)}, \qquad x \in \mathbf{R}^n.$$

From the minimizing property of Fekete points mentioned in §1, it is easily shown that

$$P(x_1) \leq P(x), \qquad x \in E. \tag{7.2}$$

Put $u = P(x_1) - P$. We claim that

$$M(r) = M(r, u, x_1) \leq cP(x_1) r^\sigma, \qquad 0 < r < 1, \tag{7.3}$$

where c, σ, are positive constants $(0 < \sigma < 1)$, depending only on r_0, λ, n, and α. To prove this claim let μ be the measure with mass $1/(m-1)$ at x_j, $2 \leq i \leq m-1$, and observe that $P = I_{2\alpha} * \mu$. We now repeat the argument in §§2 and 3 using the fact that $I_\alpha * I_\alpha = \gamma I_{2\alpha}$ for some $\gamma > 0$. The argument is unchanged up to the point in §§2 and 3 where we obtained

$$M(s,h) \leq cI_\alpha * (I_\alpha * \mu_1) = cI_{2\alpha} * \mu_1 \tag{7.4}$$

$(\alpha, 2)$ q.e. on $E \cap B(x_1, s_1/8)$. Let ν be capacitary measure for $E \cap B(x_1, s_1/8)$ and $H = \gamma I_{2\alpha} * \nu$ the corresponding equilibrium potential. Then from (7.4), the Tonelli Theorem, and the fact that H is bounded (see [**2**, **9**]), we obtain

$$\begin{aligned} M(s,h) R_{\alpha,2}[E \cap B(x_1, s_1/8)] &= M(s,h)\nu[E \cap B(x_1, s_1/8)] \\ &\leq c \int (I_{2\alpha} * \mu_1) \, d\nu = c \int (I_{2\alpha} * \nu) \, d\mu_1 \leq c\mu_1(\mathbf{R}^n). \end{aligned}$$

The above argument was used in place of the argument in §§2 and 3 because $I_{2\alpha} * \mu_1$ need not be bounded by $M(s,h)$ on $\operatorname{supp}\mu_1$. The rest of the argument is unchanged from the argument in §§2 and 3. Hence (7.3) is true. Let $r = 4|x_1 - x_2|$ and put

$$h(x) = P(x) - \frac{1}{(m-1)}|x - x_2|^{(2\alpha-n)}, \qquad x \in \mathbf{R}^n.$$

To prove Theorem 4 we consider two cases. If $\alpha \geq 1$ let $r = 4|x_2 - x_1|$ and observe from (7.3) and superharmonicity of h that

$$\begin{aligned} P(x_1) &\leq \min_{x\in\partial B(x_1,r)} P + cP(x_1)r^\sigma \\ &\leq \frac{1}{(m-1)}2^{(n-2\alpha)}r^{(2\alpha-n)} + \min_{x\in\partial B(x_1,r)} h + cP(x_1)r^\sigma \\ &\leq \frac{1}{(m-1)}2^{(n-2\alpha)}r^{(2\alpha-n)} + h(x_1) + cP(x_1)r^\sigma. \end{aligned} \tag{7.5}$$

Since

$$h(x_1) = P(x_1) - \frac{1}{(m-1)}|x_2 - x_1|^{(2\alpha-n)}$$

and $r = 4|x_1 - x_2|$, (7.5) implies

$$\frac{1}{(m-1)}|x_2 - x_1|^{(2\alpha-n)} \leq c|x_2 - x_1|^\sigma P(x_1),$$

or equivalently that

$$|x_2 - x_1| \geq \left[\frac{1}{(m-1)P(x_1)}\right]^{1/(n-2\alpha+\sigma)}. \tag{7.6}$$

If $0 < \alpha < 1$ let $0 < s < \frac{1}{2}$ and replace α by 2α in the definition of Q_s following (3.1). Then h is 2α superharmonic in $\mathbf{R}^n$. Hence

$$\int_{\mathbf{R}^n - B(x_1,s)} Q_s(x,y)h(y)\,dy \leq h(x), \tag{7.7}$$

whenever $x \in B(x_1, s)$. Now

$$\begin{aligned} 1 &= \int_{\mathbf{R}^n - B(x_1,s)} Q_s(x,y)\,dy = \int_{\mathbf{R}^n - B(x_1,s^{1/2})} \cdots + \int_{B(x_1,s^{1/2}) - B(x_1,s)} \cdots \\ &\leq cs^{2\alpha}\int_{s^{1/2}}^\infty t^{-(1+2\alpha)}\,dt + \int_{B(x_1,s^{1/2})-B(x_1,s)} \cdots \\ &\leq cs^\alpha + \int_{B(x_1,s^{1/2})-B(x_1,s)} Q_s(x,y)\,dy. \end{aligned} \tag{7.8}$$

Choose s_0 so that $cs_0^\alpha = \frac{1}{2}$, where c is the constant in the last inequality. Given s, $0 < s < s_0$, it follows from (7.7), (7.8), that there exists r, $s < r < s^{1/2}$, such that

$$\min_{x\in\partial B(x_1,r)} h(x) \leq \frac{1}{(1 - cs^\alpha)}h(x_1) \leq (1 + 2cs^\alpha)h(x_1).$$

If $s = 4|x_1 - x_2|$ we can use the above inequality and repeat the argument used in (7.5) to obtain

$$\frac{1}{(m-1)}|x_1 - x_2|^{(2\alpha-n)} \leq c(s^\alpha + r^\sigma)P(x_1) \leq cs^\beta P(x_1), \tag{7.9}$$

where $\beta = \min(\alpha, \sigma/2)$. From the above discussion we see that either $4|x_1 - x_2| \geq s_0$, in which case there is nothing to prove or (7.9) holds with $s = 4|x_1 - x_2|$. For this choice of s, (7.9) yields

$$(7.10) \qquad |x_1 - x_2| \geq c[1/mP(x_1)]^{1/(n-2\alpha+\beta)},$$

$m = 1, 2, \ldots$. To complete the proof of Theorem 4 it remains to show that

$$(7.11) \qquad P(x_1) \leq c.$$

To do this let τ be capacitary measure for E. Then from (7.2), the fact that $I_{2\alpha} * \tau \leq c$, and the Tonelli Theorem, we obtain

$$P(x_1)R_{\alpha,2}(E) \leq \int P d\tau = \int (I_{2\alpha} * \tau)\, d\mu \leq c\mu(\mathbf{R}^n) = c.$$

This inequality and (7.1) clearly imply (7.11). From (7.11), (7.10), and (7.6) we conclude that Theorem 4 is true.

As mentioned in §1, Theorem 4 has an analogue when $\alpha = n/2$. In this case $(x_i)_1^m$ is said to be a sequence of m Fekete points of E provided it minimizes

$$-\sum_{i \neq j} \log|y_i - y_j|$$

over all sequences $(y_i)_1^m$ consisting of m points in E. Suppose E is $(n/2, 2)$ locally uniformly fat and

$$\min_{i \neq j} |x_i - x_j| = |x_1 - x_2|.$$

As previously let

$$P(x) = -\frac{1}{(m-1)} \sum_{i=2}^{m} \log|x - x_i|, \qquad x \in \mathbf{R}^n.$$

Again it can be shown for $u = P(x_1) - P$ that

$$(7.12) \qquad M(r) \leq cP(x_1)r^{\sigma},$$

for some $\sigma > 0$. The argument is similar to the argument following (7.3), except that now the results in §4 are used instead of those in §§2 and 3. Also, one needs to use the fact that if $c > 0$ is large enough, then

$$\log\left[\frac{1}{(c|x|)}\right] \leq J_{n/2} * J_{n/2}(x) \leq \log\left[\frac{c}{|x|}\right], \qquad x \in B(0,2).$$

We leave the details to the reader. Using (7.12) the argument for $\alpha \geq 1$ can be repeated to get

$$\min_{i \neq j} |x_i - x_j| \geq cm^{-\sigma}.$$

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