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UNIFORMLY QUASIREGULAR SEMIGROUPS IN TWO DIMENSIONS

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Abstract. Let G be a semigroup of K-quasiregular or K-quasimeromorphic functions mapping a given open set U in the Riemann sphere into itself, for a fixed K, the semigroup operation being the composition of functions. We prove that if G satisfies an algebraic condition, which is true for all abelian semigroups, then there exists a K-quasiconformal homeomorphism of U onto an open set V such that all the functions in $f\circ G\circ f^{-1}$ are meromorphic functions of V into itself. In particular, if U is the whole sphere then the elements of $f\circ G\circ f^{-1}$ are rational functions. We give an example of a semigroup generated by two functions on the sphere, each quasiconformally conjugate to a quadratic polynomial, that cannot be quasiconformally conjugated to a semigroup of rational functions. We give another such example of a semigroup of K-quasiconformal homeomorphisms. These results extend and complement a similar positive conjugacy result of Tukia and of Sullivan for groups of K-quasiconformal homeomorphisms.

1. Introduction and results

Let U be a non-empty open set in the extended complex plane or Riemann sphere $\overline{\mathbf{C}} = \mathbf{C}\cup\{\infty\}$ where $\mathbf{C}$ denotes the complex plane. In two dimensions we may define a K-quasimeromorphic map of U (into $\overline{\mathbf{C}}$) by saying that f is K-quasimeromorphic if we can write $f=\varphi\circ h$ where h is a K-quasiconformal map on U (hence, by definition, a homeomorphism) and φ is a meromorphic function defined on $h(U)$. If f never takes the value ∞, we say that f is K-quasiregular. The composition of two quasiregular or quasimeromorphic maps is also quasiregular or quasimeromorphic. Hence we may talk about a semigroup of quasiregular or quasimeromorphic maps of U into itself, the semigroup operation being the composition of functions. A special case would be a group of quasiconformal maps of U onto itself. If all the elements of such a semigroup G are K-quasiregular or K-quasimeromorphic for a fixed $K\ge 1$, we say that G is a K-quasiregular or a K-quasimeromorphic semigroup. If this is true for some K, we say that G is a uniformly quasiregular semigroup or a uniformly quasimeromorphic semigroup. A special case would be a K-quasiconformal group or a uniformly quasiconformal group.

We shall assume throughout that each element of G is non-constant in each component of the open set U. If U is connected and if this restriction is not

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imposed, then the subset of G consisting of non-constant functions is a subsemigroup while any homeomorphism would conjugate the constant functions in G to constant functions. Thus, if U is connected, our results extend immediately to the case when G is allowed to contain constant functions.

Uniformly quasiconformal groups and uniformly quasiregular or quasimeromorphic semigroups can certainly arise as follows. Let U and V be open sets in $\overline{\mathbf{C}}$, let f be a K-quasiconformal map of U onto V, and let G' be a group of conformal self-mappings of V (for suitable V, this could be a group of Möbius transformations of V onto itself), or let G' be a semigroup of meromorphic functions of V into itself. Then $G = f^{-1} \circ G' \circ f$ is a K^2-quasiconformal group, or a K^2-quasiregular or a K^2-quasimeromorphic semigroup of functions taking U onto or into itself. In particular, in the case of a semigroup rather than a group, we could have $U = V = \overline{\mathbf{C}}$, and G' could be a semigroup of rational functions (containing at least one function of degree at least 2, say). Tukia [4] and Sullivan [3] proved that any K-quasiconformal group arises in this way, and Tukia [4] showed furthermore that if G is a K-quasiconformal group then we may take f to be K_1-quasiconformal where K_1 depends on K only and $K_1 \le K$. In this note, we shall generalize Tukia's method so as to extend his result to K-quasiregular and K-quasimeromorphic semigroups generated by a single function, and to certain other suitable semigroups. We provide a counterexample to show that in case of a semigroup generated by two functions, each of which is individually conjugate to a quadratic polynomial, there does not have to exist a quasiconformal conjugacy of the whole semigroup to a semigroup of meromorphic (in this case, rational) functions.

Theorem 1. *Let U be a non-empty open set in $\overline{\mathbf{C}}$, and let G be a semigroup of K-quasiregular or K-quasimeromorphic functions of U into itself, generated by a single function. Then there is a K_1-quasiconformal map f of $\overline{\mathbf{C}}$ onto itself, taking U onto an open set V such that every element in the semigroup $f \circ G \circ f^{-1}$ is a meromorphic function of V into itself. In particular, if $U = \overline{\mathbf{C}}$ then G is quasiconformally conjugate to a cyclic semigroup of rational functions. Here K_1 depends on K only and $K_1 \le K^2$.*

We may take $K_1 = \big(\sqrt{K^2 + 1/K^2} + K - 1/K\big)/\sqrt{2} \le \min\{K^{\sqrt{2}}, K\sqrt{2}\}$ (compare [4, p. 77]). This bound is obtained from Tukia's by replacing K by K^2 in his formula.

Theorem 2. *There exist quadratic polynomials F_1 and F_2 and quasiconformal mappings φ_1 and φ_2 of $\overline{\mathbf{C}}$ such that the functions $f_j = \varphi_j \circ F_j \circ \varphi_j^{-1}$, for $j = 1, 2$, generate a uniformly quasiregular semigroup G in $\overline{\mathbf{C}}$, but there is no quasiconformal homeomorphism f of $\overline{\mathbf{C}}$ such that $f \circ G \circ f^{-1}$ consists of meromorphic* (*hence, rational*) *functions.*

As the proof of Theorem 2 will show, the reason why Tukia's method of proof for groups fails for general semigroups, is, in a sense, a lack of relations between the elements of a semigroup. If G is a group and $g, h \in G$ then there is $u \in G$ satisfying some relation involving g and h, such as $g = h \circ u$ or $g = u \circ h$. It appears that free (uniformly quasimeromorphic) semigroups are particularly poor candidates for the existence of a quasiconformal conjugacy to a semigroup of meromorphic functions. However, for a semigroup admitting sufficiently many relations, a proof can be made to work with adequate care. There are many results of that kind, and we provide the following example. We denote the identity mapping of any set by Id. For any function f we denote the iterates of f by f^n so that $f^0 = \mathrm{Id}$ and $f^{n+1} = f \circ f^n$ for all $n \geq 0$.

Theorem 3. *Let U be a non-empty open set in $\overline{\mathbf{C}}$, and let G be a K-quasimeromorphic semigroup of functions mapping U into itself. Suppose that G has the following property*:

(1.1) *for all $g, h \in G$ there are $\varphi, \psi \in G \cup \{\mathrm{Id}\}$ such that $g \circ \varphi = h \circ \psi$.*

Then there exists a K_1-quasiconformal homeomorphism f of $\overline{\mathbf{C}}$ onto itself, taking U onto an open set V, such that the semigroup $f \circ G \circ f^{-1}$ consists of meromorphic functions taking V into itself. Here K_1 depends only on K.

In particular, this holds if G is abelian, for then we may take $\varphi = h$ and $\psi = g$ in (1.1).

In Theorem 3, we get the same estimate for K_1 as given after Theorem 1.

One can ask what happens if the condition $g \circ \varphi = h \circ \psi$ in (1.1) is replaced by $\varphi \circ g = \psi \circ h$. A simple modification of the proof of Theorem 3 does not yield the desired result, basically because of the following fact: if $f \in G$ and if F is any branch of f^{-1} then $f \circ F = \mathrm{Id}$ while we need not have $F \circ f = \mathrm{Id}$. It remains an open question whether the conclusion of Theorem 3 still holds for some other reason, after modifying (1.1) in this way.

Even though, in connection with semigroups, it may be of greater interest to consider non-homeomorphic mappings, it may be worth pointing out that an easy-to-verify counterexample can be found for semigroups of homeomorphisms.

Theorem 4. *Let h be a K-quasiconformal mapping of the strip $S = \{x+iy : 0 \leq x \leq 1\}$ onto itself that fixes every boundary point of S including each of the prime ends at $+i\infty$ and $-i\infty$. Suppose that h is not the identity mapping of S. Define $f_1(z) = h(z) + 1$ when $z \in S$ and $f_1(z) = z + 1$ when $z \in \overline{\mathbf{C}} \setminus S$. Define $f_2(z) = z + 1$ for all $z \in \overline{\mathbf{C}}$. Then f_1 and f_2 generate a K-quasiconformal semigroup G, but G cannot be quasiconformally or even topologically conjugated to a semigroup of conformal mappings.*

Roughly speaking, if the semigroup G of Theorem 4 could be conjugated to a semigroup of conformal mappings, then both f_1 and f_2 would be conjugated to the same mapping, which is impossible if $f_1 \neq f_2$, that is, if $h \neq \mathrm{Id}$. The actual proof of Theorem 4 is not phrased exactly in this way, but a consideration of the elements (necessarily translations, if the point at infinity is fixed as we may assume) of the hypothetical conjugated semigroup shows that this would be the case.

Obviously, there are many other similar examples. For example, the function $f_2(z) = z + 1$ in Theorem 4 could be replaced by λz for some $\lambda > 1$, and then S would be replaced by the annulus $\{z : 1 \le |z| \le \lambda\}$.

The positive results (Theorems 1 and 3) are based on the concept of a grand orbit. If $z \in U$, we say that a point $w \in U$ is in the grand orbit of z under the semigroup G provided that $w = g(z)$ for some g belonging to the formal group generated by G. The elements of this formal group are obtained by combining finitely many elements, each of which lies in G or corresponds to a branch of the inverse of some element of G. More precisely, we require that there are $x_1, x_2, \ldots, x_n \in U$, $z_1, \ldots, z_{n-1} \in U$, and $h_1, \ldots, h_{n-1}, h'_1, \ldots, h'_{n-1} \in G \cup \{\mathrm{Id}\}$ such that $n \ge 2$, $w = x_1$, $z = x_n$, and

$$h_i(z_i) = x_i \quad \text{and} \quad h'_i(z_i) = x_{i+1} \quad \text{for } 1 \le i \le n-1.$$

Theorem 2 shows that there are essentially more uniformly quasimeromorphic semigroups than there are semigroups of meromorphic functions. For semigroups of meromorphic functions, one can develop a Fatou–Julia theory analogously to the iteration theory due to Fatou and Julia based on the concept of a normal family (for semigroups of rational functions, see [1]). Thus, for a semigroup G defined on an open set U, the set of normality or the Fatou set of G consists of those points z in U that have a neighbourhood D such that the restrictions of the elements of G to D form a normal family. Standard results for sequences of K-quasimeromorphic mappings show that the limit function of any convergent subsequence is then a constant (possibly infinity) or a non-constant K-quasimeromorphic function. The Julia set would be $J(G) = U \setminus N(G)$. (In unpublished lecture notes of the author from 1989, for a course at the University of Texas at Austin, it is shown that for the iteration of a single meromorphic function f defined in a plane domain U without the condition that $f(U) \subset U$ (for $z \in N(G)$ one requires, in addition, that all the iterates of f are defined in the neighbourhood D), a Fatou–Julia theory can be developed, and that the classification of the components of the set of normality follows the same lines as in the classical case, the new cases that arise corresponding merely to the fact that the prime ends of U need not coincide with single topological boundary points of U.) One may ask, of course, under what circumstances a uniformly quasimeromorphic semigroup is topologically (even if not quasiconformally) conjugate to a semigroup of meromorphic functions. A topological conjugacy would still preserve the structure of the dynamics.

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2. Proof of Theorem 1

Of course, Theorem 1 is a special case of Theorem 3, but we prefer to give a separate proof to fix ideas that can be expanded later. Let the assumptions of

Theorem 1 be satisfied, except that G need not be cyclic until explicitly specified, and set $k=(K-1)/(K+1)\in[0,1)$. Write $\partial g=\partial g/\partial z$ and $\overline{\partial} g=\partial g/\partial\overline{z}$. For each $g\in G$, the complex dilatation $\mu(z,g)=\overline{\partial} g/\partial g$ is defined a.e., is measurable, and satisfies $|\mu(z,g)|\le k$ for a.e. $z\in U$. We look for an essentially bounded complex-valued function μ defined in $\overline{\mathbf{C}}$ with $\|\mu\|_\infty<1$ such that if f is a quasiconformal homeomorphism of $\overline{\mathbf{C}}$ onto itself with $\mu(z,f)=\mu(z)$ for a.e. z then for every $g\in G$, the function $f\circ g\circ f^{-1}$, defined on the open set $V=f(U)$, is meromorphic in V. This will be the case if, and only if, for each $g\in G$, we have $\mu(z)=\mu(z,f)=\mu(z,f\circ g)$ for a.e. $z\in U$. The standard formula for $\mu(z,f\circ g)$ [2, (5.6), p. 183] gives

$$\mu(z,f\circ g)=\frac{\mu(z,g)+\mu\big(g(z),f\big)e^{-2i\arg\partial g(z)}}{1+\mu(z,g)\,\mu\big(g(z),f\big)e^{-2i\arg\overline{\partial} g(z)}},\tag{2.1}$$

so that with $\mu(z,f)=\mu(z)$ we require that

$$\mu(z)=\frac{\mu(z,g)+\mu\big(g(z)\big)e^{-2i\arg\partial g(z)}}{1+\mu(z,g)\,\mu\big(g(z)\big)e^{-2i\arg\overline{\partial} g(z)}}\equiv T_{g,z}\big(\mu\big(g(z)\big)\big).\tag{2.2}$$

Here $T_{g,z}$ is the Möbius transformation of the unit disk $\mathbf{D}$ onto itself given by

$$T_{g,z}(w)=\frac{a+\overline{b}w}{b+\overline{a}w}$$

where $a=\overline{\partial} g(z)$ and $b=\partial g(z)$. In particular, if g is the identity map Id then $T_{g,z}(w)\equiv w$, that is, $T_{\mathrm{Id},z}=\mathrm{Id}$, for all z. By the definitions and by the chain rule $T_{h\circ g,z}=T_{g,z}\circ T_{h,g(z)}$ so that $T^{-1}_{h\circ g,z}=T^{-1}_{h,g(z)}\circ T^{-1}_{g,z}$.

The formula (2.1) states that

$$\mu(z,f\circ g)=T_{g,z}\big(\mu\big(g(z),f\big)\big).\tag{2.3}$$

We set $\mu\equiv 0$ outside U. To define μ in U so as to satisfy (2.2), we recall the ideas of Tukia [4]. Let us first ignore any problems concerning measurability and the fact that complex dilatations are defined and formulas such as (2.2) hold only almost everywhere.

Write $B(z,r)=\{w:|w-z|\le r\}$ and $B(r)=B(0,r)$. Define for $z\in U$,

$$S(z)=\{\mu(z,g):g\in G\}\subset B(k).$$

If $h\in G$ then by (2.3),

$$S\big(h(z)\big)=\big\{\mu\big(h(z),g\big):g\in G\big\}=\big\{T^{-1}_{h,z}\big(\mu(z,g\circ h)\big):g\in G\big\}\subset T^{-1}_{h,z}\big(S(z)\big)$$

and so

$$T_{h,z}\big(S\big(h(z)\big)\big) \subset S(z). \tag{2.4}$$

The above considerations are valid in any semigroup. Let us now assume that G is generated by a function γ. Thus $G = \{\gamma^n : n \geq 1\}$ or possibly $G = \{\gamma^n : n \geq 0\}$. We may assume that $\gamma \neq \mathrm{Id}$, for otherwise there is nothing to prove. It clearly suffices to conjugate γ to a meromorphic function.

If G were a group, then we would have the equality $T_{h,z}\big(S\big(h(z)\big)\big) = S(z)$ instead of (2.4). This would allow us to continue along the lines of Tukia and complete the proof. In general, equality need not hold here even if G is a cyclic semigroup. In this case, if G contains the identity mapping, we have $S(z) = T_{h,z}\big(S\big(h(z)\big)\big) \cup \{0\}$, and usually we need not have to have $0 \in T_{h,z}\big(S\big(h(z)\big)\big)$. If G is cyclic and does not contain the identity mapping then again usually $S(z) \setminus T_{h,z}\big(S\big(h(z)\big)\big)$ consists of one point (and in any case contains at most one point). Therefore the set $S(z)$ needs to be replaced by another set with the required invariance property. It is hard to see how this could be done except by replacing $S(z)$ by a larger set. This then means that the larger set might no longer be a subset of $B(k)$. We can only hope that the larger set will be contained in $B(k_1)$ for some fixed $k_1 < 1$. The general principle is that one needs to add to $S(z)$ the complex dilatations of all the functions ("words") in the "group" generated by the elements of the semigroup and by the locally defined branches of their inverses. To ensure that all such dilatations lie in some $B(k_1)$, one should know, for example, that only words of a fixed length in this group are needed to cover all words. For a general semigroup (for example, a free semigroup with at least two generators), it is not true for any finite N that only words of length at most N will suffice to ensure the required complete invariance of the enlarged $S(z)$. If there are relations in the group then some fixed N may suffice, and this is what will happen in the proof of Theorem 3. Even if all words are needed, it could happen by coincidence, in some special case, that the enlarged set still lies in some fixed $B(k_1)$. For more discussion and examples, see the remarks after the statement of Lemma 1 in Section 4, and the proof of Theorem 3.

In the case of a cyclic group, we add to $S(z)$ all complex dilatations that would be there if G were a group generated by γ. Thus we let $S_1(z)$ consist of the complex dilatations at z of all mappings g of the form γ^n where $n \geq 0$, or a branch of γ^{-n} for some $n \geq 1$ whenever such a branch is defined in a neighbourhood of z, or of the form $\gamma^{-m} \circ \gamma^n$ where $m, n \geq 0$. Clearly $S(z) \subset S_1(z)$. We proceed for a while without worrying about the question of for which points z, the set $S_1(z)$ is well defined, with all branches of all inverses being defined in some neighbourhood of z (depending on the branch, of course).

Pick $z \in U$ and $h \in G$. As g goes through all maps considered in the definition of $S_1\big(h(z)\big)$, the maps $g \circ h$ go through exactly all the maps considered in the definition of $S_1(z)$. As the γ^n and the branches of their inverses are locally quasiconformal, the formula (2.3) still applies, and we see that (2.4) holds with equality.

We note that each inverse branch of any γ^n is K-quasiconformal since γ^n is K-quasimeromorphic. Hence any function $\gamma^{-m} \circ \gamma^n$ is K^2-quasimeromorphic. Thus $S_1(z) \subset B(k_1)$ where $k_1 = (K^2 - 1)/(K^2 + 1) < 1$. So in the cyclic case, $S(z)$ can be enlarged without having to enlarge the hyperbolically bounded set $B(k)$ that is guaranteed to cover every ($S(z)$ and here also) $S_1(z)$ to more than this $B(k_1)$.

We now complete the proof of Theorem 1 following Tukia's method. For each $S_1(z)$, there is a unique point $P(z)$ such that among all hyperbolic disks (in the hyperbolic metric of the unit disk) containing $S_1(z)$, there is one centred at $P(z)$ with the smallest possible hyperbolic radius [4, p. 75]. Since Möbius transformations such as $T_{h,z}$ that map the unit disk onto itself, preserve hyperbolic distances and hyperbolic disks, it follows from (2.4) with equality that $P(z) = T_{h,z}\big(P\big(h(z)\big)\big)$. Thus defining $\mu(z) = P(z)$, we find a function μ satisfying (2.2). The estimate for K_1 follows directly from Tukia's paper according to our reference, since $S_1(z) \subset B(k_1)$.

We finally have to address the question of definability and measurability of μ. The set that formally contains G and, for each element of G, its at most countably many inverse branches, is countable. Each γ^n can be written as $\gamma^n = \varphi_n \circ h_n$, where h_n is a quasiconformal homeomorphism and φ_n is meromorphic in $h_n(U)$. Hence, for each n, there are only countably many points $z \in U$ at which some branch of the inverse of γ_n has a branch point (preventing us from defining a single-valued branch of the inverse in some neighbourhood of z). For a.e. $z \in U$ it is therefore true that the sets $S(z)$ and $S_1(z)$ are completely well defined, that is, all conceivable inverse branches can be considered and all functions involved have a well defined complex dilatation at z. It follows as in Tukia's paper that the function μ defined above is measurable. Thus we have the estimate on $|\mu(z)|$ for almost every z, giving rise to the cited upper bound for K_1. In view of the remarks made at the beginning of Section 2, this completes the proof of Theorem 1.

3. Proof of Theorem 3

We first dispose of some technicalities. We proceed to define a complex dilatation by following a variant of Tukia's method. This leads to a measurable function provided that we are dealing with countably many elements, including the elements of the semigroup and branches of their inverses. This situation arises when the semigroup itself is countable. Suppose that the case of a countable semigroup has been dealt with.

Let then G be any semigroup satisfying the assumptions of Theorem 3. As quasimeromorphic mappings belong locally to certain Sobolev spaces that are separable, it follows that G has a countable dense subset L. So for any $g \in G$ there is a sequence $h_n \in L$ such that $h_n \to g$ locally uniformly on U with respect to the spherical metric. Now L generates a countable semigroup $\langle L \rangle$. If $\langle L \rangle$ does not satisfy the assumptions of Theorem 3, then set $L_0 = \langle L \rangle$. Suppose that L_n is a countable subsemigroup of G, and let M_n consist of all elements of L_n and all the elements φ, ψ of G that would have to be added to L_n so as to satisfy (1.1) for any pair of functions $g, h \in L_n$. Define L_{n+1} to be the semigroup generated by M_n. Then L_{n+1} is a countable subsemigroup of G. Set $L_\infty = \bigcup_{n=0}^{\infty} L_n$. Then L_∞ is a countable subsemigroup of G that satisfies the assumptions of Theorem 3. Thus, by our assumption concerning the countable case, there is a quasiconformal homeomorphism f of $\overline{\mathbf{C}}$ mapping U onto an open set V such that $f \circ L_\infty \circ f^{-1}$ consists of meromorphic functions mapping V into itself. But then, if $g \in G$, $h_n \in L \subset L_\infty$, and $h_n \to g$ locally uniformly on U, the functions $f \circ h_n \circ f^{-1}$ are meromorphic and tend to $f \circ g \circ f^{-1}$ locally uniformly in V. Hence $f \circ g \circ f^{-1}$ is meromorphic in V, as required, and the proof of Theorem 3 will be complete after dealing with the countable case.

Suppose then that G is countable and satisfies the assumptions of Theorem 3. Consider finite words of the form $g_1 \circ g_2 \circ \cdots \circ g_n$ where each g_j is either an element of G, or the identity mapping (in case $\mathrm{Id} \notin G$), or is a branch of the inverse of some element of G. For almost every $z \in U$, it is true that for every such word, with the exceptions to be described below, there is a neighbourhood of z in which we may define the function described by this word, no matter how we choose the branches of the inverse functions at the appropriate points, and furthermore, each component function g_j is locally homeomorphic and quasiconformal at the appropriate points and has, at these points, a well defined complex dilatation. Thus the complex dilatation of each such word can be defined at z. We take $S(z)$ to be the set of all such complex dilatations at z.

There are exceptions that may occur when $h(U)$ is a proper subset of U for some $h \in G$. For example, if the word is $h^{-1} \circ g$ where $g, h \in G$ and if $g(z) \notin h(U)$, then $(h^{-1} \circ g)(z)$ is not defined. For each z, we simply ignore all such words, depending on z. Also, even if $g(z) \in h(U)$, for certain z there may be more branches of h^{-1} defined and locally homeomorphic at $g(z)$ than for other values of z, and for each z, we simply use all branches that are defined and locally homeomorphic at $g(z)$. The same principles apply to all words in an obvious way. This completes the definition of $S(z)$.

For a.e. $z \in U$ it is the case that $S(z)$ is well defined at every point on the grand orbit of z under G, where the grand orbit of z under G is the set of all points of the form $(g_1 \circ g_2 \circ \cdots \circ g_n)(z)$ for all words described above. Using (2.3), it is easily checked that for a.e. $z \in U$ it is true that $S(z) = T_{h,z}\bigl(S\bigl(h(z)\bigr)\bigr)$ for all $h \in G$. This construction can be performed for any semigroup G, and makes sense and has the above invariance property for a.e. $z \in U$ whenever G is countable.

We shall now prove that under the special conditions of Theorem 3, there is a number $k_1 < 1$ such that $S(z) \subset B(k_1)$ for a.e. z. Then we can define $P(z)$ and $\mu(z)$ as in the proof of Theorem 1, and the rest goes through in the same way as there. In particular, K_1 will depend only on K provided that k_1 depends only on K.

Suppose that (1.1) of Theorem 3 holds. The function represented by a word $g_1 \circ g_2 \circ \dots \circ g_n$ is not changed if the word is normalized in such a way that n is even with $n \geq 2$, $g_j \in G \cup \{\mathrm{Id}\}$ whenever j is even, and g_j is the branch of the inverse of an element of $G \cup \{\mathrm{Id}\}$ denoted by h_j if j is odd. So we assume that we consider a normalized word, and claim that the function it represents can be written in the form $u \circ v$ where $v \in G \cup \{\mathrm{Id}\}$ while u is the branch of the inverse of an element of $G \cup \{\mathrm{Id}\}$. We prove this claim by induction on the even integer $n \geq 2$. The claim is clearly true for $n = 2$ by definition of the normalization. Suppose that it is true for a certain $n \geq 2$. Consider a normalized word $g_1 \circ g_2 \circ \dots \circ g_{n+2}$ as above. By the induction assumption, in some neighbourhood of z, the function $g_3 \circ g_4 \circ \dots \circ g_{n+2}$ is equal to $u \circ v$ where $v \in G \cup \{\mathrm{Id}\}$ while u is the branch of the inverse of an element b of $G \cup \{\mathrm{Id}\}$. Also there is $h_1 \in G \cup \{\mathrm{Id}\}$ such that g_1 is a branch of h_1^{-1} in some neighbourhood of an appropriate point. By (1.1), there are $\varphi, \psi \in G \cup \{\mathrm{Id}\}$ with $\varphi \circ g_2 = \psi \circ b$. (Note that (1.1) can be trivially satisfied if $g = \mathrm{Id}$ or $h = \mathrm{Id}$ in (1.1).) It follows that $\varphi \circ g_2 \circ u = \psi \circ b \circ u = \psi$. Thus $\psi \circ v = \varphi \circ g_2 \circ u \circ v = \varphi \circ h_1 \circ g_1 \circ g_2 \circ u \circ v$. Now define $\Psi = \psi \circ v \in G \cup \{\mathrm{Id}\}$ and $\Phi = \varphi \circ h_1 \in G \cup \{\mathrm{Id}\}$. We see that

$$\Psi = \Phi \circ (g_1 \circ g_2 \circ u \circ v) = \Phi \circ (g_1 \circ g_2 \circ \dots \circ g_{n+2}).$$

This shows that for a suitably chosen branch of Φ^{-1} in some neighbourhood of an appropriate point, which branch we simply denote by Φ^{-1}, we have

$$g_1 \circ g_2 \circ \dots \circ g_{n+2} = \Phi^{-1} \circ \Psi,$$

as was to be proved.

Now, since Φ^{-1} and Ψ above are both locally K-quasiconformal, so that $\Phi^{-1} \circ \Psi$ is locally K^2-quasiconformal, it follows that the element of $S(z)$ generated by the word considered lies in $B(k_1)$, where $k_1 = (K^2-1)/(K^2+1)$. This together with what has been said before, completes the proof of Theorem 3.

4. Proof of Theorem 2

4.1. To keep the calculations manageable, we construct a simple counterexample. However, it is clear from the principles involved that there are many other similar examples. We first describe what the example is going to be. Choose positive numbers δ and ε smaller than, say, 10^{-2}. We also require that $6\sqrt{\delta} < \varepsilon$ and that

$$(1+\varepsilon)^{2^N} = 2 + \varepsilon + \tfrac{1}{2}\bigl(\sqrt{\delta} - \delta\bigr) \tag{4.1}$$

for some integer $N \geq 2$. We can take N as large as we like; such a choice would merely require us to choose sufficiently small numbers ε and δ. Set $\omega = \exp\bigl(\pi i/(2^N - 1)\bigr)$ and $a = 2 + \varepsilon$. Thus

$$\omega^{1+2+\dots+2^{N-1}} = \omega^{2^N - 1} = -1$$

and so the N^{th} iterate of the function $z \mapsto \omega z^2$ is $z \mapsto -z^{2^N}$. By choosing N to be large enough, we can make $|\arg \omega|$ as small as we like. Define $F_1(z) = z^2$ and $F_2(z) = a + \omega(z-a)^2$. Define a 2-quasiconformal mapping φ_1 of $\overline{\mathbf{C}}$ by taking $\varphi_1(z) = z|z|/\delta$ if $|z| < \delta$ and $\varphi_1(z) = z$ if $|z| \geq \delta$. Define a 2-quasiconformal mapping φ_2 of $\overline{\mathbf{C}}$ by taking $\varphi_2(z) = a + (z-a)|z-a|/\delta$ if $|z-a| < \delta$ and $\varphi_2(z) = z$ if $|z-a| \geq \delta$. Then $\varphi_1^{-1}(z) = z\sqrt{\delta/|z|}$ if $|z| < \delta$ and $\varphi_1^{-1}(z) = z$ if $|z| \geq \delta$. We have $\varphi_2(z) = a + \varphi_1(z-a)$ and $\varphi_2^{-1} = a + \varphi_1^{-1}(z-a)$. Finally, set $f_j = \varphi_j^{-1} \circ F_j \circ \varphi_j$ for $j = 1, 2$. Let G be the semigroup generated by f_1 and f_2. We shall prove that G has the required properties.

A calculation shows that $f_1(z) = z^2$ when $|z| \geq \sqrt{\delta}$, that $f_1(z) = z^2/\sqrt{\delta}$ when $|z| \leq \delta$, and that $f_1(z) = (z^2\sqrt{\delta})/|z|$ when $\delta < |z| < \sqrt{\delta}$. Similarly, we have $f_2(z) = a + \omega(z-a)^2$ when $|z-a| \geq \sqrt{\delta}$, $f_2(z) = a + \omega(z-a)^2/\sqrt{\delta}$ when $|z-a| \leq \delta$, and $f_2(z) = a + \bigl(\omega(z-a)^2\sqrt{\delta}\bigr)/|z-a|$ when $\delta < |z-a| < \sqrt{\delta}$. Thus $f_2(z) = a + \omega f_1(z-a)$ for all $z \in \mathbf{C}$. The complex dilatation $\mu_1(z)$ of f_1 vanishes when $|z| < \delta$ or $|z| > \sqrt{\delta}$, and is given by $\mu_1(z) = -\frac{1}{3}(z/\overline{z})$ when $\delta < |z| < \sqrt{\delta}$. For the complex dilatation $\mu_2(z)$ of f_2, we have $\mu_2(z) \equiv \mu_1(z-a)$. Hence each of f_1 and f_2 is 2-quasimeromorphic in $\overline{\mathbf{C}}$. We set $A_1 = \{z : \delta < |z| < \sqrt{\delta}\}$ and $A_2 = \{z : \delta < |z-a| < \sqrt{\delta}\}$. Thus, apart from a set of zero area, $\mu_m(z)$ is non-zero only if $z \in A_m$, for $m = 1, 2$.

To see that G is a 2-quasiregular semigroup (in $\mathbf{C}$, and 2-quasimeromorphic in $\overline{\mathbf{C}}$), consider an element $g = g_n \circ g_{n-1} \circ \cdots \circ g_1$ of G, where each g_j is equal to f_1 or f_2, in a small neighbourhood D of some point $z \in \mathbf{C}$. If the complex dilatation of g does not vanish identically in D then there is a smallest j such that $(g_j \circ \cdots \circ g_1)(D)$ intersects A_m and $g_{j+1} = f_m$ where $m = 1$ or $m = 2$ (this includes the case $j = 0$, in which case we interpret $g_j \circ \cdots \circ g_1 = \mathrm{Id}$). By symmetry, we may assume that $m = 1$. As we are looking at the fixed function g and are only interested in what happens at z, we may take D so small that $(g_j \circ \cdots \circ g_1)(D) \subset B\bigl(2\sqrt{\delta}\bigr)$. Suppose that $g_l = f_1$ for $j+1 \leq l \leq j+q$ for some $q \geq 1$ while either $j+q = n$ or $g_{j+q+1} = f_2$. The maximal dilatation of $g_j \circ \cdots \circ g_1$ restricted to D is 1, and the maximal dilatation of $g_{j+q} \circ \cdots \circ g_{j+1} = f_1^q$ does not exceed 2. (Note that if $w \in A_1$ then $|f_1^p(w)| < \delta$ for all $p \geq 1$.) If $j+q = n$, there is nothing more to prove. Suppose that $j+q < n$ and that $g_{j+q+1} = f_2$. Now $g_{j+q} \circ \cdots \circ g_{j+1}$ maps $B\bigl(2\sqrt{\delta}\bigr)$ into itself. If $w \in (g_{j+q} \circ \cdots \circ g_1)(D)$ then $|w-a| \geq 2 + \varepsilon - 2\sqrt{\delta} > 2 + 2\varepsilon/3$. Thus $|f_2(w) - a| > 4 + 8\varepsilon/3$ and so $|f_2(w)| > 2 + 5\varepsilon/3 > a + 2\sqrt{\delta}$. Now f_2 is conformal at w, and each of f_1 and f_2 is conformal at $f_2(w)$. Furthermore, applying either f_1 or f_2 at $f_2(w)$ increases the modulus so much that after this, no matter which elements of G are then applied, we are dealing with only locally conformal mappings. This shows that the maximal dilatation of any $g \in G$ cannot be more than 2 in any sufficiently small neighbourhood of z. Thus G is a 2-quasimeromorphic semigroup in $\overline{\mathbf{C}}$.

4.2. The proof that G cannot be conjugated by a quasiconformal mapping to a semigroup of rational functions is based on the following lemma.

Lemma 1. *Suppose that G is a uniformly quasimeromorphic semigroup acting on an open set U in $\overline{\mathbf{C}}$ such that for some K-quasiconformal homeomorphism f, the semigroup $f \circ G \circ f^{-1}$ consists of meromorphic functions. If $z \in U$ and if $g = g_1 \circ g_2 \circ \cdots \circ g_n$ is a function defined in a neighbourhood of z, where each g_j is an element of G or a branch of the inverse of some element of G defined in a neighbourhood of an appropriate point, then the maximal dilatation of g does not exceed K^2.*

Remark 1. Lemma 1 also shows that if G is a uniformly quasimeromorphic semigroup that can be quasiconformally conjugated to a semigroup of meromorphic functions, then the set $S(z)$ considered in the proof of Theorem 3 is uniformly bounded for all z for which it is well defined (that is, for almost every z, we have $S(z) \subset B(k_1)$ for some fixed $k_1 < 1$). Hence, if it is possible to prove that a quasiconformal conjugacy exists at all, then, in a sense, it must be possible to obtain such a conjugacy by the extension of Tukia's method given in the proof of Theorem 3. However, even if this were theoretically possible for a given G, it need not be the case that it is clear from the *algebraic structure* of G that the sets $S(z)$ are relatively compact in the unit disk $\mathbf{D}$. For example, if G' is a free semigroup of rational functions and if f is a quasiconformal self-map of $\overline{\mathbf{C}}$ then $G = f^{-1} \circ G' \circ f$ is a free uniformly quasimeromorphic semigroup which can be quasiconformally conjugated to a semigroup of rational functions. However, the fact that the sets $S(z)$ are relatively compact in $\mathbf{D}$ for this G, is by no means clear from the algebraic properties of G, and to see in some way that the $S(z)$ are bounded away from the unit circle (which they are), it would be necessary to have some different type of information concerning the complex dilatations of the elements of G.

Remark 2. Theorem 3 gives a sufficient condition on the algebraic structure of G that forces Tukia's method to work. From the proof of Theorem 3 we see that another sufficient condition can be given as follows: there exists a fixed positive integer N such that the function corresponding to any word $g_1 \circ \cdots \circ g_n$, for any $n \geq 1$, where each g_j is an element of $G \cup \{\mathrm{Id}\}$ or a branch of the inverse of such an element, can also be represented by a word of this type with $n \leq N$ (this obviously guarantees that we find a suitable $k_1 < 1$ depending only on K and N). In the proof of Theorem 3 we saw that this is true with $N = 2$ when the assumptions of Theorem 3 are satisfied. However, this assumption, when formulated as above for an arbitrary N with no other connection to the algebraic structure of G, even if it is more general than the assumption of Theorem 3, seems to be less natural than the condition (1.1) of Theorem 3.

4.3. Proof of Lemma 1. Let the set-up be as in the assumptions of Lemma 1. For each j, write $h_j = g_j$ if $g_j \in G$, and otherwise let g_j be a branch of the inverse of $h_j \in G$. There are meromorphic functions $p_j \in f \circ G \circ f^{-1}$ such that $p_j = f \circ h_j \circ f^{-1}$. Hence we may locally write $g_j = f^{-1} \circ p_j^{-1} \circ f$ when g_j is an inverse of h_j. Note that under these circumstances we are considering a branch of p_j^{-1}

which is meromorphic in the small neighbourhood where it is defined. If $g_j \in G$ then $g_j = h_j = f^{-1} \circ p_j \circ f$. We see that $g = g_1 \circ g_2 \circ \cdots \circ g_n = f^{-1} \circ p \circ f$ for some meromorphic function p defined in a small neighbourhood of $f(z)$. This shows that the maximal dilatation of g in a neighbourhood of z does not exceed K^2. Lemma 1 is proved.

4.4. We return to the proof of Theorem 2. To complete the proof, we only need to show that for any $k_1 < 1$, no matter how close to 1, there exists a word g generated by the elements of our particular semigroup G and their locally defined inverses, and a point $z \in \mathbf{C}$ such that the complex dilatation of g has modulus at least k_1 in some small neighbourhood of z. We choose $\eta_1 \in (0, 10^{-2})$, say, and take $z \in B(\eta_1, \eta_1^2)$. Then for all $z \in B(\eta_1, \eta_1^2)$, we have $|\arg z| < \eta_1/(1-\eta_1) < 2\eta_1$. We consider words $g = g_n \circ \cdots \circ g_1$ where each g_j is of the form

$$g_j = f_1 \circ f_2^N \circ f_1^{-q_2} \circ f_2 \circ f_1^N \circ f_2^{-q_1}$$

for some large positive integers q_1 and q_2 depending on and varying with j. The main idea is that each individual complex dilatation will be of the form $-\frac{1}{3}(z/\overline{z})$, or such a number multiplied by several factors of the form $z/\overline{z}$. Here all the numbers z occurring are almost real so that any factor $z/\overline{z}$ is very close to 1. When we calculate complex dilatations for composite functions we are therefore applying the formula (2.1) to numbers that are very close to $-\frac{1}{3}$. Unlimited composition leads to a sequence of numbers tending to the unit circle (if all the individual complex dilatations involved were equal to $-\frac{1}{3}$, the sequence would tend to -1). This shows that the complex dilatations will not remain in a fixed disk $B(k_1)$ for any $k_1 < 1$.

4.5. The function f_2 has the circle $\{z : |z-a| = 1\}$ as its Julia set $J(f_2)$, so that for some sufficiently large integer q, a suitable branch of f_2^{-q} maps any preassigned disk outside this circle, such as the disk $B(\eta_1, \eta_1^2)$, into a prescribed neighbourhood of the point $a - 1 = 1 + \varepsilon \in J(f_2)$. We choose $q = q_1$ so large that for all $z \in B(\eta_1, \eta_1^2)$, for such a branch of f_2^{-q}, we have $|\arg f_2^{-q}(z)| < \eta_2$ and $|f_2^{-q}(z) - (a-1)| < \eta_2$ for some $\eta_2 > 0$ to be determined later. Since by (4.1),

$$f_1^N(a-1) = 2 + \varepsilon + \tfrac{1}{2}\bigl(\sqrt{\delta} - \delta\bigr) = a + \tfrac{1}{2}\bigl(\sqrt{\delta} - \delta\bigr) \equiv b,$$

it follows that if $\eta_3 > 0$ is given, we may choose η_2 so that $\bigl|f_1^N\bigl(f_2^{-q}(z)\bigr) - b\bigr| < \eta_3$ and $\bigl|\arg\bigl(f_1^N\bigl(f_2^{-q}(z)\bigr) - a\bigr)\bigr| < \eta_3$ for all $z \in B(\eta_1, \eta_1^2)$. We certainly take $\eta_3 < \frac{1}{2}\bigl(\sqrt{\delta} - 3\delta\bigr)$, so that $\delta < \bigl|f_1^N\bigl(f_2^{-q}(z)\bigr) - a\bigr| < \sqrt{\delta}$, and possibly take η_3 to be even smaller. Then for all $z \in B(\eta_1, \eta_1^2)$, the point $f_1^N\bigl(f_2^{-q}(z)\bigr)$ lies in the region where the complex dilatation of f_2 has modulus $\frac{1}{3}$. Also

$$\Bigl|\arg\Bigl(f_2\Bigl(f_1^N\bigl(f_2^{-q}(z)\bigr)\Bigr) - a\Bigr) - \arg\omega\Bigr| < 2\eta_3$$

for all $z \in B(\eta_1, \eta_1^2)$. We next take a suitable branch of $f_1^{-q_2}$ to ensure that it maps the set $(f_2 \circ f_1^N \circ f_2^{-q})\big(B(\eta_1,\eta_1^2)\big)$ into a small neighbourhood of the point $1 = a - (1+\varepsilon)$, consequently with a very small spread in the argument. Then we apply f_2^N and map the resulting set into a small neighbourhood of $a - \big(2 + \varepsilon + \frac{1}{2}(\sqrt{\delta} - \delta)\big) = -\frac{1}{2}\big(\sqrt{\delta} - \delta\big)$. Now $\delta < \frac{1}{2}\big(\sqrt{\delta} - \delta\big) < \sqrt{\delta}$ since δ is small. Applying next f_1 we operate in a set where the complex dilatation of f_1 has modulus $\frac{1}{3}$. We end up in a set which is contained in a small neighbourhood of the point $(\frac{1}{2}\delta)\big(1-\sqrt{\delta}\,\big) \approx \frac{1}{2}\delta$, with a very small spread in the argument, and hence also in a small neighbourhood of the origin (this neighbourhood of $(\frac{1}{2}\delta)\big(1-\sqrt{\delta}\,\big)$ is, in a sense, comparable to the disk $B(\eta_1,\eta_1^2)$), and we are ready to apply a similar word g_j. We have now described the operation of one word g_j. We next calculate the derivatives with respect to z and $\overline{z}$ of such a word g_j, say g_1, at a point $z \in B(\eta_1,\eta_1^2)$. A similar result is obtained for a subsequent word g_j at any point z in a suitable neighbourhood of $(\frac{1}{2}\delta)\big(1-\sqrt{\delta}\,\big)$.

4.6. We introduce some notation. We write $w_0 = z \approx \eta_1 \approx 0$, $w_1 = f_2^{-q_1}(z) \approx 1+\varepsilon$, $w_2 = f_1^N(w_1) \approx a$, $w_3 = f_2(w_2) \approx a$, $w_4 = f_1^{-q_2}(w_3) \approx 1$, $w_5 = f_2^N(w_4) \approx 0$, and $w_6 = g_1(z) = g_1(w_0) = f_1(w_5) \approx 0$. We have $|\arg(w_3 - a) - \arg\omega| < 2\eta_3$, and $|\arg w_3 - \arg c| < 3\eta_3$, say, where

$$c = a + \omega\sqrt{\delta}\,\tfrac{1}{2}(\sqrt{\delta}-\delta) = a + \tfrac{1}{2}\omega\delta\big(1-\sqrt{\delta}\,\big). \tag{4.2}$$

For $0 \le j \le 6$ with $j \ne 3$, both w_j and $w_j - a$ have argument very close to either 0 or π, with π occurring exactly for $w_0 - a$, $w_1 - a$, $w_4 - a$, $w_5 - a$, $w_6 - a$, and w_5. Next, note that

$$\begin{aligned}
(f_2^{q_1})'(w_1) &= \omega^{2^{q_1}-1} 2^{q_1} (w_1 - a)^{2^{q_1}-1} = 2^{q_1}(w_0 - a)/(w_1 - a),\\
(f_1^N)'(w_1) &= 2^N (w_1)^{2^N-1} = 2^N w_2/w_1,\\
(f_1^{q_2})'(w_4) &= 2^{q_2} w_3/w_4, \quad \text{and}\\
(f_2^N)'(w_4) &= -2^N (w_4 - a)^{2^N-1} = 2^N (w_5 - a)/(w_4 - a).
\end{aligned}$$

We have $\mu_2(w_2) = -\frac{1}{3}(w_2 - a)/\overline{(w_2 - a)}$ and $\mu_1(w_5) = -\frac{1}{3}(w_5/\overline{w_5})$. For $0 \le j \le 6$, we set $b_j = w_j/\overline{w_j}$.

To make the absolute value of the argument of every one of the points w_1, w_2, w_4, w_5, $w_1 - a$, $w_2 - a$, $w_4 - a$, $w_5 - a$ small, we only need to choose q_1 and q_2 sufficiently large. This can be done for any initially given η_1, even though the precise choice of q_1 and q_2 will then depend on (an upper bound for) η_1. The magnitude of $|\arg z| = |\arg w_0|$ and of $|\arg(w_0 - a)|$ is controlled by the choice of η_1. If we perform the same estimate for a function g_j other than g_1 then the role of w_0 is taken by a point of the form w_6 arising from a previous calculation, and we can get $|\arg w_6|$ and $|\arg(w_6 - a)|$ to be as small as we like by taking the previous q_1 and q_2 sufficiently large. To handle this, we may choose for q_1 and

q_2 values that increase to infinity with j. Thus we may consider all of the above arguments to be $o(1)$ as $j \to \infty$, and the rate of convergence to zero implied by the $o(1)$-notation can be as fast as we like. Similarly, $\arg w_3 = \arg c + o(1)$ so that with $c_3 = c/\overline{c}$, we have $b_3 = c_3\bigl(1 + o(1)\bigr)$.

4.7. Define $f = f_1 \circ f_2^N \circ f_1^{-q_2}$ and $g = f_2 \circ f_1^N \circ f_2^{-q_1}$. Then $f \circ g = f_1 \circ f_2^N \circ f_1^{-q_2} \circ f_2 \circ f_1^N \circ f_2^{-q_1}$ is the function we are looking at, one of the g_j. We have

$$\partial f_2(w_2) = \tfrac{3}{2}\omega\sqrt{\delta}\,\frac{w_2 - a}{|w_2 - a|}, \qquad \overline{\partial} f_2(w_2) = \tfrac{-1}{2}\omega\sqrt{\delta}\,\frac{(w_2 - a)^3}{|w_2 - a|^3}.$$

Thus

$$\partial g(z) = (\partial f_2)(w_2)\frac{(f_1^N)'(w_1)}{(f_2^{q_1})'(w_1)} = \tfrac{3}{2}\omega\sqrt{\delta}\,\frac{w_2 - a}{|w_2 - a|}2^{N-q_1}\frac{w_2}{w_1}\frac{w_1 - a}{w_0 - a}$$

and

$$\overline{\partial} g(z) = (\overline{\partial} f_2)(w_2)\overline{\frac{(f_1^N)'(w_1)}{(f_2^{q_1})'(w_1)}} = \tfrac{-1}{2}\omega\sqrt{\delta}\,\frac{(w_2 - a)^3}{|w_2 - a|^3}2^{N-q_1}\frac{\overline{w_2}}{\overline{w_1}}\overline{\frac{w_1 - a}{w_0 - a}}.$$

Further, we have

$$\begin{aligned}
\partial f\bigl(g(z)\bigr) &= (\partial f_1)(w_5)\frac{(f_2^N)'(w_4)}{(f_1^{q_2})'(w_4)} = \tfrac{3}{2}\sqrt{\delta}\,\frac{w_5}{|w_5|}2^{N-q_2}\frac{w_5 - a}{w_4 - a}\frac{w_4}{w_3},\\
\overline{\partial} f\bigl(g(z)\bigr) &= (\overline{\partial} f_1)(w_5)\overline{\frac{(f_2^N)'(w_4)}{(f_1^{q_2})'(w_4)}} = \tfrac{-1}{2}\sqrt{\delta}\,\frac{w_5^3}{|w_5|^3}2^{N-q_2}\overline{\frac{w_5 - a}{w_4 - a}}\,\overline{\frac{w_4}{w_3}}.
\end{aligned}$$

Hence

$$\begin{aligned}
\partial(f_1 \circ f_2^N \circ f_1^{-q_2} \circ f_2 \circ f_1^N \circ f_2^{-q_1})(z) &= \partial(f \circ g)(z)\\
&= \partial f\bigl(g(z)\bigr)\partial g(z) + \overline{\partial} f\bigl(g(z)\bigr)\partial\overline{g}(z)\\
&= \tfrac{3}{2}\sqrt{\delta}\,\frac{w_5}{|w_5|}2^{N-q_2}\frac{w_5 - a}{w_4 - a}\frac{w_4}{w_3}\tfrac{3}{2}\omega\sqrt{\delta}\,\frac{w_2 - a}{|w_2 - a|}2^{N-q_1}\frac{w_2}{w_1}\frac{w_1 - a}{w_0 - a}\\
&\quad + \tfrac{-1}{2}\sqrt{\delta}\,\frac{w_5^3}{|w_5|^3}2^{N-q_2}\overline{\frac{w_5 - a}{w_4 - a}}\,\overline{\frac{w_4}{w_3}}\tfrac{-1}{2}\omega^{-1}\sqrt{\delta}\,\frac{\overline{(w_2 - a)^3}}{|w_2 - a|^3}2^{N-q_1}\frac{w_2}{w_1}\frac{w_1 - a}{w_0 - a}\\
&= \tfrac{9}{4}\omega\delta 2^{2N-q_1-q_2}\frac{w_5}{|w_5|}\frac{w_2}{w_1}\frac{w_1 - a}{w_0 - a}\frac{w_5 - a}{w_4 - a}\frac{w_4}{w_3}\frac{w_2 - a}{|w_2 - a|}\\
&\quad \times \left(1 + \tfrac{1}{9}\omega^{-2}\frac{w_5^2}{|w_5|^2}\frac{\overline{w_5 - a}}{w_5 - a}\frac{w_4 - a}{\overline{w_4 - a}}\frac{b_3}{b_4}\frac{\overline{(w_2 - a)^4}}{|w_2 - a|^4}\right)\\
&= \tfrac{9}{4}\omega\frac{|c|}{c}\delta 2^{2N-q_1-q_2}\frac{|w_2|}{|w_1|}\frac{|w_1 - a|}{|w_0 - a|}\frac{|w_5 - a|}{|w_4 - a|}\frac{|w_4|}{|w_3|}\bigl(1 + \tfrac{1}{9}\omega^{-2}c_3\bigr)\bigl(1 + o(1)\bigr)
\end{aligned}$$

since $\partial\overline{g}(z) = \overline{\overline{\partial} g(z)}$. Also

$$
\begin{aligned}
&\overline{\partial}(f_1 \circ f_2^N \circ f_1^{-q_2} \circ f_2 \circ f_1^N \circ f_2^{-q_1})(z) = \overline{\partial}(f \circ g)(z) \\
&\quad = \partial f\big(g(z)\big)\overline{\partial} g(z) + \overline{\partial} f\big(g(z)\big)\overline{\partial}\overline{g}(z) \\
&\quad = \tfrac{3}{2}\sqrt{\delta}\,\frac{w_5}{|w_5|} 2^{N-q_2} \frac{w_5-a}{w_4-a}\frac{w_4}{w_3}\tfrac{-1}{2}\omega\sqrt{\delta}\,\frac{(w_2-a)^3}{|w_2-a|^3} 2^{N-q_1}\frac{\overline{w_2}}{\overline{w_1}}\frac{\overline{w_1-a}}{\overline{w_0-a}} \\
&\qquad + \tfrac{-1}{2}\sqrt{\delta}\,\frac{w_5^3}{|w_5|^3} 2^{N-q_2}\frac{\overline{w_5-a}}{\overline{w_4-a}}\frac{\overline{w_4}}{\overline{w_3}}\tfrac{3}{2}\omega^{-1}\sqrt{\delta}\,\frac{\overline{w_2-a}}{|w_2-a|} 2^{N-q_1}\frac{\overline{w_2}}{\overline{w_1}}\frac{\overline{w_1-a}}{\overline{w_0-a}} \\
&\quad = \tfrac{-3}{4}\omega\delta 2^{2N-q_1-q_2}\frac{w_5}{|w_5|}\frac{\overline{w_2}}{\overline{w_1}}\frac{\overline{w_1-a}}{\overline{w_0-a}}\frac{w_5-a}{w_4-a}\frac{w_4}{w_3}\frac{(w_2-a)^3}{|w_2-a|^3} \\
&\qquad \times \bigg(1 + \omega^{-2}\frac{w_5^2}{|w_5|^2}\frac{\overline{w_5-a}}{w_5-a}\frac{w_4-a}{\overline{w_4-a}}\frac{b_3}{b_4}\frac{\overline{(w_2-a)^4}}{|w_2-a|^4}\bigg) \\
&\quad = \tfrac{-3}{4}(1+\omega^{-2}b_3)\omega\frac{|w_3|}{w_3}\delta 2^{2N-q_1-q_2}\frac{|w_2|}{|w_1|}\frac{|w_1-a|}{|w_0-a|}\frac{|w_5-a|}{|w_4-a|}\frac{|w_4|}{|w_3|}\big(1+o(1)\big).
\end{aligned}
$$

Note that $(1+\omega^{-2}b_3)\omega|w_3|/w_3 = 2\,\mathrm{Re}(\omega|w_3|/w_3)$ is real and positive.

We conclude that

$$
\arg\partial(f_1 \circ f_2^N \circ f_1^{-q_2} \circ f_2 \circ f_1^N \circ f_2^{-q_1})(z) = \arg\Big(\frac{\omega}{c} + \frac{\overline{\omega}}{9\overline{c}}\Big) + o(1)
$$

and

$$
\arg\overline{\partial}(f_1 \circ f_2^N \circ f_1^{-q_2} \circ f_2 \circ f_1^N \circ f_2^{-q_1})(z) = \pi + o(1).
$$

Further, we have

$$
\begin{aligned}
&\partial(f_1 \circ f_2^N \circ f_1^{-q_2} \circ f_2 \circ f_1^N \circ f_2^{-q_1})(z) \\
&\qquad = \tfrac{9}{4}\omega\delta 2^{2N-q_1-q_2}\frac{b^2}{(1+\varepsilon)^2 c\kappa}\big(1+\tfrac{1}{9}\omega^{-2}c_3\big)\big(1+o(1)\big) \\
&\qquad = AC_1\big(1+o(1)\big),
\end{aligned}
$$

say, where $A>0$ and $|C_1|=1$ with $\mathrm{Re}\,C_1>0$, since $a-1=1+\varepsilon$, and since in subsequent applications the role of $|w_0-a|$ is taken by $|w_6-a|$ which stabilizes to

$$
a - \tfrac{1}{2}\delta(1-\sqrt{\delta}) \equiv \kappa.
$$

Also

$$
\begin{aligned}
&\overline{\partial}(f_1 \circ f_2^N \circ f_1^{-q_2} \circ f_2 \circ f_1^N \circ f_2^{-q_1})(z) \\
&\qquad = \tfrac{-3}{4}(1+\omega^{-2}c_3)\omega\delta 2^{2N-q_1-q_2}\frac{b^2}{(1+\varepsilon)^2\kappa c}\big(1+o(1)\big) \\
&\qquad = BD_1\big(1+o(1)\big),
\end{aligned}
$$

say, where $B > 0$ and $|D_1| = 1$, in fact, $D_1 = -1$. We have now obtained estimates applicable to ∂g_j and $\overline{\partial} g_j$. The terms denoted by $o(1)$ in these estimates can be made arbitrarily small as $j \to \infty$. Hence we may assume in subsequent calculations that these terms are made so small as to make the resulting estimates for $g_j \circ \cdots \circ g_1$ sufficiently accurate for our purposes.

4.8. We use again the formulas $\partial(f \circ g)(z) = \partial f\big(g(z)\big)\partial g(z) + \overline{\partial} f\big(g(z)\big)\partial \overline{g}(z)$ and $\overline{\partial}(f \circ g)(z) = \partial f\big(g(z)\big)\overline{\partial} g(z) + \overline{\partial} f\big(g(z)\big)\overline{\partial}\,\overline{g}(z)$, replacing f and g by g_j and $g_{j-1} \circ \cdots \circ g_1$. We write $\partial(g_{j-1} \circ \cdots \circ g_1)(z) = A_{j-1}C_{j-1}\big(1+o(1)\big)$ so that $A_1 = A$, and $\overline{\partial}(g_{j-1} \circ \cdots \circ g_1)(z) = B_{j-1}D_{j-1}\big(1+o(1)\big)$ so that $B_1 = B$. We obtain

$$A_jC_j\big(1+o(1)\big) = \partial(g_j \circ \cdots \circ g_1)(z) = (AC_1A_{j-1}C_{j-1} + BD_1B_{j-1}\overline{D_{j-1}}\,)\big(1+o(1)\big)$$

and

$$B_jD_j\big(1+o(1)\big) = \overline{\partial}(g_j \circ \cdots \circ g_1)(z) = (B_{j-1}D_{j-1}AC_1 + BD_1A_{j-1}\overline{C_{j-1}}\,)\big(1+o(1)\big).$$

Thus we define A_j, B_j, C_j, and D_j by the formulas

$$\begin{aligned} A_jC_j &= AC_1A_{j-1}C_{j-1} + BD_1B_{j-1}\overline{D_{j-1}}, \\ B_jD_j &= B_{j-1}D_{j-1}AC_1 + BD_1A_{j-1}\overline{C_{j-1}} \end{aligned}$$

and the requirement that $|C_j| = |D_j| = 1$ while $A_j \ge 0$ and $B_j \ge 0$. Next define $A_j' = A_j/A^j$ and $B_j' = B_j/A^j$, and set $\rho_j = B_j/A_j = B_j'/A_j'$. We have $\rho_1 = B/A$, which can be made as close to $\frac{3}{5}$ as we like. Further, set $\alpha_j = 1 - \rho_j^2$ and $\beta_j = (2 - \alpha_j)/\alpha_j$. We have

$$\begin{aligned} |\mu(z, g_j \circ \cdots \circ g_1)| &= \left|\frac{\overline{\partial}(g_j \circ \cdots \circ g_1)(z)}{\partial(g_j \circ \cdots \circ g_1)(z)}\right| = \frac{B_j}{A_j}\big(1+o(1)\big) \\ &= \frac{B_j'}{A_j'}\big(1+o(1)\big) = \rho_j\big(1+o(1)\big) < 1 \end{aligned} \tag{4.3}$$

so that we may assume that $\rho_j < 1$ for all $j \ge 1$. Thus $0 < \alpha_j \le 1$ and $\beta_j \ge 1$ for all $j \ge 1$.

We set $\theta_j = C_1C_jD_j$ so that $|\theta_j| = 1$, and write $\lambda_j = -\operatorname{Re}\theta_j \in [-1, 1]$. A calculation now shows that (since $D_1 = -1$)

$$A_j'C_j = A_{j-1}'C_1C_{j-1} - \rho_1B_{j-1}'\overline{D_{j-1}}, \tag{4.4}$$

$$B_j'D_j = B_{j-1}'C_1D_{j-1} - \rho_1A_{j-1}'\overline{C_{j-1}}, \tag{4.5}$$

so that

$$\begin{aligned} {A_j'}^2 &= |A_j'C_j|^2 = (A_{j-1}')^2 + \rho_1^2(B_{j-1}')^2 + 2\rho_1A_{j-1}'B_{j-1}'\lambda_{j-1}, \\ {B_j'}^2 &= (B_{j-1}')^2 + \rho_1^2(A_{j-1}')^2 + 2\rho_1A_{j-1}'B_{j-1}'\lambda_{j-1}. \end{aligned}$$

This gives, by induction,

$$\begin{aligned}{A'_j}^2\alpha_j &= {A'_j}^2(1-\rho_j^2) = {A'_j}^2 - {B'_j}^2 = (1-\rho_1^2)\big((A'_{j-1})^2-(B'_{j-1})^2\big)\\ &= (1-\rho_1^2)^{j-1}({A'_1}^2-{B'_1}^2) = (1-\rho_1^2)^{j-1}(1-\rho_1^2) = (1-\rho_1^2)^j = \alpha_1^j\end{aligned}$$

so that $A'_j > 0$, and

$$ {A'_j}^2 + {B'_j}^2 = (1+\rho_1^2)\big((A'_{j-1})^2+(B'_{j-1})^2\big) + 4\rho_1 A'_{j-1}B'_{j-1}\lambda_{j-1}. \tag{4.6}$$

Noting that $1+\rho_j^2 = 2-\alpha_j$ and applying ${A'_j}^2\alpha_j = \alpha_1^j$ in (4.6) we obtain, after dividing through by α_1^{j-1} and recalling the definition of β_j, that

$$\beta_j = \beta_1\beta_{j-1} + 4\rho_1\rho_{j-1}\lambda_{j-1}/(\alpha_1\alpha_{j-1}). \tag{4.7}$$

We shall show that $\lambda_j \ge 0$ for all $j\ge 1$. Then (4.7) yields $\beta_j \ge \beta_1\beta_{j-1}$, so that by induction, $\beta_j \ge \beta_1^j$. Since $\beta_1 \approx 17/8 > 1$, we then have $\beta_j\to\infty$ as $j\to\infty$. We deduce that as $j\to\infty$, we also have $\alpha_j\to 0$ and thus $\rho_j\to 1$, and hence by (4.3), $|\mu(z,g_j\circ\cdots\circ g_1)|\to 1$ for $z\in B(\eta_1,\eta_1^2)$, as required. This then completes the proof of Theorem 2.

To prove that $\lambda_j\ge 0$ for all $j\ge 1$ by induction on j, first note that $\lambda_1 = \operatorname{Re}(C_1^2) \ge 1 - E/N$ for some positive absolute constant E, since

$$C_1 = \omega\frac{|c|}{c}\,\frac{1+\{c/(9\omega^2\overline{c})\}}{|1+\{c/(9\omega^2\overline{c})\}|}$$

where c is as in (4.2). Thus $\lambda_1>0$ if N is large enough. We also have $|\operatorname{Im}(C_1^2)| \le E/N$ if E is suitably chosen.

Suppose that $j\ge 2$ and that $\lambda_{j-1}\ge 0$. Multiplying the product of (4.4) and (4.5) by C_1 we obtain

$$\begin{aligned}A'_jB'_j(A'_{j-1})^{-2}\theta_j &= C_1^2\{\rho_{j-1}\big(\theta_{j-1}+\rho_1^2\overline{\theta_{j-1}}\,\big) - \rho_1(1+\rho_{j-1}^2)\}\\ &= C_1^2\{-\rho_{j-1}(1+\rho_1^2)\lambda_{j-1} - \rho_1(1+\rho_{j-1}^2) + i\rho_{j-1}(1-\rho_1^2)\operatorname{Im}\theta_{j-1}\}\end{aligned}$$

so that

$$\begin{aligned}A'_jB'_j(A'_{j-1})^{-2}\lambda_j &= \big(\operatorname{Re}(C_1^2)\big)\{\rho_{j-1}(1+\rho_1^2)\lambda_{j-1}+\rho_1(1+\rho_{j-1}^2)\}\\ &\qquad + \big(\operatorname{Im}(C_1^2)\big)(\operatorname{Im}\theta_{j-1})\rho_{j-1}(1-\rho_1^2)\\ &\ge (1-E/N)\rho_1(1+\rho_{j-1}^2) - (E/N)\rho_{j-1}(1-\rho_1^2).\end{aligned}\tag{4.8}$$

Here we have used the assumption that $\lambda_{j-1}\ge 0$ and the fact that $\operatorname{Re}(C_1^2)>0$ while $\operatorname{Im}\theta_{j-1}\in[-1,1]$.

The rightmost expression in (4.8) is positive provided that

$$\frac{1+\rho_{j-1}^2}{\rho_{j-1}} > \frac{E}{N-E}\,\frac{1-\rho_1^2}{\rho_1}. \tag{4.9}$$

Since $(1+\rho_{j-1}^2)/\rho_{j-1}\ge 2$ and $\rho_1\approx\frac{3}{5}$, it follows that (4.9) holds provided that N is large enough. Thus $A'_jB'_j(A'_{j-1})^{-2}\lambda_j>0$. This also shows that $B'_j\ne 0$ (we have already seen that $A'_j\ne 0$). Since $A'_j>0$ and $B'_j>0$, we now deduce from (4.8) that $\lambda_j>0$. This completes the induction proof that $\lambda_j\ge 0$, and hence the proof of Theorem 2 is also complete.

5. Proof of Theorem 4

Let the assumptions of Theorem 4 be satisfied. Every element of G is a composition of finitely many functions, each being f_1 or f_2. When applying such a function at some $z \in \mathbf{C}$, we apply the translation $z+1$ every time, except that at most once, we apply h. Thus every element of G is K-quasiconformal. Let G' be the *group* generated by f_1 and f_2. By Lemma 1, G is quasiconformally conjugate to a semigroup of conformal mappings if, and only if, G' is quasiconformally conjugate to a group of conformal mappings. Suppose that there is a quasiconformal mapping f of $\overline{\mathbf{C}}$ such that $f \circ G' \circ f^{-1}$ is a Möbius group. (Of course, even without reference to Lemma 1, one can observe that the group generated by a semigroup of Möbius transformations, consists of Möbius transformations only.) Let φ be a conformal mapping of $f(S)$ onto S. Since h is the restriction of $f_2^{-1} \circ f_1$ to S, it follows that $f \circ h \circ f^{-1}$ is a conformal mapping of $f(S)$ onto itself. Thus with $F = \varphi \circ f$, the map $F \circ h \circ F^{-1}$ is a conformal mapping of S onto itself. But $F(\partial S) = \partial S$ and $h|\partial S = \mathrm{Id}$. Thus $F \circ h \circ F^{-1} = \mathrm{Id}$ and so $h = \mathrm{Id}$, which is a contradiction. Hence, when $h \neq \mathrm{Id}$, the semigroup G cannot be quasiconformally conjugate to a semigroup of conformal mappings.

The same argument works if f is just a homeomorphism of $\overline{\mathbf{C}}$ onto itself, including an obvious extension of Lemma 1 to this particular situation, where the fact that f is merely a homeomorphism, not necessarily quasiconformal, is compensated for by the greatly simplifying fact that all the elements of the semigroup $f \circ G \circ f^{-1}$ are Möbius transformations in $\overline{\mathbf{C}}$. This proves Theorem 4.

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